Notes on sphere-based universal extra dimensions

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Abstract

We review the six dimensional universal extra dimension models compactified on the sphere S^2, the orbifold S^2/Z_2, and the projective sphere, which are based on the spontaneous compactification mechanism on the sphere. In particular, we spell out the application of the Newman-Penrose eth-formalism on these models with some technical details on the derivation of the Kaluza-Klein modes and their interactions, and revisit the problem in the existence of the zero mode of $U(1)_X$ additional gauge boson required for the spontaneous compactification. We also explain the theoretical background on the vacuum stability argument for the upper bound on the ultraviolet cutoff scale.

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1 Introduction

The universal extra dimension (UED) scenario is an interesting possibility, where the Kaluza-Klein (KK) scale of the compactified extra dimension(s) can be as small as TeV, without contradicting the electroweak precision test thanks to the fact that all the fields are propagating in the bulk of the extra dimensional space [1, 2]. In the model, lightest KK particle is stable and provides a good candidate for the dark matter [3, 4, 5]; see the recent review for general topics [6].[1]

The six dimensional (6D) UED models are of particular interest since the three number of the matter generation is required in order to cancel the global $SU(2)$ anomaly cancellation [30]. Proposed models are on two-torus, T^2/Z_2 [1], T^2/Z_4 (chiral square) [31, 32], $T^2/(Z_2 \times Z_2')$ [33], on two-sphere S^2/Z_2 [34], on S^2 with a Stückhelbarg field [35, 36], and on the nonorientable spaces, the real projective plane RP^2 [37] and the projective sphere (PS) [38].

Among them, the models on the sphere-based space [34, 38, 35, 36] share the feature that the compactification radius is spontaneously stabilized by the monopole configuration of the extra $U(1)_X$ gauge field [39]. The monopole background naturally leads to a four dimensional (4D) chiral fermion as a KK zero mode of the 6D fermion, thanks to the non-vanishing spin connection under the curved spacetime background [39].

However, this $U(1)_X$ gauge field yields a KK zero mode which necessarily couple to the Standard Model (SM) fermions in order to let them have the chiral zero modes [38]. In this article, we critically reconsider how this problem is treated in the sphere-based models, also providing some technical details which have not been spelled out in the literature.

The UED models are formulated as a gauge theory in the higher dimensions, and hence they are necessarily effective field theories, being cut off at a high scale Λ. If Λ is too close to the KK scale, then the meaning of the higher dimensional theory is lost. Also there can be dangerous contribution of the higher dimensional operators to the S and T parameters.[2] Therefore it is important how large Λ can be. The most stringent upper bound on Λ is obtained from the vacuum stability of the Higgs potential [46]; see also Refs. [49, 50, 51, 52, 53, 54, 55] for the stability analysis on the five dimensional model and/or the T^2/Z_4 model.[3] We give more detailed explanation on the theoretical background of the vacuum stability argument in Ref. [46].

This article is organized as follows. In Section 2, we provide a basic tools for the formulation of the sphere-based UED models. In Section 3, we review the KK expansions under the $U(1)_X$ monopole background with some technical details which have not been spelled out so far. In Section 4, we critically review the known UED models compactified

[1] Other possibilities of generalization of these models by an introduction of the bulk mass term and/or the brane-localized Lagrangians have been studied in Refs. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

[2] The constraint on UED models via the LHC Higgs search has also been discussed in Refs. [40, 41, 42, 43, 44, 45, 46, 47, 48].

[3] The renormalization group evolutions of parameters in UED models have been studied in Refs. [56, 57, 58, 59, 60, 54, 61, 62, 55].

on the sphere-based space, S^2, S^2/Z_2, and PS. We show that they need some modification in order to remove the $U(1)_X$ zero mode, except for the PS model. In Section 5, we explain the theoretical background on the previous vacuum stability analysis. In the last section, we summarize this article.

2 Basic formulation

In this section, we present general framework for the KK expansions on sphere, under the $U(1)_X$ monopole configuration that is necessary for the spontaneous compactification.

2.1 Spontaneous compactification

Let us first review the spontaneous compactification mechanism on two-sphere S^2 [39]. The starting metric ansatz is:

$$\mathrm{d}s^2 = g_{MN}\mathrm{d}z^M\mathrm{d}z^N = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{1}$$

where $(\eta_{\mu\nu})_{\mu,\nu=0,1,2,3} = \mathrm{diag}\,(-1,1,1,1)$. Throughout this paper, lower case greek indices $\mu,\nu,\dots$ run for $0,1,2,3$, while upper case roman ones $M,N,\dots$ for $0,1,2,3,\theta,\phi$.

By this metric, the vacuum Einstein equation cannot be satisfied except for a trivial solution with $R\to\infty$. In order to stabilize the radius R, Randjbar-Daemi, Salam and Strathdee have introduced a classical monopole configuration of a $U(1)_X$ gauge field $\mathcal{X} = \mathcal{X}_M\mathrm{d}z^M$ in the S^2 extra dimensions [63, 39]:

$$\mathcal{X}^{N}_{S} = \frac{n}{2g_X}\left(\cos\theta \mp 1\right)\mathrm{d}\phi, \tag{2}$$

where the integer n is the monopole number and g_X is the $U(1)_X$ coupling constant. Throughout this paper, superscripts N and S stand for north and south charts containing $\theta = 0$ and π, respectively, and correlate with $\pm$ signs when indicated. The transition function for the $U(1)_X$ gauge field is

$$\mathcal{X}^S_\phi = U_t\left(\mathcal{X}^N_\phi - \frac{i}{g_X}\partial_\phi\right)U_t^\dagger, \tag{3}$$

with $U_t = e^{-in\phi}$.

The field strength $\mathcal{X}_{MN} := \partial_M\mathcal{X}_N - \partial_N\mathcal{X}_M$ reads

$$\mathcal{X}_{\theta\phi} = -\mathcal{X}_{\phi\theta} = -\frac{n}{2g_X}\sin\theta, \qquad \text{others} = 0. \tag{4}$$

The energy momentum tensor $T_{MN} = -\frac{1}{4}g_{MN}F_{KL}F^{KL} + F_{MK}F_N{}^K$ becomes

$$T_{\mu\nu} = -\frac{n^2}{8g_X^2R^4}\eta_{\mu\nu}, \qquad T_{\theta\theta} = \frac{n^2}{8g_X^2R^2}, \qquad T_{\phi\phi} = \frac{n^2}{8g_X^2R^2}\sin^2\theta, \qquad \text{others} = 0. \tag{5}$$

Under this monopole configuration, the Einstein equation can be satisfied by tuning the cosmological constant, and the radius is fixed to be

$$R = \frac{\sqrt{8\pi G_6}\,|n|}{2g_X}, \tag{6}$$

where G_6 is the six dimensional (6D) Newton constant.

As the volume of the extra dimension is $4\pi R^2$, the 4D Planck scale and gauge coupling become $M_P = 1/\sqrt{8\pi G} = \sqrt{4\pi}R/\sqrt{8\pi G_6}$ ($= 2.4 \times 10^{18}$ GeV) and $g_{X4} = g_X/\sqrt{4\pi}R$, respectively. With Eq. (6), we get $g_{X4} = |n|/2RM_P$. If the Kalza-Klein (KK) scale is around TeV, then $RM_P \sim 10^{15}$, and the 4D $U(1)_X$ coupling: $g_{X4} \sim 10^{-15}\,|n|$ must be very small unless the monopole number $|n|$ is huge.[4]

2.2 Newman-Penrose eth formalism

Let us review the Newman-Penrose eth-formalism to get the spin-weighted spherical harmonics [67]. Consider a rotation by an angle α of an orthonormal basis of the tangent space at a point on S^2. A quantity η has spin weight s if it transforms under the rotation as

$$\eta' = e^{is\alpha}\eta. \tag{7}$$

If η has a spin weight s, its complex conjugate $\bar{\eta}$ has spin weight $-s$. A product of two quantities with spin weights s and s' has the spin weight $s + s'$. A derivative of a quantity with a definite spin weight may not have a well-defined spin weight. However if η has a spin weight s, the following quantities have well-defined spin weights

$$ð\eta = -\left(\sin\theta\right)^s\left[\left(\boldsymbol{e}_\theta + i\boldsymbol{e}_\phi\right)\cdot\boldsymbol{L}\right]\left(\sin\theta\right)^{-s}\eta = -\left[\frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi} - s\cot\theta\right]\eta, \tag{8}$$

$$\bar{ð}\eta = +\left(\sin\theta\right)^{-s}\left[\left(\boldsymbol{e}_\theta - i\boldsymbol{e}_\phi\right)\cdot\boldsymbol{L}\right]\left(\sin\theta\right)^{s}\eta = -\left[\frac{\partial}{\partial\theta} - i\csc\theta\frac{\partial}{\partial\phi} + s\cot\theta\right]\eta, \tag{9}$$

where

$$\boldsymbol{L} := -i\left(-\boldsymbol{e}_\phi\frac{\partial}{\partial\theta} + \boldsymbol{e}_\theta\csc\theta\frac{\partial}{\partial\phi}\right), \tag{10}$$

is the angular momentum operator. $ð$ and $\bar{ð}$ are read “eth” and “eth bar”, respectively. One can find that $ð\eta$ has spin weight $s+1$ and $\bar{ð}\eta$ has $s-1$. That is, $ð$ ($\bar{ð}$) raise (lower) the spin weight by unity.

[4] The weak gravity conjecture [64] suggests an existence of a UV cutoff of the $U(1)_X$ gauge theory in four dimensions: $\Lambda \lesssim g_{X4}M_P \sim$ TeV, and also in six dimensions [65, 66]: $\Lambda \lesssim \sqrt{\frac{g_{X4}}{M_P R}}M_P \sim$ TeV. We note that the vacuum stability gives us a similar bound $\Lambda \sim \text{few}/R$ [46].

Using eth(-bar) operators, spherical harmonic Y_{jm} can be generalized to the spin-weighted spherical harmonics:

$$
{}_sY_{jm} := \begin{cases} \sqrt{\frac{(j-s)!}{(j+s)!}}\, \eth^s\, Y_{jm} & \text{for } 0 \le s \le j, \\ (-1)^s \sqrt{\frac{(j+s)!}{(j-s)!}}\, \bar{\eth}^{-s}\, Y_{jm} & \text{for } -j \le s \le 0. \end{cases} \tag{11}
$$

The (spin-weighted) spherical harmonics Y_{jm} (${}_sY_{jm}$) has a spin weight 0 (s). The concrete form of spin-weighted spherical hamonics is

$$
\begin{aligned}
{}_sY_{jm} = (-1)^m &\sqrt{\frac{2j+1}{4\pi}\,(j+m)!\,(j-m)!\,(j+s)!\,(j-s)!} \\
&\times \sum_{k=\max\{0,\,-m-s\}}^{\min\{j-s,\,j-m\}} \frac{(-1)^k \left(\sin\frac{1}{2}\theta\right)^{m+s+2k} \left(\cos\frac{1}{2}\theta\right)^{2j-m-s-2k}}{k!\,(j-m-k)!\,(j-s-k)!\,(m+s+k)!} e^{im\phi}.
\end{aligned} \tag{12}
$$

We see that this expression for ${}_sY_{jm}$ properly reduces to the ordinary spherical harmonics Y_{jm} for $s=0$. Note that for a fixed s, $\{{}_sY_{jm}\}$ form a complete orthonormal basis on sphere S^2 and any function with spin weight s defined on it can be expanded by them. The inner product of two spin-weighted spherical harmonics satisfies the following orthonormality condition:

$$
\int \mathrm{d}\Omega\, \left[{}_sY_{jm}(\theta,\phi)\right]^* \, {}_sY_{j'm'}(\theta,\phi) = \delta_{jj'}\delta_{mm'}, \tag{13}
$$

where $\mathrm{d}\Omega := \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. Following relations are useful:

$$
\left[{}_sY_{jm}(\theta,\phi)\right]^* = (-1)^{(m+s)}\ {}_{-s}Y_{j,-m}(\theta,\phi), \tag{14}
$$

$$
{}_sY_{jm}(\pi-\theta,\phi+\pi) = (-1)^j\ {}_{-s}Y_{jm}(\theta,\phi) = (-1)^{j-s+m}\ \left[{}_sY_{j,-m}(\theta,\phi)\right]^*, \tag{15}
$$

$$
{}_sY_{jm}(\pi-\theta,-\pi) = (-1)^{j-s}\ {}_sY_{j,-m}(\theta,\phi), \tag{16}
$$

$$
{}_sY_{jm}(0,\phi) = \begin{cases} 0 & \text{for } m \neq -s, \\ (-1)^{-s}\sqrt{\frac{2j+1}{4\pi}} e^{-is\phi} & \text{for } m = -s, \end{cases} \tag{17}
$$

$$
{}_sY_{jm}(\pi,\phi) = \begin{cases} 0 & \text{for } m \neq s, \\ (-1)^{j}\sqrt{\frac{2j+1}{4\pi}} e^{is\phi} & \text{for } m = s. \end{cases} \tag{18}
$$

In particular for the $s=0$ mode,

$$
\int \mathrm{d}\Omega\, Y_{jm}(\theta,\phi)\, Y_{j'm'}(\theta,\phi) = (-1)^m \delta_{jj'}\delta_{m+m'}, \tag{19}
$$

where $\delta_{m+m'}$ follows the notation:

$$
\delta_M = \begin{cases} 1 & (M=0), \\ 0 & (M \neq 0). \end{cases} \tag{20}
$$

Each renormalizable interaction term in the Lagrangian consists of three or four fields, a KK-expanded 4D interaction includes three or four spin-weighted spherical harmonics. When calculating three and four point interactions, we use, respectively,

$$\begin{aligned}
&\int \mathrm{d}\Omega\, {}_{s_1}Y_{j_1m_1}(\theta,\phi)\, {}_{s_2}Y_{j_2m_2}(\theta,\phi)\, {}_{s_3}Y_{j_3m_3}(\theta,\phi) \\
&\quad = \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{4\pi}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix}, \qquad (21)\\
&\int \mathrm{d}\Omega\, {}_{s_1}Y_{j_1m_1}(\theta,\phi)\, {}_{s_2}Y_{j_2m_2}(\theta,\phi)\, {}_{s_3}Y_{j_3m_3}(\theta,\phi)\, {}_{s_4}Y_{j_4m_4}(\theta,\phi) \\
&\quad = \sum_{J=|j_1-j_2|}^{j_1+j_2} \frac{(-1)^{m_1+m_2+s_1+s_2}(2J+1)}{4\pi} \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)} \\
&\qquad \times \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -(m_1+m_2) \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ -s_1 & -s_2 & s_1+s_2 \end{pmatrix} \\
&\qquad \times \begin{pmatrix} J & j_3 & j_4 \\ m_1+m_2 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} J & j_3 & j_4 \\ -(s_1+s_2) & -s_3 & -s_4 \end{pmatrix}, \qquad (22)
\end{aligned}$$

where all j, m and s are integers and

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} := \frac{1}{\sqrt{2j_3+1}}(-1)^{j_1-j_2-m_3} \left\langle j_1 j_2; m_1 m_2 | j_3, -m_3 \right\rangle, \qquad (23)$$

is the Wigner's $3j$ symbol with $\langle j_1 j_2; m_1 m_2 | j_3, -m_3\rangle$ being the Clebsch-Gordan coefficient. Further details can be found in Ref. [68, 69].

We also write $\eth_s$ ($\bar{\eth}_s$) when we want to make explicit the spin weight s of the quantity on which the eth(-bar) operator act. For quantities η_s and $\kappa_{s'}$ with spin weights s and s', respectively, we have

$$\eth_{s+s'}\left(\eta_s \kappa_{s'}\right) = \eta_s \eth_{s'} \kappa_{s'} + \kappa_{s'} \bar{\eth}_s \eta_s. \qquad (24)$$

Let us define the following "K-operator":

$$\begin{aligned}
\mathbf{K}_s &:= -\bar{\eth}_{s+1}\, \eth_s + s(s+1) = -\eth_{s-1}\, \bar{\eth}_s + s(s-1) \\
&= -(\csc\theta\, \partial_\theta \sin\theta\, \partial_\theta + \csc^2\theta\, \partial_\phi^2 + 2is\csc\theta\cot\theta\, \partial_\phi - s^2\csc^2\theta), \qquad (25)
\end{aligned}$$

which satisfies

$$\mathbf{K}_s\, {}_sY_{jm}(\theta,\phi) = j(j+1)\, {}_sY_{jm}(\theta,\phi), \qquad (26)$$

for $-j \leq m \leq j$. More explicitly,

$$\begin{aligned}
\eth_{s+1}\eth_s &= \partial_\theta^2 - (2s+1)\cot\theta\,\partial_\theta + 2i\csc\theta\,\partial_\theta\partial_\phi - 2i\,(s+1)\csc\theta\cot\theta\,\partial_\phi \\
&\quad - \csc^2\theta\,\partial_\phi^2 + s\csc^2\theta + s\,(s+1)\cot^2\theta, \\
\overline{\eth}_{s-1}\overline{\eth}_s &= \partial_\theta^2 + (2s-1)\cot\theta\,\partial_\theta - 2i\csc\theta\,\partial_\theta\partial_\phi - 2i\,(s-1)\csc\theta\cot\theta\,\partial_\phi \\
&\quad - \csc^2\theta\,\partial_\phi^2 - s\csc^2\theta + s\,(s-1)\cot^2\theta, \\
\overline{\eth}_{s+1}\eth_s &= \csc\theta\,\partial_\theta\sin\theta\,\partial_\theta + \csc^2\theta\,\partial_\phi^2 + 2is\csc\theta\cot\theta\,\partial_\phi - s^2\csc^2\theta + s(s+1), \\
\eth_{s-1}\overline{\eth}_s &= \csc\theta\,\partial_\theta\sin\theta\,\partial_\theta + \csc^2\theta\,\partial_\phi^2 + 2is\csc\theta\cot\theta\,\partial_\phi - s^2\csc^2\theta + s(s-1).
\end{aligned} \tag{27}$$

We also define generalized eth, eth-bar, and K operators for later use:

$$\eth_s^\chi\,\eta_s := -e^{-i(s+1)\chi}\left[\frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi} - s\cot\theta\right]\left(e^{is\chi}\eta_s\right), \tag{28}$$

$$\overline{\eth}_s^\chi\,\eta_s := -e^{-i(s-1)\chi}\left[\frac{\partial}{\partial\theta} - i\csc\theta\frac{\partial}{\partial\phi} + s\cot\theta\right]\left(e^{is\chi}\eta_s\right), \tag{29}$$

$$\begin{aligned}
\mathbf{K}_s^\chi &:= -\overline{\eth}_{s+1}^\chi\,\eth_s^\chi + s(s+1) = -\eth_{s-1}^\chi\,\overline{\eth}_s^\chi + s(s-1) \\
&= -e^{-is\chi}\left(\csc\theta\,\partial_\theta\sin\theta\,\partial_\theta + \csc^2\theta\,\partial_\phi^2 + 2is\csc\theta\cot\theta\,\partial_\phi - s^2\csc^2\theta\right)e^{is\chi},
\end{aligned} \tag{30}$$

where χ is an arbitrary regular function of θ and ϕ. The generalized K-operator satisfies

$$\mathbf{K}_s^\chi\left({}_sY_{jm}(\theta,\phi)\,e^{-is\,\chi(\theta,\phi)}\right) = j(j+1)\left({}_sY_{jm}(\theta,\phi)\,e^{-is\,\chi(\theta,\phi)}\right). \tag{31}$$

2.3 Six dimensional gauge theory

In general, the 6D action for a gauge field $\hat{A} = \hat{A}_M\,\mathrm{d}z^M = \hat{A}_M^a T^a\,\mathrm{d}z^M$ is written as

$$S_A = \int \mathrm{d}^6 z\,\sqrt{-g}\,\left[-\frac{1}{2}\,\mathrm{tr}\left(\hat{F}_{MN}\hat{F}^{MN}\right)\right], \tag{32}$$

where indices M, N, ... are raised and lowered by the metric (1) and

$$\hat{F}_{MN} = \partial_M\hat{A}_N - \partial_N\hat{A}_M + ig_A[\hat{A}_M, \hat{A}_N], \tag{33}$$

with g_A being the 6D gauge coupling constant. Throughout this paper, "tr" is replaced by 1/2 for a $U(1)$ case. Note that since we are working in a torsion free space we have $\nabla_M\hat{A}_N - \nabla_N\hat{A}_M = \partial_M\hat{A}_N - \partial_N\hat{A}_M$, where $\nabla_M\hat{A}_N = \partial_M\hat{A}_N - \Gamma^L{}_{MN}\hat{A}_L$ is the general covariant derivative.

Following the background field method, we separate the gauge field into the classical and quantum parts $\mathcal{A}$ and A, respectively, $\hat{A}_M = \mathcal{A}_M + A_M$.[5] We also define the classical field

[5] Throughout this paper, curly and normal letters denote a classical background and a quantum fluctuation, respectively.

strength $\mathcal{F}_{MN} := \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M + ig_A[\mathcal{A}_M, \mathcal{A}_N]$ and the background-covariant derivative $\mathcal{D}_M := \partial_M + ig_A[\mathcal{A}_M, \ \cdot \ ]$ so that

$$\hat{F}_{MN} = \mathcal{F}_{MN} + \mathcal{D}_M A_N - \mathcal{D}_N A_M + ig_A[A_M, A_N]. \tag{34}$$

The actions linear and quadratic in the quantum part A_M become

$$S_A^{\text{linear}} = -\int \mathrm{d}^6 z \sqrt{-g}\ \mathrm{tr}\left[\mathcal{F}^{MN}\left(\mathcal{D}_M A_N - \mathcal{D}_N A_M\right)\right], \tag{35}$$

$$S_A^{\text{quad}} = -\int \mathrm{d}^6 z \sqrt{-g}\ \mathrm{tr}\left[ig_A\,\mathcal{F}^{MN}[A_M, A_N] + \frac{1}{2}\left(\mathcal{D}^M A^N - \mathcal{D}^N A^M\right)\left(\mathcal{D}_M A_N - \mathcal{D}_N A_M\right)\right]. \tag{36}$$

We see that the linear term vanishes for a pure gauge configuration $\mathcal{F}^{MN} = 0$. In contrast, the monopole configuration (2) gives $\mathcal{F}^{\theta\phi} \neq 0$, and more careful treatment is necessary. We will come back to this point in Section 3.3.

For the gauge fixing action,

$$S_f = -\int \mathrm{d}^6 z \sqrt{-g}\,\frac{1}{\xi}\ \mathrm{tr}\left(ff\right), \tag{37}$$

we choose the following gauge fixing function

$$\begin{aligned} f &= g^{\mu\nu}\left(\nabla_\mu A_\nu + ig_A[\mathcal{A}_\mu, A_\nu]\right) + \xi\left[g^{\theta\theta}\left(\nabla_\theta A_\theta + ig_A[\mathcal{A}_\theta, A_\theta]\right) + g^{\phi\phi}\left(\nabla_\phi A_\phi + ig_A[\mathcal{A}_\phi, A_\phi]\right)\right] \\ &= g^{\mu\nu}\,\mathcal{D}_\mu A_\nu + \xi\left[g^{\theta\theta}\,\mathcal{D}_\theta A_\theta + g^{\phi\phi}\left(\mathcal{D}_\phi A_\phi - \Gamma^M{}_{\phi\phi} A_M\right)\right]. \end{aligned} \tag{38}$$

The infinitesimal gauge transformation of A_M:

$$\delta A_M = \mathcal{D}_M \epsilon - ig_A\left[\epsilon,\, A_M\right], \qquad \delta\mathcal{A}_M = 0, \tag{39}$$

gives the ghost Lagrangian:

$$S_{\text{gh}} = \int \mathrm{d}^6 z\, 2\sqrt{-g}\,\mathrm{tr}\left(\bar{\omega}\left[\frac{\delta f}{\delta A_M}\right]\left(\mathcal{D}_M \omega + ig_A\left[A_M,\, \omega\right]\right)\right), \tag{40}$$

where ω and $\bar{\omega}$ are the ghost and anti-ghost fields and the factor -1 in (39) is absorbed by normalization of ghost field.

3 KK expansion

3.1 Free scalar on sphere

The general quadratic action for scalar, relevant to its KK expansion, is:

$$S_\Phi = -\int \mathrm{d}^6 z \sqrt{-g}\left[\left(D_M \Phi\right)^\dagger\left(D^M \Phi\right) + M_\Phi^2 \Phi^\dagger \Phi\right], \tag{41}$$

$$D_M := \partial_M + i g_X Q_\Phi \mathcal{X}_M, \tag{42}$$

where Q_Φ is the $U(1)_X$ charge of Φ. On the north and south charts,

$$S_\Phi := \int \mathrm{d}^6 z \sqrt{-g}\, \Phi^\dagger \left[\Box - \left\{ \frac{1}{R^2} \left(\mathbf{K}^{\mp\phi}_{nQ_\Phi/2} - \left(\frac{nQ_\Phi}{2} \right)^2 \right) + M_\Phi^2 \right\} \right] \Phi, \tag{43}$$

where upper and lower signs are for north and south charts, respectively. We see that we have the spin-weight $s_\Phi := nQ_\Phi/2$ and $\chi = \mp\phi$. The KK-expansion of Φ on the north and south charts are, respectively,

$$\Phi^{N}_{S}(x, \theta, \phi) = \sum_{j=|s_\Phi|}^{\infty} \sum_{m=-j}^{j} \frac{\phi^{j,m}(x)}{R}\ {}_{s_\Phi}Y_{jm}(\theta, \phi) e^{\pm i s_\Phi \phi}. \tag{44}$$

From this expression, we find that the four-dimensional free action is

$$S_{\Phi,\,\mathrm{free}} = \int d^4x \sum_{j=|s_\Phi|}^{\infty} \sum_{m=-j}^{j} \phi^{\dagger jm}(x) \left(\Box - m_{\phi,\,j}^2 \right) \phi^{jm}(x), \tag{45}$$

$$m_{\phi,\,j}^2 = \frac{j(j+1)}{R^2} + M_\Phi^2 - \frac{|s_\Phi|^2}{R^2}. \tag{46}$$

Especialy, for the lowest $j = |s_\Phi|$ mode, the mass squared is

$$m_{\phi,\,|s_\Phi|}^2 = \frac{|s_\Phi|}{R^2} + M_\Phi^2. \tag{47}$$

Therefore, there is no massless mode even if $M_\Phi = 0$, unless the scalar field is neutral under $U(1)_X$.

The following transition of scalar field between the north and south charts makes the action invariant:

$$\Phi^S = e^{-2 i s_\Phi \phi} \Phi^N. \tag{48}$$

3.2 Free spinor on sphere

Let us review the KK expansion of spinors on sphere. We summarize the Clifford algebra and Lorentz transformation properties of spinor field in six dimensions in Appendix B. The free spinor action under the $U(1)_X$ background on the sphere is:

$$S = S_+ + S_-, \qquad S_\pm = -\int \mathrm{d}^6 z \sqrt{-g}\ \overline{\Psi_\pm} \left[\Gamma^M \left(\partial_M + i g_X Q_\pm \mathcal{X}_M + \Omega_M \right) \right] \Psi_\pm, \tag{49}$$

where $\Psi_\pm$ are 6D spinors with plus and minus chiralities, $Q_\pm$ are their $U(1)_X$ charges, Γ^M are 6D gamma matrices, and Ω_M are spin-connections; see Appendix B for details. We note that the 6D theory is chiral and that the spinors Ψ_+ and Ψ_- are independent of each other.[6] Putting the classical configuration $\mathcal{X}_\phi{}^{N}_{S} = \frac{n}{2g_X}(\cos\theta \mp 1)$, the free spinor action can be recasted into the following form:

$$S = -\int \mathrm{d}^6 z \sqrt{-g} \left\{ \overline{\Psi_+} \left[\Gamma^{\underline{\mu}} \partial_\mu + \frac{i}{R} \eth^{\mp\phi}_{s_{+R}} \Gamma^+ - \frac{i}{R} \overline{\eth}^{\mp\phi}_{s_{+L}} \Gamma^- \right] \Psi_+ \right.$$
$$\left. + \overline{\Psi_-} \left[\Gamma^{\underline{\mu}} \partial_\mu + \frac{i}{R} \eth^{\mp\phi}_{s_{-L}} \Gamma^+ - \frac{i}{R} \overline{\eth}^{\mp\phi}_{s_{-R}} \Gamma^- \right] \Psi_- \right\}, \tag{50}$$

where the upper and lower signs are for north and south charts, respectively; we have also defined the integers $N_\pm := nQ_\pm$, the half integers

$$s_{+L} := \frac{N_+ + 1}{2}, \qquad s_{+R} := \frac{N_+ - 1}{2}, \qquad s_{-L} := \frac{N_- - 1}{2}, \qquad s_{-R} := \frac{N_- + 1}{2}, \tag{51}$$

and the combinations of the six dimensional gamma matrices

$$\Gamma^+ := \frac{\Gamma^{\underline{5}} + i\Gamma^{\underline{6}}}{2} = \begin{bmatrix} & P_L \\ -P_R & \end{bmatrix}, \qquad \Gamma^- := \frac{\Gamma^{\underline{5}} - i\Gamma^{\underline{6}}}{2} = \begin{bmatrix} & -P_R \\ P_L & \end{bmatrix}, \tag{52}$$

where $P_L = (1+\gamma^{\underline{5}})/2$ and $P_R = (1-\gamma^{\underline{5}})/2$ are the 4D chirality projections. In terms of the four component spinors $\psi_\pm$ given in Eq. (148) in Appendix B, we can rewrite

$$S = -\int \mathrm{d}^6 z \sqrt{-g} \left[\overline{\psi_+} \left\{ \partial\!\!\!/ - \frac{i}{R} \left(\overline{\eth}^{\mp\phi}_{s_{+L}} P_L + \eth^{\mp\phi}_{s_{+R}} P_R \right) \right\} \psi_+ + \overline{\psi_-} \left\{ \partial\!\!\!/ + \frac{i}{R} \left(\overline{\eth}^{\mp\phi}_{s_{-R}} P_R + \eth^{\mp\phi}_{s_{-L}} P_L \right) \right\} \psi_- \right]. \tag{53}$$

Further with the two component spinors

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}, \tag{54}$$

we get

$$S = \int \mathrm{d}^6 z \sqrt{-g} \left[\left\{ i\,(\chi_{+L})^\dagger \overline{\sigma}^\mu \partial_\mu \chi_{+L} + i\,(\chi_{+R})^\dagger \sigma^\mu \partial_\mu \chi_{+R} + \frac{i}{R} \left((\chi_{+L})^\dagger \eth^{\mp\phi}_{s_{+R}} \chi_{+R} + (\chi_{+R})^\dagger \overline{\eth}^{\mp\phi}_{s_{+L}} \chi_{+L} \right) \right\} \right.$$
$$\left. + \left\{ i\,(\chi_{-L})^\dagger \overline{\sigma}^\mu \partial_\mu \chi_{-L} + i\,(\chi_{-R})^\dagger \sigma^\mu \partial_\mu \chi_{-R} - \frac{i}{R} \left((\chi_{-L})^\dagger \overline{\eth}^{\mp\phi}_{s_{-R}} \chi_{-R} + (\chi_{-R})^\dagger \eth^{\mp\phi}_{s_{-L}} \chi_{-L} \right) \right\} \right], \tag{55}$$

where again upper and lower signs are for the north and south charts, respectively.

[6] In principle, we can put a bulk mass term between Ψ_+ and Ψ_- if both has completely the same charges, but it is not the case in our application.

The KK expansion is given by

$$\chi_{\pm L(R)}{}^{N}(x,\theta,\phi) = \sum_{j=\left|s_{\pm L(R)}\right|}^{\infty} \sum_{m=-j}^{j} \frac{\chi^{jm}_{\pm L(R)}(x)}{R}\, {}_{s_{\pm L(R)}}Y_{jm}(\theta,\phi)\, e^{is_{\pm L(R)}\phi}, \tag{56}$$

$$\chi_{\pm L(R)}{}^{S}(x,\theta,\phi) = \sum_{j=\left|s_{\pm L(R)}\right|}^{\infty} \sum_{m=-j}^{j} \frac{\chi^{jm}_{\pm L(R)}(x)}{R}\, {}_{s_{\pm L(R)}}Y_{jm}(\theta,\phi)\, e^{-is_{\pm L(R)}\phi}, \tag{57}$$

for the north and south charts, respectively, where the upper and lower signs are the 6D chiralities which are uncorrelated with charts. The resultant action is

$$\begin{aligned} S = \int d^4x \Bigg[\Bigg\{ & \sum_{j=|s_{+L}|}^{\infty} \sum_{m=-j}^{j} i \left(\chi^{jm}_{+L}\right)^{\dagger} \overline{\sigma}^{\mu} \partial_{\mu} \chi^{jm}_{+L} + \sum_{j=|s_{+R}|}^{\infty} \sum_{m=-j}^{j} i \left(\chi^{jm}_{+R}\right)^{\dagger} \sigma^{\mu} \partial_{\mu} \chi^{jm}_{+R} \\ & + \sum_{j=s_{+\max}}^{\infty} \sum_{m=-j}^{j} i m^{j}_{+} \left\{ \left(\chi^{jm}_{+L}\right)^{\dagger} \chi^{jm}_{+R} - \left(\chi^{jm}_{+R}\right)^{\dagger} \chi^{jm}_{+L} \right\} \Bigg\} \\ & + \Bigg\{ \sum_{j=|s_{-L}|}^{\infty} \sum_{m=-j}^{j} i \left(\chi^{jm}_{-L}\right)^{\dagger} \overline{\sigma}^{\mu} \partial_{\mu} \chi^{jm}_{-L} + \sum_{j=|s_{-R}|}^{\infty} \sum_{m=-j}^{j} i \left(\chi^{jm}_{-R}\right)^{\dagger} \sigma^{\mu} \partial_{\mu} \chi^{jm}_{-R} \\ & + \sum_{j=s_{-\max}}^{\infty} \sum_{m=-j}^{j} i m^{j}_{-} \left\{ \left(\chi^{jm}_{-L}\right)^{\dagger} \chi^{jm}_{-R} - \left(\chi^{jm}_{-R}\right)^{\dagger} \chi^{jm}_{-L} \right\} \Bigg\} \Bigg], \end{aligned} \tag{58}$$

where $s_{\pm\max} := \max\left\{|s_{\pm L}|, |s_{\pm R}|\right\}$ and

$$m^{j}_{\pm} = \frac{\sqrt{\left(j+\frac{1}{2}\right)^2 - \left(\frac{N_{\pm}}{2}\right)^2}}{R}. \tag{59}$$

The lowest-j mode of the spinor $\Psi_{\pm}$ is given by the half integer

$$j_{\pm\min} = \min\left\{|s_{\pm L}|, |s_{\pm R}|\right\} = \left|\frac{|N_{\pm}|-1}{2}\right|. \tag{60}$$

For a general mode $j = j_{\pm\min} + \ell$ with an integer $\ell \geq 0$, we get

$$m^{j}_{\pm} = \begin{cases} \frac{\sqrt{\ell(\ell+|N_{\pm}|)}}{R} & \text{for } |N_{\pm}| \geq 1, \\ \frac{\sqrt{(\ell+1)(\ell+1-|N_{\pm}|)}}{R} & \text{for } 0 \leq |N_{\pm}| \leq 1, \end{cases} \tag{61}$$

for each 6D chirality. We see that $\Psi_{\pm}$ can have a zero mode when and only when $|N_{\pm}| = 1$. When $N_{+} = 1$ (-1) for Ψ_{+}, we have a zero mode at $s_{+R} = 0$ $(s_{+L} = 0)$; when $N_{-} = 1$ (-1) for Ψ_{-}, we have a zero mode at $s_{-L} = 0$ $(s_{-R} = 0)$.

Let us check the equivalence of spinor actions in north and south charts. We see that the KK-expanded two-components spinor in both charts are related by

$$\chi^S_{\pm L(R)} = e^{-2is_{\pm L(R)}\phi}\chi^N_{\pm L(R)}. \tag{62}$$

Defining the spin-weght operator $\hat{S} := \hat{N}/2 - i\Sigma^{\underline{56}}$ with $\hat{N}\Psi_\pm = N_\pm\Psi_\pm$, we get

$$\hat{S}\Psi = \begin{bmatrix} s_{+L}\,\chi_{+L} \\ s_{+R}\,\chi_{+R} \\ s_{-L}\,\chi_{-L} \\ s_{-R}\,\chi_{-R} \end{bmatrix}, \tag{63}$$

where the local Lorentz generator $\Sigma^{\underline{56}}$ is defined in Eq. (141) in Appendix B. Therefore, we can write the transition function between north and south charts in terms of the eight component spinor as

$$\Psi^S = e^{-2i\phi\hat{S}}\Psi^N. \tag{64}$$

The transition of spinors is given by the combination of local $U(1)_X$ gauge transformation and the local Lorentz transformation in $\underline{5}$-$\underline{6}$ plane.

In the realistic model construction, we assume $N_\pm = -1$ so that $s_{+L} = 0$, $s_{+R} = -1$, $s_{-L} = -1$, and $s_{-R} = 0$ and that Ψ_+ (Ψ_-) has a 4D left (right) zero mode.[7] The four-dimensional Dirac mass of an integer $j \geq 0$ mode is now

$$m_j = \frac{\sqrt{j(j+1)}}{R}. \tag{65}$$

In terms of the four component spinors, we get

$$S_+ = -\int d^4x \left[\overline{\psi^{00}_{+L}}\gamma^\mu\partial_\mu\psi^{00}_{+L} + \sum_{j=1}^{\infty}\sum_{m=-j}^{j}\overline{\psi^{jm}_+}\left(\gamma^\mu\partial_\mu + i\gamma^{\underline{5}}m_j\right)\psi^{jm}_+\right], \tag{66}$$

$$S_- = -\int d^4x \left[\overline{\psi^{00}_{-R}}\gamma^\mu\partial_\mu\psi^{00}_{-R} + \sum_{j=1}^{\infty}\sum_{m=-j}^{j}\overline{\psi^{jm}_-}\left(\gamma^\mu\partial_\mu + i\gamma^{\underline{5}}m_j\right)\psi^{jm}_-\right]. \tag{67}$$

As usual, the $i\gamma^{\underline{5}}$ can be removed by a chiral rotation. Details can be found in Appendix C.

3.3 Free vector on sphere

Up to this point, the formalism is applicable for any metric and background configuration $\mathcal{A}_M$. From now on, let us specify the space-time background to be that of the sphere (1) and assume $\mathcal{A}_\mu = 0$ on physical ground. The gauge fixing function (38) becomes

$$f = \eta^{\mu\nu}\mathcal{D}_\mu A_\nu + \frac{\xi}{R^2\sin\theta}\mathcal{D}_\theta\sin\theta A_\theta + \frac{\xi}{R^2\sin^2\theta}\mathcal{D}_\phi A_\phi, \tag{68}$$

[7] We note that we need the same number of degrees of freedom for the plus and minus chiralities in order to cancel the gravitational anomaly in 6D [30].

In obtaining the KK expansions, it is convenient to rewrite the action in terms of the tangent space vectors $A_{\underline{M}} := e^{N}{}_{\underline{M}}\, A_N$, namely

$$A_{\underline{\mu}} = \delta^{\nu}_{\underline{\mu}} A_\nu, \qquad A_{\underline{\theta}} = \frac{A_\theta}{R}, \qquad A_{\underline{\phi}} = \frac{A_\phi}{R\sin\theta}. \tag{69}$$

The relation of the 4D part is trivial, and we do not distinguish $A_{\underline{\mu}}$ and A_μ hereafter. Under the metric (1), we obtain[8]

$$\begin{aligned} S^{\text{quad}}_{A+f} = \int \mathrm{d}^6 z \sqrt{-g}\ \mathrm{tr} \Bigg\{ & A_\mu \left[\eta^{\mu\nu} \left(\Box + \frac{1}{R^2 \sin\theta} \mathcal{D}_\theta \sin\theta\, \mathcal{D}_\theta + \frac{1}{R^2 \sin^2\theta} \mathcal{D}_\phi^2 \right) - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu \\ & + 2A_- \left[\Box + \frac{1}{R^2} \left(\frac{1}{\sin\theta} \mathcal{D}_\theta \sin\theta\, \mathcal{D}_\theta + \frac{1}{\sin^2\theta} \left(\mathcal{D}_\phi^2 - 1 \right) + \frac{2i\cos\theta}{\sin^2\theta} \mathcal{D}_\phi \right) \right] A_+ \\ & + \frac{(\xi - 1)}{R^2} \Bigg[\frac{1}{2} A_+ \left(\mathcal{D}_\theta^2 + \cot\theta\, \mathcal{D}_\theta - \frac{1}{\sin^2\theta} \left(\mathcal{D}_\phi^2 + 1 \right) - \frac{2i}{\sin\theta} \mathcal{D}_\theta \mathcal{D}_\phi \right) A_+ \\ & \qquad + \frac{1}{2} A_- \left(\mathcal{D}_\theta^2 + \cot\theta\, \mathcal{D}_\theta - \frac{1}{\sin^2\theta} \left(\mathcal{D}_\phi^2 + 1 \right) + \frac{2i}{\sin\theta} \mathcal{D}_\theta \mathcal{D}_\phi \right) A_- \\ & \qquad + A_- \left(\frac{1}{\sin\theta} \mathcal{D}_\theta \sin\theta\, \mathcal{D}_\theta + \frac{1}{\sin^2\theta} \left(\mathcal{D}_\phi^2 - 1 \right) + \frac{2i\cos\theta}{\sin^2\theta} \mathcal{D}_\phi \right) A_+ \Bigg] \\ & - \frac{2 g_A}{R^2 \sin\theta} \mathcal{F}_{\theta\phi} \left[A_+,\, A_- \right] \Bigg\}, \end{aligned} \tag{70}$$

where we have defined the new tangent space vectors: $A_\pm := (A_{\underline{\theta}} \pm i A_{\underline{\phi}})/\sqrt{2}$.

With a non-Abelian gauge field, the linear term (35) does not vanish under the monopole configuration (2), and hence such a non-Abelian monopole configuration leads to a classical instability [70]. Therefore we assume that the only $U(1)_X$ gauge field develops the monopole VEV (2). Later, we will see that the linear term vanishes for the $U(1)_X$ gauge field.

For a $U(1)$ gauge field, we have $[A_M, A_N] = 0$ and can replace $\mathcal{D}_M$ by ∂_M; for a non-Abelian gauge field, we do not consider the monopole configuration as said above, that is, $\mathcal{A}_M = 0$. In both cases, the background covariant derivative is replaced by the ordinary derivative, and we get

$$\begin{aligned} S^{\text{quad}}_{A+f} = \int \mathrm{d}^6 z \sqrt{-g}\ \mathrm{tr} \Bigg\{ & A_\mu \left[\eta^{\mu\nu} \left(\Box + \frac{1}{R^2} \bar{\eth}_1 \eth_0 \right) - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu \\ & + 2A_- \left[\Box + \frac{1}{R^2} \eth_0 \bar{\eth}_1 \right] A_+ \\ & + \frac{\xi - 1}{R^2} \left[\frac{1}{2} A_+ \bar{\eth}_0 \bar{\eth}_1 A_+ + \frac{1}{2} A_- \eth_0 \eth_{-1} A_- + A_- \eth_0 \bar{\eth}_1 A_+ \right] \Bigg\}. \end{aligned} \tag{71}$$

[8] In deriving Eq. (70), following identity is useful: $\mathcal{D}_\theta \frac{1}{\sin\theta} \mathcal{D}_\theta \sin\theta = \frac{1}{\sin\theta} \mathcal{D}_\theta \sin\theta\, \mathcal{D}_\theta - \frac{1}{\sin^2\theta}$.

We see that A_μ and $A_\pm$ have spin weights 0 and ± 1, respectively.

Since $\bar{\eth}_1 \eth_0 = -\mathbf{K}_0$ and $\mathbf{K}_0 Y_{jm} = j\,(j+1)\,Y_{jm}$, we can trivially expand the four dimensional component A_μ in terms of the ordinary spherical harmonics:

$$A_\mu(x,\theta,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{1}{R} A_\mu^{jm}(x) Y_{jm}(\theta,\phi). \tag{72}$$

As the six dimensional field A_μ is real, the complex fields $A_\mu^{jm}(x)$ subject to the reality condition:

$$(-1)^m\, A_\mu^{j,-m\dagger}(x) = A_\mu^{jm}(x). \tag{73}$$

In particular, $m=0$ modes become real due to the reality condition:

$$A_\mu^{j0\dagger}(x) = A_\mu^{j0}(x). \tag{74}$$

Putting this expansion into the quadratic action, we get

$$S_{A+f}^{\text{quad, vector}} = \sum_{j=0}^{\infty} \int \mathrm{d}^4 x \,\mathrm{tr}\left\{A_\mu^{j0} D^{\mu\nu} A_\nu^{j0}\right\} + \sum_{j=0}^{\infty}\sum_{m=1}^{j} \int \mathrm{d}^4 x\,\mathrm{tr}\left\{2A_\mu^{jm\dagger}\left(D^{\mu\nu} - \eta^{\mu\nu}\frac{j\,(j+1)}{R^2}\right) A_\nu^{jm}\right\}, \tag{75}$$

where $D_{\mu\nu} = \eta_{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu$.

Let us move on to the scalar part. For a general gauge parameter ξ, we may rewrite the action by using a complex scalar field Ξ_A defined by

$$A_+ =: -i\eth_0 \Xi_A^\dagger, \qquad\qquad A_- =: i\bar{\eth}_0 \Xi_A. \tag{76}$$

Note that Ξ_A has the spin weight $s=0$. The action is now

$$\begin{aligned} S_{A+f}^{\text{quad, scalar}} = \int \mathrm{d}^6 z\,\sqrt{-g}\;\mathrm{tr}\Bigg[&2\Xi_A^\dagger \mathbf{K}_0\left(\Box - \frac{1}{R^2}\mathbf{K}_0\right)\Xi_A \\ &+ \frac{\xi-1}{R^2}\left(\frac{1}{2}\Xi_A\left(\mathbf{K}_0\right)^2\Xi_A + \frac{1}{2}\Xi_A^\dagger\left(\mathbf{K}_0\right)^2\Xi_A^\dagger - \Xi_A^\dagger\left(\mathbf{K}_0\right)^2\Xi_A\right)\Bigg], \end{aligned} \tag{77}$$

where we have integrated by parts:

$$\int \mathrm{d}\Omega f_s^* \eth_{s-1} g_{s-1} = -\int \mathrm{d}\Omega\left(\eth_{-s} f_s^*\right) g_{s-1}, \tag{78}$$

$$\int \mathrm{d}\Omega f_s^*\,\bar{\eth}_{s+1} g_{s+1} = -\int \mathrm{d}\Omega\left(\bar{\eth}_{-s} f_s^*\right) g_{s+1}. \tag{79}$$

Decomposing Ξ_A into the real and imaginary parts,

$$\Xi_A := \frac{\Phi_A + i\Theta_A}{\sqrt{2}}, \tag{80}$$

we get

$$A_{\underline{\theta}} = \partial_\theta \Theta_A - \csc\theta\, \partial_\phi \Phi_A, \tag{81}$$
$$A_{\underline{\phi}} = \partial_\theta \Phi_A + \csc\theta\, \partial_\phi \Theta_A. \tag{82}$$

We note that Φ_A and Θ_A are the same as ϕ_1/R and ϕ_2/R defined in Eqs. (74) and (75), respectively, in Ref. [34]. We can write the action in terms of Φ_A and Θ_A as

$$S^{\text{quad, scalar}}_{A+f} = \int \mathrm{d}^6 z\, \sqrt{-g}\ \mathrm{tr}\left[\Phi_A \mathbf{K}_0 \left(\Box - \frac{1}{R^2}\mathbf{K}_0\right)\Phi_A + \Theta_A \mathbf{K}_0 \left(\Box - \frac{\xi}{R^2}\mathbf{K}_0\right)\Theta_A\right]. \tag{83}$$

From ξ dependence, we see that Φ_A and Θ_A are the physical and Nambu-Goldstone modes, respectively.

Analogously to the KK expansion of the vector, we expand as

$$\Phi_A(x,\theta,\phi) = \sum_{j=1}^{\infty}\sum_{m=-j}^{j} \frac{1}{R\sqrt{j\,(j+1)}}\, \phi_A^{jm}(x)\, Y_{jm}(\theta,\phi),$$
$$\Theta_A(x,\theta,\phi) = \sum_{j=1}^{\infty}\sum_{m=-j}^{j} \frac{1}{R\sqrt{j\,(j+1)}}\, \theta_A^{jm}(x)\, Y_{jm}(\theta,\phi), \tag{84}$$

where ϕ_A^{jm} and θ_A^{jm} are four dimensional adjoint scalars subject to the reality condition, the same as in Eq. (73). Note that $j = 0$ mode drops out because of the overall $\mathbf{K}_0$ in the action (83). The extra factor $1/\sqrt{j\,(j+1)}$ in the above expansion is to adjust the overall normalization. We note that $A_\pm$ has the spin weight ± 1, and hence have a ϕ-dependence at north and south poles; some of the KK modes of $A_\pm$ are not single valued there; see Eq. (18). Therefore we should regard Φ_A and Θ_A as the fundamental degrees of freedom, rather than $A_\pm$.

Let us now check that the linear term (35) vanishes for the $U(1)_X$ gauge field $\hat{X}_M = \mathcal{X}_M + X_M$ with the classical configuration (2). The linear action (35) reads

$$S^{\text{linear}} = \int \mathrm{d}^4 x\, \mathrm{d}\Omega \frac{in}{2\sqrt{2} g_X R}\left(\overline{\eth}_1 X_+ - \eth_{-1} X_-\right), \tag{85}$$

where $X_\pm := (X_{\underline{\theta}} \pm i X_{\underline{\phi}})/\sqrt{2}$. Putting again as in Eq. (76), we get

$$S^{\text{linear}} = \int \mathrm{d}^4 x\, \mathrm{d}\Omega \frac{n}{2\sqrt{2} g_X R}\left(\overline{\eth}_1 \eth_0 \Xi_X^\dagger + \eth_{-1}\overline{\eth}_0 \Xi_X\right). \tag{86}$$

Noting that $\bar{\eth}_1\eth_0 = \eth_{-1}\bar{\eth}_0 = -\mathbf{K}_0$ and that Ξ_X only has $j \geq 1$ mode in the expansion (80) and (84), we see that the angular integral in Eq. (86) always gives

$$\int \mathrm{d}\Omega\, Y_{00} Y_{j\neq 0,m} = 0, \tag{87}$$

and hence the linear term (86) vanishes.

Finally, we spell out the KK expansions of the ghost field Ω. Since the only $U(1)_X$ gauge field has the background configuration, we can write both for Abelian and non-Abelian cases:

$$S_{\text{gh}}^{\text{quad}} = \int \mathrm{d}^6 z\, \sqrt{-g}\, 2\,\mathrm{tr}\left\{ \bar{\Omega} \left[\Box - \frac{\xi}{R^2} \mathbf{K}_0 \right] \Omega \right\}. \tag{88}$$

Recall that "tr" reads 1/2 for a $U(1)$ field throughout this note. We see that the ghost fields can be expanded exactly the same as the gauge fields:

$$\Omega(x,\theta,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{1}{R} \omega^{jm}(x)\, Y_{jm}(\theta,\phi), \quad \bar{\Omega}(x,\theta,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{1}{R} \bar{\omega}^{jm}(x)\, Y_{jm}^*(\theta,\phi), \tag{89}$$

where the reality condition reads

$$(-1)^m\, \omega^{j,-m\dagger}(x) = \omega^{jm}(x), \qquad (-1)^m\, \bar{\omega}^{j,-m\dagger}(x) = \bar{\omega}^{jm}(x). \tag{90}$$

The KK expanded action for ghost is

$$S_{\text{gh}}^{\text{quad}} = \int \mathrm{d}^4 x\, 2\,\mathrm{tr} \left\{ \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \bar{\omega}^{jm} \left(\Box - \xi \frac{j\,(j+1)}{R^2} \right) \omega^{jm} \right\}. \tag{91}$$

4 Six-dimensional UED models on sphere

As shown above, the compactification on two-sphere automatically yields the chiral fermion zero mode. Therefore, it is tempting to use this to realize a universal extra dimension (UED) model on it. An obstacle is the existence of the massless $U(1)_X$ gauge boson. Since it must have a Yukawa coupling to each pair of the SM fermion zero modes in order to make them chiral, it necessarily transmits a long range force among them; see e.g. Ref. [71]. Therefore this possibility is excluded unless we somehow project out the massless $U(1)_X$ gauge boson or make it massive.

The former possibility is realized in Ref. [38] by applying a projection on the sphere: $(\theta,\phi) \sim (\pi-\theta,\phi+\pi)$. The resultant manifold is nothing but the real projective plane, but in order to distinguish with the UED model based on torus [37], we call it the projective sphere (PS) here.

On the other hand, the latter possibility is considered in Ref. [34] where the $U(1)_X$ is broken by an anomaly (induced by chiral bulk fermions) and the gauge boson is supposed to acquire a mass of the order of a UV cutoff scale Λ via the Green-Schwarz mechanism [72].[9] As another solution, one may also imagine to realize the gauge boson mass via the Stückelberg mechanism; see Sec. C.2 (or 2.3.2 in the preprint version) of Ref. [36]. We note however that both cases have pathology:

- Even if we assume that the anomaly indeed generates the bulk mass term $\Lambda^2 \hat{X}_M \hat{X}^M$, it changes the equation of motion for the classical monopole configuration by of the order of Λ and spoils the spontaneous compactification mechanism itself, as is pointed out in Ref. [38].

- Suppose one adds the Stückelberg mass m_X

$$\Delta S = \int \mathrm{d}^6 z \sqrt{-g} \left[-\frac{1}{2} \left(\partial_M \hat{\chi} + m_X \hat{X}_M \right) \left(\partial^M \hat{\chi} + m_X \hat{X}^M \right) \right], \tag{92}$$

  where $\hat{\chi}$ is the Stückelberg field.[10] Under the presence of the field configuration (2) and the vanishing classical background, $\chi = 0$, this mass term gives extra contribution to the classical stress-energy tensor:

$$(\Delta T_{MN})^{\,N}_{\,S} = \left(-\frac{g_{MN}}{2} g^{\phi\phi} + \delta^\phi_M \delta^\phi_N \right) m_X^2 \left(\frac{n}{2 g_X} \right)^2 (\cos\theta \mp 1)^2 . \tag{93}$$

  The stress-energy tensor is a physical quantity, and is unacceptable to depend on charts. It is nontrivial whether there can be a modified monopole solution to the Einstein-Maxwell equation with the Stückelberg extension (92).[11]

Therefore, the model with the $U(1)_X$ anomaly (S^2/Z_2 model) or the one with the Stückelberg mass (S^2 model) should be treated with caution.

4.1 S^2 UED Model with a Stückelberg Field

First we briefly comment on the S^2 UED model, where the $U(1)_X$ gauge field is made massive by the Stückelberg mechanism [75, 76] as in Eq. (92). The Stückelberg field χ behaves as the Nambu-Goldstone boson which is absorbed by the $U(1)_X$ gauge field, and makes it massive.

We note that the the KK-modes of each 6D field are not modified from those in Section 3 by the Stückelberg field. The only difference is that each KK mass of the $U(1)_X$ gauge field is lifted up by the Stückelberg mass term.

[9] The orbifold Z_2 projection in the model [34] does not remove the $U(1)_X$ zero mode, contrary to the Z_2 in the projective sphere model which is discussed in Section 4.3.

[10] The mass induced by the anomaly can be viewed as the Stückelberg mass; see e.g. Ref. [73].

[11] This problem would reside in the analysis [74] too. We thank Muneto Nitta and Makoto Sakamoto for illuminating discussions on this matter.

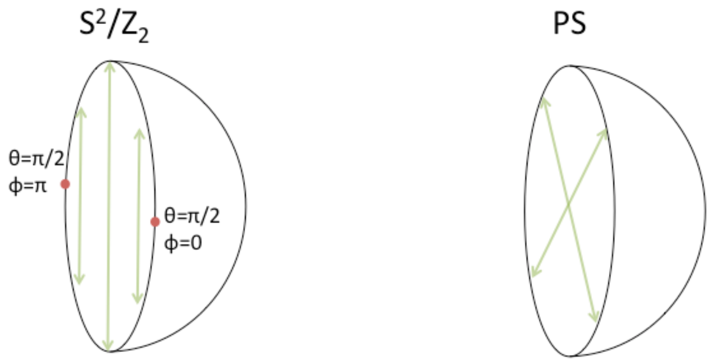

Figure 1: S^2/Z_2 orbifold and PS manifold for left and right panels, respectively, when we take the fundamental domain as $0 \le \phi \le \pi$. Arrows denote the identification on the boundary. Dots on the left panel indicate the orbifold fixed points.

4.2 S^2/Z_2 Orbifold UED Model

Let us review the orbifold S^2/Z_2 UED model [34]. In the S^2/Z_2 model, a point (θ, ϕ) on the two sphere is identified with the point $(\pi - \theta, -\phi)$. There are two fixed points $(\pi/2, 0)$ and $(\pi/2, \pi)$ in contrast to the S^2 and PS UED models which are compactified on smooth backgrounds. See the left panel in Fig. 1 for a schematic view.

In the S^2/Z_2 model, we can add the localized terms at the orbifold fixed points:

$$\Delta S = \int \mathrm{d}^6 z \sqrt{-g} \left[\delta\left(\theta - \frac{\pi}{2}\right) \delta(\phi) \mathcal{L}_{(\pi/2,0)}(x) + \delta\left(\theta - \frac{\pi}{2}\right) \delta(\phi - \pi) \mathcal{L}_{(\pi/2,\pi)}(x) \right]. \tag{94}$$

This situation is the same as in the 5D UED model compactified on the orbifold S^1/Z_2. In the minimal version of the 5D model, the localized terms are assumed to be zero at the UV cutoff scale Λ, and are generated via the RGE running at lower scales. Recently, the one-loop mass correction under the same assumption is obtained for the 6D S^2/Z_2 orbifold UED [77].

The KK-mode function of spin-weight s becomes

$$f_{s,t}^{(j,m)}(\theta,\phi)^{\,N}_{\,S} = \begin{cases} \dfrac{1}{2R}\left[{}_sY_{jm}(\theta,\phi) + (-1)^{j-s}{}_sY_{j-m}(\theta,\phi)\right] e^{\pm is\phi} & \text{for } t = +1, \\ \dfrac{1}{2R}\left[{}_sY_{jm}(\theta,\phi) - (-1)^{j-s}{}_sY_{j-m}(\theta,\phi)\right] e^{\pm is\phi} & \text{for } t = -1, \end{cases} \tag{95}$$

where $t = \pm 1$ is the Z_2 parity. The mode function $f_{s,t}^{(j,m)}$ has the Z_2 symmetry:

$$f_{s,t=\pm1}^{(j,m)}(\pi - \theta, -\phi)^{\,N}_{\,S} = \pm f_{s,t=\pm1}^{(j,m)}(\theta,\phi)^{\,S}_{\,N}. \tag{96}$$

The number of degrees of freedom for each KK mode is reduced by the Z_2-symmetry, namely, the independent m modes are not $-j \le m \le j$ but $0 \le m \le j$ for each j-th level.

Under the translation $(\theta, \phi) \to (\theta, \phi + \pi)$, the KK-mode functions transform as

$$
\begin{aligned}
f^{(j,m)}_{s=0,t=+1}(\theta, \phi + \pi)^{\,N}_{\,S} &= (-1)^m f^{(j,m)}_{s=0,t=+1}(\theta, \phi)^{\,N}_{\,S}\,, \\
f^{(j,m)}_{s=\pm1,t=-1}(\theta, \phi + \pi)^{\,N}_{\,S} &= -(-1)^m f^{(j,m)}_{s=\pm1,t=-1}(\theta, \phi)^{\,N}_{\,S}\,.
\end{aligned}
\tag{97}
$$

We find that each KK-mode has the KK-parity $(-1)^m$, which is the remnant of the KK angular momentum conservation.

Let us consider the $m = 0$ modes of each j-th KK-level. The $m = 0$ modes of a field with spinweight $s = 0, \pm1$ are

$$
f^{(j,\,m=0)}_{s=0,\,t=+1}(\theta, \phi)^{\,N}_{\,S} = \frac{1}{2R}\left(1 + (-1)^j\right)\,{}_0Y_{j0}(\theta, \phi), \tag{98}
$$

$$
f^{(j,\,m=0)}_{s=+1,\,t=-1}(\theta, \phi)^{\,N}_{\,S} = \frac{1}{2R}\left(1 + (-1)^j\right)\,{}_1Y_{j0}(\theta, \phi)\,e^{\pm i\phi}, \tag{99}
$$

$$
f^{(j,\,m=0)}_{s=-1,\,t=-1}(\theta, \phi)^{\,N}_{\,S} = \frac{1}{2R}\left(1 + (-1)^j\right)\,{}_{-1}Y_{j0}(\theta, \phi)\,e^{\mp i\phi}. \tag{100}
$$

We find that the $m = 0$ mode appears only in an even j-th level and that the degeneracy number of each KK-level is

$$
\begin{array}{ccc}
j+1 & \text{for} & j : \text{even}, \\
j & \text{for} & j : \text{odd}.
\end{array}
\tag{101}
$$

4.3 Projective Sphere UED Model (PS)

Let us review the 6D UED model compactified on the Projective Sphere (PS) [38], which is the manifold obtained by the identification of the antipodal points

$$
(\theta, \phi) \sim (\pi - \theta, \phi + \pi) \tag{102}
$$

from the two-sphere S^2; see Fig. 1. PS has no fixed points unlike the S^2/Z_2 described above, and is non-orientable. The KK mass spectra of the gauge and scalar fields are distinctive from other compactifications as we will see below. We will see that as a result of the identification condition, the zero-mode of the $U(1)_X$ gauge field is eliminated.

Let us see how the antipodal identification (102) relates the fields on the north and south charts. We first note that the identification must leave the monopole configuration (2) intact. For that, it suffices to identify them with the twist of the 6D CP-transformation:

$$
\hat{X}^N_M(x,\, \pi - \theta,\, \phi + \pi) = \left[\hat{X}^S_M(x,\, \theta,\, \phi)\right]^{\mathrm{CP}}, \tag{103}
$$

where

$$
\left[\hat{X}_M\right]^{\mathrm{CP}} = (-1) \times \begin{cases} \hat{X}_M & (M \neq \theta), \\ -\hat{X}_M & (M = \theta). \end{cases} \tag{104}
$$

More concretely,

$$\begin{aligned}
\hat{X}^N_\mu(x,\,\pi-\theta,\,\phi+\pi) &= -\hat{X}^S_\mu(x,\,\theta,\,\phi),\\
\hat{X}^N_\theta(x,\,\pi-\theta,\,\phi+\pi) &= \hat{X}^S_\theta(x,\,\theta,\,\phi),\\
\hat{X}^N_\phi(x,\,\pi-\theta,\,\phi+\pi) &= -\hat{X}^S_\phi(x,\,\theta,\,\phi).
\end{aligned} \tag{105}$$

We note that the chart dependence exists only for $\hat{X}_\phi$; see Eq. (3), where the classical part $\mathcal{X}$ and the total field $\hat{X}$ should obey the same gauge transformation; in particular, $\hat{X}^N_\mu(\theta,\phi) = \hat{X}^S_\mu(\theta,\phi)$.

As we have seen, the existence of the zero-mode of $U(1)_X$ gauge field was the major problem of the sphere-based UED models. It is important that the identification (105) removes the zero mode of the X_μ field: The gauge field is expanded as Eq. (72), and hence $\hat{X}^{00}_\mu(x) = -\hat{X}^{00}_\mu(x) = 0$ because $Y_{00}(\pi-\theta,\,\phi+\pi) = Y_{00}(\theta,\,\phi)$. Surviving modes are odd ones: $X^{jm}_\mu(x)$ with $j = 1, 3, 5, \ldots$ and $-j \le m \le j$; see Eq. (15) with $s = 0$. The result is shown in Fig. 2.

The standard model fermion is realized as a zero mode of a 6D fermion. Note that the 6D fermion with chirality plus (minus) yields a 4D left (right) handed Weyl fermions as a massless zero mode, as shown in Section 3.2. We assign the following 6D chiralities to the 6D spinor fields for anomaly cancellation:

$$Q_+,\ U_-,\ D_-,\ L_+,\ E_-,\ N_-. \tag{106}$$

Q and L are $SU(2)_L$ quark and lepton doublets, respectively; U, D and E, N are $SU(2)_L$ singlet (up, down) quarks and (charged, neutral) leptons, respectively. It is remarkable that three generations of fermions are required by the cancellation of the 6D gravitational and $SU(2)_L$ global anomalies [30], which cannot be removed by the Green-Schwarz mechanism.

In order to allow a zero mode, the fermion must couple to $U(1)_X$; see Eq. (61). Since the $U(1)_X$ field is identified with the 6D CP twist (103), it is natural to identify the fermion the same way:

$$\Psi^N(x,\,\pi-\theta,\,\phi+\pi) = \left[\Psi^S(x,\,\theta,\,\phi)\right]^{\mathrm{CP}}, \tag{107}$$

where the 6D CP transformation is summarized in Appendix B. Note that the 6D CP transformation alters the 6D chirality; see Eq. (159). In order to let fermions have 6D CP invariant gauge interaction with $U(1)_X$, we introduce "mirror fermions"

$$\mathcal{Q}_-,\ \mathcal{U}_+,\ \mathcal{D}_+,\ \mathcal{L}_-,\ \mathcal{E}_+,\ \mathcal{N}_+, \tag{108}$$

which have the opposite 6D chiralities and opposite SM and $U(1)_X$ charges compared to original fermions (106).[12] Note that we identify these mirror fermions with the original ones under the antipodal projection:

$$Q^N_+(x,\,\pi-\theta,\,\phi+\pi) = \left[\mathcal{Q}^S_-(x,\,\theta,\,\phi)\right]^{\mathrm{CP}}, \tag{109}$$

[12] For fermions, the curly letters are used for mirrors and not for a classical configuration.

6D field	$U(1)_X$	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	zero mode
Q_+	-1	3	2	1/6	q_L
U_-	-1	3	1	2/3	u_R
D_-	-1	3	1	$-1/3$	d_R
L_+	-1	1	2	$-1/2$	l_L
N_-	-1	1	1	0	ν_R
E_-	-1	1	1	-1	e_R
$\mathcal{Q}_-$	-1	3^*	2^*	$-1/6$	$-(q_L)^c$
$\mathcal{U}_+$	-1	3^*	1	$-2/3$	$(u_R)^c$
$\mathcal{D}_+$	-1	3^*	1	1/3	$(d_R)^c$
$\mathcal{L}_-$	-1	1	2^*	1/2	$-(l_L)^c$
$\mathcal{N}_+$	-1	1	1	0	$(\nu_R)^c$
$\mathcal{E}_+$	-1	1	1	1	$(e_R)^c$
H	0	1	2	1/2	H^{00}

Table 1: Assignment of charges and zero mode. The upper six spinors are physically independent fields and lower ones are mirrors. $\pm$ denotes the 6D chirality, while c denotes 4D charge conjugation which interchanges four-dimensional chiralities L and R.

and similarly for others. Therefore these mirrors do not lead to extra degrees of freedom.

Contrary to $U(1)_X$, there must remain the zero-modes of the SM gauge fields. Therefore the identification conditions of these two classes of gauge fields must be different from each other. This difference implies that the gauge interactions of the spinor fields to the $U(1)_X$ and SM gauge fields must be different too. We impose the following identification for the SM gauge fields:

$$\hat{A}^N_M(x,\,\pi-\theta,\,\phi+\pi)=\left[\hat{A}^S_M(x,\,\theta,\,\phi)\right]^{\mathrm{P}}, \tag{110}$$

where

$$\left[\hat{A}_M\right]^{\mathrm{P}}=\begin{cases}\hat{A}_M & (M\neq\theta),\\ -\hat{A}_M & (M=\theta).\end{cases} \tag{111}$$

More Concretely,

$$\begin{aligned}\hat{A}^N_\mu(x,\,\pi-\theta,\,\phi+\pi)&=\hat{A}^S_\mu(x,\,\theta,\,\phi),\\ \hat{A}^N_\theta(x,\,\pi-\theta,\,\phi+\pi)&=-\hat{A}^S_\theta(x,\,\theta,\,\phi),\\ \hat{A}^N_\phi(x,\,\pi-\theta,\,\phi+\pi)&=\hat{A}^S_\phi(x,\,\theta,\,\phi).\end{aligned} \tag{112}$$

The zero mode survives under this projection.

The covariant derivative on the SM fermion (106) is

$$D_M=\partial_M+ig\hat{A}_M+ig_XQ_X\hat{X}_M+\Omega_M, \tag{113}$$

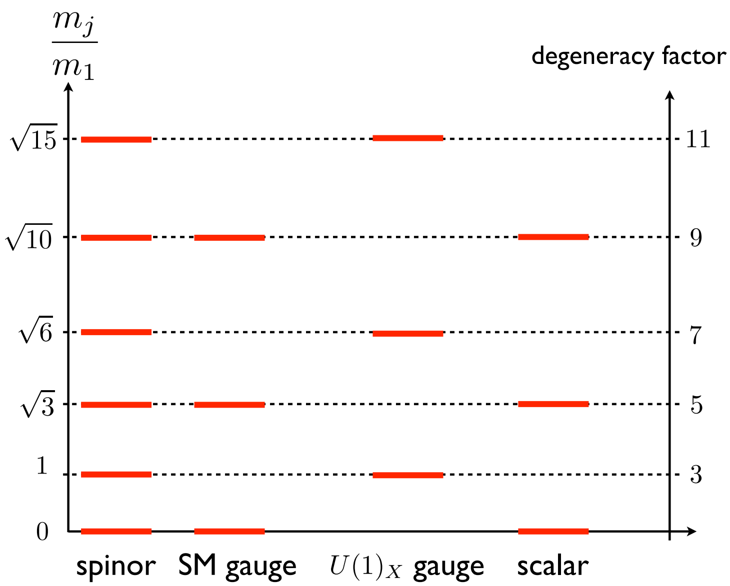

Figure 2: Tree level KK mass spectrum for the spinor, SM gauge, $U(1)_X$ gauge, and scalar fields. The mass splitting due to the electroweak symmetry breaking is neglected.

where Ω_M is the spin connection (163), g and $\hat{A}_M$ are the SM gauge coupling and field, respectively, and Q_X is the $U(1)_X$ charge which we have taken $Q_X = -1/n$; see Section 3.2. On the other hand, the covariant derivative on the mirror fermion (108) is

$$D_M = \partial_M + ig\left[\hat{A}_M\right]^C + ig_X Q_X \hat{X}_M + \Omega_M, \tag{114}$$

where

$$\left[\hat{A}^a_M T^a\right]^C = \hat{A}^a_M \left(-T^a\right)^{\mathrm{T}}, \tag{115}$$

as usual. The extra 6D charge conjugation is put so that the identification (110) leads to the CP transformation that matches Eq. (109). We summarize our charge assignment in Table 1.

The SM Higgs field must have a zero-mode, and we impose the identification:

$$\hat{H}^N(x, \pi - \theta, \phi + \pi) = \hat{H}^S(x, \theta, \phi). \tag{116}$$

It is obvious that there remains a zero-mode.

The Yukawa interaction is given by

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} = -\Big[& y_D\left(\overline{Q_+}HD_- - \overline{\mathcal{Q}^C_-}H\mathcal{D}^C_+\right) + y_U\left(\overline{Q_+}\epsilon H^* U_- - \overline{\mathcal{Q}^C_-}\epsilon H^*\mathcal{U}^C_+\right) \\ &+ y_E\left(\overline{L_+}HE_- - \overline{\mathcal{L}^C_-}H\mathcal{E}^C_+\right) + \text{h.c.}\Big],\end{aligned} \tag{117}$$

where y_x are Yukawa couplings for the field x. The invariance under the antipodal projection follows from

$$\begin{aligned}
&\overline{Q_+^N(x,\,\pi-\theta,\,\phi+\pi)}\,H^N(x,\,\pi-\theta,\,\phi+\pi)\,D_-^N(x,\,\pi-\theta,\,\phi+\pi)\\
&\quad=-\overline{\left(\mathcal{Q}_-^S\right)^C(x,\,\theta,\,\phi)}\,H^S(x,\,\theta,\,\phi)\,\left(\mathcal{D}_+^S\right)^C(x,\,\theta,\,\phi),
\end{aligned}\tag{118}$$

etc.

We have spelled out the KK modes for the $U(1)_X$ gauge field and the SM particles, namely, the Higgs boson, fermions, and the SM gauge bosons. Only the KK-modes with odd (even) j survive the antipodal projection for the $U(1)_X$ gauge (SM gauge and Higgs) boson. On the other hand, no fermion KK modes are projected out because we have doubled the number of modes by introducing the mirror fermions. As the result, the number of fermion degrees of freedom is the same as the S^2 UED model. In Fig. 2, we summarize the KK spectrum of the PS model.

5 Vacuum stability constraint

The vacuum stability leads to the most stringent upper bound on the ultraviolet cutoff scale Λ of the UED models [46]. Since the idea is only briefly sketched in Ref. [46], we clarify the argument more in detail here.

In D space-time dimensions, the Higgs action is written as

$$S=\int\sqrt{-g}\,\mathrm{d}^Dx\left[-\left(D_MH\right)^\dagger D^MH-V(H)\right],\tag{119}$$

with

$$V=m^2\left|H\right|^2+\frac{\hat\lambda}{\Lambda^{D-4}}\left|H\right|^4+\frac{\hat\lambda'}{\Lambda^{2D-6}}\left|H\right|^6+\cdots,\tag{120}$$

where the hatted $\hat\lambda$, $\hat\lambda'$, ... are dimensionless coupling constants. We note that $\hat\lambda$ contains linear (quadratic) divergence in the 5D (6D) model. At the one-loop level:

$$\hat\lambda(\mu)=\begin{cases}\lambda_{B,1}\,\Lambda+b\ln\frac{\mu}{\Lambda}+c & (D=5),\\ \lambda_{B,2}\,\Lambda^2+\lambda_{B,1}\,\Lambda+b\ln\frac{\mu}{\Lambda}+c & (D=6),\end{cases}\tag{121}$$

where the mass dimension of the field and the bare couplings is $[H]=(D-2)/2$ and $[\lambda_{B,n}]=-n$, respectively.

The zero mode Higgs h is constant in the extra dimension if neglect the electroweak symmetry breaking effects, and hence

$$H=\frac{h}{\sqrt{\mathrm{vol}}}+\cdots,\tag{122}$$

where vol is the volume of the extra dimension(s), with mass dimension [vol] $= 4 - D$. Therefore the 4D potential for the zero mode is

$$V_{\rm 4D} = m^2 \left|h\right|^2 + \frac{\hat{\lambda}}{(\text{vol})\,\Lambda^{D-4}} \left|h\right|^4 + \frac{\hat{\lambda}'}{(\text{vol})^2\,\Lambda^{2D-6}} \left|h\right|^6 + \cdots . \tag{123}$$

We see that the four dimensional Higgs quartic coupling $\lambda_{\rm 4D}$ is given by

$$\lambda_{\rm 4D}(\Lambda) = \frac{\hat{\lambda}(\Lambda)}{(\text{vol})\,\Lambda^{D-4}} = \begin{cases} \dfrac{\lambda_{B,1}}{\text{vol}} + \dfrac{c}{(\text{vol})\,\Lambda} & \text{for } D = 5, \\ \dfrac{\lambda_{B,2}}{\text{vol}} + \dfrac{\lambda_{B,1}}{(\text{vol})\,\Lambda} + \dfrac{c}{(\text{vol})\,\Lambda^2} & \text{for } D = 6. \end{cases} \tag{124}$$

The left hand side can be estimated from the low energy inputs through the 4D renormalization group running, that is, from the running coupling $\lambda(\mu)$ at the scale $\mu = \Lambda$.

Even though we can never know the bare coupling $\lambda_{B,i}$ from the low energy data, what matters for the stability of the potential is the quantity $\hat{\lambda}$, which can be evaluated within the low energy (KK reduced) 4D effective theory.[13] The bare constants are screened from the effective potential in the low energy theory at the scales $\mu < \Lambda$; see e.g. Appendix B in Ref. [78].

If we have negative $\lambda_{\rm 4D}(\mu)$ at some scale μ, then it necessarily requires the higher dimensional terms $\hat{\lambda}'$ etc. suppressed by $\Lambda \sim \mu$, in order to avoid the unbounded potential. Therefore we can read off the cutoff Λ from the scale where the running quartic coupling $\lambda_{4D}(\mu)$ becomes negative.

6 Summary

We have presented a review on the 6D UED models compactified on sphere, namely on S^2, S^2/Z_2, and PS. We have spelled out the basic techniques to treat the fields on sphere in terms of the Newman-Penrose eth formalism. KK expansion of the scalar, spinor, and vector fields are given. We have reviewed how the various fields are projected on the S^2/Z_2 orbifold and on the PS. We have critically reconsidered the $U(1)_X$ problem of the sphere-based UED models. We point out that the S^2 and the orbifold S^2/Z_2 models need a modification. We have explained the conceptual background of our previous work on the vacuum stability bound.

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[13] If one likes the bottom-up approach, the bare quantity can be regarded as totally unphysical. If one is interested in the ultraviolet completion, then the bare quantity itself becomes of interest, together with the specification of the regularization scheme. What we argue here is that no matter which viewpoint one takes, the stability argument can be done solely by the running coupling $\lambda_{\rm 4D}(\mu)$, which is reliable up to the cutoff scale $\mu < \Lambda$.

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Appendix

A Sphere metric

From the metric (1), the non-zero components of the Christoffel symbol,

$$\Gamma^{M}{}_{NL} := \frac{g^{MK}}{2}(-\partial_K g_{NL} + \partial_N g_{LK} + \partial_L g_{KN}), \tag{125}$$

are

$$\Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta, \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta. \tag{126}$$

The Riemann tensor

$$\mathcal{R}^{M}{}_{NKL} := -\,\partial_L\Gamma^{M}{}_{NK} + \partial_K\Gamma^{M}{}_{NL} - \Gamma^{P}{}_{NK}\Gamma^{M}{}_{LP} + \Gamma^{P}{}_{NL}\Gamma^{M}{}_{KP} \tag{127}$$

has the non-zero components:

$$\mathcal{R}^{\theta}{}_{\phi\theta\phi} = -\mathcal{R}^{\theta}{}_{\phi\phi\theta} = \sin^2\theta, \qquad \mathcal{R}^{\phi}{}_{\theta\phi\theta} = -\mathcal{R}^{\phi}{}_{\theta\theta\phi} = 1. \tag{128}$$

The Ricci tensor $\mathcal{R}_{MN} := \mathcal{R}^{K}{}_{MKN}$ are

$$\mathcal{R}_{\theta\theta} = 1, \qquad \mathcal{R}_{\phi\phi} = \sin^2\theta, \qquad \text{others} = 0. \tag{129}$$

The Ricci scalar $\mathcal{R} := \mathcal{R}^{M}{}_{M}$ reads

$$\mathcal{R} = 2/R^2. \tag{130}$$

B Six-dimensional spinor on sphere

In this section, we summarize our notations on the 6D spinor on sphere which we use in Section 3. We write the vielbein as

$$\left[e_M{}^{\underline{N}}\right]_{M=0,\dots,3,\theta,\phi;\,\underline{N}=\underline{0},\dots,\underline{3},\underline{\theta},\underline{\phi}} = \mathrm{diag}(1,1,1,1,R,R\sin\theta)\,. \tag{131}$$

Underlined indices $\underline{M},\underline{N},\dots$ run for $\underline{0},\dots,\underline{3}$; $\underline{\theta},\underline{\phi}$ on tangent space. Using the vielbein 1-form

$$e^{\underline{M}} = e^{\underline{M}}{}_N \mathrm{d}z^N, \tag{132}$$

we define the following new basis:

$$\begin{pmatrix} e^{\underline{5}} \\ e^{\underline{6}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \pm\sin\phi \\ \mp\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^{\underline{\phi}} \\ e^{\underline{\theta}} \end{pmatrix}, \tag{133}$$

where the upper and lower signs are for the north and south charts, respectively. We impose the 6D Clifford algebra on the gamma matrices on the new basis:

$$\left\{\Gamma^{\underline{A}}, \Gamma^{\underline{B}}\right\} = 2\eta^{\underline{AB}}, \tag{134}$$

where $\underline{A}, \underline{B}, \ldots$ run for $\underline{0}, \ldots, \underline{3}$; $\underline{5}, \underline{6}$, and the flat space metric is

$$\left[\eta^{\underline{AB}}\right]_{\underline{A},\underline{B}=\underline{0},\ldots,\underline{3};\underline{5},\underline{6}} = \left[\eta_{\underline{AB}}\right]_{\underline{A},\underline{B}=\underline{0},\ldots,\underline{3};\underline{5},\underline{6}} = \mathrm{diag}(-1, 1, \ldots, 1)\,. \tag{135}$$

Our choice for 6D gamma matrices are

$$\begin{aligned} \Gamma^{\underline{\mu}} &:= \gamma^{\underline{\mu}} \otimes \sigma_1 = \begin{bmatrix} & \gamma^{\underline{\mu}} \\ \gamma^{\underline{\mu}} & \end{bmatrix}, \\ \Gamma^{\underline{5}} &:= \gamma^{\underline{5}} \otimes \sigma_1 = \begin{bmatrix} & \gamma^{\underline{5}} \\ \gamma^{\underline{5}} & \end{bmatrix}, \\ \Gamma^{\underline{6}} &:= \mathrm{I}_4 \otimes \sigma_2 = \begin{bmatrix} & -i\mathrm{I}_4 \\ i\mathrm{I}_4 & \end{bmatrix}, \end{aligned} \tag{136}$$

where I_n is the $n \times n$ identity matrix and the 4D gamma matrices are given by

$$\gamma^{\underline{\mu}} := -i \begin{pmatrix} & \sigma^\mu \\ \overline{\sigma}^\mu & \end{pmatrix}, \qquad \gamma^{\underline{5}} := -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathrm{I}_2 & \\ & -\mathrm{I}_2 \end{pmatrix}, \tag{137}$$

$$\left(\sigma^\mu\right)_{\mu=0,\ldots,3} = \left(\mathrm{I}_2, \sigma_1, \sigma_2, \sigma_3\right), \qquad \left(\overline{\sigma}^\mu\right)_{\mu=0,\ldots,3} = \left(\mathrm{I}_2, -\sigma_1, -\sigma_2, -\sigma_3\right), \tag{138}$$

with σ_1, σ_2, σ_3 being the Pauli matrices. A slot left blank is understood to be filled with 0. Recall that under the infinitesimal local Lorentz transformation

$$\Lambda^{\underline{A}}{}_{\underline{B}}(z) = \delta^{\underline{A}}_{\underline{B}} + \omega^{\underline{A}}{}_{\underline{B}}(z), \tag{139}$$

the 6D spinor transforms as

$$\Psi(z) \to S(\Lambda(z))\Psi(z) = \left[1 + \frac{1}{2}\omega_{\underline{AB}}(z)\Sigma^{\underline{AB}}\right]\Psi(z), \tag{140}$$

where

$$\Sigma^{\underline{AB}} := \frac{1}{4}\left[\Gamma^{\underline{A}}, \Gamma^{\underline{B}}\right], \tag{141}$$

are the local Lorentz generators. The gamma matrices on the original basis is dependent on chart, in particular on the coordinate ϕ:

$$\begin{pmatrix}\Gamma^{\underline{\phi}}(\phi)\\ \Gamma^{\underline{\theta}}(\phi)\end{pmatrix} = \begin{pmatrix}\cos\phi & \mp\sin\phi\\ \pm\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\Gamma^{\underline{5}}\\ \Gamma^{\underline{6}}\end{pmatrix}, \tag{142}$$

where the upper and lower signs are for the north and south charts, respectively. These gamma matrices also satisfy the Clifford algebra in both charts:

$$\left\{\Gamma^{\underline{M}}(\phi), \Gamma^{\underline{N}}(\phi)\right\} = 2\eta^{\underline{MN}}. \tag{143}$$

In this notation, the 6D chirality operator

$$\Gamma^{\underline{7}} := -\Gamma^{\underline{0}}\Gamma^{\underline{1}}\Gamma^{\underline{2}}\Gamma^{\underline{3}}\Gamma^{\underline{5}}\Gamma^{\underline{6}} = \begin{bmatrix}\mathrm{I}_4 & \\ & -\mathrm{I}_4\end{bmatrix}, \tag{144}$$

commutes with all the local Lorentz generators:

$$\left[\Gamma^{\underline{7}}, \Sigma^{\underline{AB}}\right] = \left[\Gamma^{\underline{7}}, \Sigma^{\underline{MN}}(\phi)\right] = 0 \tag{145}$$

in both charts, where

$$\Sigma^{\underline{MM}}(\phi) := \frac{1}{4}\left[\Gamma^{\underline{M}}(\phi), \Gamma^{\underline{M}}(\phi)\right]. \tag{146}$$

The eigenspinors of $\Gamma^{\underline{7}}$:

$$\Psi_{\pm} := \frac{\mathrm{I}_8 \pm \Gamma^{\underline{7}}}{2}\Psi, \qquad\qquad \Gamma^{\underline{7}}\Psi_{\pm} = \pm\Psi_{\pm} \tag{147}$$

form an irreducible representation of the 6D Lorentz group so that

$$\Psi_+ = \begin{bmatrix}\psi_+\\ \\ \end{bmatrix}, \qquad\qquad \Psi_- = \begin{bmatrix}\\ \psi_-\end{bmatrix}, \tag{148}$$

are independent of each other.

The 6D Dirac adjoint spinors are defined as

$$\overline{\Psi} := \Psi^\dagger B = \left(\overline{\psi_-}\ \overline{\psi_+}\right), \tag{149}$$

with

$$B := i\Gamma^{\underline{0}} = \begin{pmatrix} & \beta\\ \beta & \end{pmatrix}, \qquad\qquad \beta := i\gamma^{\underline{0}} = \begin{pmatrix} & \mathrm{I}_2\\ \mathrm{I}_2 & \end{pmatrix}, \tag{150}$$

which transforms as

$$\overline{\Psi}(z) \to \overline{\Psi}(z)\, S^{-1}(\Lambda(z))\,. \tag{151}$$

Note that

$$S^{-1}(\Lambda(z))\,\Gamma^{\underline{A}}\,S\,(\Lambda(z))=\Lambda^{\underline{A}}{}_{\underline{B}}(z)\,\Gamma^{\underline{B}}. \tag{152}$$

The 6D charge conjugation of a spinor field $\Psi^C := \eta_\Psi C\Psi^*$ is defined so that it transforms as

$$\Psi^C(z)\to S(\Lambda(z))\;\Psi^C(z), \tag{153}$$

where $|\eta_\Psi|=1$ is the intrinsic charge conjugation parity and C should satisfy $C\left(\Sigma^{\underline{AB}}\right)^*=\Sigma^{\underline{AB}}C$ to realize the transformation (153). One can check that

$$C:=\eta\Gamma^{\underline{2}}\Gamma^{\underline{5}}=\eta\begin{bmatrix} & \epsilon & & \\ \epsilon & & & \\ & & & \epsilon \\ & & \epsilon & \end{bmatrix}, \tag{154}$$

satisfies the requirement, where η is an arbitrary phase factor, $|\eta|=1$, and ϵ is the antisymmetric matrix

$$\epsilon:=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{155}$$

Hereafter, we take $\eta=1$. Note that the 6D charge conjugation does not change the 6D chirality $\left(\Psi^C\right)_\pm=\left(\Psi_\pm\right)^C=:\Psi^C_\pm$, unlike the four-dimensional charge conjugation: $\left(\psi_L\right)^c=\left(\psi^c\right)_R$.

We can for example choose the parity transformation of the 6D fermion as

$$\Psi^{\mathrm{P}}=\xi_\Psi\Gamma^{\underline{5}}\Psi, \tag{156}$$

where $|\xi_\Psi|=1$ is the intrinsic parity, so that we get

$$\left[\overline{\Psi}\Gamma^{\underline{A}}\Psi\right]^{\mathrm{P}}=\begin{cases}+\overline{\Psi}\Gamma^{\underline{A}}\Psi & (A\neq 5),\\ -\overline{\Psi}\Gamma^{\underline{A}}\Psi & (A=5).\end{cases} \tag{157}$$

Then the CP transformation becomes

$$\Psi^{\mathrm{CP}}=\xi_\Psi\Gamma^{\underline{5}}\Psi^C=\xi_\Psi\eta_\Psi\Gamma^{\underline{5}}C\Psi=-\xi_\Psi\eta_\Psi\Gamma^{\underline{2}}\Psi. \tag{158}$$

Note that the 6D CP transformation alters the 6D chirality (as well as P does):

$$\left(\Psi_\pm\right)^{\mathrm{CP}}=\left(\Psi^{\mathrm{CP}}\right)_\mp. \tag{159}$$

We define the spin-connection

$$\Omega_M:=\frac{1}{2}\Omega_{M\underline{AB}}\Sigma^{\underline{AB}}, \tag{160}$$

where

$$\Omega_{M\underline{A}}{}^{\underline{B}} := e^N{}_{\underline{A}} \nabla_M e_N{}^{\underline{B}} = e^N{}_{\underline{A}} \left(\partial_M e_N{}^{\underline{B}} - \Gamma^L{}_{MN} e_L{}^{\underline{B}} \right). \tag{161}$$

In our notations,

$$\Omega_{\phi\underline{5}}{}^{\underline{6}} = -\Omega_{\phi\underline{6}}{}^{\underline{5}} = \cos\theta \mp 1, \qquad \text{others} = 0, \tag{162}$$

that is,

$$\Omega_\phi = (\cos\theta \mp 1)\ \Sigma^{\underline{56}} = \frac{i}{2} (\cos\theta \mp 1)\ \gamma^{\underline{5}} \otimes \sigma_3, \qquad \text{others} = 0, \tag{163}$$

where upper and lower signs are for north and south charts, respectively.

C Six-dimensional Bulk Dirac mass and The Higgs Mechanism

Finally, we consider the bulk Dirac mass term. Even in the 6D case, spinors with plus and minus 6D chiralities Ψ_+ and Ψ_- can have a Dirac mass if both of them have equal charges to each other for all the unbroken gauge interactions, similar to the four-dimensional case:

$$\begin{aligned} S &:= -\int \mathrm{d}^6 z \sqrt{-g} M_\Psi \left(\overline{\Psi_+}\Psi_- + \overline{\Psi_-}\Psi_+ \right) \\ &= -\int d^4x \sum_{j=j_{\min}}^{\infty} \sum_{m=-j}^{j} M_\Psi \left(\overline{\psi^{jm}_{+,4D}} \psi^{jm}_{-,4D} + \overline{\psi^{jm}_{-,4D}} \psi^{jm}_{+,4D} \right). \end{aligned} \tag{164}$$

We can diagonalize the mass matrix of the KK-modes,

$$\begin{aligned} \mathcal{L}^{4D\,jm}_{\text{mass}} &= -\begin{pmatrix} \overline{\psi^{jm}_{+,4D}} & \overline{\psi^{jm}_{-,4D}} \end{pmatrix} \begin{pmatrix} i m_j \gamma^{\underline{5}} & M_\Psi \\ M_\Psi & i m_j \gamma^{\underline{5}} \end{pmatrix} \begin{pmatrix} \psi^{jm}_{+,4D} \\ \psi^{jm}_{-,4D} \end{pmatrix} \\ &= -\begin{pmatrix} \overline{\psi^{jm}_{1,4D}} & \overline{\psi^{jm}_{2,4D}} \end{pmatrix} \begin{pmatrix} -\sqrt{m_j^2 + M_\Psi^2} & \\ & +\sqrt{m_j^2 + M_\Psi^2} \end{pmatrix} \begin{pmatrix} \psi^{jm}_{1,4D} \\ \psi^{jm}_{2,4D} \end{pmatrix}, \end{aligned} \tag{165}$$

$$\begin{pmatrix} \psi^{jm}_{+,4D} \\ \psi^{jm}_{-,4D} \end{pmatrix} = \begin{pmatrix} e^{\frac{\pi}{4} i \gamma^{\underline{5}}} \cos\alpha_j & -e^{\frac{\pi}{4} i \gamma^{\underline{5}}} \sin\alpha_j \\ e^{-\frac{\pi}{4} i \gamma^{\underline{5}}} \sin\alpha_j & e^{-\frac{\pi}{4} i \gamma^{\underline{5}}} \cos\alpha_j \end{pmatrix} \begin{pmatrix} \psi^{jm}_{1,4D} \\ \psi^{jm}_{2,4D} \end{pmatrix}, \qquad \tan 2\alpha_j := -\frac{M_\Psi}{m_j}. \tag{166}$$

We have obtained mass eigen-values $\pm\sqrt{{m_j}^2 + M_\Psi^2}$.

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