T1 theorem for homogeneous Triebel-Lizorkin spaces

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Abstract

For a singular integral operator T, G. David and J. L. Journé gave the necessary and sufficiently condition for T to be bounded on $L^2(\mathbb{R}^n)$([5]). This condition is called the T1 theorem. In this paper, we extend the T1 theorem to homogeneous Triebel-Lizorkin spaces in terms of generalized $\mathcal{Q}$-spaces.

1 Introduction

For a linear continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, we say $T \in \mathrm{CZO}(\delta)$ if T satisfies

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \qquad (x \notin \mathrm{supp}(f))$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, where $K(x,y)$ is a standard kernel, that is, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ and satisfying the following conditions for all $x, x_0, y, y_0 \in \mathbb{R}^n$:

$$(1)\ |K(x,y)| \le \frac{A_1}{|x-y|^n}, \tag{1.1a}$$

$$(2)\ |K(x,y) - K(x,y_0)| \le \frac{A_2|y-y_0|^\delta}{|x-y|^{n+\delta}} \qquad \left(|y-y_0| < \frac{|x-y|}{2}\right), \tag{1.1b}$$

$$(3)\ |K(x,y) - K(x_0,y)| \le \frac{A_3|x-x_0|^\delta}{|x-y|^{n+\delta}} \qquad \left(|x-x_0| < \frac{|x-y|}{2}\right). \tag{1.1c}$$

Here $\delta \in (0,1]$ is a positive constant. It is well known that if $T \in \mathrm{CZO}(\delta)$ is bounded on $L^2(\mathbb{R}^n)$, then T is also bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Operators in $\mathrm{CZO}(\delta)$ play an important role in harmonic analysis (see [14]). The Riesz transform and the Hardy transform are typical examples of $\mathrm{CZO}(\delta)$.

In this paper, we assume there exists a enough large constant $N > 0$ and the function $\Phi \in C_c^\infty(\mathbb{R}^n)$ satisfies the following conditions.

$$(1)\ \operatorname{supp}\Phi \subset B(0,1), \tag{1.2a}$$

$$(2)\ \int_{\mathbb{R}^n} \Phi(x)x^\alpha dx = 0 \quad \text{for all } \alpha \in (\mathbb{N}\cup\{0\})^n \text{ with } |\alpha| < N, \tag{1.2b}$$

$$(3)\ \Phi \text{ is sphere symmetric}, \tag{1.2c}$$

$$(4)\ \int_0^\infty [\hat{\Phi}(t\xi)]^2 \frac{dt}{t} = 1 \quad \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1.2d}$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ. Here and below, we define $\Phi_t(x) = \frac{1}{t^n}\Phi\left(\frac{x}{t}\right)$, $Q_t f = \Phi_t * f$ and $Q_t^2 f = Q_t(Q_t f)$. It is well known that for Φ satisfying (1.2d), we have

$$\int_0^\infty Q_t^2(f)\frac{dt}{t} = f, \tag{1.3}$$

for any $f \in L^2(\mathbb{R}^n)$. We refer to [8] for (1.3) and the existence of the function Φ satisfying (1.2a)-(1.2d), for example. Here and below, let $\mathcal{P}$ be the set of all polynomial functions.

Lemma 1.1. *For $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying* (1.2b)*, we have*

$$|\hat{\Phi}(\xi)| \lesssim \min\{|\xi|, \frac{1}{|\xi|}\}.$$

We refer [14] for Lemma 1.1.

Definition 1.2. Let $\Phi \in C_c^\infty(\mathbb{R}^n)$ be a function satisfying (1.2a)-(1.2d) and $1 < p,\ q < \infty$. Then, we define

$$||f||_{\dot{F}^0_{p,q}} = \left(\int_{\mathbb{R}^n}\left(\int_0^\infty |Q_t f(x)|^q \frac{dt}{t}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}},$$

$$\dot{F}^0_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} \mid ||f||_{\dot{F}^0_{p,q}} < \infty \right\}.$$

Remark that $\dot{F}^0_{p,q}(\mathbb{R}^n)$ is independent of the choice of Φ satisfying (1.2a)-(1.2d). Furthermore, $\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} f(x)dx = 0 \right\}$ is dense in $\dot{F}^0_{p,q}(\mathbb{R}^n)$. Next, we define the space $\mathrm{BMO}(\mathbb{R}^n)$, which is a very important space in harmonic analysis, and $\mathcal{Q}$-spaces, which generalize $\mathrm{BMO}(\mathbb{R}^n)$. Here and below, let $\mathcal{D}_0(\mathbb{R}^n)$ be the set of all functions $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\int \phi = 0$ and its topology be the restricted topology of $C_c^\infty(\mathbb{R}^n)$. The topology of $C_c^\infty(\mathbb{R}^n)$ is determined by the inductive limit of the topology of $C_c^r(\mathbb{R}^n)$ $(r > 0)$. and $\mathcal{D}_0'(\mathbb{R}^n)$ be the dual space of $\mathcal{D}_0(\mathbb{R}^n)$.

Definition 1.3. Let $1 < p < \infty$. Then, we define

$$||f||_{\mathrm{BMO}} = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx,$$

where $f_B = \frac{1}{|B|}\int_B f(x)dx$ and the supremum taken over all balls $B \subset \mathbb{R}^n$. Let $\mathrm{BMO}(\mathbb{R}^\mathrm{n})$ be the set of all measurable functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $||f||_{\mathrm{BMO}} < \infty$.

Note that $\mathrm{BMO}(\mathbb{R}^n) \subset \mathcal{D}_0'(\mathbb{R}^n)$. Actually, we can verify that by the following sense. It is well known that

$$||f||_{\mathrm{BMO}} \sim \inf_{c \in \mathbb{C}} \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - c| dx. \tag{1.4}$$

See [14] for example.

Definition 1.4. Let $\Phi \in C_\mathrm{c}^\infty(\mathbb{R}^n)$ be a function satisfying (1.2a)-(1.2d), $s \geq 0$ and $1 \leq q \leq p < \infty$. Then, we define

$$||f||_{\mathcal{Q}^s_{p,q}} = \sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|^{1-\frac{ps}{n}}} \int_B \left(\int_0^{r_B} |Q_t f(y)|^q \frac{dt}{t^{1+qs}} \right)^{\frac{p}{q}} dy \right)^{\frac{1}{p}},$$

where r_B is the radius of B and the supremum is taken over all balls. Let $\mathcal{Q}^s_{p,q}(\mathbb{R}^n)$ be the set of all functions $f \in \mathcal{D}_0'(\mathbb{R}^n)$ on $\mathbb{R}^n$ such that $||f||_{\mathcal{Q}^s_{p,q}} < \infty$.

Proposition 1.5. *Let $1 \leq p < \infty$ and $1 \leq q < \infty$, and $s \geq 0$.*

(1) $\mathcal{Q}^0_{2,2}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$,

(2) $\mathcal{Q}^\alpha_{p,q_1}(\mathbb{R}^n) \subset \mathcal{Q}^\beta_{p,q_2}(\mathbb{R}^n)$ *for* $0 \leq \beta < \alpha < \infty$, $1 \leq q_2 \leq q_1 < \infty$,

(3) $\mathcal{Q}^s_{p_1,q}(\mathbb{R}^n) \subset \mathcal{Q}^s_{p_2,q}(\mathbb{R}^n)$ *for* $1 \leq p_2 \leq p_1 < \infty$.

Proof. (1) is well known as the Carleson measure expression of BMO($\mathbb{R}^n$) (see [14, IV.4.3, Theorem 3]). (2) and (3) are easy to prove. In fact, for $\beta \le \alpha$ and $q_2 \le q_1$, putting $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q_3}$ for some $1 < q_3 \le \infty$, we have

$$\begin{aligned}
||f||_{\mathcal{Q}^{\beta}_{p,q_2}} &= \sup_{B\in\mathbb{R}^n} \left(\frac{1}{|B|^{1-\frac{p\beta}{n}}} \int_B \left(\int_0^{r_B} |Q_t f(y)|^{q_2} \frac{dt}{t^{1+q_2\beta}} \right)^{\frac{p}{q_2}} dy \right)^{\frac{1}{p}} \\
&= \sup_{B\in\mathbb{R}^n} \left(\frac{1}{|B|^{1-\frac{p\beta}{n}}} \int_B \left(\int_0^{r_B} |Q_t f(y)|^{q_2} t^{q_2(\alpha-\beta)} \frac{dt}{t^{1+q_2\alpha}} \right)^{\frac{p}{q_2}} dy \right)^{\frac{1}{p}} \\
&\le \sup_{B\in\mathbb{R}^n} \left(\frac{1}{|B|^{1-\frac{p\beta}{n}}} \int_B \left(\int_0^{r_B} |Q_t f(y)|^{q_1} \frac{dt}{t^{1+q_1\alpha}} \right)^{\frac{p}{q_1}} \left(\int_0^{r_B} t^{q_3(\alpha-\beta)} \frac{dt}{t} \right)^{\frac{p}{q_3}} dy \right)^{\frac{1}{p}} \\
&\lesssim \sup_{B\in\mathbb{R}^n} \left(\frac{r_B^{p(\alpha-\beta)}}{|B|^{1-\frac{p\beta}{n}}} \int_B \left(\int_0^{r_B} |Q_t f(y)|^{q_1} \frac{dt}{t^{1+q_1\alpha}} \right)^{\frac{p}{q_1}} dy \right)^{\frac{1}{p}} \\
&= \omega_n^{\frac{(\beta-\alpha)}{n}} ||f||_{\mathcal{Q}^{\alpha}_{p,q_1}},
\end{aligned}$$

where ω_n denotes the volume of the unit ball on $\mathbb{R}^n$. As a result, we have (2). Next, we prove (3). For $1 \le p_2 \le p_1 < \infty$, let $1 < p_3 \le \infty$ satisfy $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_3}$. From Hölder's inequality,

$$\begin{aligned}
||f||_{\mathcal{Q}^{s}_{p_2,q}} &= \sup_{B\in\mathbb{R}^n} \left(\frac{1}{|B|^{1-\frac{p_2 s}{n}}} \int_B \left(\int_0^{r_B} |Q_t f(y)|^{q} \frac{dt}{t^{1+qs}} \right)^{\frac{p_2}{q}} dy \right)^{\frac{1}{p_2}} \\
&\le \sup_{B\in\mathbb{R}^n} \frac{1}{|B|^{\frac{1}{p_2}-\frac{s}{n}}} \left(\int_B \left(\int_0^{r_B} |Q_t f(y)|^{q} \frac{dt}{t^{1+qs}} \right)^{\frac{p_1}{q}} dy \right)^{\frac{1}{p_1}} \cdot ||\chi_B||_{p_3} \\
&= \sup_{B\in\mathbb{R}^n} \frac{1}{|B|^{\frac{1}{p_2}-\frac{1}{p_3}-\frac{s}{n}}} \left(\int_B \left(\int_0^{r_B} |Q_t f(y)|^{q} \frac{dt}{t^{1+qs}} \right)^{\frac{p_1}{q}} dy \right)^{\frac{1}{p_1}} \\
&= ||f||_{\mathcal{Q}^{s}_{p_1,q}}.
\end{aligned}$$

That is the conclusion of (3). □

Lemma 1.6. *Let $s > 0$, $1 < q \le p < \infty$ and $\Phi \in C_c^{\infty}(\mathbb{R}^n)$ be a function satisfying* (1.2a) − (1.2d)*. Then, we have $\mathcal{Q}^s_{p,q}(\mathbb{R}^n) \subset \mathcal{D}'_0(\mathbb{R}^n)$.*

Proof. From Proposition 1.5, we have

$$\mathcal{Q}^s_{p,q}(\mathbb{R}^n) \subset \mathcal{Q}^0_{p,p}(\mathbb{R}^n).$$

It suffices to show that $\mathcal{Q}^0_{p,p}(\mathbb{R}^n) \subset \mathcal{D}'_0(\mathbb{R}^n)$ for any $1 < p < \infty$. We shall invoke the following inequality

$$\left|\int_0^\infty \int_{\mathbb{R}^n} f(x,t)g(x,t)dx\frac{dt}{t}\right| \lesssim \int_{\mathbb{R}^n} A_{p'}(f)(x)C_p(g)(x)dx, \tag{1.5}$$

for measurable functions on $\mathbb{R}^{n+1}_+$, f, g and $1 < p < \infty$. Here, we define

$$A_p(f)(x) = \left(\int_0^\infty \int_{B(x,t)} |f(y,t)|^p dy\frac{dt}{t^{n+1}}\right)^{1/p},$$

and

$$C_p(g)(x) = \sup_{x\in B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_0^{r_B}\int_B |g(y,t)|^p dy\frac{dt}{t}\right)^{1/p}.$$

See [2] for (1.5). Let $f \in \mathcal{Q}^0_{p,p}(\mathbb{R}^n)$ and $\phi \in \mathcal{D}_0(\mathbb{R}^n)$. From (1.5), we have

$$\begin{aligned}
|\langle f,\phi\rangle| &= \left|\left\langle f, \int_0^\infty Q_t^2(\phi)\frac{dt}{t}\right\rangle\right| \\
&= \left|\int_0^\infty \langle Q_t(f), Q_t(\phi)\rangle\frac{dt}{t}\right| \\
&\lesssim ||f||_{\mathcal{Q}^0_{p,p}}||A_{p'}(Q_t(\phi))||_1.
\end{aligned}$$

It suffices to show that $||A_{p'}(Q_t(\phi))||_1 < \infty$. We assume $\operatorname{supp}\phi \subset B = B(x_0, r)$ for a ball $B \subset \mathbb{R}^n$. We split $||A_{p'}(Q_t(\phi))||_1$ so that

$$||A_{p'}(Q_t(\phi))||_1 = ||A_{p'}(Q_t(\phi))||_{L^1(2B)} + \sum_{k=1}^\infty ||A_{p'}(Q_t(\phi))||_{L^1(2^{k+1}B\setminus 2^kB)}.$$

Keeping in mind that $||\mathcal{F}^{-1}f||_\infty \leq (2\pi)^{-n/2}||f||_1$ and $\mathcal{F}[f*g] = (2\pi)^{n/2}\mathcal{F}f\cdot\mathcal{F}\phi$, we have

$$\begin{aligned}
A_{p'}(Q_t(\phi))(x) &= \left(\int_0^\infty \int_{B(x,t)} |(\Phi_t*\phi)(y)|^{p'} dy\frac{dt}{t^{n+1}}\right)^{1/p'} \\
&= (2\pi)^{n/2}\left(\int_0^\infty \int_{B(x,t)} \left|\mathcal{F}^{-1}[\hat{\Phi}(t\xi)\cdot\hat{\phi}(\xi)](y)\right|^{p'} dy\frac{dt}{t^{n+1}}\right)^{1/p'} \\
&\lesssim \left(\int_0^\infty \int_{B(x,t)} \left|\int_{\mathbb{R}^n} |\hat{\Phi}(t\xi)|\cdot|\hat{\phi}(\xi)|d\xi\right|^{p'} dy\frac{dt}{t^{n+1}}\right)^{1/p'} \\
&= \omega_n^{1/p'}\left(\int_0^\infty \left|\int_{\mathbb{R}^n} |\hat{\Phi}(t\xi)|\cdot|\hat{\phi}(\xi)|d\xi\right|^{p'} \frac{dt}{t}\right)^{1/p'}
\end{aligned}$$

Here, the constant $\omega_n > 0$ is the volume of the unit ball on $\mathbb{R}^n$. From Lemma 1.1, we observe

$$
\begin{aligned}
&\int_0^\infty \left| \int_{\mathbb{R}^n} |\hat{\Phi}(t\xi)| \cdot |\hat{\phi}(\xi)| d\xi \right|^{p'} \frac{dt}{t} \\
&\qquad \lesssim \int_0^1 \left| \int_{\mathbb{R}^n} |t\xi| |\hat{\phi}(\xi)| d\xi \right|^{p'} \frac{dt}{t} + \int_1^\infty \left| \int_{\mathbb{R}^n} \frac{1}{|t\xi|} |\hat{\phi}(\xi)| d\xi \right|^{p'} \frac{dt}{t} \\
&\qquad \lesssim \left| \int_{\mathbb{R}^n} |\xi| |\hat{\phi}(\xi)| d\xi \right|^{p'} + \left| \int_{\mathbb{R}^n} \frac{1}{|\xi|} |\hat{\phi}(\xi)| d\xi \right|^{p'} \\
&\qquad \lesssim \left| \int_{\mathbb{R}^n} \frac{|\xi|}{(1+|\xi|)^{n+2}} d\xi \right|^{p'} + \left| \int_{B(0,1)} d\xi \right|^{p'} + \left| \int_{\mathbb{R}^n \setminus B(0,1)} \frac{1}{|\xi|(1+|\xi|)^n} d\xi \right|^{p'} \\
&\qquad < \infty.
\end{aligned}
$$

As a consequence, we conclude there exists a constant $C > 0$ and we have $A_{p'}(Q_t(\phi))(x) \le C$ for any $x \in \mathbb{R}^n$, which implies $||A_{p'}(Q_t(\phi))||_{L^1(2B)} \lesssim |2B| < \infty$. Next we estimate for the case $x \in 2^{k+1}B \setminus 2^k B$. From $\operatorname{supp} \Phi \subset B(0,1)$, we have $\operatorname{supp} Q_t(\phi) \subset B(x_0, r+t)$.Thus, if $t < 2^{k-1}r$, then $\operatorname{supp} Q_t(\phi) \cap B(x,t) = \emptyset$ for $x \in 2^{k+1}B \setminus 2^k B$. Furthermore, $\int \phi = 0$ implies that

$$
Q_t(\phi)(y) = \int_{\mathbb{R}^n} \Phi_t(y-z)\phi(z)dz = \int_{\mathbb{R}^n} \left(\Phi_t(y-z) - \Phi_t(y-x_0)\right) \phi(z)dz.
$$

Thus, for $x \in 2^{k+1}B \setminus 2^k B$, we observe

$$
\begin{aligned}
&A_{p'}(Q_t(\phi))(x) \\
&\qquad = \left(\int_0^\infty \int_{B(x,t)} |(\Phi_t * \phi)(y)|^{p'} dy \frac{dt}{t^{n+1}} \right)^{1/p'} \\
&\qquad = \left(\int_{2^{k-1}r}^\infty \int_{B(x,t)} \left| \int_{\mathbb{R}^n} \left(\Phi_t(y-z) - \Phi_t(y-x_0)\right) \phi(z)dz \right|^{p'} dy \frac{dt}{t^{n+1}} \right)^{1/p'} \\
&\qquad \lesssim \left(\int_{2^{k-1}r}^\infty \int_{B(x,t)} \left(\int_B \frac{|z-x_0|}{t^{n+1}} |\phi(z)| dz \right)^{p'} dy \frac{dt}{t^{n+1}} \right)^{1/p'} \\
&\qquad \lesssim r||\phi||_1 \left(\int_{2^{k-1}r}^\infty t^{-p'(n+1)} \frac{dt}{t} \right)^{1/p'}. \\
&\qquad \lesssim ||\phi||_1 \frac{1}{2^{k(n+1)} r^n}
\end{aligned}
$$

Here, we have used $|\Phi(y-z)-\Phi(y-x_0)| \lesssim |z-x_0|$ for the third inequality. As a result, we have

$$\begin{aligned}\sum_{k=1}^{\infty} ||A_{p'}(Q_t(\phi))||_{L^1(2^{k+1}B\setminus 2^k B)} &\lesssim \sum_{k=1}^{\infty} \frac{|2^{k+1}B|}{2^{k(n+1)}r^n} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &< \infty.\end{aligned}$$

□

2 Main theorem

Let $T \in \mathrm{CZO}(\delta)$. At first, we realize "$T1$", that is the image of the function 1 by T as an element of of $\mathcal{D}'_0(\mathbb{R}^n)$. Let $C^\infty_{\mathrm{c},0}(\mathbb{R}^n)$ be the set of all functions $\phi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ satisfying $\int \phi = 0$. The operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ naturally maps $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. For a function $\phi \in C^\infty_{\mathrm{c},0}(\mathbb{R}^n)$ and $f \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, let $\mu \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ be a function such that $0 \le \mu \le 1$ and $\mu = 1$ on a neighborhood of $\operatorname{supp}\phi$. We put

$$\begin{aligned}\langle \tilde{T} f, \phi\rangle &= \langle T(f\mu), \phi\rangle \\ &\quad + \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y)\phi(x)dx \right) f(y)(1-\mu(y))dy. \qquad (2.1)\end{aligned}$$

We claim that $\tilde{T} f$ is well defined as an element of $\mathcal{D}'_0(\mathbb{R}^n)$. The first part of the right-hand side of the above inequality is well defined from $f\mu \in \mathcal{D}(\mathbb{R}^n)$. For the second part, fix $x_0 \in \operatorname{supp}\phi$. Since $\int \phi = 0$,

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y)\phi(x)dx \right) f(y)(1-\mu(y))dy \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (K(x,y)-K(x_0,y))\phi(x)dx \right) f(y)(1-\mu(y))dy \\ &= \int_{\mathbb{R}^n} \left(\int_{|x-x_0|\ge|y-x_0|/2} (K(x,y)-K(x_0,y))\phi(x)dx \right) f(y)(1-\mu(y))dy \\ &\quad + \int_{\mathbb{R}^n} \left(\int_{|x-x_0|<|y-x_0|/2} (K(x,y)-K(x_0,y))\phi(x)dx \right) f(y)(1-\mu(y))dy \\ &=: I_1 + I_2.\end{aligned}$$

Putting $d = \mathrm{dist}(\mathrm{supp}\,\phi, \mathrm{supp}(1-\mu)) > 0$, we learn (1.1a) implies $|K(x,y)|$, $|K(x_0,y)| \lesssim d^{-n}$ for $x \in \mathrm{supp}\,\phi$ and $y \in \mathrm{supp}(1-\mu)$. Thus, we get

$$\begin{aligned}|I_1| &\lesssim d^{-n} \int_{\mathbb{R}^n} |\phi(x)| \left(\int_{|x-x_0| \geq |y-x_0|/2} |f(y)(1-\mu(y))| dy \right) dx \\ &\lesssim d^{-n} ||f||_\infty \int_{\mathbb{R}^n} |x-x_0|^n |\phi(x)| dx \\ &\lesssim d^{-n} ||f||_\infty ||\phi||_\infty (\mathrm{diam}(\mathrm{supp}\,\phi))^{2n}.\end{aligned}$$

On the other hand, keeping in mind (1.1b), we have

$$\begin{aligned}|I_2| &\lesssim \int_{\mathbb{R}^n} |x-x_0|^\delta \left(\int_{|x-x_0|<|y-x_0|/2} \frac{|f(y)|}{|y-x_0|^{n+\delta}} dy \right) |\phi(x)| dx \\ &\lesssim ||f||_\infty ||\phi||_1.\end{aligned}$$

Next, we verify that $\langle \tilde{T} f, \mu \rangle$ is independent of the choice of μ. Let $\mu, \zeta \in \mathcal{D}(\mathbb{R}^n)$ satisfy the same condition. Then, from $f(\mu-\zeta), \phi \in \mathcal{D}(\mathbb{R}^n)$ and $\mathrm{supp}[f(\mu-\zeta)] \cap \mathrm{supp}\,\phi \neq \emptyset$, we have

$$\langle T f(\mu-\zeta), \phi \rangle = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y) f(y)(\mu(y)-\zeta(y)) dy \right) \phi(x) dx.$$

Thus, we obtain

$$\begin{aligned}&\langle T(f\mu), \phi \rangle + \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y) f(y)(1-\mu(y)) dy \right) \phi(x) dx \\ &\qquad = \langle T(f\zeta), \phi \rangle + \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x,y) f(y)(1-\zeta(y)) dy \right) \phi(x) dx.\end{aligned}$$

As a result, $\langle T f, \phi \rangle$ does not depend on the choice of μ and $\tilde{T} f \in \mathcal{D}_0'(\mathbb{R}^n)$ for $f \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, which implies $\tilde{T}1 \in \mathcal{D}_0'(\mathbb{R}^n)$.
We will show that $Tf = \tilde{T} f$ for $f \in \mathcal{D}(\mathbb{R}^n)$. If $f \in \mathcal{D}(\mathbb{R}^n)$, then $f\mu, f(1-\mu) \in \mathcal{D}(\mathbb{R}^n)$ and we have

$$\langle Tf, \phi \rangle = \langle T(f\mu), \phi \rangle + \langle T(1-\mu), \phi \rangle.$$

We also have

$$\langle T(1-\mu), \phi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) f(y)(1-\mu(y)) dy \phi(x) dx,$$

which follows by $\operatorname{supp} f(1-\mu)\cap\phi=\emptyset$. Thus, we obtain $\langle\tilde{T}f,\phi\rangle=\langle Tf,\phi\rangle$ for $f\in\mathcal{D}(\mathbb{R}^n)$ and $\phi\in\mathcal{D}_0(\mathbb{R}^n)$. Here and below, we rewrite $\tilde{T}f$ by Tf.

A $C^{2[n/2]+2}$-function ϕ is said to be a bump function if it satisfies following conditions:

$$(1)\operatorname{supp}\phi\subset B(0,10),\qquad (2)|\partial_x^\alpha\phi(x)|\le 1\ \ (|\alpha|\le 2[n/2]+2).$$

We define the operator $\tau_{x_0}f(x)=f(x-x_0)$ for $x,x_0\in\mathbb{R}^n$.

Definition 2.1. A linear operator $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ is said to be weakly bounded (write $T\in WB$) if there exists a constant C such that:

$$|\langle T\tau_{x_0}(f_R),\tau_{y_0}(g_R)\rangle|\le CR^{-n}$$

for all bump functions f,g and $x_0,y_0\in\mathbb{R}^n$.

The following are the classical T1 theorem and our main theorem.

Theorem 2.2 ([5])**.** *Let $T\in\mathrm{CZO}(\delta)\cap WB$ with $0<\delta\le 1$. If $T1,T^*1\in BMO(\mathbb{R}^n)$, then T is bounded on $L^2(\mathbb{R}^n)$.*

Theorem 2.3. *Let $T\in\mathrm{CZO}(\delta)$ with $0<\delta\le 1$ and $1<q<\infty$. If $T\in WB$ and*

$$T1\in\mathcal{Q}^{\varepsilon}_{q+\varepsilon,q}(\mathbb{R}^n),\ T^*1\in\mathcal{Q}^{\varepsilon}_{q'+\varepsilon,q'}(\mathbb{R}^n)$$

for some $\varepsilon>0$, then T is bounded on $\dot{F}^0_{p,q}(\mathbb{R}^n)$ for any $1<p<\infty$.

3 Proof of Theorem 2.3

Let $f,g\in\mathcal{S}_0(\mathbb{R}^n)$. From $\mathcal{S}_0(\mathbb{R}^n)\subset\mathcal{S}(\mathbb{R}^n)\subset L^2(\mathbb{R}^n)$, (1.3) implies that

$$f=\int_0^\infty Q_t^2f\frac{dt}{t},\qquad g=\int_0^\infty Q_s^2g\frac{ds}{s}.$$

Thus, we have

$$\begin{aligned}\langle Tf,g\rangle&=\langle T\left(\int_0^\infty Q_t^2f\frac{dt}{t}\right),\int_0^\infty Q_s^2g\frac{ds}{s}\rangle\\&=\int_0^\infty\int_0^\infty\langle TQ_t^2f,Q_s^2g\rangle\frac{dt}{t}\frac{ds}{s}\\&=\int_0^\infty\int_0^t\langle Q_sTQ_t^2f,Q_sg\rangle\frac{ds}{s}\frac{dt}{t}\\&\qquad+\int_0^\infty\int_0^s\langle Q_tf,Q_tT^*Q_s^2g\rangle\frac{dt}{t}\frac{ds}{s}.\end{aligned}$$

Here and below, we write $T_{st} = Q_s T Q_t$ and let $K_{st}(x, y)$ be the kernel of T_{st}. Note that $\operatorname{supp} Q_t f = \{ x \in \mathbb{R}^n \mid |x - y| < t,\ y \in \operatorname{supp} f \}$ and that $\operatorname{supp} TQ_t = (\operatorname{supp} Q_t)^c$. Thus, for $T_{st} f$, $K_{st}(x, y)$ satisfies

$$y \in \operatorname{supp} f, \qquad x \in (\{ x \in \mathbb{R}^n \mid |x - y| < t - s,\ y \in \operatorname{supp} f \})^c.$$

Using T_{st}, we have

$$\int_0^\infty \int_0^t \langle Q_s T Q_t^2 f, Q_s g \rangle \frac{ds}{s} \frac{dt}{t} = \int_0^\infty \int_0^t \langle T_{st} Q_t f, Q_s g \rangle \frac{ds}{s} \frac{dt}{t}.$$

Proposition 3.1. *For $0 < s \leq t < \infty$, we have*

$$|K_{st}(x, y)| \lesssim \frac{s^\delta}{(t + |x - y|)^{n+\delta}} + \frac{1}{t^n} |Q_s(T1)(x)| \chi_{\{|x-y| \leq t\}}(x, y),$$

for any $x, y \in \mathbb{R}^n$.

Proof of Proposition 3.1. Since Q_s, T and Q_t are integral operators, we have

$$\begin{aligned} K_{st}(x, y) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_s(x - z) K(z, u) \Phi_t(u - y) du dz \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_s(x - z) K(z, u) [\Phi_t(u - y) - \Phi_t(x - y)] du dz \\ &\qquad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_s(x - z) K(z, u) \Phi_t(x - y) du dz \\ &=: A + B. \end{aligned}$$

Then, it is known that

$$|A| \leq \frac{s^\delta}{(t + |x - y|)^{n+\delta}}.$$

See [11, Lemma 3.2] for the detail. On the other hand, from $\Phi \in C_c^\infty(\mathbb{R}^n)$ and (2.1), we observe

$$\begin{aligned} B &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_s(x - z) K(z, u) \Phi_t(x - y) du dz \\ &= \Phi_t(x - y) \left(\langle T\mu, \Phi_s(x - \cdot) \rangle + \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(z, u) \Phi_s(x - z) dz \right) (1 - \mu(u)) du \right) \\ &= \Phi_t(x - y) \langle T1, \Phi_s(x - \cdot) \rangle \\ &= \Phi_t(x - y) Q_s(T1)(x), \end{aligned}$$

where $\mu \in C_c^\infty(\mathbb{R}^n)$ is a function satisfying $0 \leq \mu \leq 1$ and $\mu = 1$ on $B(x, s)$. Since $\Phi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp} \Phi \subset B(0, 1)$, $|\Phi_t(x - y)| \lesssim t^{-n} \chi_{\{|x-y|<t\}}(x, y)$. Thus, we have the conclusion. □

The following inequality is well known as the Fefferman-Stein vector-valued maximal inequality (see [6]).

Lemma 3.2. *Let* $1 < p, q < \infty$. *For a measurable function* $f_t = f_t(x)$ *on* $\mathbb{R}^{n+1}_+$, *we have*

$$\left\| \left(\int_0^\infty |M(f_t)|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p \lesssim \left\| \left(\int_0^\infty |f_t|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p ,$$

where M *denotes the maximal operator, which is defined by the following:*

$$Mf(x) = \sup_{x \in B} \frac{\chi_B}{|B|} \int_B |f(y)| dy.$$

and M *is taken over variables on* $\mathbb{R}^n$.

Proof of Theorem 2.3. From Proposition 3.1, we obtain

$$\begin{aligned} &\left| \int_0^\infty \int_0^t \langle T_{st} Q_t f, Q_s g \rangle \frac{ds}{s} \frac{dt}{t} \right| \\ &\quad \le \int_0^\infty \int_0^t \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |K_{st}(x,y)| |Q_t f(y)| dy \right) |Q_s g(x)| dx \frac{ds}{s} \frac{dt}{t} \\ &\quad \lesssim \int_0^\infty \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{s^\delta}{(t + |x-y|)^{n+\delta}} |Q_t f(y)| |Q_s g(x)| dy dx \frac{ds}{s} \frac{dt}{t} \\ &\qquad + \int_0^\infty \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{t^n} |Q_s(T1)(x)| \chi_{\{|x-y| \le t\}}(x,y) |Q_t f(y)| |Q_s g(x)| dx dy \frac{ds}{s} \frac{dt}{t} \\ &\quad =: I + I\!I. \end{aligned}$$

It is known that

$$\int_{\mathbb{R}^n} \frac{s^\delta}{(t + |x-y|)^{n+\delta}} |Q_t f(y)| dy \lesssim \left(\frac{s}{t} \right)^\delta M(|Q_t f|)(x), \tag{3.1}$$

where M denotes the maximal operator. See [10] for the detail of (3.1). (3.1) and Hölder's inequality yield that

$$\begin{aligned} I &\le \int_0^\infty \int_0^t \int_{\mathbb{R}^n} \left(\frac{s}{t} \right)^\delta |M(Q_t f)(x)| |Q_s g(x)| dx \frac{ds}{s} \frac{dt}{t} \\ &\le \int_{\mathbb{R}^n} \left(\int_0^\infty \int_0^t \left(\frac{s}{t} \right)^{q\delta} [M(Q_t f)(x)]^q dx \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\qquad \times \left(\int_0^\infty \int_0^t \left(\frac{s}{t} \right)^{q'\delta} |Q_s g(x)|^{q'} dx \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{q'}} dx. \end{aligned}$$

As a result, thanks to Fubini's theorem and the Fefferman-Stein vector-valued inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^t\left(\frac{s}{t}\right)^{q\delta}[M(Q_tf)(x)]^q dx\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\left(\int_0^\infty\int_0^t\left(\frac{s}{t}\right)^{q'\delta}|Q_sg(x)|^{q'}dx\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q'}}dx\\
&\le\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^t\left(\frac{s}{t}\right)^{q\delta}\frac{ds}{s}[M(Q_tf)(x)]^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\left(\int_0^\infty\int_s^\infty\left(\frac{s}{t}\right)^{q'\delta}\frac{dt}{t}|Q_sg(x)|^{q'}dx\frac{ds}{s}\right)^{\frac{1}{q'}}dx\\
&\lesssim\int_{\mathbb{R}^n}\left(\int_0^\infty|M(Q_tf)(x)|^q\frac{dt}{t}\right)^{\frac{1}{q}}\cdot\left(\int_0^\infty|Q_sg(x)|^{q'}\frac{ds}{s}\right)^{\frac{1}{q'}}dx\\
&\lesssim||f||_{\dot{F}^0_{p,q}}||g||_{\dot{F}^0_{p',q'}}.
\end{aligned}$$

On the other hand, from Hölder's inequality, we have

$$\begin{aligned}
I\!I&=\int_0^\infty\int_0^t\int_{\mathbb{R}^n}\int_{B(y,t)}\frac{1}{t^n}|Q_s(T1)(x)||Q_tf(y)||Q_sg(x)|dxdy\frac{ds}{s}\frac{dt}{t}\\
&\le\int_0^\infty\int_{\mathbb{R}^n}\int_{B(y,t)}\frac{1}{t^n}|Q_tf(y)|\left(\int_0^t|Q_s(T1)(x)|^q\frac{ds}{s^{1+\varepsilon q}}\right)^{\frac{1}{q}}\\
&\qquad\qquad\times\left(\int_0^t|Q_sg(x)|^{q'}\frac{ds}{s^{1-\varepsilon q'}}\right)^{\frac{1}{q'}}dxdy\frac{dt}{t}\\
&\le\int_0^\infty\int_{\mathbb{R}^n}|Q_tf(y)|\left(\frac{1}{t^{n-\varepsilon(q+\varepsilon)}}\int_{B(y,t)}\left(\int_0^t|Q_s(T1)(x)|^q\frac{ds}{s^{1+\varepsilon q}}\right)^{\frac{q+\varepsilon}{q}}dx\right)^{\frac{1}{q+\varepsilon}}\\
&\qquad\times\left(\frac{1}{t^{n+\varepsilon(q+\varepsilon)'}}\int_{B(y,t)}\left(\int_0^t|Q_s(g)(x)|^{q'}\frac{ds}{s^{1-\varepsilon q'}}\right)^{\frac{(q+\varepsilon)'}{q'}}dx\right)^{\frac{1}{(q+\varepsilon)'}}dy\frac{dt}{t}
\end{aligned}$$

Here, remark that

$$\left(\frac{1}{t^{n-\varepsilon(q+\varepsilon)}}\int_{B(y,t)}\left(\int_0^t|Q_s(T1)(x)|^q\frac{ds}{s^{1+\varepsilon q}}\right)^{\frac{q+\varepsilon}{q}}dx\right)^{\frac{1}{q+\varepsilon}}\lesssim||T1||_{\mathcal{Q}^{\varepsilon}_{q+\varepsilon,q}},$$

and that

$$\left(\frac{1}{t^{n+\varepsilon(q+\varepsilon)'}}\int_{B(y,t)}\left(\int_0^t |Q_s(g)(x)|^{q'}\frac{ds}{s^{1-\varepsilon q'}}\right)^{\frac{(q+\varepsilon)'}{q'}}dx\right)^{\frac{1}{(q+\varepsilon)'}}$$
$$\leq t^{-\varepsilon}\left(M(F_t)(y)\right)^{\frac{1}{(q+\varepsilon)'}},$$

where $F_t(x) = \left(\int_0^t |Q_s(g)(x)|^{q'}\frac{ds}{s^{1-\varepsilon q'}}\right)^{\frac{(q+\varepsilon)'}{q'}}$. Then, applying the Fefferman-Stein vector-valued inequality together with the Hölder's inequality, we obtain

$$\begin{aligned}
I\!I &\leq ||T1||_{\mathcal{Q}^{\alpha}_{q+\varepsilon,q}}\int_0^\infty\int_{\mathbb{R}^n}|Q_tf(y)|\,t^{-\alpha}(M(F_t)(y))^{\frac{1}{(q+\varepsilon)'}}dy\frac{dt}{t}\\
&\lesssim\left(\int_{\mathbb{R}^n}\left(\int_0^\infty|Q_tf(y)|^q\frac{dt}{t}\right)^{\frac{p}{q}}dy\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_{\mathbb{R}^n}\left(\int_0^\infty[M(F_t)(y)]^{\frac{q'}{(q+\varepsilon)'}}\frac{dt}{t^{1+q\alpha}}\right)^{\frac{p'}{q'}}dy\right)^{\frac{1}{p}}\\
&\leq||f||_{\dot F^0_{p,q}}\cdot\left(\int_{\mathbb{R}^n}\left(\int_0^\infty|F_t(y)|^{\frac{q'}{(q+\varepsilon)'}}\frac{dt}{t^{1+q\alpha}}\right)^{\frac{p'}{q'}}dy\right)^{\frac{1}{p}}\\
&=||f||_{\dot F^0_{p,q}}\cdot\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_0^t|Q_s(g)(x)|^{q'}\frac{ds}{s^{1-\alpha q'}}\frac{dt}{t^{1+q\alpha}}\right)^{\frac{p'}{q'}}dy\right)^{\frac{1}{p}}\\
&=||f||_{\dot F^0_{p,q}}\cdot\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\int_s^\infty\left(\frac{s}{t}\right)^{q\alpha}\frac{dt}{t}\right)|Q_s(g)(x)|^{q'}\frac{ds}{s}\right)^{\frac{p'}{q'}}dy\right)^{\frac{1}{p}}\\
&\sim||f||_{\dot F^0_{p,q}}||g||_{\dot F^0_{p',q'}}.
\end{aligned}$$

Thus, we get

$$\left|\int_0^\infty\int_0^t\langle T_{st}Q_tf,Q_sg\rangle\frac{ds}{s}\frac{dt}{t}\right|\lesssim\ I+I\!I\ \lesssim||f||_{\dot F^0_{p,q}}||g||_{\dot F^0_{p',q'}}.$$

Treating similarly $\int_0^\infty\int_0^s\langle Q_tf,Q_tT^*Q_s^2g\rangle\frac{dt}{t}\frac{ds}{s}$ by symmetry, we also have

$$\left|\int_0^\infty\int_0^s\langle Q_tf,Q_tT^*Q_s^2g\rangle\frac{dt}{t}\frac{ds}{s}\right|\lesssim||f||_{\dot F^0_{p,q}}||g||_{\dot F^0_{p',q'}}.$$

As a consequence, we have

$$\begin{aligned}
|\langle Tf, g\rangle| &= \left|\int_0^\infty\int_0^\infty \langle Q_s^2 T Q_t^2 f, g\rangle \frac{dt}{t}\frac{ds}{s}\right| \\
&\leq \left|\int_0^\infty\int_0^t \langle Q_s T Q_t^2 f, Q_s g\rangle \frac{ds}{s}\frac{dt}{t}\right| \\
&\qquad + \left|\int_0^\infty\int_0^s \langle Q_t f, Q_t T^* Q_s^2 g\rangle \frac{dt}{t}\frac{ds}{s}\right| \\
&\lesssim ||f||_{\dot{F}^0_{p,q}} ||g||_{\dot{F}^0_{p',q'}}.
\end{aligned}$$

Since $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot{F}^0_{p,q}(\mathbb{R}^n)$ and $(\dot{F}^0_{p,q}(\mathbb{R}^n))^* = \dot{F}^0_{p',q'}(\mathbb{R}^n)$, we have the desired result. □

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