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Real-World Modelling to Increase Mathematical Creativity

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Synopsis

Modelling could be characterised as one of the core activities in secondary mathematics education in Austria. However, when learning and teaching mathematics, mathematical modelling is mostly used to apply and deepen mathematical knowledge and competencies. Our educational case study aims to explore how mathematical modelling, using real objects and high-quality mathematical technologies, could be utilised to acquire mathematical knowledge and competencies, and how learners could creatively use their existing knowledge. To discover the potential of mathematical modelling using real objects and high-quality mathematical technologies to acquire mathematical knowledge and competencies, and to stimulate learners' creativity, first, we combined cognitive and creative spirals and mathematical modelling cycles. Then, in an explorative case study, we tested this combination of cognitive and creative spirals and mathematical modelling cycles in a Viennese secondary school and in mathematics teacher education in Austria. Applying the combination of cognitive and creative spirals and mathematical modelling cycles, we discovered that collaboration among learners, sharing technological knowledge and skills of learners determine whether knowledge can be acquired in mathematical modelling.

Keywords: mathematical modelling, technology-enhanced mathematics education

1. Introduction

"There are as many definitions of mathematical modelling as there are authors writing about it [...]" [1]. This quote demonstrates clearly that mathematical modelling is both a fashionable and nebulous buzzword in mathematics education. Although there are many varying descriptions of mathematical modelling, most descriptions have in common that mathematical modelling is interpreted as reactive in an educational context. Reactive mathematical modelling means that there is a real problem or fact which is then described by learners using familiar mathematical concepts. The responsive character of modelling is also evident from the fact that mathematical modelling and application often form a seemingly inseparable symbiosis. For example, this symbiosis can be found in the first sentence of the introduction by Kaiser, Lederich & Rau [16]: "The relevance of promoting applications and mathematical modelling in schools is widely accepted.", in the introduction by Blum [1]:

> "Now, as a basis for the following parts, I shall give some pragmatic working definitions which have been widely accepted in mathematics education in recent years (see the survey article by Blum and Niss, 1991). Let me quote the well-known simple model of applied mathematical problem-solving."

or in the first sentences of Niss [26]:

> "A major reason why mathematics is the world's single largest educational subject is the fact that mathematics is applied in a multitude of different ways in a huge variety of extra-mathematical subjects, fields and practice areas. Every time mathematics is used to deal with issues, problems, situations and contexts in domains outside of mathematics, mathematical models and modelling are necessarily involved, [...]".

However, our idea of mathematical modelling is that both are possible: that mathematical competencies are applied and that such mathematical competencies are acquired or deepened. Depending on the age of learners and levels of mathematical competenciesassociated with it, acquiring new mathematical concepts and competencies, and applying and deepening mathematical concepts and competencies should be central. The questions which we have addressed while preparing and conducting our explorative modelling experiment were:

(a) How can learners acquire new mathematical knowledge and competencies through mathematical modelling? and

(b) How can learners use increased mathematical knowledge and competencies in modelling throughout their educational careers?

In our paper, we define the distinction between knowledge and competencies to the extent that knowledge could be described as a more passive element, and competencies could be described as a more active element. According to Bloom's taxonomy and the revision of Bloom's taxonomy by Krathwohl [21], in this paper, we locate knowledge in the knowledge dimensions "Factual" and "Conceptual" as well as in the process dimensions "Remember" and "Understand". Competencies can be assigned in the knowledge dimensions to the areas "Procedural" and "Metacognitive", and in the process dimensions "Apply" or higher.

To investigate these questions, we have focused our modelling experiment on creating and investigating physical models of real objects with the help of higher-quality educational technologies. In this context, the term "higher-quality educational technologies" refers to mathematical software applications (in our case GeoGebra) on the one hand and Internet-based information databases on the other. The reason why we utilised higher-quality mathematical software products, such as GeoGebra, in our modelling experiment is that, on the one hand, few innovations such as new technologies have influenced western societies and thus also schools and educational institutions in recent years so much as technologies and digitalisation [31]. On the other hand, according to Ferchhoff [12], young adults can no longer envisage a decent life without technologies. Since authenticity also means that learning processes are linked to reality and because technologies are a central element of students' realities, to realise authentic mathematical modelling, using modern technologies could be inevitable [35]. Another reason for including higher quality mathematical software products into our case study is that utilising higher-quality mathematical software products has become an obligatory part of secondary mathematics education and mathematics teacher education in Austria. The mathematics curriculum of the lower secondary level [A] indicates that secondary students should learn to use technologies, and that students should use calculators and computers for problem-solving, research and experimental learning. The mathematics curriculum of the upper secondary level [B] emphasises that technologies should be used when students should acquire new knowledge in all areas of mathematics. Besides, using mathematical software products has become mandatory at the standardised written school-leaving examination recently. Following the curriculum of the mathematics teacher education in Austria [C], one of the main learning outcomes of mathematics teacher education is that pre-service mathematics teachers learn to use subject-specific software for corresponding mathematical problems. Furthermore, the curriculum of the mathematics teacher education highlights that pre-service mathematics teachers have to learn to use dynamic and didactic geometry software as well as CAS (Computer Algebra Systems).

To investigate how learners can develop mathematical knowledge and competencies through modelling and how learners can develop their enhanced mathematical knowledge and competencies in modelling throughout their educational careers, we have developed a technology-enhanced learning environment that combines real objects and mathematical modelling. In our explorative educational case study, we utilised this technology-enhanced learning environment following a horizontal temporal approach. Using the learning environment following a horizontal temporal approach means that different groups of learners with different progress in their educational careers where investigating mathematics in such learning environments at the same time. In our educational case study, the different groups of learners of different progress in educational careers consisted of 9th grade secondary school students, 4th term mathematics teacher students and 6th term mathematics teacher students. For two reasons, a horizontal and not a longitudinal temporal approach was feasible to examine our research questions. On the one hand, there is no class or cohort at a secondary school in which all students decide to become mathematics teachers after their secondary education. Thus, collecting the points would only be possible backwards in time, which is not a scientifically appropriate approach. On the other hand, a longitudinal temporal study from grade 9 at a secondary school until the end of teacher education would last from 9 to 10 years. If this long period is compared with the rapid development of educational software, it could be assumed that the technological framework conditions would change several times fundamentally. Consequently, the learning products developed by learners in such an educational case study would reflect the development of the educational software used rather than the development of mathematical modelling competencies. For this reason, we have decided to conduct our case study at one time and simultaneously involving mathematics learners from different educational levels into our case study. By conducting our case study at one point in time, it should be possible to focus on the mathematical competencies and how mathematics learners from different educational levels use their mathematical competencies and not on the development of the educational software used.

We have chosen a case study approach in our research because this research method is appropriate for investigating learners' solution processes and methods in mathematics learning and has a long tradition in mathematical educational research [7]. A typical element of case studies is that researchers investigate a limited system of real people in real situations in which specific interventions are performed [8]. The limited system of our case study was formed by the lessons and the learners who should create real objects and then investigate them in technology-enhanced learning environments. Furthermore, our case study can be characterised as an explorative case study, since one goal of our study is to generate hypotheses regarding the development of mathematical modelling competencies of mathematics learners in the course of educational careers [36].

2. Mathematical modelling in school contexts

As our explorative educational case study intends to discover how learners could develop new mathematical knowledge and competencies through mathematical modelling and how learners could develop their enhanced mathematical knowledge and competencies throughout their educational careers in technology-enhanced learning environments, which combine real models and mathematical modelling, mathematical modelling in learning settings forms an essential part of our theoretical framework. Carreira & Baioa [6] summarise that it is often pseudorealistic real problems in textbooks or other learning materials that should trigger mathematical modelling. Pseudorealistic problems as learning triggers could cause that mathematical models based on them are pseudorealistic as well. However, it should be emphasised that using pseudorealistic problems in mathematical modelling is not a negative approach. Especially if learners use new mathematical concepts for mathematical modelling for the first time, it could be useful if pseudorealistic problems form the educational framework. The composition of the educational framework based on pseudorealistic problems could be justified by the assumption that otherwise, students could get overwhelmed by the first application attempts of new concepts. But, only discovering and learning mathematics based on real problems could lead to real mathematical models. In this context, real mathematical models are those mathematical models which are based on real objects (see Figure 1). These real objects have not been simplified, as would be the case with pseudorealistic objects and associated pseudorealistic problems.

Figure 1: left: real model of a bridge and right: model of a pseudorealistic door arch Source right: Malle et al., 2017: 164.

According to Heck [13], it could be real problems and associated real mathematical models that enable learners to learn like real scientists—even in an educational context. Learning and researching like real scientists are also associated with learning by doing closely. Learning by doing in mathematical modelling also indicates a processual nature of mathematical modelling. The processual character of mathematical modelling suggests that mathematical modelling is an essential method of experimental science. Therefore, Kertil & Gurel [19] propose considering mathematical modelling as a possible bridge to integrated STEM education, in which physical tools play an essential role. Physical, real and digital tools, problems or objects from reality and thus connecting STE and M could also be a key to authenticity in mathematical modelling, which will be explained in more detail in the next section. In mathematical modelling, vital aspects are (A) physical modelling by learners and (B) numerical analysis and understanding of everyday artefacts. By physical modelling of real objects by learners, we understand in our case study that learners create a scaled-down and simplified model of an object. In this context of modelling, physical means that the scaled-down and simplified model of an object is tangible and not merely a virtual construct (see Figure 1, left). To facilitate modelling, analysis and understanding of everyday artefacts, we have considered using GeoGebra in our modelling experiment. In our case study, we opted for GeoGebra as the mathematical software framework because GeoGebra was developed for teaching and learning mathematics, and GeoGebra interactively combines algebra and 2D and 3D geometry [15]. Another reason for choosing GeoGebra in our case study was that Zulnaidi, Oktavika, and Hidayat [37] were able to demonstrate the positive effect of GeoGebra on students' mathematical learning outcomes. Specifically for the modelling aspects our case study it was important that GeoGebra could facilitate developing mathematical assumptions, and that GeoGebra could make it easier for students to test and verify their mathematical assumptions.

2.1. Authenticity in mathematical modelling

As real models, mathematical modelling and utilising high-quality educational software are essential pillars of our explorative case study, authenticity in mathematical modelling is also a key aspect of our research. If an object or physical model is a copy of reality that faithfully simulates reality, then, according to Vos [35], objects or physical models could increase authenticity in learning. In our paper, by authentic models, we mean those models which are simulations or copies of reality containing the properties and genuineness of the real objects. Vos [35] summarises that authentic models or authenticity in mathematics learning could link the learning process with reality, make the learning process more relevant and the activities associated with the learning process more meaningful and important to learners. In addition to authenticity, when dealing with complex problems, such as mathematical modelling of real objects, it should be considered that planning and implementation cannot be achieved temporally separated. If such a separation is made, mathematics education runs the risk of being formalised too much and of nipping creative solutions and approaches in the bud. So, lab-like learning environments could be helpful to facilitate creative solutions and approaches to the mathematical modelling of real objects. To provide learners with lab-like learning environments in our case study, learners were provided with sufficient time and physical space and were free to move back and forth between the stages of research, construction and investigation of the bridge.

Additionally, when researching, building or investigating the bridges, learners were enabled to use those analogue or digital tools which met their needs and which learners considered to be best suited.

According to Noss & Hoyles [28] and Noss, Healy & Hoyles [29], utilising geometric software, computer algebra systems, or graphics calculators could contribute to learning environments becoming laboratories in which learners could discover mathematics creatively and experimentally. In our paper we use Torrance's definition of creativity: creativity is "a process of becoming sensitive to problems, deficiencies, gaps in knowledge, missing elements [...] identifying the difficulty; searching for solutions, making guesses, or formulating hypotheses about the deficiencies: testing and retesting these hypotheses and possibly modifying and retesting them; and finally communicating the results" [34, page 6]. According to Torrance [34], it is fluency, flexibility, originality and elaboration, which could characterise students' creative process. In our educational experiment, we opted for Torrance's definition because this definition of creativity focuses on problem-solving, formulating and testing hypotheses, and communicating the results. This interpretation of creativity fits with our explorative educational experiment, as learners are supposed to solve problems (building a model of a bridge and mathematically investigating the model of the bridge), formulate and test hypotheses (selecting functions to model a bridge and then adjusting the function), and share their results in an electronic portfolio with other learners.

Especially in formulating and testing hypotheses in mathematical modelling of real objects, using technologies was important in our explorative case study. Also Borba, Meneghetti & Hermini [3] and Borba & Bovo [4] were able to illustrate in their studies concerning relationships between modelling, graphical computing and interdisciplinarity that an experimental-with-calculator approach could be vital for curve or function fitting. Concerning learning and teaching mathematics in general and learning and teaching mathematics in technology-enhanced learning environments in particular, curve or function fitting is considered a subset of mathematical modelling [22]. Since in our explorative educational case study the mathematical modelling of real objects in technology-enhanced learning environments could not be fully investigated and analysed, we have focused our study on the development of learners' competencies in the area of curve or function fitting.

According to this focus, it follows that in our study and our paper, mathematical modelling refers almost exclusively to curve or function fitting.

Hereby, it should be considered that according to Doerr and Lesh [10], it is the case that competent individuals not only do things differently but also see, interpret or conceptualise things differently. These differences also apply to modelling or modelling perspectives in mathematics education in the 21st century when modern technologies are involved. Associated to that, Kanematsu & Barry [18] emphasise that productive and creative thinkers like Einstein or Darwin consider all different ways of solving problems. In our research, considering all different ways of solving problems means that learners should be facilitated by the technology-enhanced learning environment, classmates and the teacher when building and investigating the bridge.

The aim of our experiment was not to identify the Einstein or Darwin of the 21st century among mathematics learners, but to discover how many different mathematical modelling approaches are available to learners in the course of their educational careers and which solutions are applied by learners in mathematical modelling of a physical and thus complex object. In this context, we have assumed that the number of modelling approaches available to and applied by students in the 21st century depends on students' mathematical competencies, students' technological competencies, students' competencies to acquire information and their ability to combine these competencies.

2.2. Mathematical modelling as a creative activity

Combining mathematical competencies, technological competencies and competencies to acquire information to solve real problems and to create learning artefacts could be described as a creative process. To investigate a development of available and utilised approaches in mathematical modelling while conducting such complex and creative tasks, we have used ideas of a cognitive and creative spiral. To incorporate the potential significance of utilising modern educational technologies when learning mathematics into our research, we have also considered Buchberger's spiral [5]. By combining these three spirals, it should be facilitated to consider students' mathematical, technological, and information acquisition competencies as well as students' skills to combine these three competencies when students develop mathematical modelling competencies. Ebert [11] summarises creative thinking as a multidimensional phenomenon of cognitive functions.

In this context, problem-solving, flexibility or originality are described as creative activities. According to Sternberg [32], creative thinking could be subsumed as a complex interplay of intelligence, intellectual style and personality. Furthermore, Torrance [33], Mayer [25] and DeVito [9] mention problem-solving in their definitions of creative thinking explicitly. Also, Mayer [25] offers an equally problem-solving oriented description of creative thinking, as Mayer [25] described creative thinking as a cognitive activity that should lead to new results or solutions of a problem.

Consequently, we assume that creative thinking and problem-solving have a common basis, and we see problem-solving as a part of cognitive processing. Similar to other scientists from the field of education (e.g. [14, 30]), we also adopted basic structures of the knowledge spiral [27] when developing our mathematical modelling spiral. By considering the knowledge spiral in our mathematical modelling spiral, the importance of interacting communities and learning in social structures should be emphasised. Considering utilising technologies when modelling, Kovács [20] elaborates that according to Buchberger's workflow or creativity spiral [5], a continuous workflow could be achieved based on computational results and conjectures based on them. Building on these conjectures, new algorithms are usually clarified by programming—in our case, using educational technologies—which is especially true in our digital era. Applying such algorithms should lead to new computational results, and the spiral continues with further inventions or conjectures. To provide a continuous workflow for as many students as possible and as combining creative thinking, problem solving and using technologies could be characterised as challenging, we decided that in our case study, students should work in small groups and thus support and motivate each other.

2.3. Merging creative spirals and modelling spirals

There is a multitude of descriptions and visualisations of mathematical modelling processes. What most of these descriptions and visualisations have in common is that mathematical modelling leads to connections and links between the real world and mathematics. A synthesis of the real world and mathematics in modelling descriptions is established usually by a linear sequence of a real situation, real model, mathematical model, and numerical results. In recent decades, mathematics education research has often focused on mathematical work, i.e. the transition from a mathematical model to mathematical results (e.g. [2, 17]). In our mathematical education experiment, we have concentrated on mathematising, i.e. the transition from the real model to the mathematical model.

$$f(x) = -10,6 \cdot \cosh\left(\frac{x}{-11,4}\right) + 14,5$$

Figure 2: Real object, real model, and mathematical model of a Da Vinci Bridge. Source left: https://previews.123rf.com/images/andreykr/andreykr1506/andreykr150600112/41007685-mathematische-br%C3%BCcke-am-queens-college-in-cambridge-gro%C3%9Fbritannien.jpg [27 Nov 2019]

Through our educational experiment, we aimed to investigate how mathematising competencies of mathematics learners develop in the course of their educational careers. To represent this development of mathematization competencies graphically, we interpret mathematising in our model (see Figure 3) as part of a spiral and not as part of a circle. The basis or XY-plane of our mathematising spiral is essential components of the modelling circle of Blum & Leiss [2].

We assume that higher competencies in the field of mathematising could be achieved in the course of learners' educational development or progress. In our paper, we understand educational development as the progress of learners in their educational careers and, along with higher grades, learners are familiarised with more complex mathematical concepts and develop higher competencies. These higher competencies in mathematising should also make it possible to develop higher-quality and more complex mathematical models. In our view, learners' mathematising competencies depend on learners' mathematical knowledge and competencies, and their creativity. Mathematical knowledge in a mathematising process depends on mathematical educational knowledge (acquired in schools or universities) and competencies in using knowledge sources (acquired through formal or non-formal education). The creativity of learners—in our case, the competencies to create something or to develop creative strategies of solutions for a posed question—is vital in our model, as creative competencies are needed to apply mathematical knowledge and competencies to physical models using modern educational technologies. The more mathematical knowledge and competencies, as well

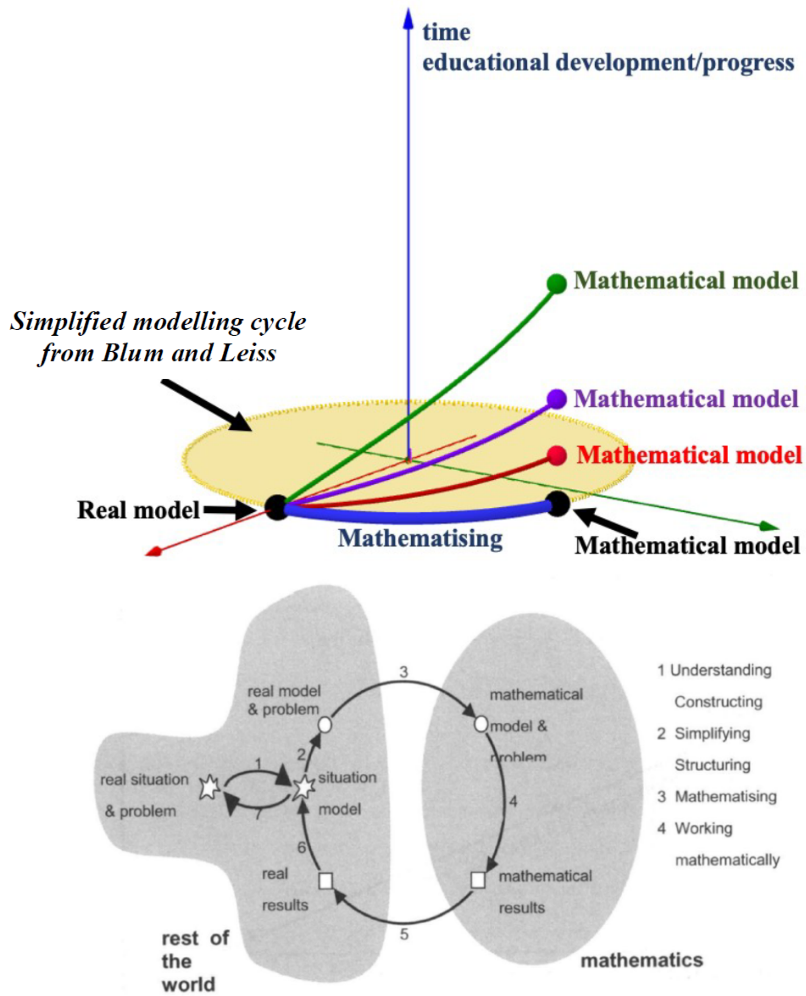

Figure 3: Mathematising Spiral (top) and *The modelling cycle* from Blum and Leiss [2] as the basis-cycle for the Mathematising Spiral.

as creativity of learners and, above all, a link between these two areas, is available, the higher the value of mathematical models (from red to purple to green) should be achievable. Utilising modern educational technologies should enable students to test their models or artefacts in real-time and adapt them if necessary. This real-time feedback through using technologies should also have a positive effect on students' workflow and thus on students' mathematising competencies.

In order to explore the potential applicability of our Mathematising Spiral, we have designed our educational experiment to be as open as formal learning settings allow and involve learners who are at different stages of their educational careers. As open as formal learning settings allow means that the researchers have made few regulations according to the curricula or course descriptions in which our educational experiment was conducted. Involving learners at different levels in their educational careers means in our educational case study that secondary school students from the 9th grade, mathematics teacher students from the 4th term and mathematics teacher students from the 6th term were involved.

3. Description of tasks and issues

To explore how students could acquire new mathematical knowledge and competencies by mathematical modelling, if our Mathematising Spiral could be applicable in formal learning settings, and thus how learners could use their increased mathematical knowledge and competencies in mathematical modelling throughout their education, we conducted an explorative case study with secondary school students and mathematics teacher students. Our experiment was conducted with secondary school students in the 9th grade as well as with mathematics teacher students in the 4th and 6th term. We choose secondary school students in the 9th grade as well as the mathematics teacher students in the 4th and 6th term because: (A) secondary school students in the 9th grade should already have knowledge about linear, quadratic and piecewise-defined functions according to the curriculum, (B) mathematics teacher students in the 4th term should have attended mathematical basic courses and (C) mathematics teacher students in the 6th term should have completed mathematical basic courses as well as computer and school practical courses. This selection of participants in our modelling experiment enabled us to investigate different groups with different mathematical competencies (9th grade secondary school students versus mathematics teacher students) and different technological competencies (4th term mathematics teacher students versus 6th term mathematics teacher students). In the modelling experiment and based on the developed mathematical models and artefacts of the learners, it should then be possible to examine how different formal competencies as well as non-formal competencies, such as creativity or information acquisition skills, could affect the values of the mathematical models (see Figure 3) of the different groups of learners.

According to the level of education, secondary school students were aged from 14 to 16 and mathematics teacher students were in their early twenties. In terms of gender distribution, female dominance could be observed in all three groups. Among mathematics teacher students, slightly more than half were female and among secondary school students, more than three quarters were female. Both the secondary school and the teacher training university are located in the city centre of large Austrian cities. Due to the location of the secondary school, it could be concluded that the majority of secondary school students have an urban and high socio-cultural background. The high socio-cultural background of the students of our case study can be explained by the circumstance that the residential area in the Austrian city centres is the most expensive residential area and therefore, a high socio-cultural background prevails in this region. Since the most crucial factor in secondary school enrolment procedures in Austria is the students' proximity to school, it is assumed that many students come from the city centre and thus from a region with a high socio-cultural background. The site of the teacher education university does not allow any conclusions to be drawn about students' urban or rural background. No conclusion on an urban or rural background of students based on the location of teacher education university can be justified by the fact that all teacher education universities in Austria are located in provincial capitals or the metropolitan area of provincial capitals. By this concentration of teacher education universities in provincial capitals, it results that all future teachers have to commute to these places for their education—regardless of whether students come from urban or rural areas.

Learners' tasks and activities of learners could be divided into three phases in our educational experiment. In all three phases of our educational experiment, learners should investigate mathematical or da Vinci bridges. A mathematical or da Vinci bridge is an arch construction consisting of rigid components. These components stake each other by cleverly interlocking and braiding the components; therefore, no nails, screws or adhesives are needed to build a mathematical or da Vinci bridge (see Figure 1).

First, secondary school students and mathematics teacher students should research what lies behind the term "mathematical bridge" or "da Vinci bridge" as well as which characteristics and peculiarities mathematical or da Vinci bridges possess. To explore the peculiarities of mathematical or da Vinci bridges, learners should conduct Internet research. It was required to perform an Internet search to discover the peculiarities of mathematical bridges, since neither secondary school students' nor mathematics teacher students' learning materials contain information about mathematical or da Vinci bridges. To obtain more detailed information about mathematical or da Vinci bridges and to have more confidence when doing Internet research, secondary school students were encouraged to conduct internet research in small groups. The mathematics teacher students were free to do internet research in small groups or individually. For internet research, both secondary school students and mathematics teacher students could use school's or university's computers or carry out internet research on their own devices.

In a next step, learners should also discover how to build such a bridge and secondary school students should create a shopping list to build a mathematical or da Vinci bridge themselves. After purchasing building materials for the bridges, physical bridges were constructed with these building materials. While building mathematical or da Vinci bridges, secondary school students formed small groups (2 to 4 learners). Not only did secondary school students support each other within a group, but there was also cross-group support when building physical bridges. Secondary school students were given a two-lesson period (100 minutes) to create mathematical or da Vinci bridges and secondary school students could use the attic of the school. The attic of the school was chosen as the learning environment so that each group would have enough space and school furniture would not make it difficult to build bridges. Building the model of a mathematical bridge was not an explicit mathematical process, but the discussions and loud thinking of the learners indicated that building the bridges were activities in which the learners implicitly used mathematical thinking and arguing. For example, a group of learners argued or wanted to know from us, the instructors, whether their assumption is correct that if the bridge is built flatter and the individual elements of the bridge are at a greater angle to each other, the bridge becomes more stable.

Mathematics teacher students were provided with the same building materials as secondary school students. Mathematics teacher students could use a university teaching lab to build the bridge, or they could spread out in university corridors to build the bridges.

Finally, when learners finished constructing mathematical or da Vinci bridges, learners were encouraged to take several photos of these bridges. Learners were informed that these photos would be used to examine their bridges mathematically afterwards—i.e. mathematising of the bridge. When examining physical models of the bridges mathematically, secondary school students and mathematics teacher students were instructed to utilise functions for the mathematical model. The framework that learners should use functions when modelling bridges was chosen by the researchers to provide learners with a first orientation and preventing learners from getting lost. Since our educational experiment was carried out in formal educational settings, it was also necessary to provide links to the curriculum or course descriptions; i.e. making restrictions, such as that functions should be used. Despite pretending to use functions in modelling, learners should have enough freedom in our case study to develop and use creativity as described above (see Section 2.2). Since there were no further restrictions for students when modelling the bridges, this freedom for learners should have a positive effect on flexibility and originality when students are solving the given problem. In mathematical investigations of bridges, secondary school students and mathematics teacher students used GeoGebra, a dynamic geometry program. The mathematical investigation should not lead to one correct mathematical model, but secondary school students and mathematics teacher students were encouraged to model their physical bridge mathematically in various ways. Encouraging learners to model their bridges in a variety of ways should prevent learners from trying desperately to find the right solution. Instead, this approach should encourage learners discovering new results and solutions to a given problem, which would correspond to the description of creativity according to Mayer [25].To develop as many mathematical models as possible, secondary school students and mathematics teacher students could use both their learning materials and the internet as reference tools.

In our explorative educational case study, we investigated learners when examining real objects or problems and mathematical modelling in a technology-enhanced learning environment, since the Austrian mathematics curriculum [A, B] also requires generals or supra-disciplinary competencies that need to be taught and learned. Among other things, these generals or supra-disciplinary competencies include that learners have to deal with real-world issues and questions or that learners should use new technologies to solve problems and tasks. In order to explore the potential of implementing these generals or supra-disciplinary competencies in the teaching and learning of mathematics, we have decided to focus our exploratory educational case study on the mathematical modelling of self-built bridges in technology-enhanced learning environments.

This tripartite division of tasks and activities of learners was undertaken to stimulate (a) non-formal or non-curriculum competencies (phase 1 and phase 2), (b) creative competencies (phase 2 and phase 3), and mathematical competencies (phase 3). Stimulating these three competency areas in mathematical modelling of real objects by different groups of learners with different competencies should also enable making conclusions about which competency areas could have a high impact on the values of mathematical models.

4. Results

When examining mathematical models of secondary school students and mathematics teacher students, we were able to deduce three groups of models:

(a) Mathematical models based on the secondary curriculum up to the 9th grade

(b) Mathematical models based on the upper secondary curriculum

(c) Mathematical models based on mathematical concepts which are not covered in secondary education.

These three groups of mathematical models were developed by comparing corresponding mathematical models when learners finished investigating mathematical or da Vinci bridges. To be able to compare mathematical models, secondary school students and mathematics teacher students sent us the mathematical models of their mathematical or da Vinci bridges. In identifying differences or similarities between different mathematical models of the bridges, the secondary mathematics curriculum formed the basis; see Table 1.

To reduce possible prejudices when comparing and grouping mathematical models, all mathematical models were anonymised before grouping. After all mathematical models could be assigned to a group, the complexity of mathematical models of particular examples of the groups was assessed. When assessing mathematical models, the mathematics curriculum of secondary school was the yardsticks. By comparing groups of mathematical models, it could be possible to derive the above groups of models.

Secondary curriculum concepts up to the 9th grade, relevant for our study	Secondary curriculum concepts after the 9th grade, relevant for our study	Mathematical concepts which were relevant for our study and used by the learners, but not covered in the secondary curriculum.
- Students can study linear growth and decline processes with different assumptions using electronic computational tools. - Students can recognise functional dependencies, represent them formulaically and graphically. - Students are introduced to linear and quadratic equations in a single variable. - Students can describe and study linear and quadratic functions. - Students can describe dependencies that can be captured by real functions in a single variable using terms, spreadsheets, and graphs, and reflect on the modelling nature of functions. - Students can describe and investigate some other nonlinear functions, e.g. broken rational functions or piece-wise defined functions. - Students can work with functions in application-oriented areas; can understand functions as mathematical models.	- Students can define and represent functions of the following types; sketch typical shapes of their graphs; specify characteristic properties and interpret them in context: power functions, polynomial functions, exponential functions, logarithm functions, angle functions. - Students can investigate real functions (monotony, local and global extremes, symmetry, periodicity) - Students can utilise the above types of real functions, mainly exponential functions, in mathematical and non-mathematical situations; interpret functions as models, compare models, and reflect boundaries of model building. - Students can solve simple polynomial equations of degree ≤ 4 in the range of real numbers (if they are used in differential calculus). - Students know the concept of derivation function; know higher derivations. - Students can carry out further applications of differential calculus, in particular from economics and science. - Students can use and investigate polynomial functions in mathematical and non-mathematical areas. - Students are familiar with circles, spheres, conic section lines and other curves.	The "mathematical concepts which are not covered in secondary education" consist of those mathematical concepts which were used by the learners when modelling the bridges an which are not listed in column one or column two.

Table 1: Mathematical frameworks and concepts for grouping the models. Source [A, B].

In the next section, individual groups of mathematical models are presented, and concrete examples of respective groups of mathematical models are provided.

4.1. Mathematical models based on the secondary curriculum up to the 9th grade

Those mathematical models that were created using secondary curriculum content up to the 9th grade were polynomial functions and piecewise-defined functions. A closer look at these functions in terms of learners' mathematical backgrounds indicated that models of 9th-grade secondary school students and 4th-term mathematics teacher students do not differ.

Both 9th-grade secondary school students and 4th-term mathematics teacher students used only second-degree polynomial functions (see Figure 4) to model mathematical bridges.

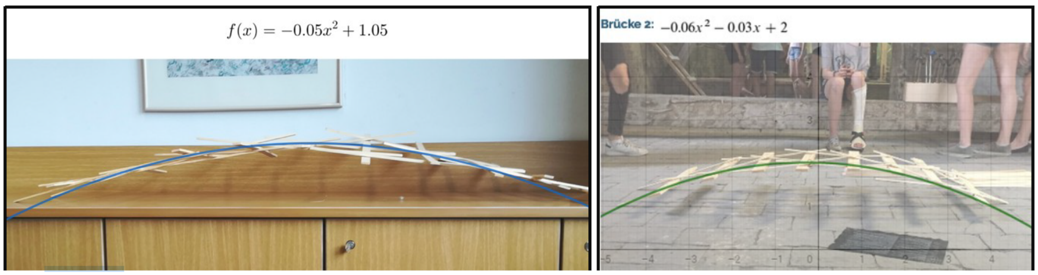

Figure 4: Second-degree polynomial models of 4th-term students and 9th-grade pupils.

The 6th-term mathematics teacher students utilised second-degree polynomial functions as well, but also used fourth, sixth and eighth- degree polynomial functions (see Figure 5) to model mathematical bridges. The reason why 9th-grade secondary school students chose second-degree polynomial functions only could be because the curriculum and textbooks of the 9th grade pay much attention to such functions. Higher-grade functions are treated—if at all—only marginally in the 9th grade. It could be why 4th-term mathematics teacher students utilised second-degree polynomial functions merely because they have not yet completed a computer internship. A group of 4th-term mathematics teacher students wanted to use a sixth-degree polynomial function to model a bridge. In this modelling attempt, students started to use sliders for parameters of the function. When the 4th-term mathematics teacher students realised that this approach was very laborious and that parameter values had to be changed in opposite directions, they dropped this approach and began to search for a new way of modelling the bridge. When 6th-term mathematics teacher students utilised highergrade polynomial functions to model a mathematical bridge, students set a number of points on the image of the bridge corresponding to the degree of the function. Then, students applied technologies to deduce the functional term of the mathematical model based on the set of points (see Figure 5). Therefore, different competencies of 4th-term mathematics teacher students and 6th-term mathematics teacher students concerning utilising educational technologies could explain most significant differences between 4th-term and 6th-term mathematics teacher students' mathematical models.

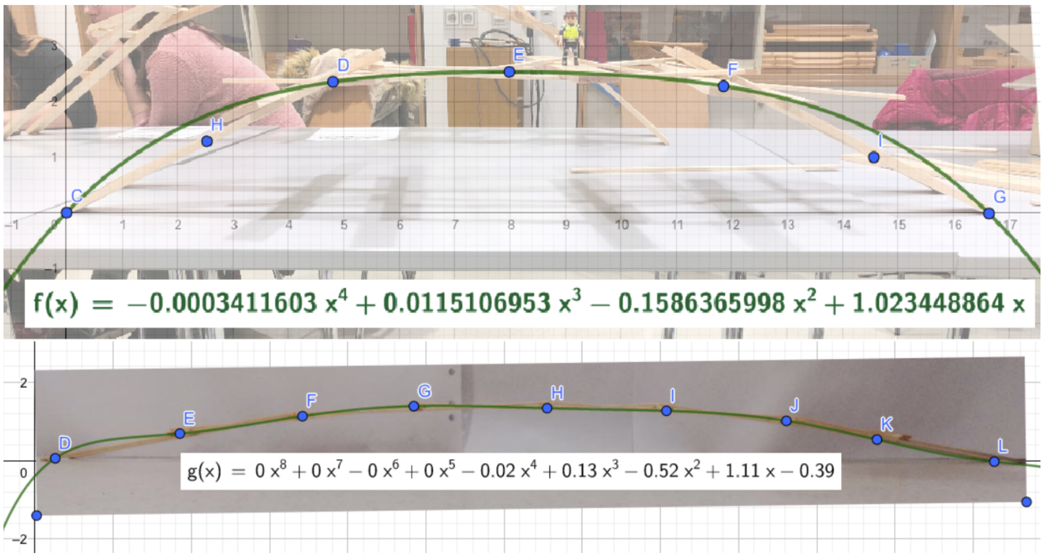

Figure 5: Fourth- and eighth-degree polynomial models of 6th-term students.

In the mathematical modelling based on piecewise-defined functions, 9th-grade secondary school students' and 4th- term mathematics teacher students' models were limited to linear functions. In contrast, 6th-term mathematics teacher students also utilised non-linear functions (see Figure 6) when describing mathematical bridges using piecewise-defined functions. It was not surprising that 9th-grade secondary school students applied linear piecewise-defined functions only when utilising piecewise-defined functions. Using linear piecewise-defined functions could be explained by the fact that principles of piecewise-defined functions are superficially treated in the 9th grade and linear functions are almost always applied in these treatments. What was more remarkable is that 4th-term mathematics teacher students used linear piecewise defined functions only. A difference between linear piecewise-defined functions of 4th-term mathematics teacher students and nonlinear piecewise-defined functions of 6th-term mathematics teacher students could be explained by an increase of technological competencies or to be more precise in the further development of techno-mathematical competencies of 6th-term mathematics teacher students. By the term technomathematical competencies, we understand in our paper learners' abilities to use technological tools and mathematical software to solve mathematical tasks.

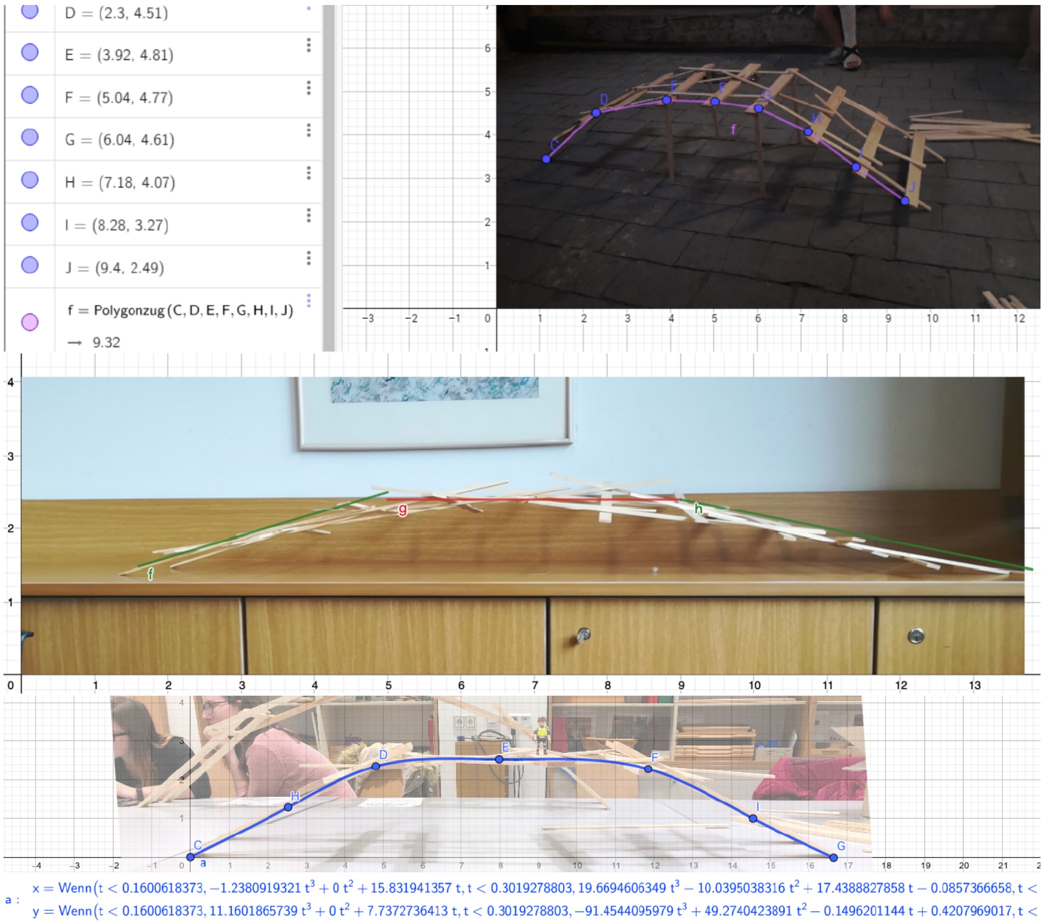

Figure 6: Piecewise-defined functions by 4th-term mathematics teacher students, 9th-grade secondary students and 6th-term mathematics teacher students.

These competencies could be particularly critical when modelling the bridge with piecewise-defined non-linear functions. On the one hand, developing a piecewise-defined non-linear model of the bridge with paper and pencil could be described as very laborious. Too laborious approaches in modelling the bridge could lead to learners dropping a chosen approach and thus limiting creativity in modelling. On the other hand, developing a piecewise-defined non-linear model of the bridge could be seen as a simple task, if learners have sufficient techno-mathematical competencies.

GeoGebra's spline command enables plotting higher-degree piecewise-defined functions through a list of points (see Figure 6). Consequently, techno-mathematical competencies could be seen as boosters for creativity in the mathematical modelling process because Ebert [11] describes creative thinking as a multidimensional phenomenon of cognitive functions and Sternberg [32] defines creative thinking as a complex interplay of intelligence. Concerning the piecewise-defined functions for modelling the bridge, mathematical intelligence and technological intelligence interact. Corresponding to mathematical intelligence, learners have recognised that non-linear piecewisedefined functions lead to a better mathematical model of the bridge than linear piecewise-defined functions. However, this knowledge could only be utilised by those groups which already had sufficient technological intelligence to achieve this with the help of mathematical software.

In the argumentations of 6th-term mathematics teacher students regarding higher-grade polynomial functions as well as non-linear piecewise-defined functions, it was interesting that students thought and argued beyond the actual task. Thinking and arguing beyond the task became apparent since 6th-term mathematics teacher students stated, that, in reality, an abrupt transition from a flat road to a bridge is not desirable. To avoid an abrupt transition, 6th-term mathematics teacher students tried to model the bridge in such a way that "the bridge goes up slowly at the beginning and then faster". Verifying a mathematical model of a bridge against reality or rationality could be an essential sign of a learning process when mathematically examining the bridges. However, when trying to avoid an abrupt transition, only one of 6th-term mathematics teacher students groups argued that the curvature of the function had to be taken into account. The other groups of 6th-term mathematics teacher students tried to solve this problem by trial and error approaches. Also, with mathematical argumentations, a trial and error approach was chosen by some 6th-term mathematics teacher students groups. For example, it was stated that (only) the value of the leading coefficient is essential or that it has to be a complete polynomial function of higher degree. By the term "complete polynomial function", the students meant (following the German technical terminology) the following function:

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x^1 + a_0 \cdots x^0;\ n \in \mathbb{N}, a_i \neq 0.$$

4.2. Mathematical models based on the upper secondary curriculum

Models of the bridge which were based on mathematical concepts of the upper secondary curriculum were applied only by 4th-term and 6th-term mathematics teacher students. These models used conic sections, trigonometric functions and the Gaussian bell curve. Interestingly, when utilising conic sections to model mathematical bridges, all groups of mathematics teacher students used equations (see Figure 7), but not functions.

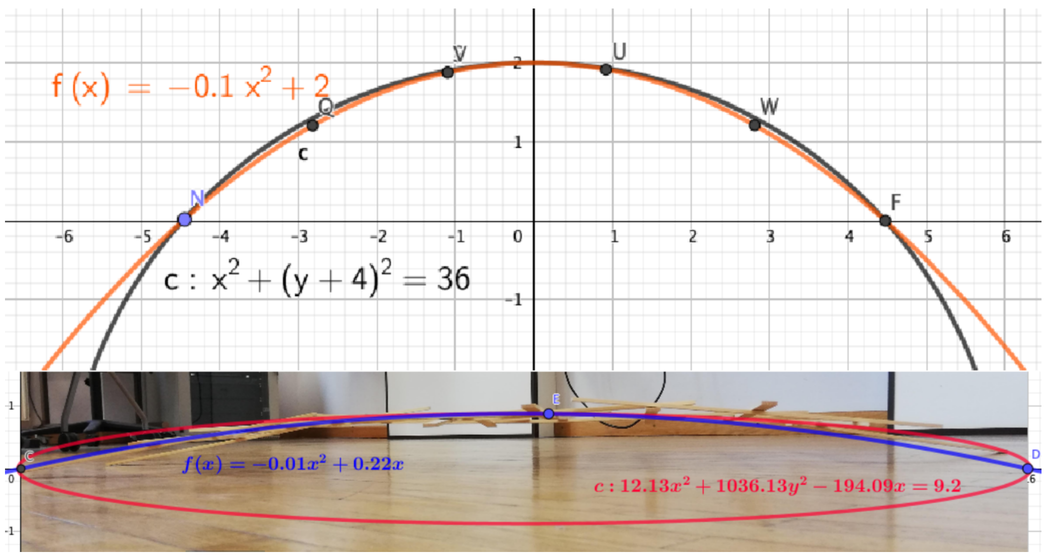

Figure 7: Using conic sections to model a mathematical bridge.

Using equations, not functions, could be grounded in the fact that if conic sections were used to model mathematical bridges, GeoGebra would render the conic section as an equation. Even after lecturers re-emphasised the task (i.e., that functions should be applied), only three groups of mathematics teacher students were able to convert equations into functions. The circumstance that only three groups of mathematics teacher students were able to convert conic equations into functions could be related to the situation that these types of tasks are only performed in the 11th grade of secondary schools and are then only superficially dealt with in mathematics teacher education (only if one chooses corresponding courses in area of "school mathematics"). It was interesting to note that only one group of mathematics teacher students knew that GeoGebra itself could perform this operation.

The other groups used paper and pencil to express equations in functional forms. In these manual operations, transforming an ellipse equation into a function, caused apparent problems for mathematics teacher students. These problems when transforming equations into functional terms could lead to conclude that using technologies when learning mathematics in secondary schools and universities could contribute to declining manual mathematical competencies of learners. However, this assumption would have to be examined separately and in greater depth in future work. Declining manual mathematical competencies of learners could contribute to concluding that a well-chosen combination of manual operations, argumentations and utilising technologies could be purposeful for learning mathematics.

Using trigonometric functions to model mathematical bridges has been done in most groups. In this context, approaches of two 4th-term mathematics teacher students groups were interesting: These two groups each wanted to adopt a sine function to the image of the built bridge by stretching and/or compressing the axes. Since stretching or compressing of the axes also caused that the inserted image was distorted accordingly, an alternative approach was sought, while retaining the "stretching or compressing of the axes approach". One alternative was to stretch or compress the axes first and then insert the image. When this approach was equally unsuccessful, one group obtained assistance from other groups—the second group stopped using trigonometric modelling. When lecturers informed both groups that stretching or compressing the axes would not change the actual function, students could not follow this argument or these remarks.

Only one 6th-term mathematics teacher student group thought of using the Gaussian bell curve to model the mathematical bridge (Figure 8). However, as it was difficult for the group to adapt the Gaussian bell curve to the image of the mathematical bridge and as students did not know immediately

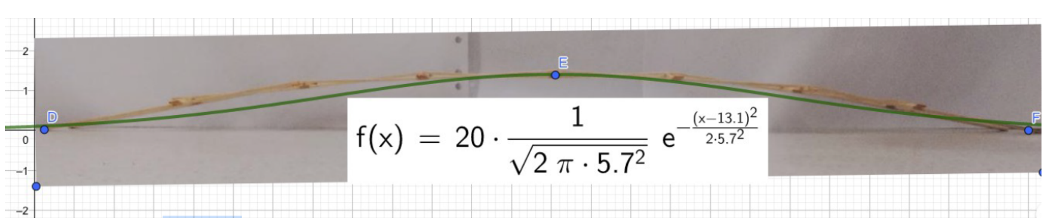

Figure 8: Gaussian bell curve as a model of a mathematical bridge.

how and which parameters were to be changed, this approach was quickly rejected.

4.3. Mathematical models based on mathematical concepts which are not covered in secondary education

The models which were based on mathematical concepts that are not covered in secondary education were catenary lines (see Figure 9) and Fourier series (see Figure 10). What was interesting about these mathematical models, which contain mathematical concepts that are dealt with at university level, was that these models were used by 6th-term mathematics teacher students (catenary line and Fournier series) and 9th-term secondary school students (catenary lines), and in seeking these mathematical models, neither the software used nor the lecturers have guided the learners.

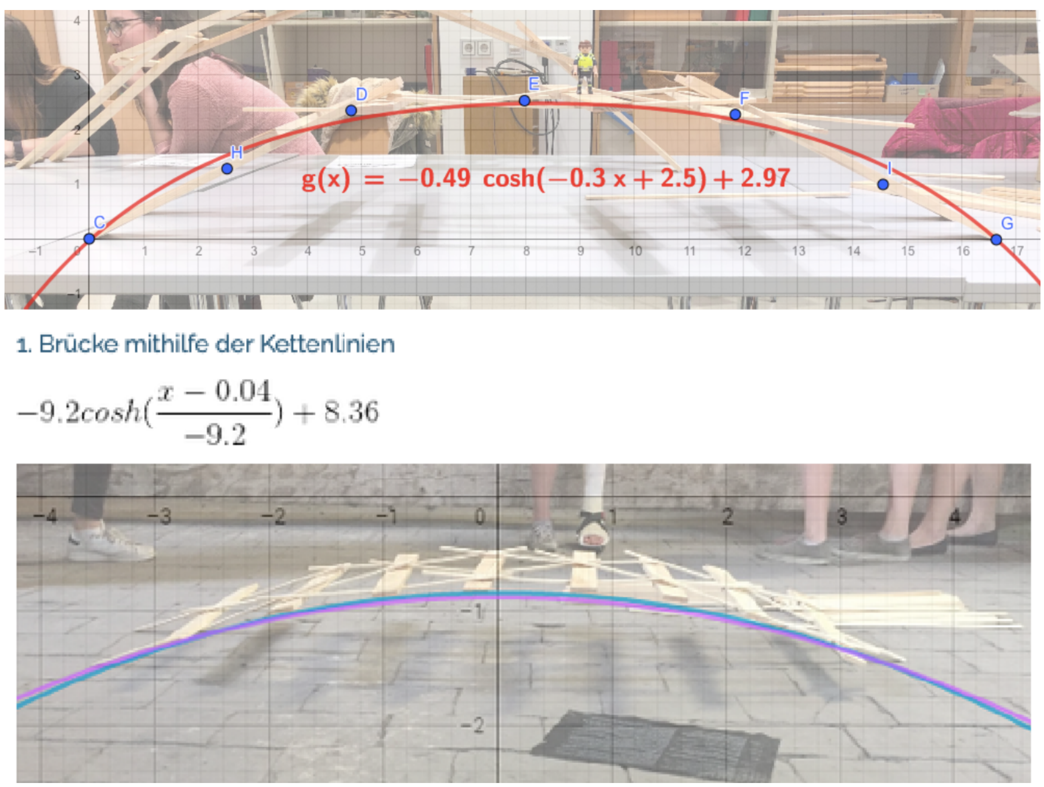

Figure 9: Using catenaries to model bridges by 6th-term students and 9th-grade pupils.

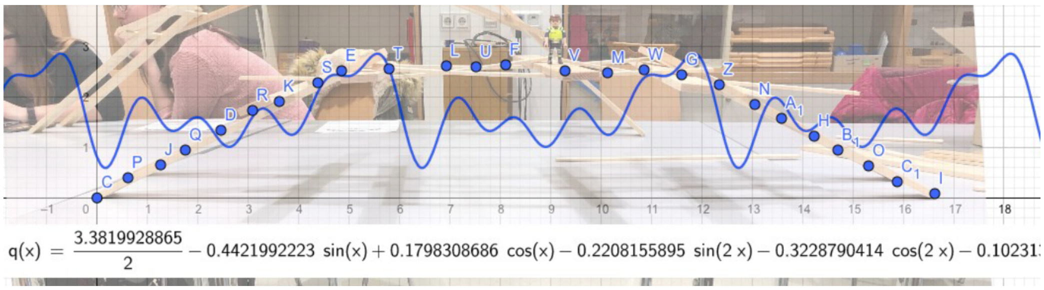

Figure 10: Utilising Fournier series for modelling bridges by 6th-term students.

At this point, it should be emphasised that when applying mathematical concepts to modelling, 9th-grade secondary school students were required to be able to explain the mathematics behind the model. By explaining the mathematics behind the respective models, it should be averted that students just copied functions, which are found on the internet, into a dynamic geometry program. The joint collection of pupil information on catenary lines (see Figure 11) indicated that pupils were working in-depth on the underlying mathematical concepts, even if this was not yet the subject matter of the 9th-grade secondary curriculum.

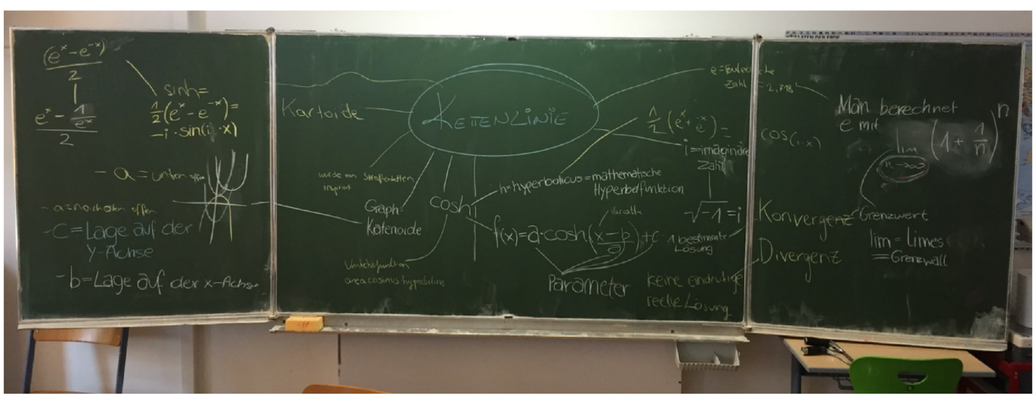

Figure 11: Joint collection of 9th-grade secondary students' information on catenary lines.

Discovering how learners could acquire new mathematical knowledge and competencies through mathematical modelling, how learners could use increased mathematical knowledge and competencies in modelling throughout their educational careers and how or if our Mathematising Spiral is feasible for mathematical modelling in formal learning settings were the goals of our explorative educational case study.

When investigating how learners could acquire new mathematical knowledge and competencies through mathematical modelling, it turned out that working in groups, no fear of making mistakes and increased technological skills could be decisive elements for this.

However, when learners developed new mathematical knowledge and competencies through mathematical modelling, it was not the elements considered individually but the interaction of these three elements. The interaction of these three elements should be understood in such a way that, on the one hand, working and learning in groups led to a reduction of the potential fear of individual group members of making mistakes. On the other hand, working together increased the technological skills of group members. Both a high level of technical skill and little or no fear of making mistakes were important for learners when new hypotheses (i.e., functions) were used to model the bridge mathematically.

Our case study indicated that even more critical than the learners' mathematical knowledge was their technological knowledge or skills when mathematical modelling in a technology-enhanced learning environment. The importance of learners' technological knowledge or skills was demonstrated by the circumstance that the most mathematically complex models were developed by 6th term mathematics teacher students and 9th grade secondary school students. Both 6th term mathematics teacher students and 9th grade secondary school students could be characterised as groups with high knowledge and competencies in the field of using mathematical educational technologies. High knowledge and competencies in the field of using mathematical education technologies could be justified by the circumstance that 6th term mathematics teacher students had a course on using mathematical education technologies before our case study. Concerning 9th grade secondary school students, using mathematical education technologies is a vital element of the curriculum and the regulations of the written school-leaving examination. These requirements also result in using technologies being a central element of current secondary mathematics teaching. This increase in knowledge and competencies in utilising mathematical education technologies from 9th grade secondary school students compared to 4th term mathematics teacher students led to some groups of 9th grade secondary school students developing mathematically more complex models than 4th term mathematics teacher students.

Regarding the applicability of Mathematising Spiral, our explorative case study indicated that when mathematising real models or problems in a technology-enhanced learning environment, the creativity and technological competencies of learners are crucial. Creativity and the technological competencies of learners as decisive factors when mathematising could be justified by the circumstance that the mathematical models based on the most sophisticated mathematical concepts were developed by learners who had high skills in using mathematical technologies.

5. Discussion and Conclusions

Our explorative educational case study focused on discovering how learners could acquire new mathematical knowledge and competencies through mathematical modelling, how learners could use increased mathematical knowledge and competencies in modelling throughout their educational careers and how or if our Mathematising Spiral is feasible for mathematical modelling in formal learning settings.

To investigate these questions, we have developed a technology-enhanced learning environment which combines physical or real models and mathematical modelling. This learning environment has been made available to 9th grade secondary school students, 4th term mathematics teacher students and 6th term mathematics teacher students. Our results are based on the learning products and the mathematical concepts behind these learning products of the different groups of learners. Combining real objects and mathematical modelling in our technology-enhanced learning environment led to mathematical models of physical or real models being linked to reality. Thus, according to Vos [35], our learning environment could be characterised as an authentic learning environment. In this context, it was interesting to note that for two groups of learners (6th term mathematics teacher students) this authenticity or connection of the learning environment to reality led to an extension of the learning task. The extension of the learning task was such that the learners not only tried to create a mathematical model of a real object, but two groups of learners also required their mathematical model to be feasible or applicable in reality.

This claim that the mathematical model should also correspond to the requirements of reality could be demonstrated by the circumstance that the 6th term mathematics teacher students wanted to prevent an abrupt transition in the mathematical model. Extending the task and connecting the mathematical model with the requirements of reality could also be an indicator that the authenticity in our technologyenhanced learning environment in which physical or real models and mathematical models were combined could trigger or facilitate discovery-based or inquiry-based learning, which would be material for new research.

Another interesting aspect of our explorative case study was the way learners utilised educational technologies when working on the tasks. According to Noss & Hoyles [28] and Noss, Healy & Hoyles [29], using educational technologies should contribute to making learning environments have laboratory-like characteristics. Our educational case study indicated that using educational technologies and integrating authenticity into the learning environment could further lead to a learning environment having laboratory-like characteristics. The circumstance that learners switched between mathematical modelling and real objects could justify the reinforcement of the laboratory-like properties of learning environments by combining authenticity and utilising educational technologies. This switching between mathematical modelling and the real object occurred, especially when the pictures of the real object were not suitable for mathematical modelling using educational technologies. Learners then returned to the real object and created a new picture of the real object. This new picture of the real object could be interpreted as a new or additional sample, which would then be the subject of further investigations (i.e., mathematical modelling).

One goal of the technology enhancement of learning environments in our research was to facilitate learners to adjust their mathematical models to real objects or pictures of real objects. This adjustment of mathematical models to real objects or pictures of real objects could be interpreted as a further development of the experimental-with-calculator approach in curve or function fitting by Borba, Meneghetti & Hermini [3] and Borba & Bovo [4]. However, our explorative educational case study took place almost twenty years after the research of Borba, Meneghetti & Hermini [3] and Borba & Bovo [4] and thus we were able to integrate high-quality technologies as calculators into the research design. These high-quality technologies and the increased confidence of today's learners in using high-quality technologies could contribute to the conclusion that an experimental-with-dynamic geometry system approach could be even better suited for curve or function fitting than an experimental-with-calculator approach.

6. Impacts on teaching and learning, and limitations

Learners' written feedback that was received after each unit of our modelling experiment and the joy of learners to be observed during work phases could be interpreted as most learners involved in the modelling experiment enjoyed discovering mathematics following an approach which combines real-world models and technology-enhanced learning. The key to motivation was probably both: hands-on experiences and an explorative discovery of mathematics in our modelling experiment.

The hands-on experiences of the modelling experiment were highlighted as very positive in the written feedback by learners from all three groups. Recurring patterns of learner responses were that learning was enjoyable and that mathematics was filled with meaning and life. This joy in working hands-on could also be observed in the participation of the learners during the units. It was interesting that especially the 6th-term mathematics teacher students and the 9th-grade secondary school students were very enthusiastic that they could work independently when learning. Working independently when learning means that concrete solutions were not required and that learning paths was not prescribed. Independent work could be recognised both during construction phases of the bridge and during the modelling and examination of the bridge. Working independently during construction means that different groups of learners had implemented different construction plans for a mathematical bridge. In the mathematical modelling and investigation of the bridge, independence was demonstrated by different groups applying different mathematical concepts to characterise the bridge. These representations indicated that 6th-term mathematics teacher students mainly used mathematical concepts they had learned during their secondary and university education. In the beginning, the 9th-grade secondary school students also applied the mathematical concepts known to them for modelling and describing the bridge. However, after some time, 9th-grade secondary school students left the known terrain and searched for further possibilities to characterise and investigate the bridge.

In this further research for mathematical concepts, pupils formed groups of two to four learners. The ideas discovered here were shared by student groups with their classmates and thus also examined and deepened more closely. A joint work within and between secondary student groups was evident in the technological implementation of the mathematical concepts. Concerning the question of whether new mathematical concepts could be developed through mathematical modelling of physical objects by learners, it turned out that this issue could be confirmed for 9th-grade secondary students. Particularly during a follow-up investigation of mathematical concepts used in mathematical modelling, it became evident that 9th-grade secondary school students were intensively concerned with new mathematical concepts (see Figure 9). The 9th-grade secondary school students were able to explain correctly and comprehensibly concepts of the limit of mathematical sequences or fundamental principles of complex numbers. Since these mathematical concepts are treated according to the secondary curriculum after a 9th grade, mathematical modelling could also be used to discover new mathematical concepts by students actively.

It was also noticeable that in mathematical models of the 4th-term mathematics teacher students, only content of the secondary school curriculum was used. The 4th-term mathematics teacher students neither applied newly learned mathematical concepts (at university) to mathematical models of the bridge during their studies nor searched for an alternative and higher-value modelling possibilities.

Potential limitations of the results of our case study should be made, especially concerning secondary education. The limitations of the results concerning secondary education could be justified by the circumstance that the secondary school of our case study is located in the centre of Vienna and that the proximity to residential areas in Austria is an essential factor in the secondary schools' enrolment process. According to these enrolment processes circumstances, it could be assumed that many of the secondary school students in our case study live close to the centre and therefore have a high socioeconomic status and that education has a high priority in these families. The socioeconomically high status, as well as the high value of education, should have a positive effect on the availability of a multitude of high-quality technologies in the families of secondary school students and thus have a positive effect on the familiarity of secondary students using high-quality technologies.

The potentially high value of education in the families of secondary school students should also have a positive effect on the curiosity and motivation of secondary school students in our explorative case study. In future research, it could be interesting to examine how the learning environment of our case study affects less privileged students and to what extent the results of our present study need to be adapted or changed.

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References

[1] Blum, W., "Mathematical modelling in mathematics education and instruction," pages 3–14 in *Teaching and learning mathematics in context* edited by Breiteig *et al.* (Ellis Horwood Limited, Chichester, 1993).

[2] Blum, W. & Leiss, D., "How do students and teachers deal with mathematical modelling problems?" pages 222–231 in *Mathematical Modelling* edited by Haines *et al.* (Elsevier, 2007).

[3] Borba, M. C., Meneghetti, R. C. G., & Hermini, H. A., "Modelagem, calculadora gráfica e interdisciplinaridade na sala de aula de um curso de ciências biológicas," *Revista de Educação Matemática da SBEM-SP*, Volume **5** Issue 3 (1997), pages 63-70.

[4] Borba, M. C., & Bovo, A. A., "Modelagem em sala de aula de matemática: interdisciplinaridade e pesquisa em biologia," *Revista de Educação Matemática*, Volume **8** Issue 6 (2002), 27-33.

[5] Buchberger, B., "Theorema: Theorem proving for the masses using Mathematica," invited talk at the Worldwide Mathematica Conference, Chicago, June 18-21, 1998.

[6] Carreira, S., & Baioa, A. M., "Mathematical modelling with hands-on experimental tasks: on the student's sense of credibility," *ZDM*, Volume **50** (2018), pages 201-2015. doi:10.1007/s11858-017-0905-1

[7] Cobb, P., "Concrete can be abstract: A case study," *Educational Studies in Mathematics*, Volume **17** Issue 1 (1986), pages 37–48. doi:10.1007/BF00302377

[8] Cohen, L., Manion, L., & Morrison, K., *Research Methods in Education*, sixth edition, Routledge, 2007.

[9] DeVito, A., *Creative Wellsprings for Science Teaching*, Creative Ventures, 1989.

[10] Doerr, H. M., & Lesh, R., "Models and modelling perspectives on teaching and learning mathematics in the twenty-first century," pages 247-268 in *Trends in Teaching and Learning of Mathematical Modelling* edited by Kaiser *et al.* (Springer, Dordrecht, 2011).

[11] Ebert, E. S., "The cognitive spiral: Creative thinking and cognitive processing," *The Journal of Creative Behavior*, Volume **28** Issue 4 (1994), pages 275–290.

[12] Ferchhoff, W., *Jugend und Jugendkulturen im 21. Jahrhundert — Lebensformen und Lebensstile*, Wiesbaden, 2007.

[13] Heck, A., "Modelling in cross-disciplinary authentic student research projects," *International Journal for Technology in Mathematics Education*, Volume **17** Issue 3 (2010), pages 115-120.

[14] Hood, P., *Perspectives on Knowledge Utilization in Education*, WestEd, San Francisco, CA, 2002.

[15] Kaenders, R., & Schmidt, R., "Zu einem tieferen Mathematikverständnis," pages 1–11 in *Mit GeoGebra mehr Mathematik verstehen* edited by Kaenders *et al.* (Springer, 2014).

[16] Kaiser, G., Lederich, C., & Rau, V., "Theoretical approaches and examples for modelling in mathematics education," pages 29–246 in *Mathematical Applications And Modelling: Yearbook 2010* (Association of Mathematics Educators, 2010).

[17] Kaiser, G., & Stender, P., "Complex Modelling Problems in Co-operative, Self-Directed Learning Environments," pages 277-293 in *Teaching Mathematical Modelling: Connecting to Research and Practice* edited by Stillman *et al.* (Springer, Dordrecht, 2013).

[18] Kanematsu, H., & Barry, D. M., *STEM and ICT Education in Intelligent Environments*, Springer, 2016.

[19] Kertil, M., & Gurel, C., "Mathematical modeling: A bridge to STEM education," *International Journal of Education in mathematics, science and Technology*, Volume **4** Issue 1 (2016), pages 44-55.

[20] Kovács, Z., "Automated Reasoning Tools in GeoGebra: A new approach for experiments in planar geometry," *South Bohemia Mathematical Letters*, Volume **25** Issue 1 (2017), pages 48-65.

[21] Krathwohl, D. R., "A revision of Bloom's taxonomy: An overview," *Theory into Practice*, Volume **41** Issue 4 (2002), pages 212–218.

[22] Lingefjärd, T., & Holmquist, M., "Mathematical modelling and technology in teacher education–Visions and reality," pages 205–215 in *Modelling and Mathematics Education* (Elsevier, 2001). doi:10.1533/9780857099655.4.205

[23] Maaß, K., "What are modelling competencies? *ZDM*, Volume **38** (2006), pages 113-142.

[24] Malle, G., Koth, M., Woschitz, H., Malle, S., Salzger, B., & Ulovec, A., *Mathematik verstehen 5*, textbook, Österreichischer Bundesverlag (öbv), Vienna, 2017.

[25] Mayer, R. E., *Thinking, Problem Solving, Cognition*, Freeman, New York, 1983.

[26] Niss, M., "Models and modelling in mathematics education," *EMS Newsletter*, Volume **86** (2012), pages 49-52.

[27] Nonaka, I., & Takeuchi, H., *The Knowledge Creating Company*, Oxford University Press, New York, 1995.

[28] Noss, R., & Hoyles, C., *Windows on Mathematical Meanings: Learning Cultures and Computers*, Springer Science & Business Media, 1996.

[29] Noss, R., Healy, L., & Hoyles, C., "The construction of mathematical meanings: Connecting the visual with the symbolic," *Educational studies in mathematics*, Volume **33** Issue 2 (1997), pages 203- 233.

[30] Rogers, E., *Diffusion of Innovations*, fourth edition, The Free Press, New York, 1995.

[31] Samuelsson, J., "ICT as a Change Agent of Mathematics Teaching in Swedish Secondary School," *Education and Information Technologies*, Volume **11** Issue 1 (2006), pages 71–81. doi:10.1007/s10639-005-5713-5

[32] Sternberg, R. J., & Sternberg, R. J. P. (editors), *The Nature of Creativity: Contemporary Psychological Perspectives*, Cambridge University Press, 1988.

[33] Torrance, E. P., *Rewarding Creative Behavior; Experiments in Classroom Creativity*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[34] Torrance, E. P., *The Torrance tests of creative thinking: Norms-technical manual*, Research edition, Verbal tests, forms A and B. Figural tests, forms A and B, Princeton, NJ: Personnel Press, Princeton, NJ, 1966.

[35] Vos, P., "What Is 'Authentic' in the Teaching and Learning of Mathematical Modelling?" pages 713–722 in *Trends in Teaching and Learning of Mathematical Modelling* edited by Kaiser *et al.* (Springer, Dordrecht, 2011).

[36] Yin, R. K., *Case Study Research: Design and Methods*, Sage, 1984.

[37] Zulnaidi, H., Oktavika, E., & Hidayat, R., "Effect of use of GeoGebra on achievement of high school mathematics students," *Education and Information Technologies* Volume **25** (2019), pages 51–72. doi:10.1007/s10639-019-09899-y

Mathematics curricula and legal framework conditions for education

[A] https://www.ris.bka.gv.at/GeltendeFassung.wxe?Abfrage=Bundesnormen&Gesetzesnummer=10008568, requested on: 21 November 2019

[B] https://www.ris.bka.gv.at/GeltendeFassung.wxe?Abfrage=Bundesnormen&Gesetzesnummer=20007845, requested on: 21 November 2019

[C] https://www.jku.at/fileadmin/gruppen/32/ZUS/Curricula/Bachelor/3_BS_Lehramt_AllgemeinB_CM_MTB33_240619.pdf, requested on: 21 November 2019