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Length Biased Burr- XII Distribution: Properties and Application

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Abstract

Burr XII distribution is widely applicable in reliability and life testing problems in engineering as well as in survival analysis. The concept of weighted and size- biased sampling and length biased distribution due to importance of this kind of distributions and it's application in many fields such as medicine, ecology, reliability and human populations. In this paper, length biased Burr XII distribution is proposed and studied. Different properties of this new distribution are discussed such as the density function and its behaviour , moments, hazard , survival functions and order statistics. The parameters of this new distribution are estimated by maximum likelihood method . The observed information matrix is derived. Finally, we provide a simulation and real data analysis to see how the new model is applicable in practice.

Keywords: Burr XII distribution; Length Biased; Survival Function; Order Statistics; Renei Entropy; Maximum Likelihood Method.

1. Introduction

The applications of weighted distributions in research related to reliability, bio-medicine, ecology and several other areas are of tremendous practical importance in mathematics, probability and statistics.The concept of weighted distributions can be traced as [23] .

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Later it was introduced and formulated in general terms by [6], in connection with modeling statistical data where the usual practiceof using standard distributions for the purpose was not found to be appropriate.Length biased distributions are special case of the more general form known as weighted distribution. In literature, many authors present length biased distributions. Researcher [14] studied the length biased of Weibull distribution and [15] discuss length biased from the Weighted Generalized Rayleigh distribution. Reference [11] present length biased Beta distribution of first kind. Reference [12] introduced Length-biased of weighted exponential and Rayleigh distributions . [20] present length biased beta Pareto distribution with application which called new generalized of Pareto distribution. The length-biased weighted Frechet distribution have been discussed by [9]. The main aim of this paper is to present another version of the Burr-XII distribution based on length biased method. So, the length biased Burr-XII distribution is proposed and also to study some structural properties of the proposed distribution are discussed. The trend of parameters induction to the baseline distribution has received increased attention in recent years to explore properties and for efficient estimation of the parameters. In the literature, some extensions of Burr-XII distribution are available such as, Beta Burr XII with statistics application by [22]. A new generalization of Burr XII distribution by [19]. The Marshall-Olkin Extended Burr XII by [2]. The geometric inverse Burr distribution by [4]. A new four parameter extension of Burr-XII distribution by [16]. Generalized log burr-XII distribution by [10].The extended Burr XII distribution with variable shapes for the hazard rate by [26]. The rest of the paper is organized as follows. The definition of length biased of Burr -XII distribution is provided in Section 2. The statistical properties of LBB- XII distribution are provide in Section 3. In Section 4, the order statistics are provided. The parameter estimation by using maximum likelihood method are considered in Section 5. Application upon three real data sets in section 6.Simulation study are performed in Section 7.Finally, conclusion in Section 8.

2. The LBB-XII Distribution

Let y be a random variable distributed according to a Burr XII distribution, with parameters c and k, the density function (PDF) as follows [10],

$$g(y) = cky^{c-1}\left[1+y^{c}\right]^{-(k+1)}, \; y>0 \tag{1}$$

where both c and k are shape parameters. The distribution function of $Y|\{c,k\}$ is

$$G(y) = 1-\left(1+y^{c}\right)^{-(k+1)}, y>0$$

The weighted distribution concept is

$$g^{w}(y) = \frac{w(y)g(y)}{w}, \quad \text{for } y>0 \tag{2}$$

Where $w = \int_{-\infty}^{\infty} w(y)g_{w}(y)dy$ and $w(y) = y$ or y^{a} which called length biased.

Let y , denotes the random variable with density function (1) and let we choose the weighted function as $w(y) = y$.

According to (2), the density function of Length Biased Burr -XII distribution is

$$g_{LBB}(y) = \frac{cy^c\left[1+y^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}, \; c, k > 0 \text{ and } y > 0 \tag{3}$$

In addition, the distribution function is:

$$G_{LBB}(y) = \frac{1}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \int_0^t \frac{cy^c}{\left[1+y^c\right]^{(k+1)}} dy = I_Y, y>0 \tag{4}$$

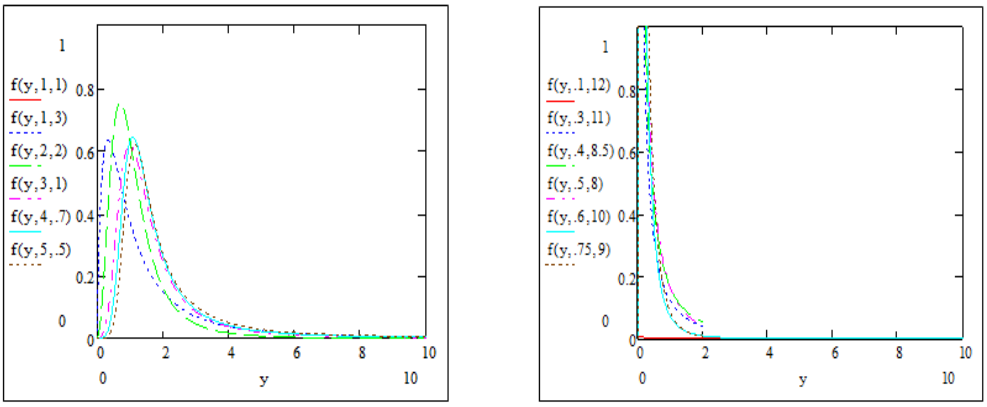

Figure 2.1: The density function of LBBXII

Table 2.1: Sub Models of LBB-XII distribution

C	***K***	**Distribution**
4.874	**6.158**	**Weighted Approximate Normal Distribution**
c	∞	**Weighted Weibull Distribution**
1	∞	**Weighted Exponential Distribution**
∞	k	**Weighted Generalized Logistic Distribution**
∞	1	**Weighted Logistic Distribution**
∞	∞	**Weighted Gompertz Distribution**
1	1	**Weighted Lomax Distribution (weighted Pareto II)**

3. Some properties of the LBB-XII distribution

In this section some properties of the LBB-XIIdistribution will be obtained as follows:

3.1 The r-th Moment

Generally, the *r-th* moment of a continuous random variable y , [21], is given by

$$E\left(y^r\right)=\mu_r'=\int_0^\infty y^r g(y)dy$$

Substituting equation (3) into last equation

$$E_{LBB}\left(y^r\right)=\int_0^\infty y^r \frac{cy^c\left[1+y^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}dy$$

$$E_{LBB}\left(y^r\right)=\frac{c}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\int_0^\infty y^{c+r+1-1}\left(1+y^c\right)^{-(k+1)}dy$$

Let $u=y^c \rightarrow du=cy^{c-1}dy$ and $u^{\frac{1}{c}}=y$

Since

$$\int_0^\infty z^{r-1}(1+z)^{-a}\,dz=\beta(r,a-r)$$

Hence

$$E_{LBB}\left(y^r\right)=\frac{1}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\int_0^\infty (u)^{\left(\frac{r+1}{c}\right)}(1+u)^{-(k+1)}du$$

One easily can find that

$$\therefore \underset{LBB}{E}\left(y^{r}\right)=\frac{\beta\left(\frac{r+1}{c}+1,k-\frac{r+1}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}, \qquad (5)$$

At $r=0$

$$\underset{LBB}{E}\left(y^{0}\right)=\frac{\beta\left(\frac{0+1}{c}+1,k-\frac{0+1}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}=1$$

In addition, the first three moments about the LBB distribution is

$$\mu_1'=\beta\left(\frac{2}{c}+1,k-\frac{2}{c}\right)\Big/\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right),$$

$$\mu_2'=\beta\left(\frac{3}{c}+1,k-\frac{3}{c}\right)\Big/\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right),$$

$$\mu_3'=\beta\left(\frac{4}{c}+1,k-\frac{4}{c}\right)\Big/\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right),$$

In addition, ,the mean $\underset{LBB}{E}(y)$, the variance $\underset{LBB}{V}(y)$, the coefficient of variation $\underset{LBB}{C.V}(y)$ respectively can acquire as

- $$\underset{LBB}{E}(y)=\frac{\beta\left(\frac{2}{c}+1,k-\frac{2}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}$$
- $$\underset{LBB}{V}(y)=\left(\mu_2'-\mu_1'^{2}\right)=\left[\frac{\beta\left(\frac{3}{c}+1,k-\frac{3}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\right]-\left[\frac{\beta\left(\frac{2}{c}+1,k-\frac{2}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\right]^{2}$$

$$\underset{LBB}{CV}(y)=\frac{\left(\mu_2'-\mu_1'^2\right)^{\frac{1}{2}}}{\mu_1'}=\left\{\left[\frac{\beta\left(\frac{3}{c}+1,k-\frac{3}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\right]-\left[\frac{\beta\left(\frac{2}{c}+1,k-\frac{2}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\right]^2\right\}^{1/2}\div\left[\frac{\beta\left(\frac{2}{c}+1,k-\frac{2}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\right]$$

3.2 The Moment generation function

If Y is a continuous random variable distributed as LBBXII distribution, the moment generation function is given by

$$\underset{LBB}{M_y}(t)=E\left(e^{ty}\right)=E\left[1+ty+\frac{(ty)^2}{2!}+\frac{(ty)^3}{3!}+\ldots\right]=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(y^r\right)$$

Substituting (5) into last equation yield,

$$\underset{LBB}{M_y}(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\beta\left(\frac{r+1}{c}+1,k-\frac{r+1}{c}\right)}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)} \tag{6}$$

3.3 The Quantile function

The well known definition of the 100 *q-th* is

$$q=p(X\le Y)=F\left(Y_q\right),\ Y_q>0,\ 0<q<1.$$

So,

$$q=\frac{1}{\beta\left(\frac{1}{c}+1,k-\frac{1}{c}\right)}\int_0^t cy^c\left(1+y^c\right)^{-(k+1)}dy \tag{7}$$

Clearly, the last equation is a nonlinear quantile function with respect to y and incomplete function and it needs a numerical solution to be solved.

3.4 The mean deviation

Generally, the mean deviation for a random variable about the mean and about y the median

y , respectively, can be given as

If Y has the LBB-XII distribution, we derive the mean deviation about the mean μ by

$$S_1(y) = \int_y |y-\mu| f(y)dy \text{ and } S_2(y) = \int_y |y-M| f(y)dy$$

Another form can be given as

$S_1(y) = 2\mu F(\mu) - 2T(\mu)$ and $S_2(y) = \mu - 2T(M)$ See [17], where $T(q) = \int_{-\infty}^{q} yf(y)dy$.

Substituting (3) into $T(q)$ gives

$$T_{LBB}(q) = \int_0^q y \frac{cy^c \left[1+y^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} dy$$

So,

$$T_{LBB}(q) = \frac{c}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \int_0^q y^{c+1-1+1} \left[1+y^c\right]^{-(k+1)} dy$$

Put $u = y^c$, so $y = u^{\frac{1}{c}}$ and $du = cy^{c-1}dy$

And let

$$I = \int_0^q u^{\frac{2}{c}} (1+u)^{-(k+1)} du$$

By using the incomplete beta function

$$\int_0^a z^{r-1}(1+z)^{-a} dz = \beta(a; r, a-r)$$

So,

$$I = \beta\left(q; \frac{2}{c}+1, k-\frac{2}{c}\right)$$

Hence,

$$T(q) = \frac{\beta\left(q; \frac{2}{c}+1, k-\frac{2}{c}\right)}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}$$

3.5 The mode

The natural logarithm of (3) is

$$\ln\left[\underset{LBB}{g}(y)\right] = \ln(c) + c\ln(y) - (k+1)\ln\left(1+y^{c}\right) - \ln\left[\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]$$

Differentiating the last equation with respect to y and equating it to zero yields

$$\frac{d}{dy}\ln\left[\underset{LBB}{g}(y)\right] = \frac{c}{y} - (k+1)\frac{cy^{c-1}}{\left(1+y^{c}\right)} = 0 \quad , \qquad (8)$$

The last equation is a nonlinear equation and it does not have an analytic solution with respect to y, therefore it have to be solved numerically.

3.6 The hazard function of LBB-XII distribution

Generally, the survival function of a random variable Y, see [27] can be given by

$$S(y) = 1 - G(y)$$

Substituting (4) into last equation gives

$$\underset{LBB}{S}(y) = 1 - \frac{1}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\int_{0}^{t} cy^{c}\left(1+y^{c}\right)^{-(k+1)}dy = 1 - I_{y} \quad . \qquad (9)$$

Simply, the hazard function, see [27], can be given by

$$H(y) = \frac{f(y)}{S(Y)}$$

Substituting (3) and (9) into last equation yields

$$\underset{LBB}{H}(y) = \frac{cy^{c}\left[1+y^{c}\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\left(1-I_{y}\right)}, \qquad (10)$$

3.7 The Rényi entropy of the LBB-XII distribution

The Rényi entropy of a random variable Y, see [27], is defined by

$$e_R(\rho) = \frac{1}{1-\rho}\log\left[\int_{-\infty}^{\infty}[g(y)]^{\rho}\right]dy$$

Substituting (3) into last equation

$$\underset{LBB}{e_R}(\rho) = \frac{1}{1-\rho}\log\int_0^{\infty}\left[\frac{cy^{c}\left(1+y^{c}\right)^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{\rho} dy$$

So,

$$e_R(\rho) = \frac{1}{1-\rho}\log\frac{1}{\left[\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\rho}}\int_0^{\infty}\frac{cy^{c\rho}}{\left(1+y^{c}\right)^{\rho(k+1)}}dy$$

Let $I_2 = \int_0^{\infty}\frac{cy^{c\rho}}{\left(1+y^{c}\right)^{\rho(k+1)}}dy$

We can say

$$I_2 = \int_0^{\infty}\frac{cy^{c-1}y^{c\rho-c+1}}{\left(1+y^{c}\right)^{\rho(k+1)}}dy$$

Let $u = y^{c} \rightarrow du = cy^{c-1}dy$ and $u^{\frac{1}{c}} = y$

Since

$$\int_0^\infty z^{r-1}(1+z)^{-a}\,dz = \beta(r,a-r)$$

$$I_2 = \int_0^\infty u^{\left(\frac{c\rho-c+1}{c}\right)}(1+u)^{-\rho(k+1)}\,du = \beta\left(\frac{c\rho-c+1}{c}+1, \rho(k+1)-\frac{c\rho-c+1}{c}-1\right)$$

Hence,

$$e_{R_{LBB}}(\rho) = \frac{1}{1-\rho}\log\left\{\frac{c^{\rho-1}}{\left[\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\rho}}\beta\left[\frac{c\rho-c+1}{c}+1, \rho(k+1)-\frac{c\rho-c+1}{c}-1\right]\right\}$$

So,

$$_{R_{LBB}}(\rho) = \frac{1}{1-\rho}\left\{(\rho-1)\log(c)-\rho\log\left[\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]+\log\beta\left[\frac{c\rho-c+1}{c}+1, \rho(k+1)-\frac{c\rho-c+1}{c}-1\right]\right\} \tag{11}$$

4. Order statistics of LBB-XII distribution

Let $Y_1, Y_2, \ldots, Y_n$ is a random sample drawn from $LBB(c,k)$. Since $Y_1, Y_2, \ldots, Y_n$ are i.i.d. continuous random variables, then the probability that any two (or more) observation in random sample take the same magnitude (the same value is equal to zero). Therefore, there exists a unique ordered arrangement of the sample observation according to magnitude.

Let $Y_{(1:n)}, Y_{(2:n)}, \ldots, Y_{(n:n)}$ be the order statistics. Then the PDF of $Y_{(r:n)}, 1 \le r \le n,$ denoted by $g_{r:n}(y)$ is given by

$$g_{r:n}(t) = C_{r:n}[G(y)]^{r-1}[S(y)]^{n-r}g(y), \tag{12}$$

where $C_{r:n} = n!/((r-1)!(n-r)!)$.

4.1 Smallest and Largest Order Statistics of LBB-XII

The smallest observation in the sample as $Y_{(1:n)} = \min(Y_1, \ldots, Y_n)$, the largest observation in the sample $Y_{(n)} = \max(Y_1, \ldots, Y_n)$ and the median order as $Y_{(m+1;n)}$, if $n = 2m+1$, thus $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ are given by *(i)* $$g_{1:n}(y) = n\left[\underset{LBB}{S(y)}\right]^{n-1} \underset{LBB}{g(y)}$$

$$g_{1:n}(y) = n[1 - I_Y]^{n-1}\left[\frac{cy^c\left[1+y^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]$$

(ii) $$g_{n:n}(y) = n\left[\underset{LBB}{G(y)}\right]^{n-1} \underset{LBB}{g(y)};$$

$$g_{n:n}(y) = n[I_Y]^{n-1}\left[\frac{cy^c\left[1+y^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]$$

(iii) $$g_{m+1;n} = \frac{(2m+1)!}{(m!)^2}\left[\underset{LBB}{G(y)}\right]^m \left[\underset{LBB}{S(y)}\right]^m \underset{LBB}{g(y)}$$

$$g_{m+1;n} = \frac{(2m+1)!}{(m!)^2}(I_Y)^m(1-I_Y)^m\left[\frac{cy^c\left(1+y^c\right)^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]$$

4.2 *Joint distribution of the r^{th} and i^{th} order statistics*

The bivariate probability density function of $Y_{(r:n)}$ and $Y_{(i:n)}, 1 \le r \le i \le n,$ from the LBB-XII distribution is given by

$$g_{r,i:n}(y_r, y_j) = C_{r:i:n}\left(\underset{LBB}{G(y_r)}\right)^{r-1}\left(\underset{LBB}{G(y_i)} - \underset{LBB}{G(y_r)}\right)^{i-r-1} \times \left(\underset{LBB}{S(y_i)}\right)^{n-i} \underset{LBB}{g(y_r)}\underset{LBB}{g(y_i)} \tag{13}$$

By using (13), we have

$$g_{r,i,n}(y_r, y_i) = C_{r,i,n}\left[I_{Y_r}\right]^{r-1}\left[I_{Y_i} - I_{Y_r}\right]^{i-r-1}\left[1 - I_i\right]^{n-i}\left[\frac{cy_i^c\left[1+y_i^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]\left[\frac{cy_r^c\left[1+y_r^c\right]^{-(k+1)}}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]$$

whereas:

1. $-\infty < y_r < y_i < \infty$;
2. $C_{r:i:n} = n!/((r-1)!(i-r-1)!(n-i)!)$;
3. $I_{Y_i} = \frac{1}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\int_0^t \frac{cy_i^c}{\left[1+y_i^c\right]^{(k+1)}}dy_i$ and $I_{Y_r} = \frac{1}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\int_0^t \frac{cy_r^c}{\left[1+y_r^c\right]^{(k+1)}}dy_r$

Consider (13), then we can conclude that the minimum and maximum bivariate probability density of the LBBXII denoted by $g_{1:n::n}(y_r, y_i)$, thatcan be investigated from equation (13) by substituting $i = n$ and $r = 1$ as follows

$$g_{1,n,n}(y_1, y_n) = \frac{n!}{(n-2)!}\left[I_{Y_1} - I_{Y_n}\right]^{n-2}\left[\frac{c^2 y_1^c y_n^c\left[1+y_1^c\right]^{-(k+1)}\left[1+y_n^c\right]^{-(k+1)}}{\left[\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^2}\right]$$

5. Estimation of the LBB-XII distribution parameters

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from the LBB-XII distribution with (3), therefore the likelihood function can be written as follows:

$$L(c, k, / y) = \prod_{i=1}^{n} g_{LBB}(y),$$

the likelihood function is

$$\ell(y_1, y_2, \ldots, y_n | c, k) = \frac{c^n \prod_{i=1}^{n} y_i^c \prod_{i=1}^{n}\left(1+y_i^c\right)^{-(k+1)}}{\left[\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^n}. \quad (14)$$

By using (14), we have

$$\ln \ell(y_1, y_2, \ldots, y_n | c, k) = n \ln c + + c \sum_{i=1}^{n} \ln y_i - (k+1) \sum_{i=1}^{n} \ln(1 + y_i^c) - n \ln \beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right) \tag{15}$$

The components of the score vector for the parameters c and k are given by

$$\frac{\partial \ln \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \ln y_i - (k+1) \sum_{i=1}^{n} \frac{y_i^c}{(1 + y_i^c)} \ln(y_i) - n \frac{1}{\beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)} \phi'_{\beta c} \tag{16}$$

where

$$\phi'_{\beta c} = \frac{\partial \beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)}{\partial c} = \beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)\left[\Psi_0\left(\frac{1}{c} + 1\right) - \Psi_0(k+1)\right]$$

And

$$\frac{\partial \ln \ell}{\partial k} = -\sum_{i=1}^{n} \ln(1 + y_i^c) - n \frac{1}{\beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)} \phi'_{\beta k} \tag{17}$$

Where

$$\phi'_{\beta k} = \frac{\partial \beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)}{\partial k} = \beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)\left[\Psi_0\left(k - \frac{1}{c}\right) - \Psi_0(k+1)\right]$$

Where $\Psi_0(a)$ is the polygamma function.

For the observed information matrix of the parameters c and k we calculate the second partial derivatives of (16) and (17) with respect to c and k respectively, then it can derivative as follows

$$\frac{\partial^2 \ln \ell}{\partial c^2} = -\frac{n}{c^2} + (k+1) \sum_{i=1}^{n} \left[\frac{y^{2c} \ln(y_i)}{(1 + y^c)^2} + \frac{y^c \ln(y_i)}{(1 + y^c)}\right] \ln(y_i) - n \left[\frac{1}{\beta\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)} \phi''_{\beta c}\right], \tag{18}$$

Where

$$\Phi''_{\beta c} = \frac{\partial^2 \beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}{\partial\left(k-\frac{1}{c}\right)^2} = \beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\left[\left[\Psi_0\left(k-\frac{1}{c}\right)-\Psi_0(k+1)\right]^2+\left[\Psi_1\left(k-\frac{1}{c}\right)-\Psi_1(k+1)\right]\right]$$

in addition

$$\frac{\partial^2 \ln \ell}{\partial k^2} = \Phi''_{\beta k} = \frac{\partial^2 \beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}{\partial\left(k-\frac{1}{c}\right)^2} = \beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\left[\left[\Psi_0\left(k-\frac{1}{c}\right)-\Psi_0(k+1)\right]^2+\left[\Psi_1\left(k-\frac{1}{c}\right)-\Psi_1(k+1)\right]\right],$$

(19)

In addition, the partial derivatives of (16) with respect to k is

$$\frac{\partial^2 \ln \ell}{\partial c \partial k} = \sum_{i=1}^{n} \frac{y_i^c}{1+y_i^c} \ln(y_i) - n \frac{1}{\beta\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \phi''_{\beta k} \tag{20}$$

Therefore, we have

$$\left(\hat{c}, \hat{k}\right)^T \in Normal\left[(c,k)^T, V_0^{-1}\right], \tag{21}$$

With the information matrix

$$V_0 = -E\begin{bmatrix} V_{cc} & V_{ck} \\ V_{kc} & V_{KK} \end{bmatrix},$$

The matrix V_0^{-1} represents the asymptotic variance and covariance matrix of the MLEs. According to (21), approximate $100(1-\phi)\%$ confidence intervals of the parameters c and k are determined respectively as $\hat{c} \pm Z_{\alpha/2}\sqrt{Var(\hat{c})}$ and $\hat{k} \pm Z_{\alpha/2}\sqrt{Var(\hat{k})}$.

Here, $Z_{\alpha/2}$ is the upper $\alpha/2$ the percentile of the standard normal distribution.

6. A numerical study

This study is about obtaining MLEs of parameters of the LBB distribution using random numbers to assess the finite sample behaviour of the MLEs. The algorithm of obtaining parameters estimates is described in the following steps:

Step (1): Generating a random sample $u_1, u_2, \ldots u_n$ of sizes $n = (10,20,30,50,100,300)$ by using the LBB distribution.

Step (2): Seven different set values of the parameters are selected as: set(1): $(c = 0.9, k = 5)$, set(2): $(c = 1.2, k = 5)$ set(3): $(c = 1.3, k = 5)$, set(4): $(c = 1.4, k = 5)$,set(5): $(c = 0.9, k = 5.2)$, set(6): $(c = 0.9, k = 5.4)$,set(7): $(c = 0.9, k = 5.6)$.

Step (3): Solving (16) and (17) by iteration to get MLEs, biases, RMSE (the root of mean squared error)Total Biases, Total RMSE and the Pearson type of parameters estimators of the LBB distribution.

Step (4): Repeating steps from 1 to 3 1000 times.

Table 6.1: the parameter estimation from LBB-XII distribution using MLE at

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=0.9**	**5.605**	**4.705**	**5.541**	**75.255**	**85.107**	**0.0069**	**IV**
	k=5	**-70.108**	**-75.108**	**84.927**			**-0.01**	**I**
20	**c=0.9**	**3.905**	**3.005**	**3.587**	**52.057**	**58.51**	**-0.441**	**I**
	k=5	**-46.971**	**-51.971**	**58.4**			**8.849**	**VI**
30	**c=0.9**	**2.789**	**1.889**	**2.213**	**36.822**	**40.029**	**4.045**	**VI**
	k=5	**-31.773**	**-36.773**	**39.968**			**-0.182**	**I**
50	**c=0.9**	**1.603**	**0.703**	**0.817**	**20.753**	**21.15**	**0.351**	**IV**
	k=5	**-15.741**	**-20.741**	**21.495**			**-0.135**	**I**
100	**c=0.9**	**0.917**	**0.017**	**0.096**	**0.018**	**0.381**	**0.071**	**IV**
	k=5	**5.005**	**0.005**	**0.369**			**-0.008**	**I**
300	**c=0.9**	**0.904**	**0.004**	**0.052**	**0.00604**	**0.22**	**0.019**	**IV**
	k=5	**5.004**	**0.0045**	**0.241**			**0.0084**	**IV**

Table (6.1) shows that, estimators, biases and RMSE are decreasing with increase sample size, also total biases and total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size.

Table 6.2: the parameter estimation from LBB-XII distribution using MLE at :

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=1.2**	**5.915**	**4.715**	**7.371**	**35.61**	**52.356**	**-0.54**	**I**
	k=5	**-30.296**	**-35.296**	**51.835**			**-4.455**	**I**
20	**c=1.2**	**4.4**	**3.2**	**4.961**	**27.957**	**39.554**	**0.812**	**IV**
	k=5	**-22.774**	**-27.774**	**39.242**			**-0.38**	**I**
30	**c=1.2**	**3.808**	**2.608**	**3.929**	**24.122**	**31.901**	**0.433**	**IV**
	k=5	**-18.981**	**-23.981**	**31.659**			**-0.0379**	**I**
50	**c=1.2**	**2.817**	**1.617**	**2.558**	**16.799**	**21.567**	**0.247**	**IV**
	k=5	**-11.721**	**-16.721**	**21.415**			**-0.012**	**I**
100	**c=1.2**	**1.894**	**0.784**	**1.35**	**11.955**	**14.148**	**0.262**	**IV**
	k=5	**-6.929**	**-11.929**	**14.083**			**0.0068**	**IV**
300	**c=1.2**	**1.207**	**0.00655**	**0.065**	**0.009**	**0.211**	**0.027**	**IV**
	k=5	**5.006**	**0.0062**	**0.201**			**0.046**	**IV**

Table (6.2) show that, estimators, biases and RMSE decrease as sample size increases, also total Biases and total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size

Table 6.3: the parameter estimation from LBB-XII distribution using MLE at : $(c = 1.3, k = 5)$

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=1.3**	**8.761**	**7.461**	**11.254**	**44.067**	**57.88**	**1.663**	**VI**
	k=5	**-38.431**	**-43.431**	**56.775**			**-3.727**	**I**
20	**c=1.3**	**6.975**	**5.675**	**7.597**	**40.282**	**48.981**	**0.348**	**IV**
	k=5	**-34.88**	**-39.88**	**48.388**			**-2.579**	**I**
30	**c=1.3**	**5.703**	**4.403**	**5.678**	**35.697**	**41.765**	**-0.887**	**I**
	k=5	**-30.424**	**-35.424**	**41.377**			**-0.592**	**I**
50	**c=1.3**	**4.167**	**2.867**	**3.627**	**26.906**	**30.302**	**-0.744**	**I**
	k=5	**-21.753**	**-26.753**	**30.084**			**-0.197**	**I**
100	**c=1.3**	**2.306**	**1.006**	**1.286**	**15.254**	**16.084**	**10.251**	**VI**
	k=5	**-10.221**	**-15.221**	**16.033**			**-0.155**	**VI**
300	**c=1.3**	**1.308**	**0.0079**	**0.07**	**0.012**	**0.213**	**0.011**	**IV**
	k=5	**5.009**	**0.0094**	**0.201**			**0.025**	**IV**

Table (6.3) show that, estimators, biases and RMSE decrease as sample size increases, also total Biases and total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size.

Table6.4: the parameter estimation from LBB-XII distribution using MLE at $(c=1.4, k=5)$

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=1.4**	**5.228**	**3.828**	**8.697**	**17.89**	**38.185**	**0.248**	**IV**
	k=5	**-12.476**	**-17.476**	**37.182**			**-0.26**	**I**
20	**c=1.4**	**4.348**	**2.948**	**7.304**	**9.054**	**19.818**	**0.356**	**IV**
	k=5	**-3.561**	**-8.561**	**18.423**			**-0.169**	**I**
30	**c=1.4**	**3.715**	**2.315**	**5.546**	**6.966**	**15.391**	**0.382**	**IV**
	k=5	**-1.57**	**-6.75**	**14.357**			**-0.132**	**I**
50	**c=1.4**	**3.28**	**1.88**	**4.108**	**4.887**	**9.682**	**0.357**	**IV**
	k=5	**0.489**	**-4.511**	**8.767**			**-0.206**	**I**
100	**c=1.4**	**3.056**	**1.656**	**3.248**	**3.864**	**6.777**	**-1.399**	**I**
	k=5	**1.509**	**-3.491**	**5.949**			**-1.552**	**I**
300	**c=1.4**	**1.408**	**0.00795**	**0.073**	**0.00945**	**0.216**	**0.00903**	**IV**
	k=5	**5.005**	**0.0051**	**0.203**			**0.00371**	**IV**

Table 6.5: the parameter estimation from LBB-XII distribution using MLE at :

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=0.9**	**5.643**	**4.737**	**5.842**	**67.269**	**76.117**	**-0.256**	**I**
	k=5.2	**-61.902**	**-67.102**	**74.424**			**3.321**	**VI**
20	**c=0.9**	**4.008**	**3.108**	**3.699**	**50.073**	**56.172**	**-3.554**	**I**
	k=5.2	**-44.74**	**-49.94**	**56.05**			**-0.16**	**I**
30	**c=0.9**	**3.096**	**2.196**	**2.578**	**38.791**	**42.347**	**-1.079**	**I**
	k=5.2	**-33.529**	**-38.729**	**42.268**			**-0.174**	**I**
50	**c=0.9**	**2.065**	**1.165**	**1.377**	**25.326**	**27**	**0.194**	**IV**
	k=5.2	**-20.099**	**-25.299**	**26.965**			**-0.155**	**I**
100	**c=0.9**	**0.916**	**0.016**	**0.09**	**0.021**	**0.376**	**0.077**	**IV**
	k=5.2	**5.214**	**0.014**	**0.365**			**0.016**	**IV**
300	**c=0.9**	**0.904**	**0.003815**	**0.054**	**0.007155**	**0.214**	**0.41**	**IV**
	k=5.2	**5.206**	**0.006053**	**0.207**			**0.0022**	**IV**

Table (6.4) show that, estimators, biases and RMSE decrease as sample size increases, also Total Biases and Total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size.

Table (6.5) show that, estimators, biases and RMSE decrease as sample size increases, also total Biases and total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size.

Table 6.6: the parameter estimation from LBB-XII distribution using MLE at :

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=0.9**	**5.044**	**4.144**	**5.042**	**91.05**	**102.231**	**-0.305**	**I**
	k=5.4	**-85.556**	**-90.956**	**102.106**			**-0.029**	**I**
20	**c=0.9**	**3.49**	**2.59**	**3.13**	**63.163**	**70.328**	**0.359**	**IV**
	k=5.4	**-57.71**	**-63.11**	**70.258**			**-0.295**	**I**
30	**c=0.9**	**2.714**	**1.814**	**2.106**	**48.521**	**52.256**	**0.405**	**IV**
	k=5.4	**-43.087**	**-48.487**	**52.213**			**-0.13**	**I**
50	**c=0.9**	**1.588**	**0.688**	**0.862**	**27.36**	**28.818**	**0.243**	**IV**
	k=5.4	**-21.951**	**-27.351**	**28.805**			**-0.175**	**I**
100	**c=0.9**	**0.853**	**-0.047**	**0.468**	**10.928**	**11.373**	**0.243**	**IV**
	k=5.4	**-5.528**	**-10.928**	**11.364**			**0.193**	**IV**
300	**c=0.9**	**0.906**	**0.00597**	**0.052**	**0.009445**	**0.216**	**0.02**	**IV**
	k=5.4	**5.407**	**0.00732**	**0.209**			**-0.006**	**I**

Table 6.7: the parameter estimation from LBB-XII distribution using MLE at

sample size	Parameters	Mean Of Estimators	Biases	RMSE	Total Biases	Total RMSE	Pearson System Coefficients	Pearson Type
10	**c=0.9**	**6.007**	**5.107**	**6.853**	**62.582**	**77.644**	**0.474**	**IV**
	k=5.6	**-56.773**	**-63.373**	**75.808**			**-1.168**	**I**
20	**c=0.9**	**4.568**	**3.668**	**4.488**	**51.513**	**58.754**	**-3.753**	**I**
	k=5.6	**-45.783**	**-51.383**	**58.583**			**-0.334**	**I**
30	**c=0.9**	**3.849**	**2.949**	**3.527**	**43.105**	**47.578**	**1.413**	**VI**
	k=5.6	**-37.404**	**-43.004**	**47.447**			**-0.119**	**I**
50	**c=0.9**	**2.663**	**1.763**	**2.301**	**28.998**	**30.896**	**0.336**	**IV**
	k=5.6	**-23.345**	**-28.945**	**30.81**			**-0.21**	**I**
100	**c=0.9**	**1.302**	**0.402**	**0.698**	**15.114**	**14.4**	**0.317**	**IV**
	k=5.6	**-9.509**	**-15.109**	**15.384**			**-0.16**	**I**
300	**c=0.9**	**0.902**	**0.001907**	**0.05**	**0.003063**	**0.228**	**0.023**	**IV**
	k=5.6	**5.602**	**0.002397**	**0.223**			**0.007428**	**IV**

Table (6.6) show that, estimators, biases and RMSE decrease as sample size increases, also total Biases and total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size.

Table (6.7) show that, estimators, biases and RMSE decrease as sample size increases, also total Biases and total RMSE decrease in case of increasing the sample size. The estimators can be consistent, specially, when sample size increases. In addition, the sampling distribution of both estimators are differ according to sample size.

From results of the study it is clear that,

- For different values of c and k parameters, as sample size increases, the mean of estimators decreases, also biases and RMSE decreases.
- For different values of c and k parameters , as sample size increases ,total biases decrease and total RMSE decrease.
- For different values of c and k parameters the sampling distribution of both estimators are differ according to sample size.
- . The estimators c and k can be consistent, specially, when sample size increases.

6. Applications

In this section, the three real data sets to illustrate that the LBBX-II distribution might fit better than a model based on the BX-II distribution .

Data Set 1: The first data sets represents the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England. Unfortunately, the units of measurements are not given in the paper, and they are taken from [25].

0.55	**0.93**	**1.25**	**1.36**	**1.49**	**1.52**	**1.58**	**1.61**	**1.64**	**1.68**	**1.73**	**1.81**	**2**
0.74	**1.04**	**1.27**	**1.39**	**1.49**	**1.53**	**1.59**	**1.61**	**1.66**	**1.68**	**1.76**	**1.82**	**2.01**
0.77	**1.11**	**1.28**	**1.42**	**1.5**	**1.54**	**1.6**	**1.62**	**1.66**	**1.69**	**1.76**	**1.84**	**2.24**
0.81	**1.13**	**1.29**	**1.48**	**1.5**	**1.55**	**1.61**	**1.62**	**1.66**	**1.7**	**1.77**	**1.84**	**1.84**
1.24	**1.3**	**1.48**	**1.51**	**1.55**	**1.61**	**1.63**	**1.67**	**1.7**	**1.78**	**1.89**		

In order to compare the proposed distribution LBB-XII with BX-II distribution we consider criteria like the Kolmogorov- Smirnov test statistic (-2ln L) , Akaike Information Criterion (AIC), Bayesian information criterion (BIC) and Consistent Akaike Information Criterion (CAIC) which are defined, respectively, by,

$AIC = -2\ln L + 2\Omega$, $BIC = \Omega \ln L - 2\ln L$, $CAIC = AIC + \frac{2\Omega(\Omega+1)}{n-\Omega-1}$, where Ω is the number of parameter in the statistical model, n denotes the sample size and $(\ln L)$ is the maximized value of the log-likelihood function. The better distribution corresponds to smaller $(-\ln L)$, AIC, BIC, $CAIC$ values. All the computations were done using the Mathcad software. Summary of all these fitted distributions is introduced in tables.

Table 7.1: Estimate of Models for the $BX-II$ and $LBBX-II$ Distributions

Model	Parameter Estimate		$\ln L(., y)$
	$\hat{c}$	$\hat{k}$	
$B-XII$	1.255	1.028	−117.753
$LBB-XII$	1.618	2.25	−101.953

Table 7.2: Goodness of Fit Criteria

Model	$-2\log L(., t)$	AIC	$CAIC$	BIC
$B-XII$	235.506	239.506	239.706	243.793
$LBB-XII$	203.906	207.906	208.106	212.193

Tables (7.1) and (7.2) shows that the LBBX-II distribution gives better fit than the BX-II distribution.

Data Set 2: The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm from [18].

1.312	**1.479**	**1.479**	**1.552**	**1.7**	**1.803**	**1.861**	**1.865**	**1.944**	**1.958**	**1.966**	**1.997**
2.006	**2.027**	**2.027**	**2.055**	**2.063**	**2.098**	**2.14**	**2.179**	**2.224**	**2.24**	**2.253**	**2.27**
2.272	**2.301**	**2.301**	**2.301**	**2.359**	**2.382**	**2.382**	**2.426**	**2.434**	**2.435**	**2.478**	**2.49**
2.511	**2.535**	**2.535**	**2.554**	**2.566**	**2.57**	**2.586**	**2.629**	**2.633**	**2.642**	**2.648**	**2.684**
2.697	**2.77**	**2.77**	**2.773**	**2.8**	**2.809**	**2.818**	**2.821**	**2.848**	**2.88**	**2.954**	**3.012**
3.067	**3.084**	**3.09**	**3.096**	**3.128**	**3.233**	**3.433**	**3.585**	**3.858**			

Table 7.3: Estimate of Models for the $BX - II$ and $LBBX - II$ Distributions

Model	Parameter Estimate		$\ln L(.,y)$
	$\hat{c}$	$\hat{k}$	
$B - XII$	1	1.821	-225.837
$LBB - XII$	1.188	2.083	-143.326

Table 7.4: Goodness of Fit Criteria

Model	$-2\log L(.,t)$	AIC	$CAIC$	BIC
$B - XII$	451.675	455.675	455.857	460.143
$LBB - XII$	286.652	290.652	290.834	295.121

Tables (7.3) and (7.4) shows that the LBBX-II distribution gives better fit than the BX-II distribution.

Data Set 3: The data set reported by [5] represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.2	**23.56**	**23.74**	**25.87**	**31.98**	**37**	**41.35**	**47.38**	**55.46**	**58.36**	**63.47**	**68.46**	**78.26**
74.47	**81.43**	**84**	**92**	**94**	**110**	**112**	**119**	**127**	**130**	**133**	**140**	**146**
155	**159**	**173**	**179**	**194**	**195**	**209**	**249**	**281**	**319**	**339**	**432**	**469**
519	**633**	**725**	**817**	**1776**								

Table 7.5: Estimate of Models for the $BX - II$ and $LBBX - II$ Distributions

Model	Parameter Estimate		$\ln L(.,y)$
	$\hat{c}$	$\hat{k}$	
$B - XII$	2.591	0.08	-516.739
$LBB - XII$	0.858	1.455	-333.971

Table 7.6: Goodness of Fit Criteria

Model	$-2\log L(.,t)$	AIC	$CAIC$	BIC
$B - XII$	1033	1037	1038	1041
$LBB - XII$	667.943	671.943	672.235	675.511

Tables (7.5) and (7.6) shows that the LBBX-II distribution gives better fit than the BX-II distribution.

7. Conclusion

The length biased Burr XII distribution has been studied. At first, the PDF of the LBB(c, k) have been obtained. Some properties of LBB have been studied. Expressions for density, minimum and maximum order statistic and moment of the order statistics are derived. The estimation of the parameters of the LBB introduced by maximum likelihood method .An simulation of the LBB distribution and a real data set.

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