Abstract

A strong foundation in students' understanding of the multifaceted nature of the angle concept is of paramount significance in understanding trigonometry and other advanced mathematics courses involving angles. Research has shown that sixth-grade students struggle understanding the multifaceted nature of the angle concept (Keiser, 2004). Building on existing work on students' understanding of angle and angle measure and instructional supports, this study asks: How do sixth-grade students conceptualize angle and angle measure before, during, and after learning through a geometry unit of instruction set in a miniature golf context? What instructional supports contribute to sixth-grade students' conceptualization of angle and angle measure in such a context? I conducted a retrospective analysis of existing data generated using design-based research methodology and guided by Realistic Mathematics Education (RME) theory. Using Cobb and Yackel's (1996) Emergent Perspective as an interpretive framework, I analyzed transcripts of video and audio recordings from nine days of lessons in a collaborative teaching experiment (CTE), focusing on two pairs of students in sixth-grade mathematics classes. I also analyzed transcripts of pre-interviews before instruction, midway interviews during instruction, and post-interviews after instruction with each student in the two pairs. To answer research question one, I developed codes from data guided by the existing literature. For research question two, I used Anghileri's (2006) levels of supports framework. Overall, the findings revealed that sixth-grade students conceptualized an angle as a static geometric figure defined by two rays meeting at a common point, and conceptualized angle measure through their body turns. In addition, Anghileri's three levels of supports, such as the use of structured tasks, teacher's use of probing questions, generation of conceptual discourse were evident in

contributing to students' conceptualization of angle and angle measure during the miniature golf geometry unit of instruction. The findings of this study have implications for the school mathematics curriculum, and how to teach and to prepare teachers to teach angle and angle measure. This study emphasizes the need to redefine the angle concept in the curriculum documents, the need to increase activities involving body turns and the use of Anghileri's (2006) levels of supports in the teaching and learning of angle and angle measure in a real-world context. Further research is needed to identify instructional supports, in particular activities that can support students' conceptualization of slopes and turns as angles in a real-world context.

An Investigation of Sixth-grade Students' Conceptualization of Angle and Angle Measure: A Retrospective Analysis of Design Research Study of a Real-world Context

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Table of Contents

List of Abbreviations and Acronyms

RME	Realistic Mathematics Education
NCTM	National Council of Teachers of Mathematics
CCSS	Common Core State Standards
NGA & CCSSO	National Governors Association Center for Best Practices & Council of Chief State School Officers
CTE	Collaborative Teaching Experiment
ISs	Instructional Supports
CSNs	Classroom Social Norms
SMNs	Sociomathematical Norms
CMPs	Classroom Mathematical Practices
RA	Researcher Assistant
CT	Classroom Teacher
RQ1	Research Question One
RQ2	Research Question Two

List of Tables

List of Figures

Chapter One: Introduction

Background of the Study

The Relationship of the Angle Concept to Geometry and Measurement

Angles are at the confluence of both geometry and measurement, in spite of the two strands being treated as separate in the mathematics standards, such as the *Common Core State Standards for Mathematics* (CCSS) in the United States (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). The word "geometry" comes from two ancient Greek words: *Geo*- meaning "earth" and -*metron* meaning "measurement". This shows clearly that measurement is woven in geometry, and thus the two are naturally conceptualized together when thinking about shape and space. As noted by Battista (2007), measurement is pertinent in understanding how shapes are structured, how coordinates are used to find locations in space, how to differentiate transformations, and how to find measures of objects. In other words, measurement is important in spatial reasoning, that is "the ability to see, inspect, and reflect on spatial objects, images, relationships, and transformations" (Battista, 2007, p. 843). Spatial reasoning is in turn an important aspect of geometrical reasoning. In this case, geometrical reasoning is perceived as the innovation and the use of angle measure and other systems that help in making sense of space and shape (Battista, 2007). This suggest that angle measure is part of geometrical reasoning, and "angles are a central component of geometric measurement … as well as a tool for earth measure" (Smith, 2017, p. 372). According to Smith (2017), studies on angle and its measure are needed as this is an area with a paucity of research.

The Importance of the Angle Concept

In school contexts, students need knowledge of angles in classifying geometric shapes such as triangles, quadrilaterals, and other polygons (Clements, Wilson & Sarama, 2004). They also need angle knowledge to understand trigonometry and its application to angles of inclination. According to Smith (2017), "angles provide graphical meaning of the slope concept, and the relations between angles and side lengths in triangles, which are the core of trigonometry" (p. 372). In addition, angle knowledge is needed in understanding of other proof-related advanced mathematics courses (Edwards et al., 2014), understanding of rotations as transformations, and being able to distinguish between congruence and similarity (Smith, 2017). In real-world contexts, students need angle knowledge, for instance, in drawing, in construction, in moving from one place to another, in aligning their bodies with respect to other objects, and locating the movement of objects (Smith, 2017). Despite the importance of understanding angles, the multifaceted nature of the angle concept poses a challenge to many students, particularly from K – 8^{th} grade. By multifaceted, this means that angle has more than one meaning.

The Multifaceted Nature of the Angle Concept

The National Council of Teachers of Mathematics (NCTM) emphasizes the need to attend to precision particularly with meanings of mathematical concepts (2000). Students are viewed as proficient in mathematics when they can communicate and understand precisely the meaning of mathematical concepts (NCTM, 2000). While attending to precision applies to geometrical concepts such as the angle concept, researchers have documented the multifaceted nature of the angle concept (Keiser et al., 2003; Keiser, 2004; Mitchelmore, 1997, 1998, 2000; Smith, 2017; Tanguay & Venant, 2016), which makes it difficult to precisely define the concept. According to Keiser (2004), meanings of an angle can be summarized into three major categories: an angle as "a measure of the turning of a ray about a point from one position to another" (i.e., a dynamic nature), an angle as a geometrical figure defined by "the union of two rays with a common endpoint" (i.e., a static nature), and an angle as "the region contained between the two rays" (i.e., a static nature) (p. 288). As a consequence of the several meanings of an angle, students particularly from the elementary into secondary school struggle to understand the multifaceted nature of the concept (Mitchelmore, 1997). For instance, in her study with sixth-grade students, Keiser et al. (2003) found that majority of participants defined an angle by its measure, while others emphasized one visible feature over others, such as the intersection of the rays over the region between the rays. Such struggles as reported by Keiser et al., point to the need for more studies on how to support students' understanding of the multifaceted nature of an angle.

Studies suggest various contexts in support of students understanding of the multifaceted nature of the angle concept. For instance, use of technology such as LOGO activities (Clements et al., 1996), use of body movements (Smith et al., 2014), use of real-world situations (Mitchelmore, 1997), and use of real-world contexts (Crompton, 2015; Fyhn, 2008; Masingila & de Silva, 1997). However, studies lack that have detailed instructional supports that contribute to the development of students' conceptualization of angle concept and angle measure, particularly in real-world contexts. Such studies can inform teachers on how to support students in understanding the multifaceted angle concept as they connect students' learning of angle with the real world. This study defines instructional supports as those that require indirect teacher's intervention, such as providing structured tasks, and those that require direct teacher's intervention such as restructuring of tasks (Anghileri, 2006).

Statement of the Problem

Research has shown that sixth-grade students' struggle with the multifaceted nature of the angle concept (Biber, Tuna, & Korkmaz, 2013; Butuner & Filiz, 2017; Devichi & Munier, 2013; Fyhn, 2008; Keiser, 2004; Tanguay & Venant, 2016). Knowing the struggle faced by sixth-grade students is important, however, studies that can investigate means of supporting this group towards conceptualization of the multifaceted nature of the concept are of paramount significance. More so, the aforementioned studies have used survey questions to investigate students understanding of the meaning of angle. As Keiser (2004) noted, individual interview questions following classroom observations would be helpful in clarifying students' meanings of an angle. This study incorporated both pre-and-post semi-structured interview questions to understand in depth how students conceptualized meanings of the multifaceted angle before and after the instructional unit.

Most studies on supporting students' conceptualization of the angle concept are conducted with third and fourth grade students (Browning & Garza-Kling, 2009); Bustang et al., 2013; Clements & Battista, 1990; Clements et al., 1996; Clements & Burns, 2000; Devichi & Munier, 2013; Mitchelmore, 1997; Mitchelmore & White, 1998; White & Mitchelmore, 2010; Smith, King & Hoyte, 2014; Wilson & Adams, 1992;

Wilson, 1990). A few studies have focused with sixth-grade students (e.g., Browning, Garza-Kling & Sundling, 2008; Mitchelmore, 1998; Mitchelmore & White, 2000a), and yet it is evident that this group struggle to understand the angle concept. In addition, Mitchelmore studies have been conducted with 2^{nd} – 8^{th} grade students, and not specifically with sixth-graders. Mathematics standards show that students are formally introduced to the angle concept at fourth grade, with less emphasis in the subsequent grades (NGA & CCSSO, 2010). This suggests the need for studies that can focus students as they prepare to transition to the secondary level (e.g., at the sixth-grade level). This study aimed at adding to research with sixth-grade students' conceptualization of angle and how the conceptualization of the concept can be supported.

A number of researchers have reported that technology activities support students' understanding of the angle measure. For instance, use of SmileMath activities, a calculator application (Browning et al., 2008; Browning & Garza-Kling, 2009), use of LOGO activities (Browning et al., 2008; Clements & Battista, 1990; Clements et al., 1996; Clements & Burns, 2000), use of mobile activities to take pictures in a real-world setting (Crompton, 2015), and use of GeoGebra activities in supporting students' conceptualizations of angle definitions (Richardson & Koyunkaya, 2017). While technology have been shown to support students' conceptualization of the angle concept and its measure, Richardson and Koyunkaya (2017) indicated the importance of including a physical angle context when learning angles. Bustang et al. (2013) made a similar suggestion that learning of angles be connected to a real-world context. With these great suggestions, providing details of the instructional supports in these contexts would be of significance to teachers in supporting students' conceptualization of the multifaceted angle concept.

Most of Mitchelmore work with scholars such as White (1997, 1998, 2000, 2010) have focused on how to support students understand the abstract angle concept from physical angle contexts, such as doorknobs, scissors, hills, and bends. Other researchers have used real-world contexts, such as playground (Crompton, 2015) and a miniature golf context (Masingila & de Silva, 1997), where they documented students' understanding of one aspect of an angle. For instance, Crompton (2015) reported that a playground supported students' conceptualization of an angle by providing angle situations of different lengths, which helped students overcome sides-of-lengths obstacle. Masingila and de Silva (1997) reported that a miniature golf context supported students in understanding that angle measure is preserved. Details on how students conceptualize the meaning of the multifaceted nature of angle concept and means of supporting such conceptualization are lacking in real-world contexts. The current study aimed at investigating how sixth-grade students conceptualize the concept of angle and its measure in a miniature golf context and instructional supports in developing the conceptualization in such a context.

Situating the Study

This study drew on data sources collected during a three-year National Science Foundation-funded project entitled "Connecting In-school and Out-of-school Mathematics Practice" (Masingila, 1995). The larger project was guided by the following three goals (Masingila, 1995):

(1) To gain insight into the goals that emerge during students' out-of-school activities by examining:

   (a) the goal structure of the activities

   (b) social interactions that occur during the activities,

   (c) conventions and artifacts that are used during the activities, and

   (d) students' prior understandings

(2) To explore what cognitive forms and functions students are constructing to accomplish these goals.

(3) To examine interplay among these various cognitive forms by studying how and if students are able to use cognitive forms in a school setting that they have appropriated and specialized in an out-of-school setting. (p. 2)

The third goal formed the background of the current study. In order to accomplish this goal, Masingila worked with a middle school teacher and a research assistant to develop ideas of creating a classroom practice that had characteristics of the students' out-of-school mathematics practice. With the help of the teacher, the research team identified a miniature golf context as a context that was meaningful to the majority of students. Following this, the research team constructed a miniature golf context as a classroom practice that they could use to investigate students' connection of their out-of-school and in-school mathematics practices. In particular, the research team engaged students in the miniature golf context to investigate geometry and measurement ideas, as well as other mathematical ideas, such as ratio.

Drawing on the data that were collected, Masingila and de Silva (1997) conducted a comparative study on students' understanding of the angle concept. In their study, Masingila and de Silva compared students who used a miniature golf geometry unit of instruction with those who largely followed a traditional way of learning, which is drawing largely on textbooks without connecting to a real-world context. They found that when students were asked to copy an angle, those who used a miniature golf context were better at preserving an angle measure than those who followed a traditional way of learning. Although Masingila and de Silva's study did shed some light on the potential of a miniature golf context in supporting students' understanding of the angle measure, the study did not provide a detailed analysis of how students conceptualized the multifaceted nature of the angle concept, and the instructional supports that contributed toward the conceptualization. In addition, while Masingila and de Silva conducted a comparative study, the current study conducted a retrospective analysis of existing data generated using a design-based research methodology with an aim of developing a learning trajectory of students' conceptualization of angle and its measure in a miniature golf context, an example of a real-world context.

According to Boaler (1993), students may interact with a context in different ways depending with an individual or the nature of the task. Thus, Boaler suggested that decisions on whether a context or an activity is effective be based on the nature of students' learning outcomes, and not simply on the ground of its familiarity to the students. As noted by Nicol and Crespo (2005), a context may be real to one student but unreal to another student. In 2007, Enyedy and Mukhopadhyay also emphasized the need to move beyond the benefits of engagement and motivation if connections are to be made between students' learning of particular mathematical concepts and particular culturally relevant contexts. This study aimed at providing detailed analysis of students' conceptualization of angle and angle measure in a miniature golf context. In addition, the study strived to provide detailed analyses of instructional supports that contributed to students' conceptualization of an angle within such a real-world context.

Purpose of the Study

The purpose of this study was to investigate a learning trajectory of students' conceptualization of angle and its measure in a real-world context. Specifically, I investigated students' conceptualization of angle and its measure before, during, and after learning through a geometry unit of instruction set in a miniature golf context. I also investigated the instructional supports that contributed to students' conceptualization of angle and its measure during the miniature golf context instructional unit.

Research Questions

In the light of the purpose of this study, the following research questions were addressed:

1. How do sixth-grade students conceptualize angle and angle measure before, during, and after learning through a geometry unit of instruction set in a miniature golf context?
2. What instructional supports (ISs) contribute to sixth-grade students' conceptualization of angle and angle measure in such a context?

Theoretical Underpinnings

This study conducted a retrospective analysis of existing data from a larger study whose instructional unit was guided by the *Realistic Mathematics Education* (RME) theory. I thereby provide a brief explanation of the RMA theory. In addition, I discuss an *Emergent perspective,* as an interpretive framework that guided the analysis of this study.

Realistic Mathematics Education (RME) Theory

The RME theory is a domain-specific instructional theory that guided the development of the instructional unit. From an instructional design perspective, the RME theory follows Freudenthal's (1971, 1973) beliefs. Freudenthal believed that mathematics is not a system of ready-made product but a result of human invention. Freudenthal's beliefs culminated into three design heuristics of RME: (1) guided reinvention, (2) didactical phenomenology, and (3) emergent models or mediating models (Gravemeijer & Cobb, 2006; Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen & Drijvers, 2014). Reinvention heuristic suggest that students need to be positioned as mathematics reinventors, while as didactical phenomenology heuristic emphasize the need to provide students with situations that are realistic to them. (Gravemeijer & Cobb, 2006). That is, situations that are from the real-world which will allow students to construct mathematical ideas through progressive mathematization (organizing from a mathematical perspective) (Gravemeijer & Doorman, 1999). In the larger study, sixth-grade students were positioned as reinventors of mathematics in a miniature golf context, which is an example of a real-world context.

The RME third heuristic, the emergent modeling suggest that a context can move beyond being used as a model of a certain activity to be used as a model for mathematical reasoning (Gravemeijer & Cobb, 2006; van den Heuvel-Panhuizen & Drijvers, 2014). That way, such a context enables bridging the gap between the mathematics learned in school and the real-world mathematics. In the larger study, a miniature golf context served as a model for students' conceptualization of mathematical ideas, shifting its role from being a context of playing a miniature golf game to becoming a context of conceptualizing angles (van den Heuvel-Panhuizen & Drijvers, 2014). The three RME guidelines relate to Cobb's (2003) key areas that an instruction serving a design-based research should have. Cobb (2003) noted that an instruction sequence should allow students to imagine and reinvent mathematics using their reasoning, as well as provide opportunities for students to develop own models. The instructional sequence through a miniature golf context provided students opportunities to coordinate a number of measurements including angle measures.

Emergent Perspective

Cobb and Yackel's (1996) emergent perspective guided this study as an interpretive framework. The emergent viewpoint is that learning is both individual and social, hence coordinates both social perspective and individual perspective in a classroom community as shown in Table 1.1 (Cobb et al., 2001; Cobb et al., 2003; Gravemeijer & Cobb, 2006). This study strived to analyze learning from a social lens.

Table 1.1

The Emergent Perspective: An Interpretive Framework (Cobb & Yackel, 1996)

Social Lens	Psychological Lens
Classroom social norms	An individual's beliefs about own role, others' roles, and the general nature of mathematical activity

Socio-mathematical norms	Specific individual mathematical beliefs and values
Classroom mathematical practices	An individual mathematical conceptions (interpretations) and actions

From a social perspective, learning happens in a community of students and teachers (Cobb et al., 2001). As students participate in the collective activities, teachers offer support directly or indirectly (Anghileri, 2006). The social perspective consists of three tenets: the classroom social norms (CSNs), the sociomathematical norms (SMNs), and the classroom mathematical practices (CMPs), as outlined in Table 1.1 (Cobb & Yackel, 1996; Cobb et al., 2001; Gravemeijer & Cobb, 2006). Classroom social norms (CSNs) describes the expectations of the structure of classroom participation as established by both teachers and students (Cobb & Yackel, 2006). For example, the classroom community expectations are that members will engage in explaining and justifying their ideas, attempt to make sense of others' ideas, agree or disagree with others' ideas, and call for alternative explanations to resolve a conflict in case of disagreements (Cobb et al., 2001). Although Cobb et al. (2001) criticized CSNs as inadequate in characterizing supports for students' mathematical learning since they are not mathematics specific, this study views applicability of CSNs to any subject matter as a strength. In this study, the classroom social norms are viewed as supports in contributing to students' conceptualization of angle and angle measure through a geometry instructional unit set in a miniature golf context. In the larger study, the research team expected that students would give detailed explanations of their Worksheets' assignment in such a way that they could make sense of their thinking.

Worksheets' assignment provided students with questions based on the activities of each day of the instructional unit. For example, during Day 1, students were given Worksheet 1 with three questions (see Appendix A1). The first question asked students to draw a rough sketch of a hole they had played or of the one modeled in the classroom. The second question asked students to describe details of the hole, such as shape, sides, angles, etc. The third question asked students to note the measurements they would consider in order to reproduce the hole. Following the feedback given on students' Worksheet 1 assignment, the researcher, who played the role as the lead teacher said:

> I think you need to give little more information, some of you. I tried to write comments on the bottom. Some of you indicated ways you got views like angles but didn't say how that would be important because I think there are angles involved. So, we would expect that you would write how angles might be involved or what it meant for that angle. So, just sort of add details and make sure you are explaining fully what you might think so … we can't read your mind and that way we can know what you are thinking about how that might be used. (Masingila, March 13, 1997)

This excerpt indicates clearly that students were expected to provide detailed explanation of their work so that those reading could make sense of it. In particular, Masingila highlighted the importance of students giving details about their thinking about angles. This clearly indicated a social norm towards supporting students' conceptualization of the angle concept.

The second category of a social perspective are sociomathematical norms (SMNs). Similar to classroom social norms, sociomathematical norms are established by both teachers and students as they negotiate mathematical activities (Cobb & Yackel, 2001). According to Cobb et al. (2001), sociomathematical norms include "what counts as acceptable mathematical explanations and what mathematical solutions counts as different, sophisticated, and efficient" (p. 124). In other words, sociomathematical norms are criteria that validate mathematical activities, such as a sufficient definition (Cobb et al., 2001). In the larger study, the research team expected students to ensure that their mathematical explanations were detailed for others to make sense of. In particular, as the above excerpt indicates, students were expected to explain in detail what angles might be needed in designing of a miniature golf hole, and what those angles meant. This way, this study viewed such a sociomathematical norm as a support towards developing students' conceptualization of what counted as acceptable explanation of angle ideas.

The third tier of a social perspective are classroom mathematical practices (CMPs) and these focuses on particular mathematical ideas (Cobb & Yackel, 1996). Cobb et al. (2001) defined classroom mathematical practices as "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (p. 126). This means that CMPs emerge as students engage in a task and as a culmination of the accepted-as-shared mathematical ideas by members of the learning community. Thus, emergence of classroom mathematics practices is a culmination of "generating conceptual discourse," which this study viewed as an instructional support. To this end, the emergent interpretive framework guides the analysis of this study, particularly in answering of research question two: what instructional supports contribute to students' conceptualization of angle and angle measure in a miniature golf context. In Chapter Three, I describe Anghileri's (2006) levels of support analytical framework and show how the emergent interpretive framework guided the analysis.

Limitations of the Study

1. Since I analyzed the transcripts of video-recorded classroom data and interview data because the videos were not available, this might have limited me in experiencing all of the "aha moments" that watching of videos might have provided. However, the available data were enough to address my research questions.

Delimitations of the Study

1. The study was based in a United States suburban school context in a school context that has high parental involvement in students' well-being.
2. The study was based on only one subject area – mathematics.

Significance of the Study

This study strived to add to the knowledge base of students' conceptualization of angle and angle measure in a real-world context and means of supporting the conceptualization in such a context. In particular, the study aimed at providing detailed analyses of the instructional support that contributed to students' conceptualization of angle and angle measure in a miniature golf context. Such analyses are significant to classroom teachers who may be striving to connect students' learning of angles to real-world contexts.

Definitions of Key Terms

Conceptualization is the process of forming a concept.

Classroom practice "is a process that involves multiple agents and their interactions within a classroom as a system" (Li & Oliveira, 2015, p. 489).

Collaborative teaching experiment "is a teaching experiment conducted in collaboration with a practicing teacher" (Cobb, 2000, p. 1).

Everyday mathematics practices are mathematics practices usually learned and used in everyday contexts (Masingila, 2002).

Formal mathematical knowledge is the knowledge acquired through school (Masingila, 2002).

In-school mathematical practices are mathematics practices learned and used in school (Masingila, 2002).

Instructional supports are direct and non-direct teacher-student interactions that enhance learning (Anghileri, 2006).

Out-of-school mathematical practices are mathematics practices learned and used outside school (Masingila, 2002).

Misconception is a partial misunderstanding.

Outline of the Remaining Chapters of the Study

Chapter Two focuses on the literature review, including the conceptual framework on which the current study drew. I reviewed literature in the following areas: (1) the angle concept with focus on the complexity in defining of the concept, its history, the implications of its multifaceted nature, and a summary of students' misconceptions and other difficulties on the concept and its measure, (2) research investigating support for understanding of the angle concept and its measure, and (3) theoretical frameworks used to study students' understanding/development of the angle concept and its measure.

Chapter Three focuses on the research methods. I focused in particular on (a) describing Design-Based Research methodology (DBR) as used in the larger study and the phase where the current study is situated, (b) classroom teaching experiment of the larger project from which the current study drew, (d) participants, (e) data collection, and (f) data analysis.

Chapter Four presents findings of the study, and Chapter Five presents an overview of the study, interpretation of the findings, limitations and future studies, recommendations, generalization of the analyses, and trustworthiness of the analyses.

Chapter Two: A Review of Literature

Background

The overarching aim of this study was to investigate how students conceptualize angle and its measure and means of supporting that conceptualization in a real-world context. Specifically, the study investigated students' conceptualization of angle and its measure before, during, and after learning through a geometry instruction unit set in a miniature golf context. The study also sought to investigate instructional supports that contributed to students' conceptualization of angle and angle measure in a miniature golf context. This chapter presents the bodies of research on what other scholars have learned from their research on angle concept and its measure. The bodies of research discussed in this chapter include: (1) the meaning of the angle concept – a historical perspective and present definitions, (2) research on students' understanding of the meaning of an angle and angle measure, and (3) research on supports for students' conceptualization of the angle concept and angle measure. I next discuss these bodies of research.

The Meaning of the Angle Concept

A Historical Perspective of the Definition of an Angle

The term angle was first invented by a Greek geometer, Aristotle during pre-Euclidian era (Matos, 1990). Aristotle's understanding of an angle followed his three principles of classifying the nature of geometric figures: a quantity, a quality, and a relation (Matos, 1990). Other scholars such as Heron, Syrianus (through Proclus), Appolonius, Plutarch, and Carpus of Antioch (Dimitri, 2012) attached different meanings on the angle concept (Matos, 1990). For instance, following Aristotle's three principles, Proclus thought of an angle as “a quantity, a magnitude that can be compared (equal, greater, or less than); a quality by way of its form; and a relation between the lines and / or planes bounding it” (Dimitri, 2012, p. 34). Consistent with this view, Keiser (2004) observed that students’ conceptions of angles could be classified into one of Aristotle’s categories.

During Euclid’s era, Euclid defined a plane angle as “the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line … and when the lines containing the angle are straight, the angle is rectilinear” (Matos, 1990, p. 7). Matos (1990) further noted that Euclid’s definition highlighted two aspects of an angle: “an angle as a set of two lines with specific characteristics, and an angle as a kind of area contained by two lines” (p. 7), which aligns with the idea of an angle as a sector in a circle. Euclid’s definitions emphasized the static nature of an angle and appeared to exclude the zero angle and angles greater than or equal to 180^{o}. Keiser (2004) noted that early definitions of an angle differed by emphasizing one facet, such as the rays than another facet such as the region between the rays. This made it difficult to precisely define an angle.

Present Definitions of an Angle

Today, researchers have termed the nature of the angle concept as multifaceted (e.g., Browning et al., 2008; Devichi & Munier, 2013; Keiser, 2004; Mitchelmore & White, 2000). By multifaceted, it means that the term has more than one meaning. Most textbooks define an angle as “the union of two rays that share a common endpoint”

(Smith, 2017, p. 372), as Figure 2.1 shows. This definition emphasizes the static nature of an angle.

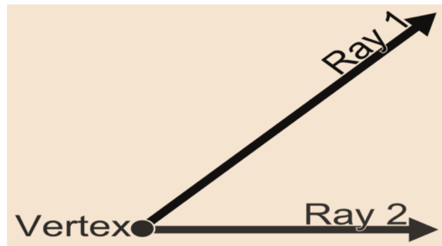


Figure 2.1. A representation of an angle (https://en.wikipedia.org/wiki/Angle).

Another static definition is an angle as a sector or a wedge in a circle, as Figure 2.2 shows.

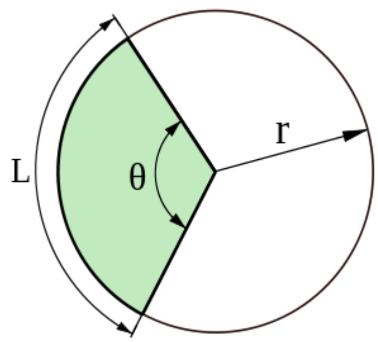


Figure 2.2. A representation of an angle (θ) as a sector or a wedge in a circle (https://en.wikipedia.org/wiki/Circular_sector).

Viewing an angle as a sector or a wedge in a circle is limited in itself because by a sector or a wedge being triangular shaped that does not define what an angle is. Tanguay and Venant (2016) noted that the perception of an angle as a sector in a circle relates to students thinking of an angle as a “slice of pizza,” which they noted as a misconception. According to Smith (2017), an angle can also be defined as a turn, that is “a rotation about a single point” (p. 372), which is the dynamic nature of an angle. Consistently, Butuner and Filiz (2017) noted that the meaning of the angle concept can be classified into either dynamic or static. More so, Kontorovich and Zazkis (2016) noted that an angle has three facets, which include: an angle as a static geometric shape, an angle as a dynamic turn, and an angle as a measurement.

From both the historical perspective and the present definitions of an angle, no doubt that several meanings have been attached to the angle concept. In fact, Devichi and Munier (2013) noted that “no formal definition can capture all aspects of our experience of what an angle is” (p. 2). This study agrees that an angle is a multifaceted concept, and students need to be supported towards that conceptualization. Students need to understand that an angle should be defined from both static and dynamic perspective, without emphasizing one feature than the other. This study sought to investigate how sixth-grade students conceptualized the multifaceted concept and what supports contributed to their conceptualization. The next section present research on students’ understanding of the meaning of angle and angle measure, and related misconceptions. In this study, a misconception refers to a partial understanding.

Students’ Understanding of the Meaning of Angle and Angle Measure

Studies show that students’ initial conceptions of an angle are usually dominated by the static nature of the angle concept (Browning et al., 2008; Butuner & Filiz, 2017; Keiser, 2004; Richardson & Koyunkaya, 2017). For example, Browning et al. (2008) noted that whenever the term angle is used many students tend to think of it as either a corner, two rays, or a vertex, which all depict the static definition of an angle. Richardson and Koyunkaya (2017) also found that students’ initial definitions of an angle only depicted an angle as a figure formed when two rays intersect at vertex. However, after students were supported through GeoGebra tasks, their definitions progressed to contain the dynamic aspect of an angle, a rotation, where students indicated an angle using an arrow or a hand motion. Although, Richardson and Koyunkaya used technology, they also suggested the need to include a physical context in learning of angles.

Students are also reported to think of an angle as a geometric figure or a measure in degree (Biber et al., 2013; Tanguay & Venant, 2016). In their study, Tanguay and Venant (2016) investigated how sixth-grade students who were about to enter into secondary school conceptualized the angle concept. They sought to understand whether the students conceptualized an angle as a magnitude (quantity or amount or measure) or a geometric figure, and how they coordinated the two aspects in their understanding. Consistent with what textbooks provide, Tanguay and Venant (2016) found that students thought of an angle as being acute, right, obtuse, or a degree. In other words, students viewed an angle as a geometric figure or a measurement in degrees. Previous studies also noted students' tendency to view an angle in terms of its standard unit of measure, the degree (e.g., Browning et al., 2008; Keiser, 2003). As noted by Smith (2017), viewing an angle as a degree is a misconception as a degree is unit of measure and has no tangible physical features. Tanguay and Venant (2016) attributed students' conceptualization of an angle as a degree as a result of the systematic use of the tool of measure, the protractor which is labeled in degrees. Thus, these studies suggest students' tendency to interpret an angle as figure in a static nature or as a unit of measure, a non-tangible thing.

When the static nature of an angle is emphasized than the dynamic, students tend to think of an angle as a figure that is simply characterized by two static visible rays (Keiser, 2004; Mitchelmore & White, 2000). According to Mitchelmore and White (2000), the presence or absence of the rays that makes up an angle influences how students interpret angles. Mitchelmore and White (2000) found that students find it easy to identify corner angle contexts (e.g., tiles, walls, scissors, and junction) where both rays of an angle are visible, but they are challenged where only one ray is visible (e.g., opening situations such as doors, sloping situations such as hills). Students also find it difficult to identify angles where there is no visible ray, but need to be imagined (e.g., turning situations such as wheels) (Browning et al., 2008; Butuner & Filiz, 2017; Keiser, 2004; Mitchelmore & White, 2000). This means that emphasizing only the static nature of angles makes it difficult for students to conceptualize angles in dynamic contexts. This suggest the need to provide students with diverse contexts that would allow them to explore the multifaceted nature of angles.

Researchers have also shown that when students are limited to only a static definition of an angle, other difficulties emerge, such as not being able to measure angles, a challenge that persist even into secondary level (Akkoc, 2008; Browning & Garza-Kling, 2009; Devichi & Munier, 2013; Keiser, 2004; Mitchelmore, 1998; Moore, 2009, 2013; Moore & LaForest, 2014; Smith, 2017; Topcu et al., 2006; Wilson, 1990). As Smith (2017) noted, “angle measurement relates the static geometric figure to rotational motion,” where “an angle’s measure is its amount of rotational sweep - the amount one ray has been rotated to coincide with the other” (p. 372). This means that the static and dynamic nature of an angle need to be conceptualized together. Consistent with Smith (2017), Dimitric (2012) had previously also suggested that the teaching of angles should include the dual nature of the angle concept. When students' understanding of angles go beyond statics to dynamic, Olson, Zenigami, and Okazaki (2008) noted that students experience less difficulty measuring angles with a protractor. Limiting students' understanding of angles to the static nature only, has also led to exclusion of some angles such as 0°, 180°, 360°, and reflex angles whose measure is between 180 degrees and 360 degrees (Keiser et al., 2003; Keiser, 2004; Tanguay & Venant, 2016).

In her study with sixth-grade students, Keiser (2004) used a historical perspective to compare the meanings of an angle that students constructed during a geometry instruction unit from the Connected Mathematics Project. Keiser's study focused on three major topics: "what exactly is being measured when referring to the size of angles, can angles contain curves, and difficulties in conceiving 0°, 180°, and 360° Angles" (p. 288). Keiser found that students' conceptualization of angle measure related to Aristotle's three ways of viewing of an angle. Keiser's participants conceptualized angle measure as a quantity (e.g., the longer the rays, the larger the measure), as a quality (e.g., the sharper the angle, the more of an angle), and as a relation (e.g., the bigger the arc making an angle , the larger the angle). Keiser (2004) also found that some students conceptualized 0°, 180°, and 360° as non-angles as these did not have lines and a point of intersection. In addition, students did not consider curves to have angles. Keiser concluded the need to support students' understanding of the multifaceted nature of the angle concept. Keiser suggested to provide students with multiple representations of an angle in order to support their conceptualization.

Emphasis of the static nature of angles has also contributed to other misconceptions about angle and angle measure. The most common misconception is the side-length obstacle, where students tend to think that the longer the rays of an angle, the bigger the angle in measure (Devichi & Munier, 2013). Devichi and Munier suggested angles to be introduced as a space described by two rays meeting at a common vertex, rather than a figure defined by the two rays. Similar sentiments were made by Tanguay and Venant (2016). Closely related to side-length obstacle is the salience of prototypical right-angle obstacle (Devichi & Munier, 2013; Smith, 2017; Wilson & Adams, 1990). A prototypical right angle is one whose horizontal ray is oriented to the right and is parallel to the horizontal edge of a paper (Devichi & Munier, 2013). Devichi and Munier found that students encountered difficulties conceptualizing angles greater or less than 90 degrees. Other misconceptions include: students' tendency to think of an angle being less when its interior measure increases and loses its "sharpness" (Keiser, 2004), thinking that the measure of an angle changes with different orientation (Keiser, 2004; Smith, 2017; Smith et al., 2014), difficulty in applying angle knowledge in real life situations (Butuner & Filiz, 2017), and unable to identify angles from various contexts such as sloping and turning situations (Mitchelmore & White, 2000).

Towards this end, there is no doubt that the several meanings of an angle pose difficulties in precisely defining the concept. In addition, emphasizing only one nature or certain features of an angle and leaving others, ends up causing difficulties and confusion about what exactly is an angle and angle measure. This emphasizes the need to provide students with contexts that will allow them to develop an understanding of angles as multifaceted. This study strived to provide detailed analyses of students' conceptualization of angles through an instructional unit in a miniature golf context and supports in such a context. In the next section, I present studies that have investigated various supports for students' understanding of the complex angle concept and its measure.

Instructional Supports for Students' Conceptualization of Angle and Angle Measure

Supporting sixth-grade students' understanding of the multifaceted nature of the angle concept is of paramount significance in understanding trigonometry and advanced math at higher levels. The static nature of an angle is highly emphasized, particularly for K-8th grades (NGA & CCSSO, 2010). Hence, researchers advocate for the need to support students to develop an understanding of an angle from both static and dynamic perspective (Dimitric, 2012; Smith, 2017; Wilson & Adams, 1990). Supports can be classified into various categories and therefore this study aimed at focusing on instructional supports as defined by Anghileri's levels of support framework (2006), described in detail in Chapter Three. Anghileri classified instructional supports into three levels, with level one comprising supports that require indirect teacher-student(s) interaction, and level two and three supports involving direct teacher-student(s) interaction. Examples of Anghileri's level one supports are learning environment affordances such as a favorable classroom atmosphere, collaboration among peers, structured tasks, encouraging feedback, among others (Anghileri, 2006). Examples of

Anghileri's level two and three supports include explaining, restructuring and reviewing of tasks, and building conceptual thinking through the use of representations and tools, as well as helping students make connections (Anghileri, 2006). Most studies on students' understanding of angles have focused mainly on level one of Anghileri's supports, such as the use of technology, the use of physical body motions, and drawing on informal angle knowledge in real-world settings. This study aimed at investigating all of Anghileri's levels of supports that contributed to students' conceptualization of the angle concept in a miniature golf real-world context. I next discuss various contexts and their support of students' understanding of the angle concept and its measure as presented in the research literature.

Use of Technology

While this study did not use technology, I include a discussion of how technology as a support has impacted students' understanding of angles. Research suggests that the use of technology can support students' development of the angle concept (Andreasen & Haciomeroglu, 2014; Browning & Garza-Kling, 2009; Browning et al., 2008; Clements & Battista, 1990; Clements et al., 1996; Crompton, 2015; Jones, 2000; Richardson & Koyunkaya, 2017). In particular, Clements and Battista (1990) showed that LOGO activities can support students in developing intuitive notions of angle, angle size, and turn to a more sophisticated and elaborative level. However, LOGO experiences have also been critiqued. Researchers have argued that, although logo activities can support students in their development of angle, they "might favor trial-and-error strategies" (Devichi & Munier, 2013, p. 3). More so, LOGO activities may not help in eliminating student held misconceptions and erroneous ideas on the angle concept (Mitchelmore, 1998; Mitchelmore & White, 2000). In another study, Clements et al. (1996) found that some students struggled to connect the LOGO turn command to their physical body turns. As Mitchelmore (1998) noted, students first need to explore various physical angle contexts in order to develop the standard angle concept for all contexts. This suggestion aligned with Clements and Battista's (1990) earlier assertion that studies need to investigate "specific geometric knowledge, processes, and misconceptions that children develop both with and without logo experiences" (p. 370). This implies the importance of exploring students' conceptions of angle in all contexts.

In a design-based research study, Crompton (2015) investigated students' understanding of angle and its measure using context-aware ubiquitous learning – "a subcategory of mobile learning that refers to mobile technologies being used while connecting with real world phenomenon" (p. 19). Crompton conducted two teaching experiments with two fourth-grade classes of 30 students each. Crompton used Scally's (1990) revised version of van Hiele levels of geometrical thinking with a focus on angles in her analysis. The theory of van Hiele levels of geometrical thinking posits that students move through different levels of geometrical thinking that are distinct, arranged in a logical qualitatively order. These levels are: *level 1, pre-recognition* – where the focus is on some parts of a shape's features such as the intersecting rays for an angle; *level 2, visual* – the focus is on a shape's appearance such as the intersecting rays for an angle meet at a common point; *level 3, descriptive/analytic* – the focus is on a shape's properties such as an angle size is determined when one ray opens an amount of turn;

level 4, abstract/relational – the focus goes beyond a shape's properties to provide formal definitions, necessary and sufficient conditions, and some logical arguments such as recognizing turns and slopes that do not have two visible rays as other angle contexts; and *level 5 rigor/mathematical* – where students provide formal mathematical systems (Battista, 2007). According to the theory of van Hiele levels of geometrical thinking, instruction is a major factor in supporting student development through these levels than maturation (Fuys, Geddes, & Tischler, 1988), a perspective adopted in this study.

Combining both technology and real-world activities, Crompton (2015) found that the use of the dynamic geometry environment allowed students to measure their photographed angles while still in the real-world setting. In this case, a dynamic protractor as a tool of support enabled them to think of an angle as a turn, rather than a static figure. In addition, the real-world setting allowed students to study angles with rays of different length, avoiding the side-length obstacle. Thus, Crompton's study highlighted the use of a tool as a support studying angles in a playground real-world setting. The current study aimed at documenting instructional supports that contributed to students' conceptualization of angles in a miniature golf real-world context. I now turn to discuss the use of physical motions, particularly body motions, in supporting students' understanding of the angle concept and its measure.

Use of Physical Motions, Particularly Body Motions

Using physical motions, particularly body motions, is another strategy found to support students' understanding of an angle as a turn and its measure. A number of researchers have conducted studies on this (Clements et al., 1996; Clements & Burns, 2000; Smith, King, & Hoyte, 2014). Clements and Burns (2000) found that besides fourth-grade students using benchmarks and guess and check to estimate turn measures, they also used their body movements to figure out about turns. These findings were consistent with the previous findings of Clements et al. (1996) with third-grade students who referenced more to physical rotations, particularly their body movements, than assigning numerical numbers to turns such as a 90 degree turn. Clements et al. (1996) noted that a critical learning point for students' learning of angles is when they are able to coordinate both turn-as-number (e.g., a 90 degree turn), and turn-as-body-motion (e.g., a right turn or left turn) (Clements et al., 1996).

The use of physical body movements in supporting students' understanding of the angle concept is also reported by Smith et al. (2014). While working with third- and fourth-grade students, Smith et al. investigated how a task that contained a coordination of both "body-based" activities and "abstract, visual representations of angles" (p. 105) would support students' understanding of angles. Smith et al. found that the task helped students to develop a strong connection of their body movements as turns, though with no visible rays meeting at a point, as related to static angles where visible rays are evident. More so, Smith et al. found that students were able to identify angles with equal measures despite being oriented in different directions. Most studies that have explored on the importance of body motion in the learning of angles have focused on younger students in third and fourth grades, and no study in particular have focused with sixth-grade students. Yet, sixth-grade students struggle to understand angles as well (Keiser, 2004; Tanguay & Venant, 2016). This study aimed at investigating the supports for sixth-grade students' conceptualization of angles through an instructional unit in a miniature golf real-world context. In the next section, I discuss more studies that have drawn on real-world settings in studying students' learning of angles.

Drawing on Students' Informal Angle Knowledge in Real-world Settings

Formal angle knowledge is the knowledge that is provided by teachers and is drawn from textbooks, with informal angle knowledge referring to all angle knowledge besides formal (Mitchelmore, 1997). A substantial amount of work has been reported on investigation of children's informal knowledge of physical angle situations (Mitchelmore, 1997, 1998; Mitchelmore & White, 2000). In his study with second-grade students, Mitchelmore (1997) found that participants were able to identify and classify the presented physical angle situations into predicted angle contexts that included: turns, slopes, crossings, bends, rebounds, and corners. However, same participants found it difficult to identify and classify angle contexts, such as slopes and turns as related to the abstract, static representation of an angle with two rays meeting at a common point. Mitchelmore (1997) suggested a critical learning point on angles as when students can relate all angle contexts, whether with visible rays intersecting or with none. Following these findings, Mitchelmore (1997) suggested students to be provided with contexts that can support them develop an understanding of angles from multiple representations.

Drawing on Piaget (1970) process of concept formation by abstraction, Mitchelmore and White (1998) proposed the theory of progressive abstraction or simply the abstraction theory as a framework for studying students' development of the angle concept. The abstraction theory posits that children progressively form angle subconcepts that they later generalize into the standard abstract angle concept (Mitchelmore, 1997; Mitchelmore & White, 1998, 2000). Abstract angle concept means the static representation of an angle as having two rays meeting at a common point (Mitchelmore, 1997). According to Mitchelmore and White (1998), the process of abstraction begins when students recognize similarities between their physical angle experiences (such as driving down hills, walking down inclined paths, etc.) to form specific physical angle situations (such as hills, cranes, etc.). Then, from specific angle situations, students move to recognizing more similarities to form specific physical angle contexts (such as slopes, corners, etc.), and finally they are able to generalize to the standard abstract angle concept (Mitchelmore, 1997; Mitchelmore & White, 1998, 2000). Abstraction theory emphasizes the importance of providing students with real-world situations when learning angles since such contexts can allow students interact with a variety of angle contexts. The current study drew from an existing data of a larger study that was situated in a miniature golf real-world context, and sought to investigate how students conceptualized angle concept in that context and supports that contributed to the conceptualization.

In another study, Mitchelmore and White (2000) investigated second- through eighth-grade students' use of the standard abstract angle concept in modelling physical angle situations into specific angle contexts. They found that it was easy for students as early as second-grade to recognize similarities for situations, such as corners and scissors where both rays of an angle were visible, but some students even at eighth-grade experienced difficulties with situations with one ray such as sloping (e.g., doors, ramps), or no ray such as turning (e.g., oven knobs, door knobs, wheels). Following this finding, Mitchelmore and White suggested the need to support students to identify angles in contexts where one or both rays are invisible. In such situations where one or no ray of an angle is visible, Prescott, Mitchelmore and White (2002), noted that students will be required to imagine the rays of the standard angle concept. There is evidence that when students learn to construct or remember the imaginary rays, they find it easy to use a protractor and to measure angles (Battista, 2007; Prescott et al., 2002).

Other studies have suggested the use of real-world contexts, such as a playground (Crompton, 2015) and a miniature golf context (Masingila & de Silva, 1997). Although these studies suggested support of students in understanding of the angle concept in such contexts, they do not provide us with detail analyses of instructional supports that supported students' conceptualization of angles in such contexts. The current study aimed at providing a detailed analysis of instructional supports of students' conceptualization of angles during an instructional unit set up in a miniature golf setting, a real-world context.

Chapter Summary

In this chapter, I presented a review of literature related to student's understanding of the angle concept and its measure in various contexts. This study sought to investigate how students conceptualize angle and its measure and the means of supporting that conceptualization in a real- world context. To situate my study, I reviewed literature on the angle concept, specifically the definition of the angle concept from a historical and present perspective, research on students' understanding of the meaning of angle and angle measure, and the difficulties experienced. I also reviewed research on supports toward students' understanding of angle and angle measure.

The reviewed literature revealed that no precise formal definition can be attached to the angle concept as the term can be viewed from several facets. The common definition presented in most textbooks of an angle as a union of two rays meeting at a common point cannot wholly define what an angle is. As a consequence, students encounter a lot of difficulties while conceptualizing the meaning of the angle concept and its measure. In this sense, researchers have used technology, body motions, real-world situations and contexts in order to support students in developing the right conceptualization of the angle concept and its measure. This study strives to add to this knowledge base by providing a detailed analysis of students' conceptualization of angle and angle measure, and means of supporting that conceptualization when students are situated in a miniature golf context, an example of a real-world context.

Chapter Three: Research Methods

This chapter explains the methods that the current study followed in order to answer the research questions stated in Chapter One: (1) How do sixth-grade students conceptualize angle and its measure before, during, and after learning through a geometry unit of instruction set in a miniature golf context? and (2) What instructional supports contribute to students' conceptualization of angle and its measure in such a context? The current study is an analysis of existing data from a larger study that was guided by design-based research methodology. I begin by giving a description of design-based research methodology as used in the larger study, and how the current study fits in. I then describe the collaborative teaching experiment that generated the data that the current study draws on. This is followed by the description of participants both in the larger study and in the current study, data collection, and data analysis.

Design-based Research and the Rationale

The current study is a retrospective analysis of existing data that was generated in a larger study that used design-based research (DBR) methodology. During the larger study, the three phases of a DBR occurred. First, the preliminary phase, where all preparation for the teaching experiment is done. Second, the experimental phase, where the actual teaching experiment is conducted. Third, the retrospective analysis, where a detailed analysis of data is done after the teaching experiment is complete (Gravemeijer & Cobb, 2006). According to Gravemeijer and van Eerde (2009), a retrospective analysis also enables a researcher an opportunity for new detailed analyses particularly when topics of interest emerge.

Through a retrospective analysis, the current study investigated how students conceptualized about angle and its measure as an important component in geometric measurement. Although the initial goals of the larger project were not focused specifically on studying angle as the only concept but the overall mathematical ideas, such as geometric representation and transformation, measurement and estimation (length, perimeter, area, angles, slope), and ratio and proportion, the data collected were rich to inform about how students conceptualized angle and angle measure.

The underlying philosophy of a design-based research (DBR) study relates to the adage “if you want to change something, you have to understand it, and if you want to understand something, you have to change it” (Gravemeijer & Cobb, 2006, p. 73). Building on this philosophy, the current study aimed at understanding a trajectory of how sixth-grade students conceptualize angle and angle measure while situated in a miniature golf context, and how their conceptualization was supported in such a context. As pointed out by Gravemeijer and van Eerde (2009), the purpose of a DBR study is to shed light on how new instructional approaches work and not to compare which works better than the other. From this perspective, the current study aimed at finding out what instructional supports contributed to students’ conceptualization of the multifaceted angle concept in a miniature golf context.

This study is of paramount significance as difficulties in understanding the multifaceted angle concept have been identified with sixth-grade students (Keiser, 2003, 2004; Tanguay and Venant, 2016). Students’ understanding of the angle concept can be supported when detailed analysis of how they conceptualize about angle and its measure in various contexts is teased out. Sixth-grade students' understanding of angles is of significance in understanding of advanced math such as trigonometry at higher levels. The next section describes the collaborative teaching experiment used to collect the larger project data on which the current study draws.

Collaborative Teaching Experiment

The major conjecture that guided the teaching experiment for the larger project was that "creating a classroom practice that has characteristics of the students' everyday mathematics practice may encourage and facilitate students to make connections between their in-school and out-of-school mathematics practice" (Masingila, 1995, p. 6). This conjecture culminated in the construction of a miniature golf geometry unit of instruction that was used during the teaching experiment. In collaboration with the classroom teacher, researchers chose a miniature golf context after gathering evidence that the context would be familiar to almost all sixth-grade students at the school and would be able to be modeled in the classroom.

The entire unit was thus centered around the "problem" of designing a miniature golf course. Although this is not a problem they would face, the researcher believed it was a situation that would appeal to their familiarity with miniature golf and their experience in designing and building models in other classes and hobbies, while providing a fertile context for their own problem posing. The unit was designed to connect with students' life, out-of-school experience by periodically having miniature golf holes set up in class (see Appendix A.1 for a miniature golf hole model set at the center of the classroom) and by taking a fieldtrip to a miniature golf course. Worksheets were designed to pose problems that arose out of their experience and instruction was designed to assist students in generalizing from their spontaneous concepts to generalized mathematical concepts. By spontaneous concepts, I mean concepts developed during interaction with the everyday world. Active student participation in terms of discussion, hands on exploration, and journal writing on assigned topics was used throughout the unit. The unit was implemented over four weeks (18 approximately 30-minute lessons) in three sixth-grade mathematics classes. One of these classes was accelerated and although the research team envisioned the enacted curriculum might differ in the three classes, the material was designed to be accessible to students in all three classes.

The big ideas and multiple mathematical concepts that the students explored in the instructional unit included:

- geometric representation (rough sketches, scale drawings, contour drawings, multiple perspectives and silhouettes of solids),
- measurement and estimation (length/perimeter, area, *angles*, slope),
- two- and three-dimensional geometrical objects (polygons, circles, prisms, cones, cylinders) and some of their properties,
- similarity and congruence,
- geometric transformations (enlargement, reduction, reflection),
- ratio and proportion (through ideas of scale drawing, enlargement, reduction, similarity, and equivalent fractions, the existence of π),

- problem solving and modelling (dealing with constraints, exploring viable strategies, weighing merits of alternate solutions, using concrete materials to help model and bridge the process of mathematizing), and
- conjecturing and verification (of relationships between similar figures, of the path of rebound of a ball)

Table 3.1 provides a summary of the nine days of the instructional sequence that focused on angle ideas.

Table 3.1

A Summary of the Course Topics During the CTE that Focused on Angle Ideas (Masingila, 1997)

Day	Lesson Focus
1	**Course Introduction**: Designing a miniature golf hole • Geometric representation (rough sketches of a miniature golf hole students have seen) • Details and measurements required to produce the holes, such as length, perimeter, area, angles, slope, etc.) • Establishing class socio norms
2	**Field Trip & Discussion**: • Making observations and measurements on an actual miniature golf course • Geometric representation (rough sketches of actual miniature golf holes) • Measurement & Estimation (length, perimeter, area, angles, slope) • Problem solving strategies for playing the hole
3	**Scale Drawing** • Discussing rough sketches • Introduction to scale drawings of the holes, requirements & differences between a rough sketch and a scale drawing
4	**Scale Drawing** • Making scale drawings of the holes • Demonstrating how to use of a protractor • Discuss strategies for finding a reflex angle

	• Giving examples of application of scale drawing in electronics and architecture
5	**Scale Drawing** • Discuss similarities and differences of scale drawings and the actual object (which measurements change [length], which measurements remain the same [angles]) • Measurements needed to measure a curved side • Similarity & congruence • Geometrical transformations (enlargement, reduction, & reflection)
10 - 12	**Examining Angles Created by the Incoming and Rebounding Paths of the Ball** • Exploring the path of a ball when it hits a wall and rebounds. What angle ideas are involved? Through making predictions; using Miras to locate the path of a rebound as a reflection and using protractors to measure angles.
17	**Culminating Activity:** Designing a Miniature Golf Hole • Engage students in designing a miniature golf hole (2-dimensional) through applying some ideas of the instructional unit, and thinking what mathematics are involved.

As Table 3.1 indicates, nine lessons (days 1 – 5, 10 – 12, and 17) provided a rich ground for students to conceptualize about angle and angle measure. For instance, the lesson focus for days 1 – 5 presented a context for students to conceptualize an angle as a measurement required in drawing of either a rough sketch or a scale drawing of a miniature golf hole. In addition, the lesson focus for days 10 – 12 provided a context for students to conceptualize angle and angle measure, while exploring path of rebounds, and angles of incidence and reflection. Finally, the lesson focus for day 17 was a culmination of the instructional unit in rethinking through the problem of designing of a miniature golf hole and the mathematics involved.

Masingila (1997) also explicitly stated the learning goals and students' expectations to adhere to during the teaching experiment in day one (for further details, see Appendix A1-A6 for lesson plans for days 1- 5, 10-12, 17). For instance, the overall goal for the teaching experiment was to have students investigate mathematical ideas while considering what is involved in designing a miniature golf hole. Consistent with RME theory, a miniature golf hole served as a model for conceptualization of angle and angle measure as students engaged with activities situated in this context.

During the collaborative teaching experiment, the research team that consisted of the researcher (R), the researcher assistant (RA), and the classroom teacher (CT) played various roles. The researcher and the teacher co-taught the lessons, although the researcher was the lead instructor. The role of the RA was to take field notes and conduct interviews that consisted of pre-interviews before the instruction, midway interviews during the instruction, and post-interviews after instruction. In this case, the pre-interviews served as preliminary assessments of students' angle knowledge before engaging in the miniature golf unit of instruction. This is consistent with Boaler's (1993) assertion that "theories which promote the use of contexts should also take on board the range and complexities of individual experience and interpretation" (p. 15). This emphasizes the importance of investigating students' experiences and interpretations of a concept before engaging them in the intended context of learning. Furthermore, as students worked in pairs, the research team also circulated in the classroom, listening to the emerging students' diverse ways of problem solving, and using probing questions to push students to communicate their reasoning. This study sought to shed light on instructional supports such as the ones played by the research team in supporting students' conceptualization of the angle context.

Participants

This section describes participants in the larger project, and their school context, as well as the participants on whom I focused for the current study.

Participants in the Larger Project Study

The participants in the larger project attended a public middle school located in the northeastern United States. The school serves students from a largely middle-class suburban community**.** Participants were students in six sixth-grade classes. The students in three classes - an accelerated (math period 9) and two non-accelerated (math periods 6 and 7) – were taught by the same teacher (T1) who was the CT in the study (see Table 3.2). These three classes participated in the miniature golf context instructional unit. Three other classes did not participate in a miniature golf context instructional unit. Two of the three, one accelerated and the other non-accelerated were taught geometry by the same teacher (T2) drawing largely from the district adopted sixth- grade mathematics textbook during the same time the miniature golf context instructional unit was going on. The sixth non-accelerated class under T3 had their geometry unit at a later time but participated in the interviews. Of all the participants from the six classes, 13 participated in both the miniature golf context instructional unit and the interviews, five participated in learning geometry through the textbook and participated in the interviews, and three participated in the interviews but did not study geometry or participate in the instructional unit. Table 3.2 shows a summary description of these classes: teacher, content, type, and data collected with the participants.

Table 3.2

Description of Classes that Participated During the Larger Project

Teacher	Content	Type of Class	Data
T1	Golf geometry	Non-accel (math period 6)	Collaborative teaching experiment and pre-, midway and post-interviews
		*Non-accel (math period 7)	
		*Accelerated (math period 9)	
T2	Textbook geometry	Non-accel	Pre- and post-interviews
		Accelerated	
T3	Non-geometry	Non-accel	

Note: T1, T2, and T3 denotes teacher 1, teacher 2, and teacher 3, respectively. Non-accel denotes non-accelerated. An asterisk * denotes the two classes that the current study drew on from the data corpus.

Participants in the Current Study

The current study drew on the data corpus from two of T1's classes, non-accelerated class (math period 7) and accelerated class (math period 9). I selected these two classes because they consisted of students who were audiotaped and video-recorded during both the miniature golf context instructional unit and interviews. Although math period 6 students participated in both the instructional unit and the interview, these students were not targeted during the instructional unit video- or audio recording. Two pairs of students (a pair of female students and a pair of male students) in each of Math 7 period, non-accelerated class, and Math 9 period, accelerated class, were audio recorded and tape recorded during each lesson. These students were recommended by the teacher as students who would be comfortable in working and talking with their partners and would be comfortable being recorded. I selected two pairs of participants for the focus of the current study, particularly when analyzing students' conceptualization of angle and angle measure for RQ1. These specific pairs comprised of students who participated in both the instructional unit and the interviews. I used the following pseudonyms for the pairs, Sarah and Emma, and Adi and Matt. While answering RQ2, I situated analysis of pairs in their class discussion, where I also referred to other students, whom I also assigned pseudonyms.

Data Collection

This study drew on the corpus of data of transcripts of video and audio recorded teaching experiment, interviews with individual students before, during, and after the collaborative teaching experiment, and copies of the students' written work and assignments on the provided worksheets. A design-based research study is informed by a variety of data (Gravemeijer & Cobb, 2006; Gravemeijer & van Eerde, 2009). I used the transcripts because I did not have access to video or audio recordings.

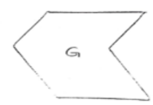
In order to address my two research questions, I used transcripts of the collaborative teaching experiment data of nine lessons of 30 minutes each (i.e., lessons for days 1 – 5, 10 – 12 and 17) (see a summary in Table 3.1). I focused on these lessons because they presented situations that enabled students to conceptualize angles. These situations included: drawing of sketches and scale drawings of a miniature golf hole and exploring paths of a ball as it hit a wall and rebound. As a supplement of what transpired during the teaching experiment, this study used students' written work that was comprised of structured tasks' worksheets and journal assignments. This study focused on Worksheet 1, Day 1; Worksheet 2, Day 5; Worksheet 10, Day 10; Worksheet 12, Day 12; Worksheet 13, Day 12; and Worksheet 18, Day 17. There were four journal assignments. Journal 1 assignment asked students to think of what mathematics might be involved in designing a miniature golf context. Journal 2 assignment asked students to describe the drawing and the measuring process of an actual miniature golf hole after the field trip. Journal 3 assignment asked students to compare five different strategies of measuring a curved side of a miniature golf hole. Journal 4 assignment asked students the angle ideas involved in the path of a ball and its rebound. Finally, I used transcripts of pre-interviews and post-interviews, as well as midway interviews, which were based on students' responses on Journal 3 and 4 assignments.

The pre-interview consisted of two parts, A and B, as it was not possible to conduct both parts in one sitting. Each part of the interview took approximately 30 minutes. The midway and post-interviews also lasted approximately 30 minutes. The pre-interview A questions revolved around describing geometrical shapes, describing a variety of angles, comparing angles, describing turns/directions, identifying angles, definition of an angle, describing situations where the word angle is used outside class, and measuring angles. The pre-interview B protocol consisted questions such as, "When, if ever, do you use the word angle? Explain. What do you think is an angle? Can you show (draw) me an angle? Explain why it is an angle. Show/draw a different angle. How is it different?" (Masingila, 1997, Appendix B1) The post-interview protocol consisted of questions such as, "What do you think is an angle? Draw me an example of an angle. Draw me a different angle. How different is it? What is the largest and the smallest angle you think of? What angles do you see in these solids?" (Masingila, 1997, Appendix B3) The interview questions were adapted from other researchers' work (e.g., Clements &

Battista, 1990). Table 3.3 gives a summary of the focus of the pre-, midway, and post interview questions.

Table 3.3

A Summary of the Focus of the Interview Questions

Interview Type	Question Focus	Examples of Questions
Pre-interview A	• Sorting out and describing 2-D shapes • Describing turns	• How would you describe such a shape as this?  • Have you ever watched a marching band on parade, what kinds of turns do they make?
Pre-interview B	• Definitions & identifying angles. • Copying an angle • Angle measure • Angles of incidence & reflection	• What do you think is an angle? Can you draw me an angle? Can you draw me a different angle? How different are they? • Look at this picture carefully and try to copy it as exactly as you can on this paper. • How could one measure an angle?
Midway interview	Journal 3 and 4 assignment	• You are provided with five different strategies that students came up with for measuring a curved side of a hole. Pick and explain which one is best one. • What angle ideas are involved in a path of a ball and its rebound?
Post-interview	• Definition & drawing of an angle • Identifying & comparing angles • Copying an angle • Angle measure	• What do you think is an angle? Draw me an example of an angle? Draw me a different angle? Explain how different the angles are. • Is this still an angle? (*extends legs of a drawn angle*) What is the inside and outside of this angle? • Tell me what you see (showing a picture). Copy the figure as exactly as you can see. • Complete this sentence in writing, *measuring an angle of a shape is similar (or is different) to measuring the side of a shape …* • *Angles and turns are similar (or different) because …*

The aim of using these interviews was to understand how students' conceptualized angle and angle measure before, during, and after engaging in a geometry unit of instruction set in a miniature golf context. For RQ2, since it sought to investigate what instructional supports contributed to students' conceptualization of angles, I focused on what happened during the actual experiment, and thus I used the collaborative teaching experiment data to address the question.

Initial Conjectures of Students' Conceptualization of Angle and Angle Measure

Since this study drew on an existing data of a larger study that used a collaborative teaching experiment, in place of hypothetical learning trajectories, I developed initial conjectures on students' conceptualization of angle and angle measure to serve as a guideline during the analysis of the CTE data. I was guided by the existing literature and the lesson plans that were previously designed in the larger study. I made my own conjectures on how I anticipated participants might conceptualize angle and its measure on Days 1 through 5, and 10 through 12 that focused on angle ideas. Table 3.4 gives an overview of the initial conjectures that I developed for students' conceptualization of angle and its measure through an instructional unit set in a miniature golf context.

Table 3.4

Overview of the Initial Conjectures of Students' Conceptualization of Angle and Angle Measure in a Miniature Golf Context

Instructional Activity & its Description	Main Goals	Conjectures of students' conceptualization

Day 1: Thinking about designing a miniature golf course. • Who has played miniature golf? • Worksheet 1 to be distributed with questions about what is involved in designing a miniature golf hole.	• Students to work in pairs in sharing ideas on what is involved in designing a miniature golf course. • Students to brainstorm on their expectations on the field trip the next day.	• Some students will mainly focus on lengths, widths, heights, perimeter, and area. • Angles and slope, might be mentioned as the last measurements required (Smith, 2017).
Day 2: Field Trip & Discussion. • Students to work in pairs to observe, measure, and discuss what is easy or difficult to measure on an actual assigned miniature golf hole. • Whole-class discussion	• Students to engage in making observations and measurements on an actual miniature golf course.	• Some students will talk of curves, reflex angles being hard to measure (Keiser, 2004) • Some students will find it difficult to use a protractor to measure angles (Keiser, 2004)
Day 3-5: Scale Drawing. • Students to discuss rough sketches of holes • The teacher to lead students in discussing what is involved in scale drawing and the differences from a sketch drawing using Worksheet 2 & 3.	• Students to be able to differentiate between a rough sketch and a scale drawing. • Students to engage in making scale drawings of holes. • Students to identify angles and their measures on a hypothesized hole.	• Some students might have an understanding that angle measure is preserved for a scale drawing (Masingila & de Silva, 1997) • Students to identify familiar angles such as acute, obtuse, and right angles (Keiser, 2003).
Day 10: Path of Rebound. • Worksheet 10 to be distributed with questions on angles created when a ball hits a wall and rebounds. • Students to take turns to play shots on the class holes and observe the path of the ball	• Students to engage in exploring the path of the ball when it hits a wall and rebounds.	• Some students will think the ball moves in a straight line or in a right angle.

after it rebounds off of a wall.		
Day 11 & 12: Exploring the idea of reflection and Angles Created by the Incoming and Rebounding Paths of the ball and the wall. • Worksheet 12 is distributed with questions of angles created when a ball hits a wall and rebounds. • The teacher to demonstrate how to use Miras, protractors and students in pairs to further the exploration.	• Students to examine angles created by the incoming and rebounding paths of the ball once it hit a wall.	• Some students might think of angles of incidence and reflection as being different due to measurement errors.

As Table 3.4 indicates, it is partitioned into three categories. The first category is about the instructional activities for Days 1-5 and Days 10-12 as described during the larger project. The second category consisted of the main goals as described during the larger project. The third category consisted of the conjectures I made of how students might think of angle and angle measure following the activities. For instance, I conjectured that when students think about measurements involved in designing a miniature golf hole, they might mention angles as the last thing. As noted by Smith (2017), students encounter measurements such as lengths, areas, volumes more time than they do for angles thus angles may be thought as a last measure needed. Another thing is that students may find it difficult to use a protractor to measure angles, particularly reflex angles and where there are curves (Keiser, 2004). I tested my proposed initial conjectures during data analysis as documented under the findings of RQ2.

Data Analysis

The data analysis for the current study consisted of a retrospective analysis, where I used the constant-comparative method of Glaser and Strauss (1967) to analyze data. This method of analysis is suitable particularly when one aims at investigating a particular aspect within a learning context (Gravemeijer & van Eerde, 2009), such as instructional supports for the current study. The constant-comparative method allowed me to constantly ask questions, compare data, and find relationship among data (Corbin & Strauss, 2015). This is consistent with Gravemeijer and van Eerde's (2009) assertion of the constant-comparative method that:

> Central to this approach is an iterative process of looking for patterns as conjectures about the data, testing those conjectures on the complete data set, and using the findings as data for a subsequent round of analysis. Final claims and assertions can then be justified by backtracking through the various phases of the analysis. (p. 517)

This implies that the constant-comparative method of analysis is consistent with a DBR methodology. I next describe the analysis that consisted of both individual students' learning and collective activity.

Analyzing Individual Students' Learning

The goal of analyzing individual students' learning was to address RQ1: how sixth-grade students conceptualize angle and its measure before, during, and after a geometry unit of instruction set up in a miniature golf context. I developed a codebook to analyze both collaborative teaching experiment (CTE) and interview data transcripts. The interview data analyzed is shown in Table 3.5.

Table 3.5

A List of Data Analyzed with the Participants

Participants Data	Adi	Matt	Sarah	Emma
Pre-interview A	⋇	⋇		
Pre-Interview B	⋇	⋇	⋇	
CTE & Midway interview	⋇	⋇	⋇	⋇
Post-interview	⋇	⋇	⋇	⋇

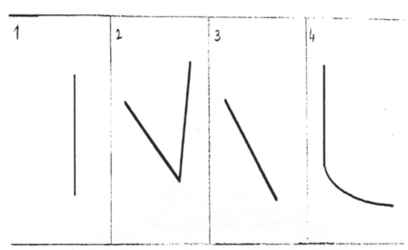
Note. The cells with ⋇ means those data were analyzed for the participants. The blank cells mean the data were missing because the participant was absent during that particular day.

The process of data analysis went through the following three cycles of coding.

Cycle one. This cycle involved exploratory coding using MAXQDA software. I read the pre- interview transcripts, followed by the collaborative teaching experiment data transcripts for the nine lessons taught in Days 1 – 5, 10 – 12, and 17 chronologically, and finally the post-interview data transcripts. Since it was during the collaborative teaching experiment when participants took the mid-way interview on Journal 3 and Journal 4 assignments, I analyzed this midway interview and students' work as part of the collaborative teaching experiment data. Using each phrase as my unit of analysis (Syed & Nelson, 2015), I read through the transcripts, identified and coded all participants' phrases that used the word angle or a language suggesting the use of the word angle. Table 3.6 shows some examples of participants' language of angle following the researcher's or the teacher's question.

Table 3.6

Examples of Participants' Language of Angle

Data	Question	Some Examples of Participant's Language of angle
Pre-interview	R: "First I want to know if you ever used the word angle."	Sarah, "Yeah, when you are using protractors, 90 degrees is a 90 degree angle."
	R: "I am going to show you some drawings and for each one, I would like you to tell me if it could be an angle and if so, why and if not, why not." 	Mark: "1, no because it is just a straight line. There is no point where the angle would be. 2, yeah because there is two lines and they meet at one point and they come out in the point. 3, no because it is basically the same as the first one, it is just put differently on the paper. 4, no because the flat part would need another flat part to make it an angle and the other side is curved."
Collaborative Teaching Experiment	R: "I know you guys never had difficulties when you were playing, but what about when you were measuring?"	Charles, "Using a protractor to measure angles … on the inclines. That were more difficult" Sarah, "it was easy to measure angles on an edge, with two straight parts."
	R: "What happens if you hit a curved surface?"	Jimmy, "If you hit it at the right speed, you make a right angle … because it is kind of round."
Post-interview	R: "Any relationship between these two angles?" (*referring to acute and reflex angles*)	Matt, "Yeah. Like let's just pretend this is like the 60 degree angle and then you subtract 60 from 360, and this reflex angle would be a 300 degree angle."

	R: Complete the following sentence, "Angles and turns are similar because and different because …"	Matt, "Angles and turns are different because an angle has straight sides (rays) and the turns have rounded sides with no straight sides or any vertices."

Note. R represents the researcher.

Cycle two. The second cycle of analysis involved reading through the new data set developed in cycle one, identifying and juxtaposing my four participants' ways of conceptualizing about angle and angle measure, during pre-interviews, during the unit of instruction set up in a miniature golf context, and during post-interviews. Table 3.7 provides an example of cycle two process.

Table 3.7

Juxtaposing Participants' Ways of Conceptualizing Angles at Cycle Two of Analysis

Task	Sarah	Emma	Adi	Matt
Pre-interview What do you think is an angle?	Two straight lines meeting.	Two straight lines meeting.	Two lines with one point. (*steeples hands*)	Like two lines from 180 degrees to like to zero degree.
Post-interview What do you think is an angle?	When two lines meet, and it is an angle.	Two lines that meet at a point.	Two rays that connect in one vertex.	Two rays that meet at a vertex.

I also developed a code book using a data-driven bottom-up approach (Syed & Nelson, 2015), in other words, codes that emerged from the data analysis. I identified three major focused codes: the meaning of an angle (MA), angle measure (AM), and the use of turn (T), and interpreted them as shown in Table 9. The focused codes later served as my major themes after refining the code book. Table 3.8 shows the initial developed unrefined code book.

Table 3.8

The Focused Codes Used for Data Analysis and Their Interpretation

Focused codes/Themes	Definition/Interpretation
The meaning of an angle (MA)	Are participants descriptions and drawings of angles suggest their conceptualization of an angle as: ♦ a point ♦ a corner ♦ a turn, ♦ two lines/rays meeting at a vertex, ♦ or any other angle context, such as slope
Angle measure/size (AM)	Are participants' conceptualization of angle measure in terms of: ♦ the measuring tool – the protractor. ♦ relation to length of rays. ♦ size appearance (smallest angle, largest angle, comparing angles). ♦ angle's orientation. ♦ Comparing a sketch and a scale drawing. ♦ the measure of angles between incoming and rebound paths of a ball and a wall. ♦ measuring a curved side of a miniature golf hole
The Use of Turn (T)	Are participants' conceptualization of turn as an angle: ♦ in relation to body movements, or ♦ in relation to giving directions

To test and refine my codes, I had an external coder code six transcripts out of a total of 30 transcripts using the code book as shown in Table 3.8. Before the coder started coding, I discussed the code book with the coder and explained my three goals of having the coder code with me. These goals were: (1) to evaluate whether the codebook provided enough clarification for someone else to use it to get the results that I was getting, (2) to refine the language used for the codes, and (3) to see if how I coded one transcript matched up with the external coding, in hope of adding reliability to my study's findings. We both coded three interview transcripts (pre-interview A and B, and a post-interview), and three collaborative teaching experiment data transcripts.

In order to reach my goals, the coding went through two phases. Both the external coder and I coded during the first phase of all of the six transcripts. After our first round of coding, we both met to compare our coding. Out of a total of 132 codes, we matched 103 codes. We discussed our coding discrepancies, where I again clarified the tasks and the interview questions in order to resolve our discrepancies. Phase two involved recoding the areas we had discrepancies. After recoding, we agreed on 27 codes and disagreed on 2 codes. Based on the coding outcomes, I used Cohen's kappa (κ) statistic to calculate our rate of coding agreement, as shown below.

		Rater 1	
		Yes	No
Rater 2	Yes	**103** **(83.53)**	2 (23.07)
	No	2 (23.07)	**27** **(6.37)**

$\kappa = \frac{P_o - P_c}{1 - P_c}$, where P_o is the observed agreement and P_c is the index of chance

P_o = (103 + 27)/ 132 = 0.9848

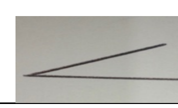
P_c = (83.52 + 6. 37)/ 132 = 0.6810

$$\kappa = (0.9848 - 0.6810) / (1- 0.6810) = 0.9524$$

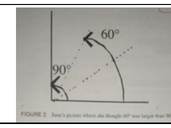
Cohen kappa statistics yielded 0.9524 with a 98% agreement, which can be interpreted as an excellent rate of agreement (Syed & Nelson, 2015). Upon establishing a consensus and out inter-rater agreement being almost perfect, with the help of the external coder, I refined my code book as shown in Table 3.9, which I used to analyze data in order to address RQ1.

Table 3.9

A Refined Code Book

Themes	Codes	Definitions	Exemplars
The meaning of an angle (MA)	1. A point	Conceptualization of an angle as a point	Matt said, “This figure … it is kind of like a square connected with a triangle and it has got five sides and five points.” Keiser (2003)
	2. A corner	Conceptualization of an angle as a corner	Matt said, “This figure, it is like a rhombus I think … it has got four sides and four corners”
	3. Two rays meeting at a vertex	Conceptualization of an angle as figure formed when two rays/lines meet at a common point (vertex)	• Sarah said, “an angle is just like two straight lines meeting.” • Adi’s drawing of an acute angle as shown in this figure (see more examples in Table 21) 

			• Identifying angles in given shapes that are defined by two lines meeting a vertex. Matt said, "I mostly see 90 degree angles." • Description of the inside/outside of an angle as an area defined by the two rays of an angle. Keiser (2003)
	4. A kind of a geometric figure	Conceptualization of an angle as a kind of geometric figure	A right angle, an acute angle, an obtuse angle, a straight angle, a reflex angle
	5. A unit of measure	Conceptualization of an angle as a degree	During the instructional unit, Maggy said, "you have to measure the degree there." Another student, Matt said, "The angle." Keiser (2003)
Angle measure (AM)	1. Geometrical figures defined by two rays meeting at a point	Conceptualization of angle measure as the space where two rays meet at a point	Matt said, "So, like a way to measure angles is like obtuse, right, acute."
	2. A linear distance between two rays (Clements et al., 1996)	Conceptualization of angle measure in terms of the width distance between the rays	Clements et al. (1996) observed that Melissa thought 60 degrees was larger than 90 degrees as the width of the arcs depicted in the following figure.

			
	3. A degree	Conceptualization of angle measure in terms of a degree as a unit of measure	Researcher: “about how much are those angles? Student: “like what degrees?”
	4. The length of rays	Conceptualization of the angle measure based on the length of the rays	“An angle is a measure of one side of a shape” (Keiser, 2003, p. 2)
	5. Size appearance	Conceptualization of angle measure in terms of size appearance	Smallest angle, largest angle, less of an angle, more of an angle
	6. Comparison of angles	Conceptualization of angle measure in terms of comparing angles	Placing one angle on top of another to tell its measure
	7. Angle’s orientation	Conceptualization of angle measure in terms of angle’s orientation	A prototypical right angle’s legs facing left-right direction
	8. Measuring of a curved side	Conceptualization of angle measure when measuring a curved side of a miniature golf hole	How the language of angle measure is being used.
	9. Measuring of an incline or slope	Conceptualization of angle measure for an incline or a slope	How the language of angle measure is being used and carried out
	10. Measuring of angles between incoming and rebound paths of a ball and a wall	Conceptualization of angle measure for angles created by the incoming and rebound paths of a ball and a wall	How the measuring process is going
The use of turn (T)	1. Body movements	Conceptualization of the use of turn as an angle in relation to body movements	Angle as turn while conceptualizing body motions.
	2. Giving directions	Conceptualization of the use of turn as an angle in relation to giving directions	Angle as turn while giving directions

Note: Numbers were used to assign codes. For example, MA1 meant meaning of an angle as a point.

As the exemplars in Table 3.9 shows, the developed codes were informed by the data as well as the existing literature.

Cycle three. The third cycle of analysis involved re-reading through the second cycle analysis, writing memos on how I was interpreting what was evolving, as I refined my code book. Table 3.10 shows an example of a written memo from a post-interview task.

Table 3.10

An Example of a Written Memo During Post-Interview Analysis

Task	Student Response & What Observed	Memo
Post-interview Look at this map of a treasure island. You are going to be starting at the point P. I am going to give you directions so that you can move around and find the point X where the treasure is buried. (Draw the path from point P as accurately as possible. Go North 200 ft. Turn 110° left. Go forward 300 ft. Turn 25° right. Go forward 100 ft to get to the spot X).	**Sarah**: *Places protractor correctly but marks the incorrect 110. Places protractor correctly, but marks at the wrong 25° mark.* **Emma**: *First turns her head imagining the turn. Then places protractor and locates the angle correctly. Places protractor correctly but marks the wrong 25° mark.* **Adi**: 90 degrees would be like here. *Turns his body and imagines pretty accurately. Places protractor correctly and finally marks 110° correctly.* **Matt**: *First places protractor so 90° is ahead, then places it correctly and*	When students were asked to turn 110 degrees left, two things that helped them to place the protractor and locate the angle correctly were: (1)Turning their bodies to conceptualize the turn and (2) using 90° angle as a reference point in order to locate 110°. Students like Sarah, Emma and Matt had difficulties marking the reading of the acute angle, the 25°. This show that although the students were able to conceptualize the turn using their body and using 90° as a reference point, they struggled with transferring that knowledge of turn to their protractor in order to tell which scale to

	marks off 110^o correctly. Places protractor correctly but marks the wrong 25^o. **Researcher**: Is that, how much did you turn … show me how you measured … if you have 25 here, where is your zero? **Matt**: On the line. **Researcher**: So, now you are turning this way. Is that left, or right?” **Matt**: …turn right. **Researcher**: 25 degrees is … how do you measure it. From which direction are you measuring? **Matt**: from here to here. Researcher: So, then you are facing this way (*showing that he would have started out facing in the opposite direction*). *Matt marks the correct 25^o.*	use. For instance, when the researcher asked, "should you be reading the top set of numbers or the bottom set?" Sarah replied, "the bottom set of numbers … the top I mean." At this moment, Sarah seemed to be guessing and unsure. Overall, students appeared to struggle to read 25°, the acute angle than they did for 110°, the obtuse angle.

Note. The words in italics is what was observed during the collaborative teaching experiment as indicated on the transcripts.

While I used the developed codes to analyze data in order to address RQ1, I used a priori interpretative framework for addressing RQ2. I next present Anghileri’s interpretative framework, which I used to analyze instructional supports in order to address RQ2.

Analyzing Instructional Supports (IS)

Anghileri (2006) proposed an analysis framework of supports of mathematical learning in a classroom. According to Anghileri’s framework, instructional supports can be viewed both as those that require indirect teacher intervention and those that need direct teacher intervention. Anghileri further categorized the two broad categories into three hierarchical levels. Anghileri's level one supports are those that do not need direct teacher intervention, such as environmental supports. Level two and three supports are those that need both teacher and student direct interaction, such as teacher explanation and student explanation and justification of their thinking. Table 3.11 provides Anghileri's instructional supports in detail.

Table 3.11

Anghileri's Instructional Supports (Anghileri, 2006, p. 39)

Levels	Category	Sub-category	Examples
1	• Environmental Affordances	• Provision of artefacts	• Use of manipulatives • Use of appropriate tools
		• Classroom organization	• Sequencing and pacing • Peer collaboration • Structured tasks through worksheets and activities
		• Emotive feedback	• Remarks and actions to gain attention and encourage students' outcomes
2	• Explaining	• Funneling	• Teacher to show and tell
	• Reviewing	• Focusing	• Teacher to focus students to look, touch, and verbalize their notices and thinking • Students explaining and justifying their thinking • Teacher interpreting students' actions and talk

	• Restructuring		• Teacher's use of prompts and probing questions to push students' thinking • Teacher's use of parallel modelling for students to imitate
		• Focusing	• Affordances of meaningful context for abstract ideas • Simplifying the problem • Rephrasing students' talk • Negotiating meanings
3	• Developing Conceptual Thinking	• Making connections	• Students' explaining their thinking • Students' listening to the thinking of others
		• Developing representational tools	• Refining informal language to formal language with time (e.g., corner to angle • Structuring practical activities
		• Generating conceptual discourse	• Characterized by norms and standards of what counts as acceptable conceptual mathematical explanation

As Table 3.11 shows, although emotive feedback is more of teacher-student interaction, it is considered as part of environmental affordances since it does not directly relate to mathematics being learned as do level 2 and 3 supports (Anghileri, 2006). Anghileri's levels 2 and 3 supports are closely related to social norms, sociomathematical norms, two of the three constructs under the emergent perspective (Cobb & Yackel, 1996). For instance, some examples of the social norms are expectations that students will explain and justify their thinking, listen to the explanations of others, which are highlighted under reviewing in level 2, and make connections in level 3 of Anghileri's supports. Examples of sociomathematical norms is "what counts as an acceptable mathematical explanation" (Cobb & Yackel, 1996, p. 126), which relates to the example under generating conceptual discourse in Anghileri's level 3. Classroom mathematical practices, which is the third construct under the emergent perspective (Cobb & Yackel, 1996), can be seen as the culmination of Anghileri's supports. Classroom mathematical practices are defined as "taken-as-shared" students' mathematical collective learning (Bowers, Cobb & McClain, 1999, p. 28). Thus, all of Anghileri's supports can be seen as contributors to emergence of classroom mathematical practices.

According to Dove and Hollenbrands (2014), technology-enhanced activities provided opportunities for teachers to showcase Anghileri's levels of support. One of the future questions that Dove and Hollenbrands posed was whether Anghileri's levels of support could be evident in other contexts that are less-technology. This study focused on investigating which of Anghileri's supports contributed to sixth-grade students' conceptualization of angle and angle measure in a non-technology, real-world context. The study sought to find out which of Anghileri's levels of support would contribute to sixth-grade students' conceptualization of angles through an instructional unit set up in a miniature golf context. These analyses will benefit teachers in knowing how to support students' understanding of angles while teaching through a real-world context.

Chapter Summary

In this chapter, I presented the design-based research (DBR) methodology as used in the larger study that the current study draws on. I described the three phases and the underlying philosophy of DBR. I noted that this study focused on the third phase of DBR, the retrospective analysis, with an interest of taking a whole new look at the data to investigate in detail how students conceptualized about the angle concept and its measure while situated in a miniature golf context, an example of a real-world context.

I also discussed the classroom teaching experiment of the larger project, which provided this study's data corpus. I discussed the participants of the larger project and described the participants that this study focused on and provided a rationale. I described data collection of the larger project and the data that fed this study. I also described data analysis and the two different coding systems that I used to address my two research questions. I described how I developed, tested, and refined my code book to use it for analyzing data to address RQ1.

I ended the chapter by describing Anghileri's (2006) interpretative framework that I used to analyze data in order to address RQ2. I also shown the way Anghileri's levels of supports correlates to the three constructs of the emergent perspective, which are social norms, sociomathematical norms, and classroom mathematical practices.

Chapter Four: Findings

This chapter presents findings of my two research questions: RQ1 – How do sixth-grade students conceptualize about angle and angle measure before, during, and after learning through a geometry unit of instruction, set in a miniature golf context? RQ2 – What instructional supports contribute to students' conceptualization of angle and angle measure in such a context? As mentioned in Chapter Three, I analyzed the data collected using two different methods in order to address the two questions. For RQ1, I developed and used codes as shown in the code book (see Table 3.9, page 56). For RQ2, I used Anghileri's levels of supports to document the instructional supports that contributed to students' conceptualization of angle and angle measure during an instructional unit set up in a miniature golf context. I next present the findings of each research question.

Research Question One (RQ1)

For RQ1, I report findings of how participants conceptualized the angle concept and angle measure, organized into three categories: before, during, and after learning through a geometry unit of instruction set in a miniature golf context. The focused codes, as shown in my code book, served as my themes in these categories. I later give a summary of the overall participants' conceptualization of the angle and its measure across the three categories. Note that two of the participants were absent on the day when some of the pre-interview data were collected (see Table 3.5 in Chapter 3, page 51, that lists all data for this study).

To clarify the way of presenting this study findings, I am using excerpts both in the form of dialogue in tables, and dialogue within narrative and indented paragraphs.

Tables are for participants' longer responses and for easy comparison across participants, while indented paragraphs are for shorter responses that do not necessarily need comparison across participants.

Category 1: Participants' Conceptualization of the Angle Concept and Angle Measure *Before the* Unit of Instruction

My themes for this category are: (1) the meaning of an angle, (2) angle measure/size, and (3) turn. The first theme consisted of four sub-themes: conceptualizing an angle as a point or a corner, conceptualizing an angle as a figure formed by two straight lines meeting at a common point, conceptualizing an angle as a right angle, and conceptualizing an angle as a tool of measure. The second theme consisted of the sub-theme: conceptualizing angle measure as related to a degree, a protractor, and measuring lengths, while the third theme did not have any sub-theme. I next elaborate each of the aforementioned themes in the next section.

The Meaning of an Angle

Due to the multifaceted nature of the angle concept, participants tend to interpret the concept in different ways. My analysis has revealed four major interpretations of the angle concept by the sixth-grade participants in this study. That is, (a) conceptualizing an angle as a point or a corner, and (b) conceptualizing an angle as a figure formed by two straight lines meeting at a point, (c) conceptualizing an angle as a right angle, and (d) conceptualizing an angle as the tool of measure. I next discuss these four angle interpretations and how participants' words and work illustrated these interpretations in this study.

Conceptualizing an angle as a point or a corner. During pre-interviews, participants were presented with the geometrical shapes in Figure 4.1. They were asked to describe the shapes to someone who does not know their names.

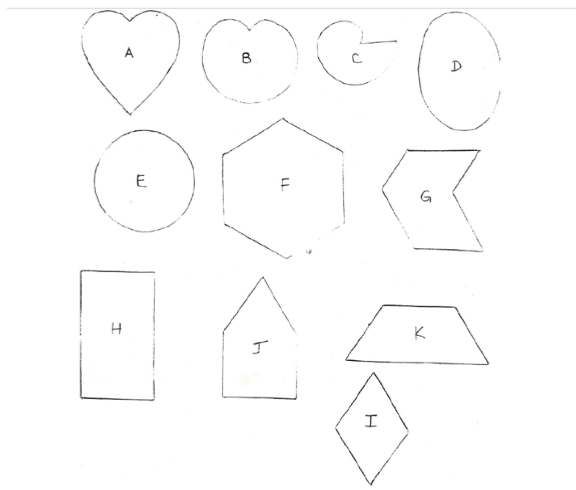


Figure 4.1: Two dimensional geometrical shapes.

The following excerpts shows how Adi and Matt described shape I in Figure 4.

Researcher: Can you describe any of them in particular?

Adi: This is diamond (*referring to shape I*). It has got four, well, it is like, it is kind of like a square, but the two sides like these are like that and there are 1, 2, 3, 4. (*counts around the shape*)

Researcher: And what are the 1, 2, 3, 4?

Adi: Points.

This excerpt show that Adi used an informal language "points" to describe shape I. At the beginning, Adi was hesitant to use any word to describe; he stated, "1, 2, 3, 4," but after being probed further what he meant by "1, 2, 3, 4", he said points. In describing shapes, students are to use angles and sides (NCTM, 2006). This suggest that Adi conceptualized an angle as a point.

For the same question, Matt's response was as follows:

Matt: This one (*referring to shape I*), it is like a rhombus, I think. Or it can be taken as a diamond if you put it this way (*turning it around*).

Researcher: How about if I didn't know what the name was? How would you describe it?

Matt: Well, it has got two, the bottom and the top are parallel to each other and so are the sides, and the sides are kind of like diagonally, going diagonally vertically. And has got four sides and four corners.

In this excerpt, Matt similarly used an informal language "corners" to describe Shape I, as stated in the CCSS for kindergarten math (NGA & CCSSO, 2010). These two excerpts suggest that participants conceptualized an angle as a point or a corner. Table 4.1 provide more evidence of both Adi's and Matt's references to an angle as a point or a corner with the other shapes in Figure 4.1.

Table 4.1

Adi's and Matt's Responses on Description of Some Geometrical Shapes in Figure 4.1

Questions/Task	Adi	Matt
• Can you describe any of these shapes for someone who doesn't know their names?	• Figure G… has six points. • Figure B … instead of connecting the two points right there, you	• Figure F … has 1, 2, 3, …, 6 corners and six sides all the same length. • Figure G … it is on the point at bottom, a point

	come over and it is kind of goes like that. • Figures E, D … are the groups that don’t have a point where it stops.”	here, a point on the left side, a point on the upper left side, a point on the upper right side, and a point on the lower right side … • Figure J … it has got five sides and points.

As Table 4.1 shows, Adi referenced the circular shapes E and D as having no angles. He said, “they don’t have a point where it stops.” This suggests that Adi conceptualized angles as points, seeming to mean that if the figures had points then that is where an angle can be located. Matt completely avoided describing all round shapes. These findings suggest that some students at the sixth-grade level may struggle to describe shapes using the word angle as the formal language, but instead reference points or corners, an informal language for angles (NGA & CCSSO, 2010). So, one may be left wondering when do students transition from an informal language to a formal language for mathematical concepts, such as the angle concept?

Conceptualizing an angle as a figure formed by two straight lines meeting at a common point. Participants relating an angle to two lines meeting at a point was revealed in a number of instances in this study. These instances were when participants were asked to: (a) sort and group given shapes, (b) define an angle, (c) draw an angle, (d) define the interior and exterior of an angle, and (e) identify angles in given pictures. I next discuss each of these instances in detail.

Sorting and grouping shapes. When participants were asked to sort the shapes in Figure 4.1 in whatever manner they wished, both Adi and Matt sorted them into two groups: those with round shapes and those with straight lines. The follow-up questions reveal that both Adi and Matt were referencing an angle as a point or a corner, but one defined by meeting of two lines.

Researcher: I would like you to take those shapes now and sort them into however many groups you want. Tell me how you are thinking about sorting them.

Adi: These (*non-polygons*) have rounded sides; these (*polygons*) have straight-angled sides.

Researcher: What do you mean by straight-angled sides?

Adi: Like, instead of having them like this and then this and then this (*pointing to points A, B, C*), like that, it is straight and right to the point, you know.

In this excerpt, Adi explained that he had two groups for the shapes in Figure 4.1. The first group with “rounded sides” and the second group with “straight-angled sides.” When Adi was probed further what he meant by “straight-angled-sides,” while pointing to shapes A, B, and C which had curved lines meeting at a point, Adi said that the straight-angled sides had straight lines which were right to the point. In this case, my interpretation is that Adi conceptualized angles not only as a point but one that was related to straight lines meeting at that specific point.

Matt response to the same question indicate that he conceptualized angles as corners.

Researcher: I would like you to take those shapes now and sort them into however many groups you want. Tell me how you are thinking about sorting them.

Matt: I have two groups and one is with round parts on the shape and the other one has just straight lines.

Researcher: Does having straight lines give these shapes anything else that these don't seem to have?

Matt: Corners.

Researcher: Are these (*non-polygons*) not corners or are they different kinds of corners?

Matt: They are kind of more like sort of corners because the corner normally has two straight sides coming together and these have curved sides and a straight side coming together or just curved sides coming together.

In this excerpt, it is clear that Matt conceptualized angles as corners. When the researcher probed him further about corners, Matt described the different kinds of corners as formed by two straight sides, a straight line and a curved side, or two curved sides. Matt's phrase "… the corner normally has two straight sides coming together" relates to the common static definition of an angle, the union of two rays meeting at a common point (Smith, 2017). In both excerpts, Adi and Matt seem to conceptualize an angle as a point or a corner, but one related to straight lines.

Definition of an angle. When participants were specifically asked to define an angle, they all conceptualized two lines meeting at a point. For instance, Sarah said, "two straight lines meeting," Adi said, "two lines with one point (*steeples hands and draws a picture of an acute almost right angle to illustrate*), and Matt said, "two radii of two lines with a point that meet together." The notion of "meeting at a point" may suggest why participants tend to term an angle as a point. Adi tried to explain how a point could be placed on a straight line in order to make it a valid 180-degree angle.

Researcher: So, what is an angle?

Adi: It is two lines with one point (*steeples hands and draws a picture of an angle*). Well, actually, it doesn't always have to be a line because there is 180 is like that, right? It is a straight line (*draws*). There can be a point. I guess you could make a point there.

This excerpt illustrates how Adi conceptualized an angle as a figure formed by lines and points. Therefore, for a straight angle, it has to have a point on its line to indicate it's an angle.

Drawings of an angle. Participants' drawings further confirmed their conceptualization of an angle as a figure formed by two lines meeting at a point. When participants were asked to draw pictures of an angle, their drawings revolved around right angle, acute angle, and obtuse angle as shown in Figure 4.2.

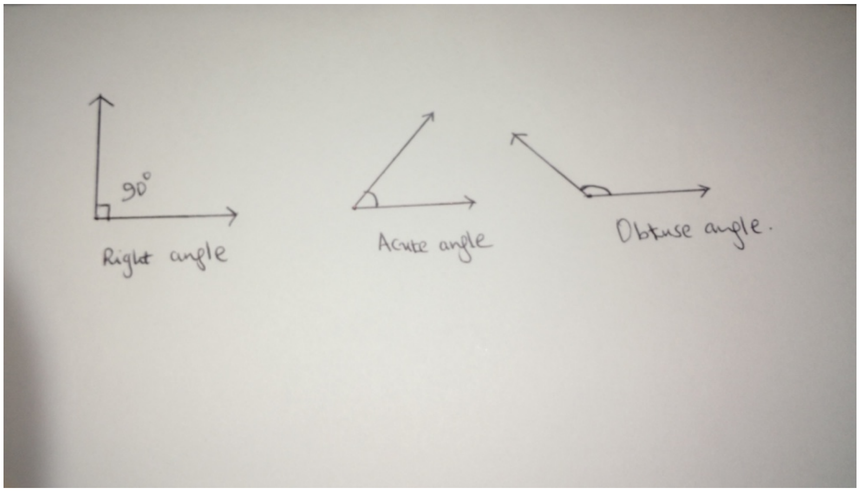


Figure 4.2. Drawings of a right angle, acute angle, and obtuse angle.

All these kinds of angles are defined by two straight lines meeting at a point. In addition, when participants were asked to show an angle in their drawings, they pointed to the vertex or circled the vertex as drawings in Figure 4.2 show. This further confirmed that participants tend to conceptualize an angle as related to a point. The questions that arose in my mind were: What about slopes and turns, that may not have a visible point or ray (s)? Did these angle contexts exist in participants' angle repertoires?

The interior and exterior of an angle. Participants were presented with a drawn obtuse angle and marked letters as shown in Table 4.2. They were asked to say whether the marked letter was inside or outside the angle. Masingila, Lester, and Raymond (2011) noted that "An angle separates a plane into three disjoint sets of points: the set of points that makes up the interior of the angle, the set of points that makes up the angle, and the set of points that makes up the exterior of an angle" (p. 261), as shown in Figure 4.3. Table 4.2 show how participants responded to this question, with some giving some justification of their conceptualization.

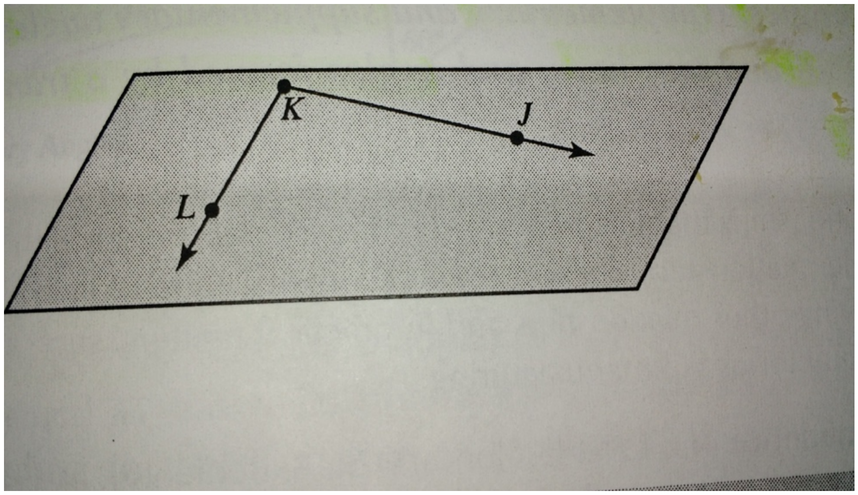
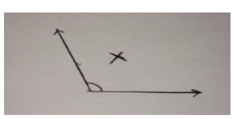


Figure 4.3: A drawing of angle JKL separating a plane into three disjoint sets of points (Masingila et al., 2011).

Table 4.2

Participants' Responses on the Interior and Exterior of an Angle

Question/Task	Sarah	Adi	Matt
• Where is X with respect to the angle? (*marked inside the partial triangle of obtuse angle*) 	• Inside, because the little line (*referring to the arc in between the rays*) is in the inside.	• Inside.	• Inside.
• Where is Y with the respect to the angle? (*marked within the sweep of the angle but outside the triangle*)	• Outside, because like if you put another line against it (*indicates connecting end of legs with a line segment* as shown in the figure below), it	• Inside, because if I kept drawing this line all the way, then it would be like if I drew a line across, and like then it would be inside.	• Inside, because if you were to keep these lines going, and it ended up being inside the lines.

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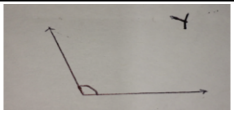
<td>
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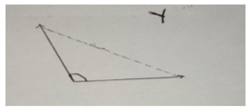
<td>furthers out, or if you did it like this way it would be part way out so it is more out than in.

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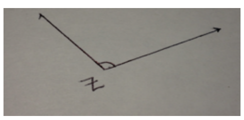
<td>• Where is Z with respect to the angle? (marked in the reflex region)

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<td>• Outside.</td>
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Note. The words in italics is what was observed as indicated on the transcripts.

Following Masingila's et al. (2011) definition of the interior and exterior of an angle noted above, the data in Table 4.3 shows that three out of four participants had a correct conceptualization of the interior and exterior of an angle. However, Sarah seemed to conceptualize the interior of an angle as only points defined by where the two lines forming an angle reached, and not all points that are on the same side as the rays making an angle. This could suggest Sarah's conceptualization of an angle as a figure defined by the two lines as segments and not as rays. Segments have two endpoints while rays have only one endpoint, implying the other end can go to infinity. Given participants' conceptualization of letter Z being outside of the angle, this indicates that they likely were not aware of reflex angles and, at least, did not consider the reflex angle at the time.

Identifying angles in given figures. Participants were presented with Figure 4.4 and were asked to explain how many angles they could see in the drawings.

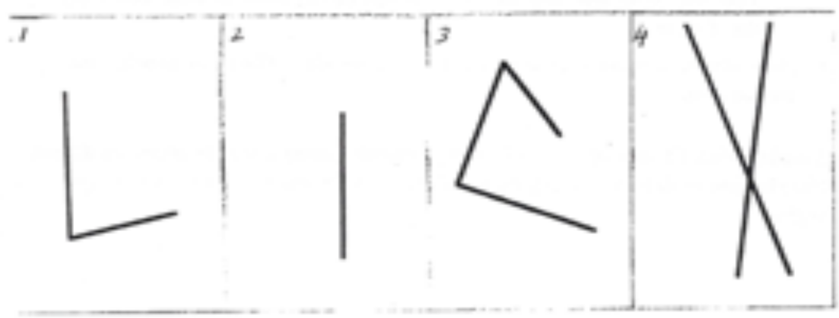


Figure 4.4: Drawings of a straight line and intersecting lines.

Sarah's response:

Researcher: How many angles do you see?

Sarah: Figure 1 has one angle. Figure 4 has four angles because they have straight lines. Figure 3 has two angles. Figure 2 is an angle because it is straight … there is only one angle and the angle is at the middle. (*indicates a point in the middle of segment*).

That is where it would be because like all straight lines are angles because you can move it to make it bent, so it can bend.

Adi's response:

Researcher: How many angles do you see?

Adi: Figure 1 has one angle (*an acute angle*). Figure 2 … there is not really an inside but like I think that like if this is 90, then like this is another 90 is 180, and 180 degree is an angle. So, it is an angle. Figure 3 has two angles (acute angles). Figure 4 has four angles (*he identifies the vertically opposite angles*).

Matt's response:

Researcher: How many angles do you see?

Matt: Figure 1 has one angle (*acute angle*). Figure 2 there is no angle. Figure 3 has two angles (*the acute angles*). Figure 4 has four angles (*he identifies the vertically opposite angles*)

In the above excerpts, all the three participants said that drawing 1, 3, and 4 had one, two, and four angles respectively. This suggest that the three participants conceptualized an angle as a figure formed when two lines intersect. For drawing 2, both Sarah and Adi identified it as a straight angle. Sarah emphasized the middle point of the drawing 2 as where to find the angle. Adi begin by noting that "there is not really an inside" for drawing 2. This suggest that Adi conceptualized an angle as related to a point on the line. Adi further explains that if two right angles are placed together, they would form a straight angle, and thus he concludes drawing 2 as an angle. Matt did not conceptualize drawing 2 to be an angle. This finding emphasizes participants conceptualization of an angle as a figure formed when lines intersect at a common point.

In a similar question, participants were given the drawings 1-10 in Figure 4.5, and were asked to identify angles and explain their choices.

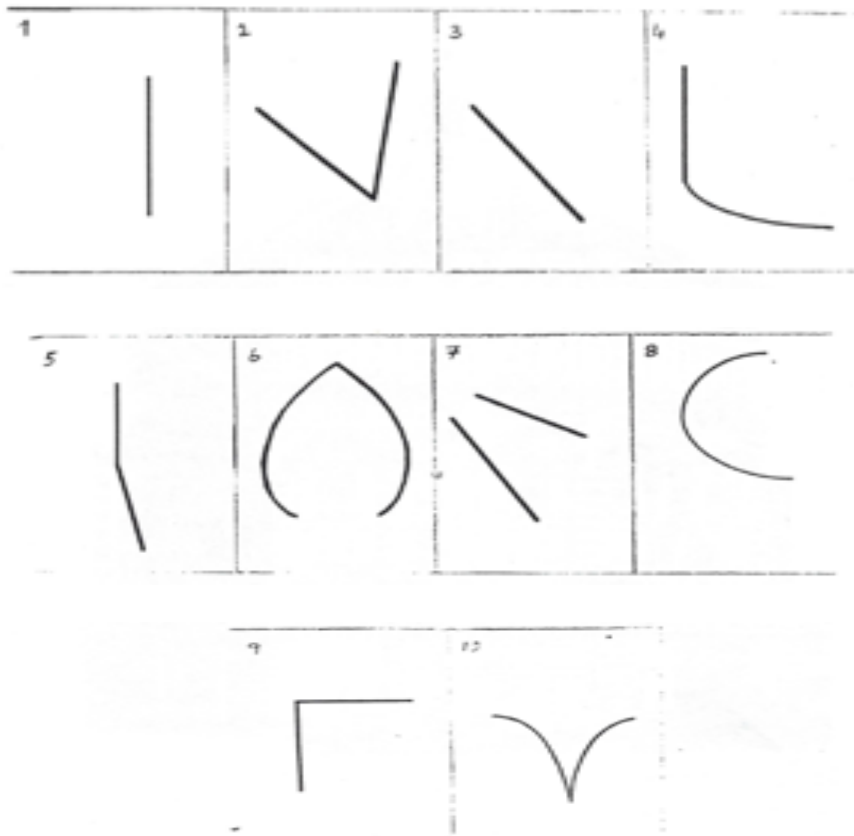


Figure 4.5: Drawing of straight lines, intersecting lines, and curved lines.

Adi's response:

Researcher: Which one could be angles and why?

Adi: Figure 1 is an angle because like 180 it goes straight across.

Researcher: What part of it is an angle? What produces the inside of the angle?

Adi: There really isn't an inside because there isn't a point.

Matt's response:

Researcher: Which one could be angles and why?

Matt: Figure 1 is not an angle because it is a straight line. There is no point where the angle would be.

Researcher: If there was a point somewhere in the middle?

Matt: Well, it still, yes, there would, it would be because it would just be a 180-degree angle. Figure 2 is an angle because there is two lines and they meet at one point and they come out in the point. Figure 3 is not an angle because it is basically the same as the first one, it is just put differently on the paper.

The above excerpts confirm the latter finding of students conceptualizing an angle as a figure formed when two lines meet at a common point. Adi identified figure 1 as a straight angle with no inside, since there was no point. Matt explained that if there was a point somewhere at the middle of figure 1, then it could be a straight angle.

Conceptualizing an angle as a right angle. When participants were asked to draw an angle, they all began by drawing right angles. Furthermore, when participants were asked to draw another angle, they used the drawn right angle as a reference point to draw either an obtuse angle or an acute angle. For instance, while referring to the first right angle drawn, Sarah said, “this one, obtuse, is farther down than this one (referring to the right angle),” as illustrated in Figure 4.6. This indicates that Sarah was using her first drawn right angle in order to conceptualize how to draw the obtuse angle.

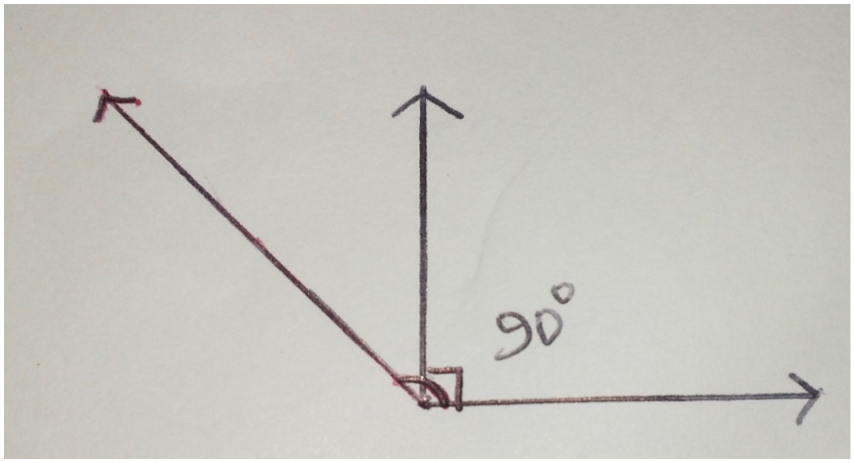


Figure 4.6. An illustration of one ray of an obtuse angle being farther down the ray of a 90-degree angle.

Matt also drew a 45-degree angle and said, “well, if this is exactly 45 and this is exactly 90, they would have the same bottom line, the top line would be on this angle, the 45 would be halfway, so halfway closer to the other one than this line,” as illustrated in Figure 4.7. Again, this suggests that Matt was using the right angle as a reference point in order to draw an easy acute angle, the 45-degree angle.

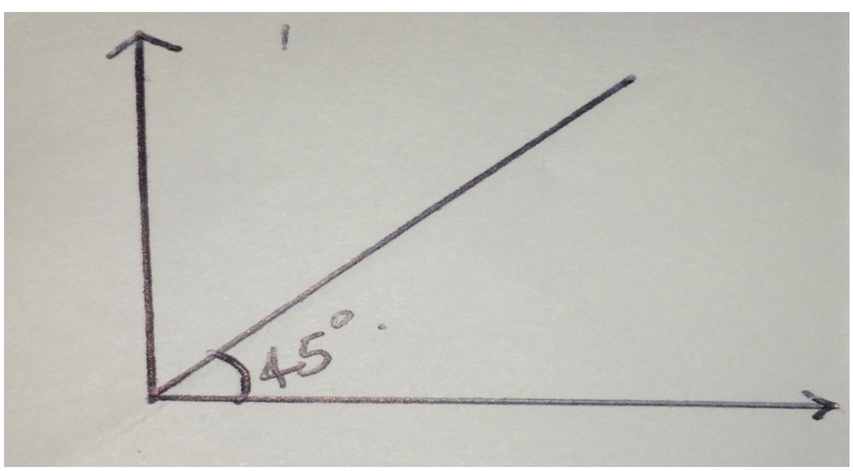
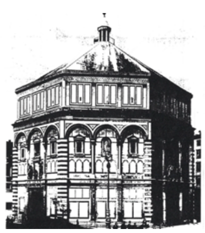


Figure 4.7: An illustration of 45-degree angle, halfway 90-degree angle.

This finding suggests the salience of right angles in drawing angles that are less than or more than 90-degree angle. The finding also suggests that participants find it easy to draw a 45-degree angle as an example of an acute angle as it is half of a 90-degree angle. The salience of right angles was also confirmed when participants were presented with a photograph of an old building izn Italy and asked to identify the angles they could see or think of. The data in Table 4.3 show participants' responses.

Table 4.3

Participants' Responses on Identifying Angles on a Given Photograph of a Building

Question/Task	Sarah	Adi	Matt
• Identify angles you can see or think of in this photograph of an old building in Italy. 	• 90 degrees angle(window). • 90 degrees angle (edges of adjacent roof sections), maybe 100, something like that. • Less than a 90-degree angle (angles at the base of roof section). • There are a lot of 90 degrees right here, right here, right here. They are just all over.	• A triangle (above the window). (**Follow up question by R**: …a name?) • A triangle has 3 angles. (**Follow up question by R**: You did you not realize that before?) • An angle is probably this point with two lines right there instead of the whole entire thing. Actually, it could be the whole entire thing because you can't just have a point.	• I mostly see 90 degrees angles (e.g., windows, pillars). • I also see 45-degree angles kind of (above the windows), these triangles above the window. • Thinks of obtuse angle because the six sides (where walls meet identified as angle).

		But there are three angles in a triangle.

As data in Table 4.3 indicate, participants identified many 90-degree angles. Column three also shows an interesting discussion that ensued between Adi and the researcher when Adi identified a triangle and said it has three angles. The researcher asked Adi, “did you not realize that before?” Adi said, "an angle is probably this point with two lines … instead of the whole entire thing." Then he continued to say, "Actually, it could be the whole entire thing because you can't just have a point, but there are three angles in a triangle." This excerpt demonstrates Adi's dilemma of articulating what is really an angle, particularly in the context of a triangle that has three angles. Adi wondered, is it the entire thing? Or is it a point? Adi concluded that it cannot be a point since there are three angles. The fact that Adi recognized the three angles in a triangle indicates that he conceptualized an angle as a figure formed when two lines intersect at a common point.

Conceptualizing an angle as a tool of measure. When participants were asked about situations in which they use the word angle, Sarah said, “when using protractors, 90-degrees is a 90-degree angle. Then there is 180 degrees, and then there are other ones that I don’t really use yet.” This suggests Sarah’s conceptualization of angles as related to the measuring tool, the protractor, as well as to angles measuring 90 and 180 degrees. It also suggests the emphasis attached to unit of measure “the degree” as related to the tool of measure, the protractor as students are being introduced to the angle measure.

Angle Measure

It is one thing to conceptualize the meaning of an angle, and it is another thing to conceptualize what angle measure means. This study revealed sixth-grade students' conceptualization of the angle measure as related to the unit of measure (degree) or the tool of measure (the protractor), the length of rays, and the angle's orientation. I next discuss each of these references to angle measure in detail.

Conceptualizing angle measure as related to a degree, a protractor, and measuring lengths. When participants were asked "how could one measure an angle?" Adi said, "with a protractor … we use degrees." On the same question, Matt said, "I don't know. Um, maybe if you could use a compass, you could measure the radius of a circle." Adi's response suggests that he had an accurate conceptualization of the tool and the unit of measure to use. However, Matt's response suggests that he did not know how he could measure an angle. Instead, Matt talked of using a compass to measure the radius of a circle. This may suggest that Matt conceptualized measuring angles to measuring lengths, as the radius of a circle is a length. The following excerpt reveals more evidence of Matt conceptualizing measuring lengths to measuring angles.

Researcher: Okay, so estimating angles, does that seem easy or more difficult than estimating length?

Matt: More difficult.

Researcher: Do you have any idea why that might be the case? Why do you feel that you are better at estimating length?

Matt: Well, I have been exposed to length a lot longer than I have to angles.

Researcher: Right.

Matt: I used rulers when I was 2 and 3 years old.

In the above excerpt, Matt noted that he had been exposed to length measures since he was a young kid than angle measures. As a consequence, Matt claimed estimating angle measure to be more difficult than estimating length measure. This claim may partly explain why some students have difficulties with angle and angle measure and tend to conceptualize angle measure in terms of length measure. Conceptualization of measuring an angle as measuring lengths was also clear with other participants. For instance, when participants were asked what they measure on an angle, Sarah said, "…the width of the part of it. You can measure the length." This might suggest Sarah's conceptualization of the angle measure as measuring the length of the arc of the angle.

When participants were asked to complete the sentence: *Measuring an angle of a shape is different from measuring the side of the shape because* … different participants responded as follows:

Matt: … you are measuring the length that is between the two sides.

Adi: … you are measuring where two sides are coming together at a point.

Sarah: … the width of the part of it.

This excerpt is a clear indication that participants conceptualized angle measure as being the measure of the space in between the two sides that meet at a point. However, they seemed to conceptualize about the space in terms of the length measure instead of the turn measure. As the following excerpt further indicates, participants conceptualized that by extending the rays of an angle that cannot change the measure of an angle.

Researcher: By extending the rays of the angle that you drew, does the angle measure change?

Sarah: Just making it how long it is. This could be like these and like you just made it bigger like longer. You have something longer and you want to measure it. I think because you just made it go out more. There is more here. Actually, I don't think.

Adi: No change.

Matt: No change.

As the excerpt indicates, both Adi and Matt seemed certain that by extending the rays of an angle that does not change the size (measure) of an angle. At first Sarah seemed uncertain about the preservation of the angle measure after extending the rays of angle, but finally she concluded the angle measure will remain the same regardless. This suggest that participants had an accurate conception about the angle measure being preserved regardless of extending the legs, and thus they challenged side-length obstacle (Devichi & Munier, 2013).

Turn

The most dynamic way to think of an angle measure is as an amount of turning. In order to investigate whether participants in this study had a sense of angle as turn, they were presented with a floorplan of a small shopping mall, as shown in Figure 4.8.

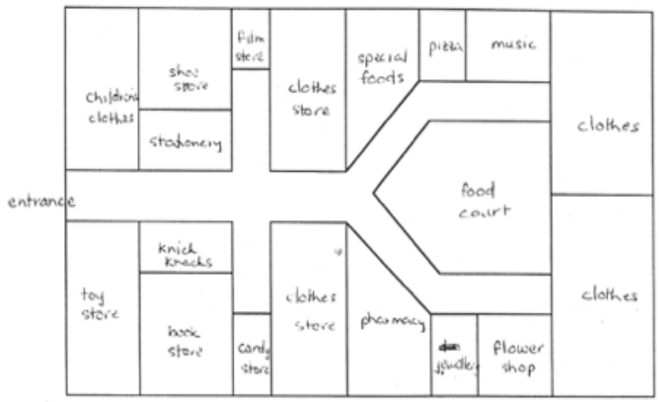


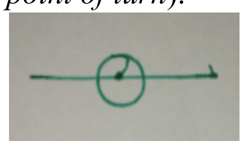
Figure 4.8: A floorplan of a small shopping mall.

Participants were then asked to explain various directions from one point to another in the shopping mall. The data in Table 4.4 show how Adi and Matt responded to this activity.

Table 4.4

Adi's and Matt's Responses to Giving Directions

Question/Task	Adi	Matt
What would you say to someone asking you for: directions from the entrance to the flower shop?	(*He oriented the paper to face the fork*) Go straight to the fork. You go to the right at the end of the pharmacy, where it connects, just go straight to the clothes. But then right before the clothes, go to the flower shop and it is on the right.	(*He does not orient the paper to face fork*) Take the hallway down to food court, then at the fork, take a right and go all the way down to the last shop on the right beside the clothes store.
What would you say to someone asking you for: directions from the flower shop to the music store?	Go outside the food court, you take a left, keep going left and don't turn into any stores. Get in front of the clothes store, then go toward the other side of the food court, then pass the pizza store and go to the music store	From food court, take a right down here and go to the last store beside the clothes store again on the left side

What would you say to someone asking you for directions from the flower shop to the music store?		Go back to the point where the food court comes into the main hall and then take a right down here and go to the last store beside the clothes store again on your left side.
I want you to think about how your body moves as you walk this path.	I go out of the flower shop. I take a left. There isn't a hall that keeps going left, so I follow the hall and keep going. That is kind of at an angle.	Well, I am taking all right turns.
What special turns do you know?	Mentions a 90-degree angle (*draws an angle*).	
Have you heard of 180 turn?	- (*draws a complete about turn with a little circle at point of turn*). 	
If you were to make 90 degree turns, how many to make in the same direction before you are facing the front again	…like you start there and there is like 1, 2, 3, and then at your fourth turn you are back where you started from.	

Note. The words in italics are what was observed during collaborative teaching experiment as indicated on the transcripts.

The data in Table 4.4 indicate that both Adi and Matt conceptualized angle as a turn, while they were encouraged to think the way their bodies were moving. At first, when Adi and Matt were asked to give directions, they used words such as “go to the right,” “on the right,” “you take a left,” “take a right,” and “on the left-hand side,” without mentioning something like make a left turn or a right turn. However, when both participants were asked to think of how their bodies were moving as they walked the path, the use of the word turn became prevalent. For example, Matt claimed that he will take all the right turns, while Adi related the turn to an angle. This finding suggests a possible way to support participants develop their conceptualization of an angle as a turn is to provide them with activities that will involve their body movements (Smith et al., 2014).

Summary of Pre-interview Findings

In summary, the pre-interview findings indicate that before participants engaged in the teaching experiment they conceptualized an angle as: (1) a point or a corner that is related to straight lines (2) a figure formed by two straight lines meeting at a common point, (3) a thing related to the tool of measure, the protractor, and simply a right angle. In addition, participants conceptualized angle measure as related to the protractor and the degree and seemed not to conceptualize angle measure as an amount of turning at the moment. However, majority of the participants had the correct conceptualization of how extending the rays of an angle cannot change the angle measure/size, but they seemed to relate angle measure to length measure. I next present findings of participants' conceptualization of the angle and it's measure during the instructional unit.

Category 2: Participants' Conceptualization of the Angle Concept and Angle Measure *During* the Unit of Instruction

This section report findings of participants' conceptualization of angle and its measure during the collaborative teaching experiment. As noted in Chapter Three, I analyzed data for nine lessons (Days 1-5, 10-12, and 17) that provided students with an opportunity to conceptualize about angle and its measure. I also provided my overview of the hypothetical learning trajectory for eight days in Chapter Three. I thus also report on the testing and refining of the conjectures that I had made before data analysis. Data analysis has revealed that throughout the nine lessons, as participants engaged with various activities, two major themes emerged: (1) the meaning of an angle, and (2) the angle measure. I next provide the findings based on the aforementioned themes.

The Meaning of an Angle

On Day 1, before the field trip to the actual miniature golf hole, students were provided with a model of a miniature golf hole inside the classroom. They were also given Worksheet 1 that had three questions: (1) to draw a rough sketch of a miniature golf hole that a student had played or the one modeled in the classroom, (2) to describe in short phrases the details of the hole (its general shape, details of its sides and angles, and obstacles), and (3) to state the measurements that should be required in order to reproduce the hole at another time. Students were instructed to work in pairs as the researcher and the classroom teacher walked around listening to students' discussions. I report findings of the two pairs who were audio recorded during the lessons and also analyzing what they wrote on their worksheets. I also report instances during the whole-class discussions in which the two pairs were situated with other students.

Angles as geometric figures. Before the field trip to an actual miniature golf hole, participants conceptualized angles in terms of geometrical figures, such as acute, right, obtuse, reflex, and straight. For instance, Matt and Adi said in the following excerpt.

Researcher: What kind of measurements will you need to do? This is actually number three …

Matt: Measuring of like angles. So, like a way to measure the angles is like an obtuse, right, acute.

Adi: Straight. Reflex.

This excerpt indicates that both Matt and Adi conceptualized angles as geometric figures, and in particular those defined by two straight lines meeting at a common point. Students' conceptualization of an angle as a geometric figure was also revealed when they were asked to describe the sketches of their imaginary holes or the hole that had been modeled in the classroom.

Researcher: Would you take about 20 or probably 30 seconds and just chat with the person nearby about what you have drawn and what you have listed?

Adi: Right here, I just drew a hole right here and I said it has one right angle, one acute angle, one obtuse angle.

Matt: Mine has eight right angles. It has a code in front of the hole, cup, the cup.

The excerpt above indicates that both Adi and Matt easily identified right, acute, and obtuse angles on their sketches. In addition, similar to Matt, Adi described his sketch as having eight right angles, as indicated on his worksheet one in Figure 4.9.

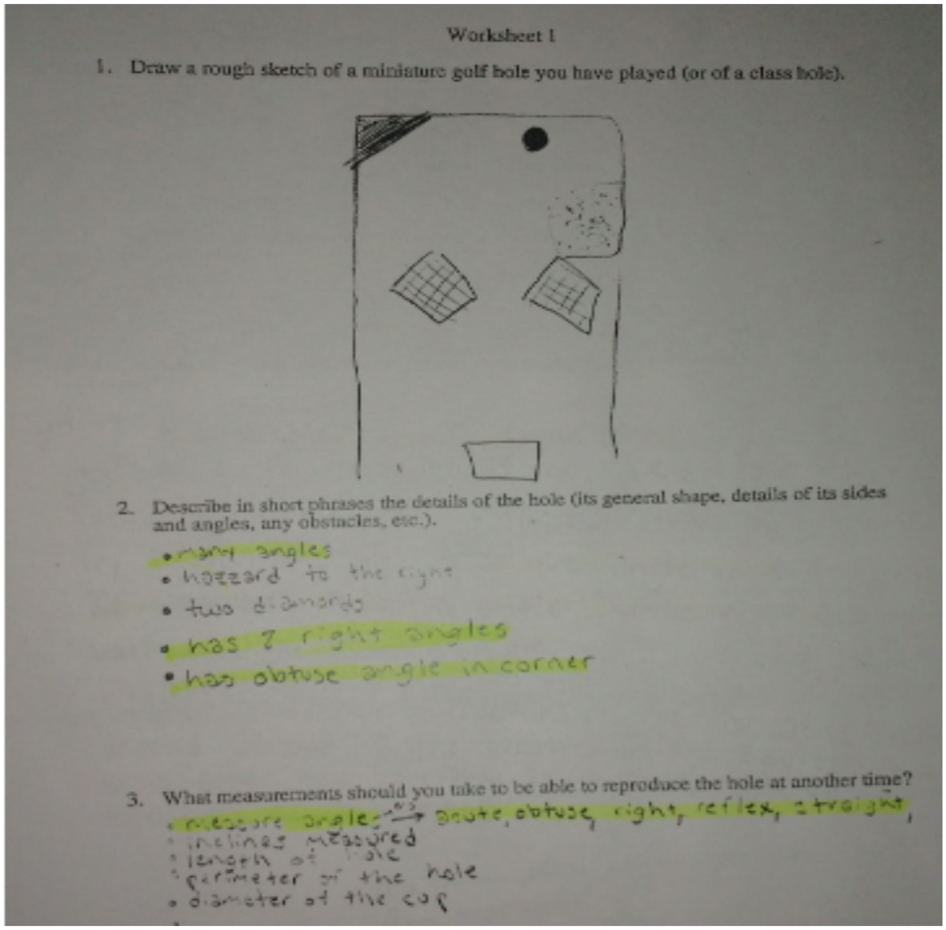


Figure 4.9: Adi's Worksheet 1 responses.

Figure 4.9 show that Adi's responses emphasized angles as geometrical figures defined by two lines meeting at a common point.

Identifying and naming angles on a sketch of a miniature golf hole. On Day 3, participants were given Worksheet 2 (see Figure 4.10), which had a sketch of a miniature golf hole with some recorded measurements of a hypothetical student named Ann. In pairs, participants were asked to discuss and identify measurements that had not been recorded.

Sarah: What other measurements might Ann have recorded? The measurement across here, except I don't know what to call it.

Emma: Wait. Did she do the, she did the 90 degrees here and here and here and here. She didn't get this measurement.

Sarah: The two corner angles. So, what will we write?

Emma: The corner at the bottom right. And the one right there.

Sarah: It should say the corner bottom right needs an angle measurement.

Emma: And then the angle right there?

Sarah: The middle corner. Um, could we just call this like the middle corner part?

Researcher: You could call it the angle …

Sarah: Obtuse angle.

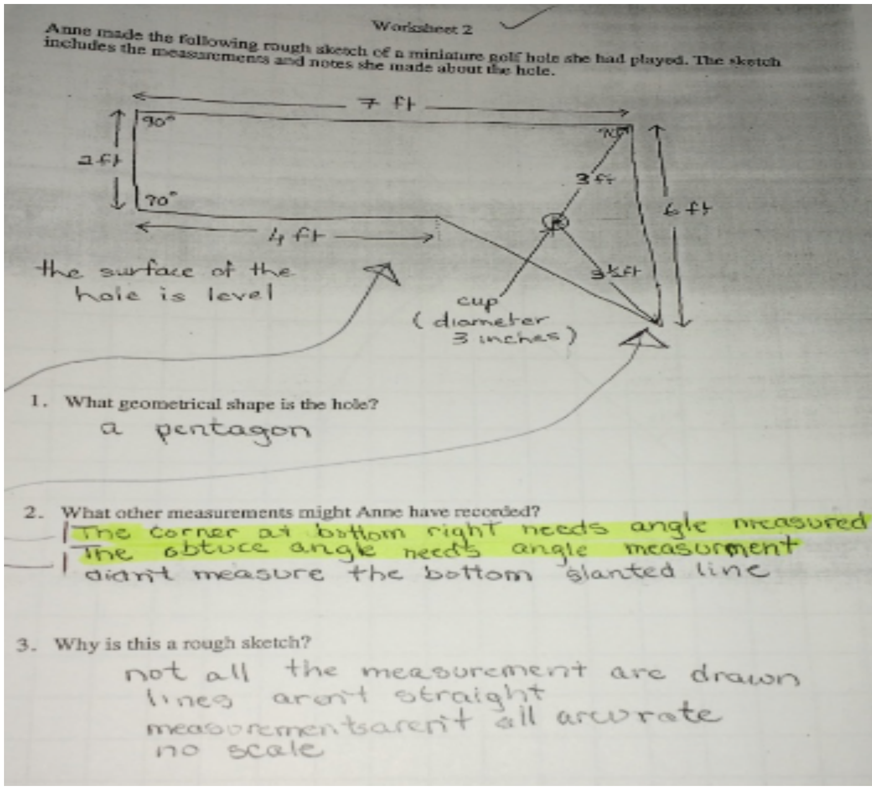


Figure 4.10: Emma's Worksheet 2. A replica of Sarah's Worksheet 2.

As the above excerpt show, Sarah and Emma identified angles whose measurements needed to be recorded as "the two corner angles," the angle at "the corner of the bottom right," and the angle at "the middle corner," instead of their names. It was after the researcher intervened that Sarah referred to the angle at the middle bottom part of the drawing in Figure 4.10 as the obtuse angle. In this case, Sarah referred to the salient angle and not the reflex angle. Based on number two responses in Figure 4.10, it might be both Emma and Sarah may not have conceptualized of the reflex angles at the time. This is also evident with both Matt and Adi in the next excerpt.

Matt: What measurements might Ann have recorded?

Adi: What? Um … oh, the angle of …

Matt: Um, the angle of … just write the angle of corner and draw arrow.

Adi: The angle of the corner to the corner. That is kind of …

Matt: Then draw an arrow to it.

Adi: Okay.

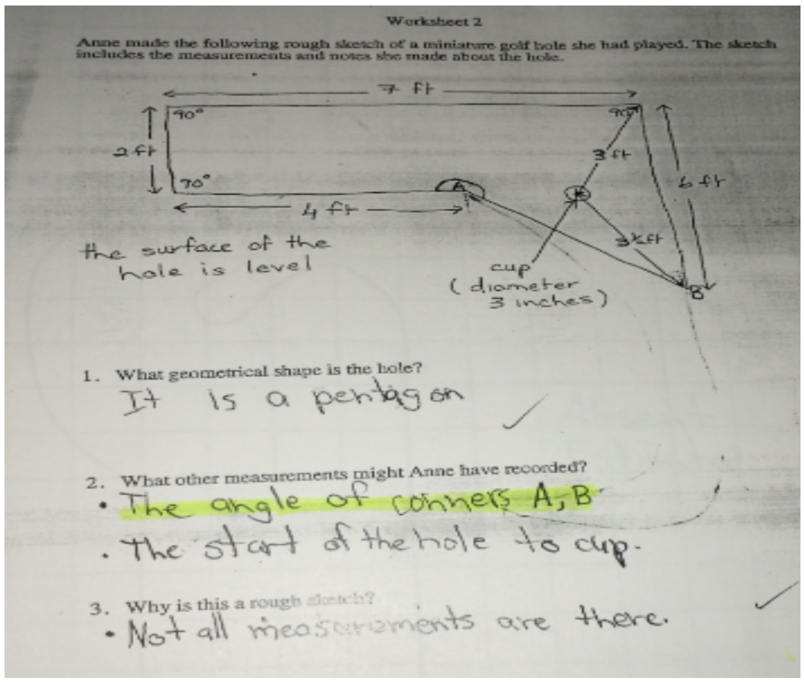


Figure 4.11: Matt's Worksheet 2. A replica of Adi's Worksheet 2.

The excerpt above also shows that both Matt and Adi identified angles whose measurements needed to be recorded as "the angle of corner to the corner." As Matt's worksheet 2 in Figure 4.11 show, Matt recorded "the angle of corners A and B, as those that needed measurement. Similar to Sarah and Emma, it might be at the time, Matt and Adi did not have the names for angles they labeled as A and B. These findings may suggest that asking participants to describe a sketch of a miniature golf context provided them with an opportunity to conceptualize reflex angles, which may not have been provided in the sixth-grade curriculum.

Angle Measure

Measurements needed to reproduce a miniature golf hole. On Day 1, participants were asked to discuss the measurements they would need to consider once they visit an actual miniature golf hole in order to reproduce a miniature golf hole another time. The following excerpt show the dialogue that ensued between Adi and Matt.

Researcher:	What kind of measurements will you need to do. This is actually number 3 on worksheet 1(see Figure 4.9). You can write some things on your paper and talk it with your partner.
Adi:	Measuring of like angles. So, like a way to measure the angles like an obtuse, right, and acute.
Matt:	Straight. Reflex.
Adi:	Well, right, but you wouldn't measure a reflex.
Matt:	Yeah, you would because like a hole goes like that has, see you would …
Adi:	Oh, right. So, the water on the left. Yeah. That is a reflex angle.
Matt:	Angle measurements.
Adi:	An arrow is going to be like acute.
Matt:	Incline.
Adi:	Obtuse. Right. Wait.
Matt:	Incline.
Adi:	Wait. Reflex, what is the other one, straight.

Matt: We aren't going to measure straight angles. There is like a million straight angles there. Yeah.

It is evident from this excerpt that both Adi and Matt conceptualized angle measure as measuring geometrical figures such as acute, obtuse, right, reflex, and straight angles. Adi related an arrow to an acute angle. Matt thought that they will not measure straight angles because there were many of them. This may suggest that Matt conceptualized measuring of straight angles differently compared to measuring of acute, right, obtuse, and reflex angles.

Things measured on an actual miniature golf hole. On Day 2, after participants visited the actual miniature golf hole, they discussed the things that they measured that were difficult and were easy to measure. The following excerpt that features Sarah's whole-class discussion shed some light on angles that students conceptualized to be easy to measure. I have used pseudonyms for all students.

Researcher: What were some things that were easy? Sammy?

Sammy: Just measuring how long it was.

Researcher: Okay, the length of the hole. Right? Kevo?

Kevo: The width of the hole.

Researcher: All right. Sarah?

Sarah: Like measuring the angles.

Researcher: Which ones were easy? Somebody thought some of them were hard. Which were easy?

Sarah: Well, the ones that went like this that you could see part, inside part.

Researcher: So, were they like on the edge?

Sarah: Yes.

Researcher: With two straight parts?

Sarah: Yes.

In this excerpt, besides the length and width of the hole being easier to measure, Sarah is captured discussing about the angles that were easy to measure. It is evident that Sarah found it easier to measure angles that were on an edge. This suggest the obvious that most students find it easier to measure angles that are formed by two visible straight rays meeting at a point, than angles, for instance, on a curve. In another instance, during Adi and Matt's whole-class discussion, a student is captured saying that reflex angles were easier to measure.

> Researcher: What were some things that were easy? Essie?
>
> Essie: On our first one, it was practically all right angles or reflex angles. So, that was really easy.

Based on this excerpt, I conjecture that it is easier to measure right angles, but not reflex angles. As had been noted previously, participants conceptualized reflex angles as "an angle at the corner" or "an angle at the middle" and not by the name reflex. I suggest that Essie could be confusing reflex angles to obtuse angles. In another question, when students were asked what measure they would consider taking if given another chance to visit an actual miniature golf hole, they noted inclines and curved sides.

> Researcher: What is one thing that you would measure that you didn't measure today, knowing what you know now?

Sarah: How like the curved sides …

Emma: Yeah, I didn't know what we were supposed to do.

Sarah: I didn't either.

Adi: Inclines.

Researcher: Inclines, right.

This excerpt suggests the likelihood of the instructional unit supporting students toward conceptualizing measuring slope and turn contexts.

Measuring of acute and reflex angles. On Day 4, participants were asked to discuss their measurements for the acute and reflex angle that needed to be recorded on Ann's hole. The findings revealed that participants had the correct measures for the acute angles (angles less than 90 degrees). However, measures for reflex angles revealed that some participants conceptualized reflex angles as obtuse angles and vice versa.

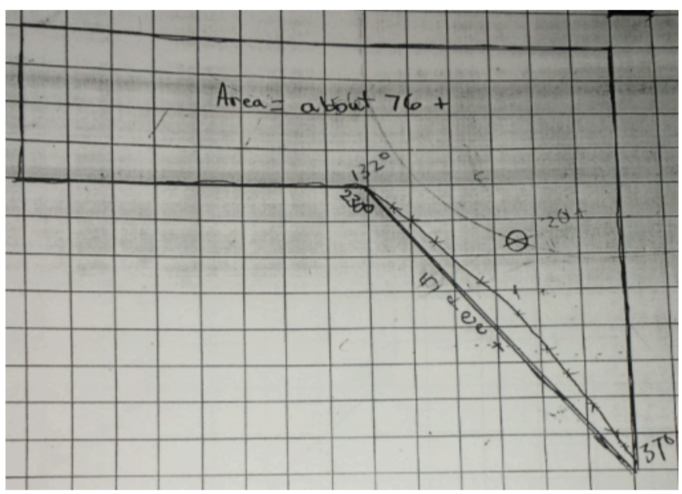


Figure 4.12: Sarah's diagram indicating angle measures.

It is evident from Sarah's drawing that she conceptualized the reflex angle (an angle greater than 180 degrees and less than 360 degrees) as the obtuse (angle greater than 90 degrees and less than 180 degrees) and the obtuse angle as the reflex angle. The following excerpt shows Sarah trying to convince Emma of the angle recorded as 230 degrees.

Sarah: It is 230 degrees, Emma.

Emma: What is?

Sarah: This.

Emma: No.

Sarah: Yes, it is! Yes, it is upside down. She does it the other way.

This excerpt shows how Sarah convinced Emma that the obtuse angle is the one measuring 230 degrees. As it can be seen in Figure 4.13, Emma got convinced and erased the angle she had recorded that appeared to be 137 degrees for the obtuse angle and wrote 233 degrees instead.

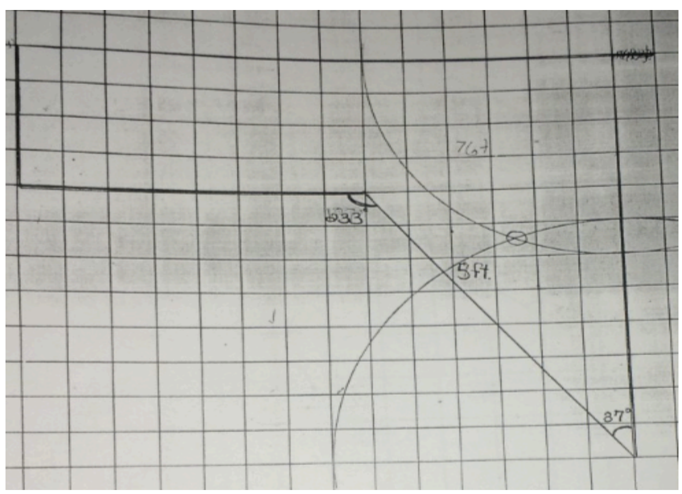


Figure 4.13: Emma's diagram indicating angle measures.

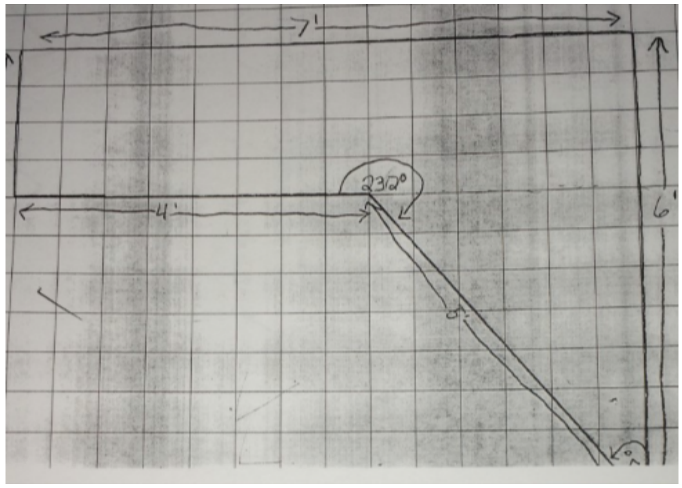


Figure 4.14: Matt's diagram indicating angle measures.

Figure 4.14 show the correct position and the measure of the reflex angle on Matt's drawing. This shows that Matt had the correct conceptualization of the measure of a reflex angle as the angle greater than 180 degrees and less than 360 degrees. The following excerpt shows Matt dialoguing with Adi about where the reflex angle is on the drawing.

Matt: I have 135. That is like right that. How did you like … this is a reflex angle.

Adi: Which one is?

Matt: This. It is not 135.

Adi: Yeah, it is a reflex because the hole is right there.

Matt: No, it is down there.

Adi: Right. But like the starting point, the starting point was right there. You had to come down to the cup, so that it could be like a reflex angle or a regular.

This excerpt shows Matt trying to convince Adi of the position of the reflex angle on the drawing. As Figure 4.15 indicates, Adi conceptualized the reflex angle as an obtuse angle with measure 131 degrees.

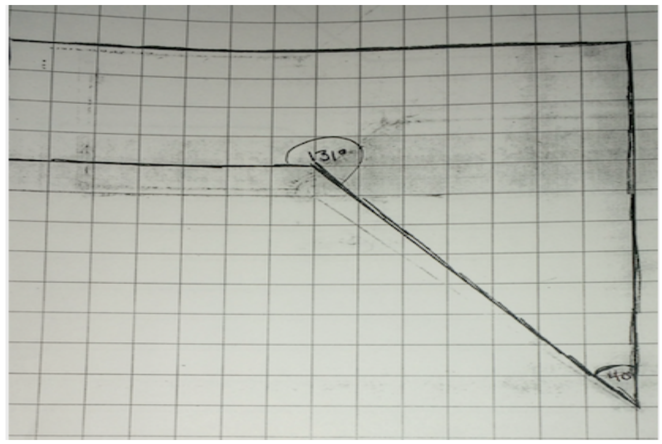


Figure 4:15: Adi's diagram indicating his angle measures.

To this end, these findings reveal that some sixth-grade students conceptualized a reflex angle as an obtuse angle and vice versa, for instance as can be seen in Figure 4.12, 4.13, and 4.15. The measure of the reflex angle was placed on the position of the obtuse angle and vice versa. Only Matt as Figure 4.14 indicates who conceptualized both the measure and the position of the reflex angle correctly. These findings suggest the need to support sixth-grade students' conceptualization of reflex angles which may not be evident in sixth-grade curriculum.

Comparing sketch and scale drawings of a miniature golf hole. On Day 5, participants were asked to compare their sketch and scale drawings of a hole. The findings revealed that all students conceptualized that angle measures are preserved during scale drawing, but length measures change based on the scale used.

Researcher: Which of the measurement that were made – the angle measurements, the side measurements, any measurements there – which are the exact same measurements that if you measured on your scale drawing, they would be the same, and which ones are different?

Matt: Did you get that?

Adi: I think that, well, the angles are the same. Like all the angles would be the same.

Matt: That I do know. But I think like if you, like … in our scale, from like this point A to point B, that would equal like the 50 in., so I mean, like … but like the angles are the same. That I do know.

This excerpt shows that both Matt and Adi were in agreement that angle measurements are preserved while side lengths are not. A similar conceptualization is also observed with both Sarah and Emma.

Researcher: Which things are exactly the same measure or degrees as they are on your scale drawing?

Emma: I don't get it.

Sarah: I don't get it, either. I don't either.

Emma: I don't know. What things do you think stayed the same? I don't get it.

Sarah: There, does that look about right?

Emma: I don't get what the question is.

Sarah: I don't either.

Emma: Um, I don't get the question.

Researcher: Okay, the question is, when you measured things at Pinescape, and you measured like a 90-degree angle, when you now draw it, are you still drawing a 90-degree angle?

Both: Yeah.

Researcher: Okay. When you measured something at Pinescape that was 50 inches long, are you drawing it 50 inches long?

Both: No.

Researcher: No. So, the angles stayed the same, but the lengths …

Both: Oh, okay.

Sarah: We got it. All right. So, the angles stayed the same. The lengths didn't.

Emma: The angles stayed the same; the measurements, the lengths don't.

This excerpt shows that both Sarah and Emma also agreed that indeed angle measures are preserved while side length measures are not for a scale drawing. However, it took them some time before they could understand the question that the researcher was posing to them. To this end, I claim that sixth-grade students conceptualize angle measure as being preserved for scale drawings.

The measure of angles formed by an incoming path of a ball and its rebound with a wall. On Day 10, students discussed Worksheet 10 to help them conceptualize about the measure of angles formed by an incoming path of a ball and its rebound with the wall. The findings revealed that initially two of the participants conceptualized the two angles to be different while two conceptualized the two angles to be the same.

Emma: When the ball hits the wall, it will bounce off the wall, because it always bounces off the wall. I mean that is a given. like it will bounce off the wall and go at another angle.

Sarah: Yes. So, when the ball hits the wall, it will … So, how do we know … Oh, oh. If you hit the ball kind of hard it makes a wider angle, like that.

This excerpt shows that both Emma and Sarah initially conceptualized the angle formed by the incoming path of a ball and the wall to be different from the angle formed by its rebound and the wall. Both Adi and Matt had a different conceptualization as shown in the next excerpt.

Matt: It bounces back at the exact same angle that is was hit to the board. So, if it comes …

Adi: The speed also affects.

Matt: The only thing that affects it, is if it has a spin … But if it has no spin it will bounce exactly at the same angle that it hit.

Adi: Right. The same angle, but it is speed.

This excerpt shows that both Matt and Adi conceptualized that when the ball hit the wall it will bounce back at the same angle it went in although with some conditions. Matt conceptualized that the measure of the angle of rebound will be affected if the ball spins, and Adi conceptualized that the speed will affect it as well. The following excerpt show a whole-class discussion that ensued in which Emma was involved.

Researcher: Do you think your prediction is true? That is number four (worksheet 10). And why or why not? Amy?

Amy: I don't think so.

Researcher: You don't think so?

Amy: No, because if you hit the ball straight and it came back the same way, if it was going at the opposite angle, then why would it come back the same way?

Researcher: Okay. Somebody who had the prediction about the opposite angle, what would you do. In fact, it did happen. We had a wall like this, and we hit the ball and it went there and came back on the same path. Is that the opposite angle?

Emma: Yeah. If it was the same angle, it would keep going straight.

Researcher: So, maybe we need to figure out what angles we are talking about.

Looking into Emma's response in this excerpt, this might explain why the researcher was prompted to delve deeper into participants' conceptualization of which angles they were referring to. Emma conceptualized that if the angle formed by the incoming path of the ball and the wall was the same with the angle formed by the rebound path of the ball and the wall, then the ball should continue going in a straight line. In other words, Emma conceptualized a rebound to indicate different angle measures. The next excerpt shows a discussion that ensued between Sarah and Emma and how Sarah managed to convince Emma that the two angles have the same measure.

Researcher: Okay. Now let's talk about a little bit what this means by the angles, the prediction is the ball rebounds at the opposite angle from which it was hit … What are the angles we are talking about?

Sarah: I think it is the two outside ones. It says it hits at the angle it bounces back at.

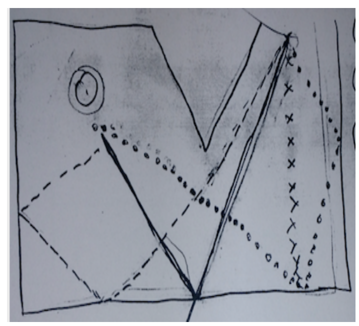


Figure 4.16: Sarah's diagram showing the incoming path of the ball and its rebound.

Emma: I always thought it was the middle angle.

Sarah: Well, that is what I put, because if you hit the ball to the wall and the path would be at a certain between the ball path and the wall. Then, if the ball hits back, it would be the same amount between the ball path and the wall on the other side.

Emma: Yeah, I guess.

Sarah: Because the real one is only one angle. We are talking about two angles.

Emma: It is not going to be the same on each side, though.

Sarah: It could.

Emma: No, because we have like a wide angle, obtuse, it will be different on the sides.

Sarah: What do you mean?

Emma: Like, um, if you went like that. Okay, here is, you hit it way over here. It is going to bounce off to that wall, so that is going to be the same angle right here.

Sarah: Or it could go like that.

Emma: It couldn't because it would have to, so I think the prediction is right.

Sarah: See, if the ball hit, like this one. Like if the ball hit there like that, then it has to bounce off the same way. So, this angle must be the same.

Emma: What?

Sarah: This angle and that angle are the same, and this angle and that angle. And these are both right angles.

Emma: Yeah, I guess you are right because you see, these two looks like the same angle.

Sarah: Right.

Emma: I didn't measure it very well.

Sarah: You can't do it like that. See, that is the …

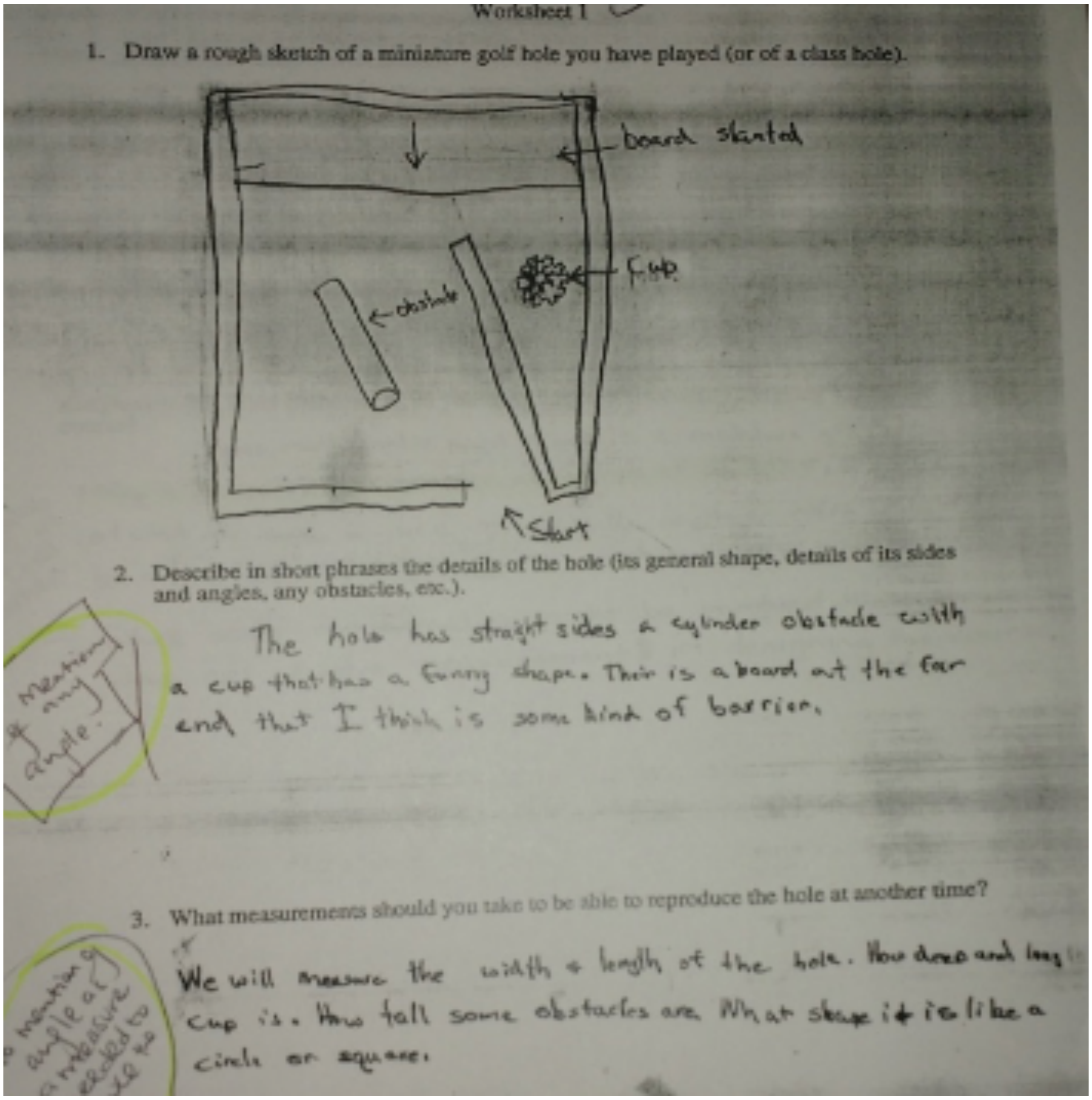
Emma: Oh, 30 degrees and then you, the other one is 30 degrees, I guess. It is about the same.

This long excerpt provides a clear evidence of how Sarah was able to convince Emma that the angle formed by the incoming path of a ball and the wall, and the angle formed by its rebound and the wall have equal measure. Initially, Emma thought that the rebound angle was the middle angle, that is the angle formed by the incoming path of a ball and its rebound. After Sarah discussed with Emma of the angles they were referring to, Emma conceptualized the angle formed by the incoming path of a ball and the wall, and the angle formed by its rebound and the wall as having equal measure.

In this same excerpt, Sarah is noted talking about there being only one real angle. I conjecture that Sarah was probably referring to the middle angle. In this case, the middle angle being the one formed by two visible lines that represents the path of the ball. This suggest that Sarah conceptualized an angle as being real if it is formed by two lines meeting at a vertex.

The Students' Actual Conceptualizations of Angle and Angle Measure Through a Miniature Golf Context Instructional Unit

On Day 1, I had conjectured that when students will be asked to say the things that they will be measuring when they visit an actual miniature golf hole, some will mainly focus on lengths, widths, heights, perimeter, and area, where angles might be the last measurement for students to think about. This conjecture held true. For instance, Figure 4:17 show Sarah's responses for worksheet one questions.


Worksheet 1

1. Draw a rough sketch of a miniature golf hole you have played (or of a class hole).

2. Describe in short phrases the details of the hole (its general shape, details of its sides and angles, any obstacles, etc.).

The hole has straight sides a cylinder obstacle with a cup that has a funny shape. Their is a board at the far end that I think is some kind of barrier.

Mention any angle.

3. What measurements should you take to be able to reproduce the hole at another time?

We will measure the width + length of the hole. How deep and long cup is. How tall some obstacles are. What shape it is like a circle or square.

Figure 4.17: Sarah's Worksheet 1 responses.

As question three in Figure 4.17 shows, when asked about the measurements that were needed to reproduce the hole another time, Sarah wrote about the width, length, depth, and shape of the hole, without mentioning of angles. Sarah's responses for question two on description of her sketch also indicates that at that moment she had not conceptualized about angles in relation to this figure. Sarah described her hole sketch as having “straight sides, a cylindrical obstacle, and a cup with funny shape,” although her sketch as Figure 4.17 shows had several angles such as straight angles, nearly 90 degree angles, obtuse angles, and 360 degree angles for the circular shapes.

On Day 2, when students visited the actual miniature golf hole, I conjectured that some students will talk of curves, reflex angles being hard to measure, as well as being hard to use a protractor. This conjecture held true. For instance, during Sarah and Emma’s whole-class discussion, the following dialogue ensued.

R: Now, I know you guys never had difficulties when you were playing, but what about when you were measuring? Sassy?

Sassy: Using a protractor to measure angles.

R: Okay, using the protractor to measure the angles just on a regular one or were there some areas for instance that you tried to measure that were more difficult?

Sassy: That were more difficult. It went down like this.

R: A curved side. Okay.

This excerpt provides evidence that some students claimed to struggle to measure curved sides using a protractor.

On Day 3 - 5, when students were asked to compare their sketch and scale drawings of a miniature golf hole, I conjectured that some students might have difficulties understanding that angle measure is preserved for a scale drawing. This conjecture was nullified. All participants conceptualized that the angle measures are preserved while the side measures are not for a scale drawing. For instance, Adi said, “I think that, well, the angles are the same. Like all the angles would be the same," and Matt responded, "That I do know. But I think that like if you, like … in our scale, from like this point A to point B, that would equal like the 50 in., so I mean, like … But like the angles are the same. That I do know." This is a clear evidence that both Matt and Adi conceptualized that angle measures are preserved for a scale drawing, while length measures change. In addition, when students were asked to identify and measure angles on a hypothesized hole, I had conjectured that students will find it easy to identify familiar angles such as acute, obtuse, and right angles. This conjecture held true, where it was challenging for students to identify reflex angles and their measures in the moment. However, the instructional unit provided opportunities for students to conceptualize reflex angles, their measures and their relationship with other familiar angles, such as acute, obtuse, and straight angles.

On Day 10-12, when students engaged in exploring the measure of the angles formed by an incoming path of a ball and a wall, and its rebound and the wall, I conjectured that some students will conceptualize that the ball will move in a straight line or in a right angle. This conjectured proved true. For instance, Emma said, "if it was the same angle, it would keep going straight." In another instance, Sarah said, "oh, look at that one. That is almost a right angle." These instances provide evidence of participants' initial conceptualization of the measure of angle formed by the path of rebound of a ball and the wall or the incoming path of the ball. However, as previously noted, students finally conceptualized that the measure of the angle formed by an incoming path of a ball and the wall will be equal to the angle formed by the rebound path of a ball and the wall.

Students' Journal 4 assignment and interview based on the assignment provided similar findings. The following excerpt shows how Emma conceptualized about angles created by incoming and rebounding path on hitting a curved side during the mid-way interview process whose questions were based on participant's responses on Journal 4 assignment.

Researcher: Let's make this extreme curved line and put in a path. (*drawing a curved hole and an incoming path approaching a very tightly curved section of the wall*) So, this is how it is coming in. Draw a dotted line about, which you think might be the way it goes out.

Emma: I think it would go out and hit that wall and then come out again that way. (*draws a dotted path for the predicted first rebound - indicates that there will be, yet another rebound as a result*)

Researcher: Okay. Now, when you drew that in, were you thinking about angles at all?

Emma: Not really. I was just kind of thinking, well, kind of. I don't know. This is kind of came into my head. That is the way the ball goes.

This excerpt confirms that Emma was sort of thinking how the ball will rebound on hitting a curved wall, without relating that to angle ideas. Towards this end, the instructional unit set up in a miniature golf context either affirmed at the moment or supported students' conceptualization of angle and its measure. Table 4.5 provides a summary of characterization of students' actual conceptualization of angle and its measure in a miniature golf context with respect to the initial conjectures I had developed (see Table 3.4, page 47)

Table 4.5

A Summary of Characterization of Students' Actual Conceptualization of Angle and its Measure

Instructional Activity & its Description	**Main Goals**	**Conjectures of students' conceptualization**	**Students Actual Conceptualization**
Day 1: Thinking about designing a miniature golf course. • Who has played miniature golf? • Worksheet 1 to be distributed with questions about what is involved in designing a miniature golf hole.	• Students to work in pairs in sharing ideas on what is involved in designing a miniature golf course. • Students to brainstorm on their expectations on the field trip the next day.	• Some students will mainly focus on lengths, widths, heights, perimeter, and area. • Angles and slope, might be mentioned as the last measurements required (Smith, 2017).	• Angle measures will be required in designing a miniature golf hole.
Day 2: Field Trip & Discussion. • Students to work in pairs to observe, measure, and discuss what is easy or difficult to measure on an actual assigned miniature golf hole. • Whole-class discussion	• Students to engage in making observations and measurements on an actual miniature golf course.	• Some students will talk of curves, reflex angles being hard to measure, and difficult to use a protractor to measure angles (Keiser, 2004)	• Angles on an edge are easier to measure and curves are challenging to measure. • Angles can be used to tell the direction the curve is taking.

Day 3-5: Scale Drawing. • Students to discuss rough sketches of holes • The teacher to lead students in discussing what is involved in scale drawing and the differences from a sketch drawing using Worksheet 2 & 3.	• Students to be able to differentiate between a rough sketch and a scale drawing. • Students to engage in making scale drawings of holes. • To identify angles and their measures on a hypothesized hole.	• Some students might have an understanding that angle measure is preserved for a scale drawing (Masingila & de Silva, 1997) • Students to identify familiar angles such as acute, obtuse, and right angles (Keiser, 2003).	• Angle measures are preserved for a scale drawing. • . • Reflex angles are greater than 180 degrees but less than 360 degrees
Day 10 - 12: Path of Rebound. • Worksheet 10 to be distributed with questions on angles created when a ball hits a wall and rebounds. • Students to take turns to play shots on the class holes and observe the path of the ball after it rebounds off of a wall.	• Students to engage in exploring the path of the ball when it hits a wall and rebounds.	• Some students will think the ball moves in a straight line or in a right angle.	• Angles created by an incoming path of a ball and a wall and its rebound and a wall have equal measures.

Summary of What Transpired During the Instructional Unit

The findings have revealed that at the beginning of the instructional unit, consistent with the pre-interview findings, students conceptualized angles in terms of geometric figures. For instance, when students were asked of the things to measure on a miniature golf hole, they said, angles that are like obtuse, acute, reflex, and straight. When students were given rough sketch drawings of miniature golf holes and were asked to identify measures that needed to be recorded, they referred to “angles at the corner,” “angle at the middle,” instead of naming the angles as reflex angles. The findings have also revealed that some sixth-grade students conceptualized reflex angles as obtuse angles, and vice versa. However, with support students were able to learn of how to correctly measure and identify the correct position of reflex angles on a sketch of a miniature golf. This observation was particularly common where obtuse and acute angles were connected to the reflex angles making a complete turn. This finding suggests the importance of introducing reflex angles as angles are being introduced and defined in a circle (Tanguay & Venant, 2016).

Besides conceptualizing angles in terms of geometric figures, the findings have also revealed that students conceptualized angle measure in terms of measuring geometric figures such as reflex, obtuse, acute, and straight angles. As a consequence, participants claimed that measuring angles on an edge being easier. That is, angles defined by two visible rays meeting at a common point. In addition, students suggested that when given another chance for a miniature golf fieldtrip, they would consider measuring inclines and curved sides. This suggest that the miniature golf context provided students an opportunity to conceptualize how to measure inclines and curved sides.

When students were asked what measurements change or are preserved for a scale drawing of a miniature golf hole, consistent with previous findings (Masingila & de Silva, 1997), they all noted that angle measurements are preserved while lengths are not. The findings have also showed that with support, students learned that the measure of angles created by the incoming path of a ball and the wall and the rebounding path of the ball and the wall are equal. I next present post-interview findings.

Category 3: Participants' Conceptualization of the Angle Concept and Angle Measure *After* the Unit of Instruction

During post-interviews, participants were asked to define an angle and its interior, to draw angles, to identify angles in solids, to describe turns and directions, and to complete sentences involving side versus angle measure and angles versus turns. My analysis revealed that the post-interview findings also fell under the three major themes identified with the pre-interviews: the meaning of an angle, angle measure/size, and turn. I next present the findings under the aforementioned three major themes.

The Meaning of an Angle

By the end of the unit of instruction, participants' conceptualization of an angle as a figure formed by two straight lines/rays meeting at a point persisted. This common definition of an angle was revealed when students were asked to: (a) define an angle, its interior, and draw an angle, and (b) identify angles on solids. The post-interviews analysis also revealed how participants conceptualized the meaning of an angle in relation to angles such as 0°, 180°, and 360°. I next discuss each of these instances in detail.

Definition of an angle, its interior and drawings of an angle. When participants were asked to say what they thought is an angle, two of them said it is when two lines meet at a point, and the other two said it is when two rays meet at a vertex. A follow-up question on what participants meant with rays and vertex, revealed a vertex as a point where the rays meet. For instance, Matt said, “this is the vertex where the rays meet … may keep going on and on but this is where they hit.” Matt response indicates he conceptualized an angle as defined by two rays meeting at a common vertex.

When participants were asked to show what constitute the inside of an angle, they all marked the space between the rays and the vertex as illustrated in Figure 4.18.

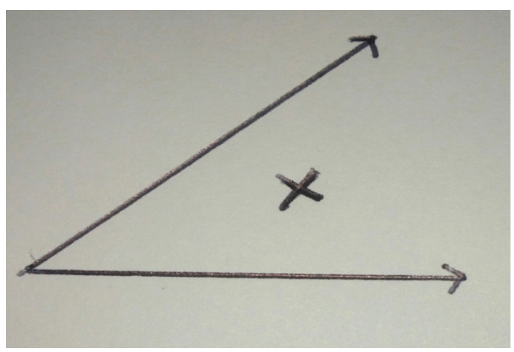


Figure 4.18: An inside of an angle marked with a letter X.

In addition, participants conceptualized that the rays of an angle implied that the angle could be extended infinitely. The next excerpt provide evidence from Adi.

Researcher: If I was to ask you to speak about the inside of that angle, what would you say would be the inside of the angle?

Adi: In between these two lines. From here and inside there. And anywhere from here. (*Shows the space in between the two rays of an angle, as Figure 4.18 illustrates*)

Researcher: So, it could go on?

Adi: Yeah.

The next excerpt also provide evidence from Matt on his conceptualization of an inside of an angle.

Researcher: Suppose we were to look at what you called was the 60 degrees. What would you say constitutes the inside of that angle?

Matt: Like inside the 60 degree angle would be like in there. (*shows the space between the two rays of an angle, as Figure 4.18 illustrates*)

Researcher: Now, is the inside of the 60 degree, does it stop here where the lines end?

Matt: No. It would keep going for like ever.

These excerpts show that both Adi and Matt conceptualized the inside of an angle as the space between the two rays of an angle. This conceptualization is significant in understanding how students define an angle.

In another question, when participants were asked to draw an angle, their drawings confirmed participants' conceptualization of an angle as a figure formed by two rays meeting at a common point as illustrated in Table 4.6.

Table 4.6

Participants' Drawings of an Angle

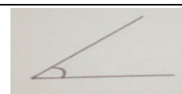
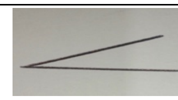
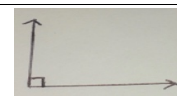
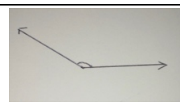
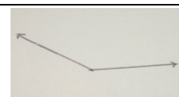
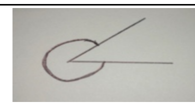
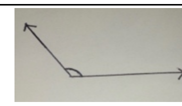
Question	Sarah	Emma	Adi	Matt
Can you draw a picture of an angle?				
Can you draw any different angle?				

Table 4.6 show that participants initial drawings were acute angles, such as Emma's and Adi's, or right angles, such as Sarah's and Matt's; while their second drawings were obtuse angles with Adi drawing a reflex angle. Consistent with pre-interviews, participants' drawings of an angle emphasized conceptualization of an angle as a figure formed by two rays meeting at a common point. However, during post-interview it is evident that participants had developed a deeper conceptualization of an angle, that constituted reflex angles. Participants' broadening their conceptualization of angles to include reflex angles is also evident in the next excerpt.

Researcher: Now, if I was to draw that picture of yours again (*redrawing 1st angle, see Table 4.6 under Adi*), sort of more or less like that, how many angles do you see in that picture? If there are two rays coming out this way, how many angles do you see?

Adi: Two.

Researcher: Can you show me the two?

Adi: There and right there. (*Identifies acute and reflex angles*).

Researcher: Okay, any relationship between these two angles?

Adi: Yeah. Like let's just pretend this is like the 60 degree angle, like … this is like 60 degree angle, and then you subtract 60 from 360, and this reflex angle would be a 300 degree angle.

Researcher: Okay, great.

This excerpt show that Adi had broaden his conceptualization of an angle to go beyond a single angle defined by two rays meeting at a common vertex to conceptualize angles in a circle. Adi identified both reflex and acute angle in his first drawing as illustrated in Table 4.6. Adi conceptualized a reflex angle in terms of its measure. In particular, Adi gave an example of how one can obtain a reflex angle by subtracting 60 from 360. This is an indication that Adi had developed his conceptualization of an angle. Emma also had a similar conceptualization of a reflex angle as the following excerpt shows.

Researcher: So actually, you told me there are two angles there.

Emma: Right.

Researcher: So, are there two angles here? (*indicating the 2nd angle she drew, see Table 4.6 under Emma*)

Emma: Yeah. One there. (*draws another arc on the first picture to indicate the reflex angle*)

Researcher: How would you compare the two angles here?

Emma: Um …

Researcher: Is one bigger than the other, or smaller than the other? Are they the same?

Emma: Actually, they are pretty …, I think this one (reflex) is bigger than this angle (obtuse) over here because it is measuring the rest of the circle besides this point. And it is not a full straight line. It is got a point too.

The above excerpt show that Emma's conceptualization of angles had broaden to constitute angles in a circle rather than conceptualizing angles as only defined by rays meeting at a point. Emma was able to identify both obtuse and reflex angles sharing two rays meeting at a point. Emma's assertion that "it is not a full straight line; it has got a point too" indicates Emma's conceptualization of an angle, whether obtuse or reflex, as formed by lines meeting at a point. Towards this end, both Adi and Emma identification of a reflex angle in their drawings showed broadening of their angle conceptualization, which I can associate with the support of the instructional unit as reflex angles were not prominent during pre-interviews or in the literature review.

Identifying angles on solids. Participants were given solids as shown in Figure 4.19 and asked to say the angles they could see.

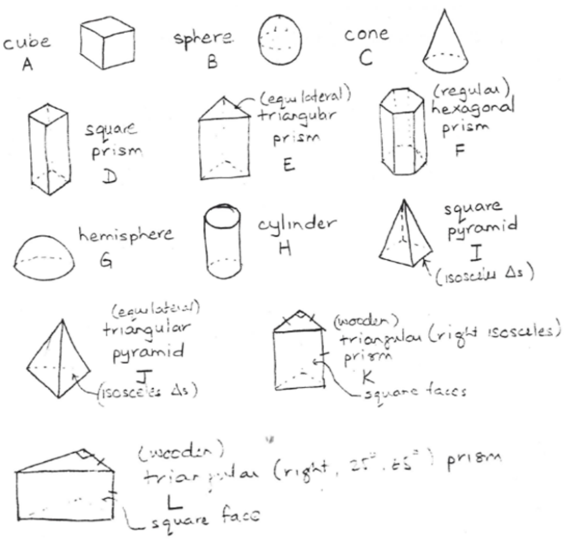


Figure 4.19: Three dimensional solids.

The following excerpt from separate interviews show what angles both Emma and Sarah identified.

Researcher: Now, I would like you to look at these solids, well, we call them solids and for each one, tell me what angles you see.

Emma: All of these angles are 90 degrees. (*points on A*) … These angles are 90 degrees (points on D) … This one (F) is the angle around like this on the sides … More than 90.

Sarah: A 90 degree angle (*on square base of I*) … This (J), I see an acute angle … This (E), I see an acute angle (*on triangular face*) and … a right angle (*on rectangular face*). This (D), I see, I'm not sure. I see right angles here … This one (F) has an obtuse. (*traces angles on hexagonal face*) … This (K) has a right angle, acute angle, straight angle. (*traces a line down the middle of the rectangular face*)

This excerpt show that Emma and Sarah identified right angles (90 degree angles), acute angles (angles less than 90 degrees), obtuse angles (angles greater than 90 degrees but less than 180 degrees) in solids A, D, E, F, I and K. One characteristic of the angles identified is that they have edges that meet at a common point. This suggest participants' conceptualization of an angle as a figure that is defined by two lines meeting at a common point. Among the solids that some participants did not identify angles were a sphere (B), a hemisphere (G), a cone (C), and a cylinder (H). For instance, Emma said solid B has "no angles, it is a circle," and Matt said, solid B has "no angles, except maybe the middle of the inside (*the circle which joins the two hemispheres*) would be 360 degree angle." This indicates that Matt conceptualized a circle as having a 360 degree angle.

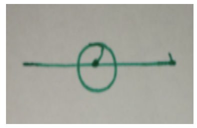
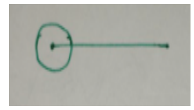
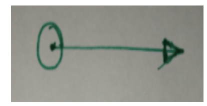
Both Emma and Sarah conceptualized solid C (a cone) as having no angles. Emma said, solid C "isn't like it is an angle, except it is going all the way around. So, it is not really straight (*indicating that there is no straight edge*). It is like that (*indicating that you could imagine a lineup the lateral side*). That would be the point of the angle." This response suggests Emma conceptualized angles in the context of straight lines. A similar conceptualization was observed with Sarah when she was asked why solids H, C, G have no angles. Looking at solid C (a cone), Sarah said, "because there is no sharp, not sharp, but no, they are all smooth. They are not all smooth (*appears to see the plane surfaces as not smooth)*, but, like I couldn't see any." Here, Sarah might have been referring to a straight edge as a sharp.

When participants were asked what was unique with the solids they had identified as angles from those they had not, Sarah said, "these have lines that are straight (*referring to the ones with angles*) and these have round (*referring to the ones with no angles*). These findings emphasize students' conceptualization of an angle as a figure that is formed by straight lines meeting at a point.

Conceptualization of 0°, 180°, and 360°. When participants were asked what they termed as the largest angle, three out of four students conceived 360 degrees as the largest angle. One student being unsure said a straight angle (180 degrees). For the smallest angle, students could not conceive an angle of zero degree. One student said that if it does exist then it has to be a line formed when the angle between two lines meeting at a vertex is really close. Other students conceptualized the smallest angle as a fraction of a degree or one degree angle, emphasizing that the smallest angle would be between two lines which are very close. Table 4.7 shows the evidence of the four participants' conceptualization of the largest and the smallest angle.

Table 4.7

Participants' Conceptualization of the Largest and Smallest Angle

Question	Sarah	Emma	Adi	Matt
What do you think is the largest angle?	A straight angle (180), but not sure	360 degree angle (*draws a single line and a circular arc going all the way around from one end*) 	360 — it would be like a circle, but it ends and starts at the same point (*Draws a single line with a tiny circle at one end* [vertex]) 	360 degrees (a circle I think) — (*emphasizes a point and ray extending from it, then draws a circle centered at start of ray, with radius less than the length of ray depicted*) 
How about the smallest angle you could have?	Draws a line…suggesting there are 2 lines very close together, which meet at a point.	Zero really isn't an angle. If it does exist, it can be represented with a line. The two lines are too close for zero such that you can't see them as two but as a single line.	Would be drawn the same way as 360 degree angle, except instead of going all the way it would be between the two.	It is like two rays and they are on top of each other. One that is really, really close. Like really close. Probably like a fraction of a degree.

As indicated in Table 4.7, the diagrams illustrate that students conceptualized 360 degrees angle as the largest angle and emphasized 360 degrees angle as a circle characterized by a ray or a line and a point. This suggests participants emphasis of a line and a vertex as characteristics of an angle. In addition, participants conceptualized the smallest angle as being represented by a single line, probably a zero degree angle or a very small fraction of a degree.

In another question, participants were asked what things they did not consider as angles but now they do after learning through the unit of instruction. Two of the participants said that they now consider straight lines as straight angles which they did not before.

Researcher: Was there anything that you used to think was an angle that you no longer think is an angle?

Matt: Not really.

Researcher: Okay, how about anything that you really didn't think was an angle but now you think is an angle?

Matt: Like a straight angle. (*Draws a horizontal line with point emphasized in center and an arc above depicting the angle measure (*see Figure 4.20*).* Like here to here, I didn't use to think that was an angle. But now I do.

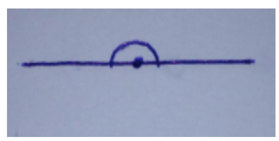


Figure 4.20: An illustration of Matt's drawing of a straight angle.

The above excerpt shows that after learning through the unit of instruction, Matt included in his angle repertoire, straight angles which initially he did not consider to be angles. Adi had a similar learning.

Researcher: Something that you didn't associate with an angle and you now do?

Adi: That is a straight line. It would be, this is an angle, it's vertex would be somewhere over there. The two rays. (*Draw a line with "vertex circle" at the center and indicates 2 rays emanating from there (*see Figure 4.21*).*

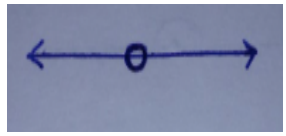


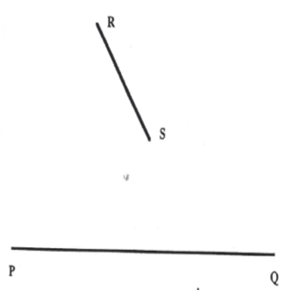
Figure 4.21: An illustration of Adi's drawing of a straight angle.

It is clear from both Matt's and Adi's drawings of a straight angle, the emphasis of a vertex at the center of extended rays or lines. This provides compelling evidence of students' conceptualization of an angle as a figure that is formed by lines meeting at a point.

Matt's and Adi's conceptualization of a straight lines as straight angles is further confirmed when participants were presented with a picture and asked to say something about the lines. While both Sarah and Emma just mentioned about lines being horizontally and diagonally placed, Matt and Adi identified the lines as straight angles. Table 4.8 shows how all participants conceptualized about the straight lines in the picture.

Table 4.8

Participants' Conceptualization of Given Straight Lines

Question	Sarah	Emma	Adi	Matt
What can you say about these two lines? 	One is placed flat and one is diagonal.	One is diagonal and one is horizontal.	Two 180 degree angles (straight angles)	If you continued this line (RS) … down to where it would meet this line and then it could be two angles, one obtuse and one acute,

	which adds up to 180 degrees.

Table 4.8 shows clearly that both Matt and Adi went beyond conceptualizing the given straight lines from the manner in which they were placed, to conceptualizing angles associated with the lines. This suggest that these participants' angle repertoire had expanded to conceptualize straight lines as straight angles something they had not before this unit of instruction.

Towards this end, even during post-interviews, participants' definitions, drawings, and identification of an angle, suggested their conceptualizations of an angle as two lines meeting at a common point. The instructional unit supported some participants to conceptualize straight angles as well as reflex angles. I next discuss the other themes and how they unfolded during post-interviews.

Angle Measure and Turn

In this section, I discuss both the angle measure and the turn as implicated by participants' responses. Conceptualization of angle measure became evident when participants were: (a) asked the relation of the angle measure/size to length of rays or measuring lengths, (b) using the protractor to determine direction and to measure angles, and (c) asked the relation between angles and turns. I discuss these instances in detail.

Conceptualizing angle measure in relation to the length of the rays or measuring side length. Participants were asked to say whether the angle measure changed when the legs of angle were extended. While during the pre-interviews some students were not sure whether extending the legs of an angle change the size of an angle, during the post-interviews, all participants conceptualized angle measure as preserved with the rays of an angle extended or changed. The following excerpt provide evidence.

Researcher: Suppose I was to take a picture of your angle and extend the lines. (*extending legs of an angle*) Has that angle changed in any way?

Sarah: No, it is still, like pretend this angle is like 150 degrees. It still would be 150 degrees like there.

Emma: The lines just got taller … it didn't change at all.

Adi: It wouldn't change the angle.

Matt: It is like just the same thing except the rays weren't represented as the longer lines … the angle is still the same.

This excerpt from separate post interviews show participants conceptualized angle measure as not affected by the change of length of the sides.

In another activity, students were asked to complete the sentence: *Measuring an angle of a shape is similar to measuring the side of a shape because … and it is different because* …. The following excerpt from separate interviews shows participants' responses to the question.

Sarah: Measuring an angle of a shape is similar to measuring the side of a shape because you are measuring both times … you are measuring the bent part of the angle and for the side of the shape you are measuring a side that could be any length or size.

Emma: Measuring an angle of a shape is similar to measuring the side of a shape because when you are measuring the angle of a shape you are also measuring a side of that shape.

Adi: Measuring an angle of a shape is different to measuring the side of a shape because if you are measuring the surface of a side of an object, you'll measure it like 3 cm^2 instead of 30°.

Matt: Measuring an angle of a shape is different to measuring the side of a shape because when measuring the side of a shape you are using measurements of length. However, when you are measuring the angle of a shape you are measuring a relationship between 2 sides, lines, etc.

As the above excerpt indicates, both Sarah and Emma conceptualized measuring an angle as similar to measuring the side of a shape since both involves measuring. Adi and Matt conceptualized the two as being different based on their different unit of measures. For the students who conceptualize angle measure as different from side measure, it is evident that the students conceptualized angle measure in terms of its unit of measure, the degree.

Conceptualization of angle measure through the use of body movements and the use of 90° as a reference point. Participants' conceptualization of the angle measure was also revealed when they were presented with a map of "Treasure Island" as shown in Figure 4.22. Participants were given directions from a specific point P, and were asked to look for a hidden treasure at point X. The findings revealed that participants used a protractor to determine direction and to measure angles and conceptualized the turn by the use of their body movements, as well as by the use of the 90° as a reference point. The evidence is presented in the following two excerpts: (1) when measuring an angle of 110° and (2) when measuring an angle of 25° as discussed next.

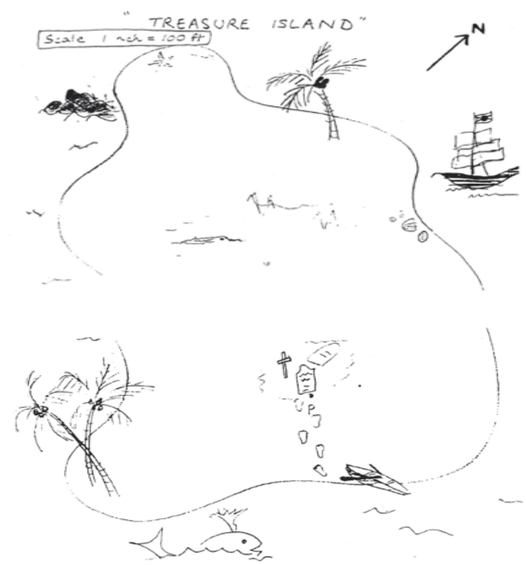


Figure 4.22: A map of Treasure Island.

Measuring an angle of 110 degrees. When students were asked to turn 110 degrees left, those who conceptualized the turn through their body movements placed their protractor and located the angle correctly. The following excerpt from separate interviews show how participants conceptualized the angle measure of 110 degrees.

Researcher: Turn 110 degrees left.

Emma: (*First turns her head imagining the turn. Then places protractor and locates the angle correctly*).

Adi: Would the angle like start right there?

Researcher: Here is how you imagine it. There is where you are facing. First imagine about how much you have to turn. Turn your body 110 degrees. About how much are you going to be turning?

Adi: 90 degrees would be like here. (*Turns his body and imagines pretty accurately*)

Researcher: So, how should you keep your protractor?

Adi: Turn it like this. (*Places protractor correctly and finally marks 110 degrees correctly*).

Matt: (*First places protractor so 90 degrees is ahead, then places it correctly and marks off 110 degrees correctly)*.

Sarah: (*Places protractor correctly but marks the incorrect 110*).

Researcher: Should you be reading the top set of numbers or the bottom set?

Sarah: The bottom set of numbers … the top I mean.

This excerpt shows how Emma and Adi conceptualized the angle measure of 110 degrees through turning their body. Adi and Matt used a 90° angle as a reference point in order to conceptualize the angle measure of 110º degrees. This finding suggests that participants conceptualization of the angle measure can be supported when they are encouraged to imagine the turn using their bodies or to use 90 degrees as a reference point. Similar findings are revealed with measuring an angle of 25 degrees.

Measuring an angle of 25 degrees. When students were asked to turn 25° right, three students out of four placed their protractor correctly, but marked the wrong 25° mark. They encountered difficulties with which scale they were to use on their protractor.

When the researcher intervened and asked students to think carefully about which direction they were facing and turning to, and to envision the actual turn of their body as they had previously done with 110 degrees, they were able to mark an angle of 25 degrees correctly.

Researcher: Again, you are coming that way and you are going to turn 25 degrees. [*pause*] (*Participant places protractor correctly but marks the wrong 25 degree mark*) Think carefully about this. I noticed the earlier one you actually turned your head and judged where 110 degrees would be. Why don't you do a similar thing here? Pretend you are here and turn and judge where 25 degrees would be. Turn right 25 degrees. Judge where that would be. Remember you are coming like this. Which way are you facing? Turn your head so you are facing that, okay, there you go. Okay, now turn your head 25 degrees. It is going to be all the way there, right? So, what should we read at that point? (*places the protractor correctly*) Is this the 25 degrees you are looking at? (*the one she first marked*)

Emma: No. No.

Researcher: If you turned all the way here, how much would you have turned?

Emma: Wait. At the top or bottom one?

Researcher: That is what I am asking. If you turn all the way here …

Emma: Is 25 on the top part?

Researcher: Would you have turned 25? Which of the two numbers should you read if you are turning from this direction?

Emma: The top one. No, the bottom one. (*marks the angle correctly*)

Researcher: The bottom one because the zero begins at that direction.

This excerpt shows that Emma was able to measure an angle of 25° using a protractor through making sense of how her body was turning. This provides further evidence that students' conceptualization of angle measure when using a protractor can be supported by encouraging them to visualize angle measure through their body turns. Figure 4.23 shows the inner and the outer scale of a protractor.

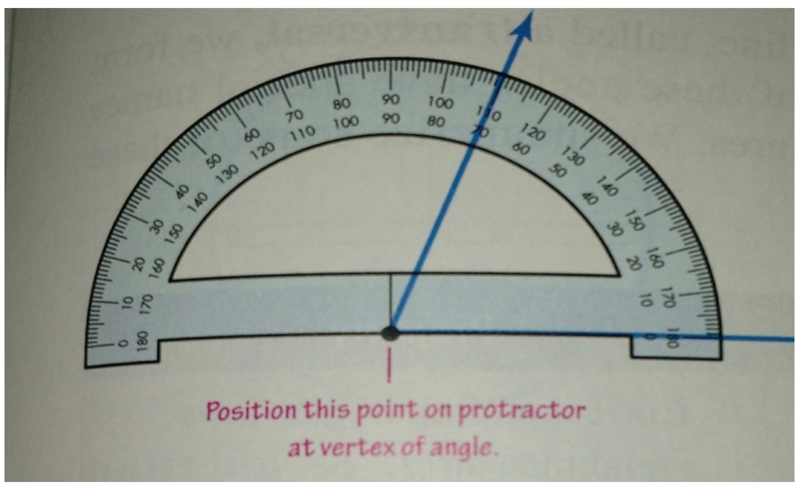


Figure 4.23: A picture of a protractor showing the inner and the outer scale (Pirnot, 2014, p. 439).

As Figure 4.23 shows, the angle marked with a blue line measure is 70 on the inner scale, but if one does not identify the initial point and make the correct turn, the angle measure might be wrongly read as 110 degrees on the outer scale.

Conceptualizing relation between angles and turns. In another activity, participants were asked to describe how they thought angles and turns were similar and how they were different. The following excerpt presents different participants' responses.

Sarah: Angles and turns are similar because angles look sort like a turn … and are different because angles have two lines and turns have a rounded side.

Emma: Angles and turns are similar because an angle is like a turn, like when a ball bounces off a wall and turns, it makes an angle that if you follow the path of the ball you can measure an angle at which it turned.

Adi: Angles and turns are different because an angle has straight sides and turns have rounded sides with no straight sides or any with vertices.

Matt: Angles and turns are similar because they both involve relationships. They are different because an angle is a measurement and a turn an action.

This excerpt shows that participants conceptualized angles alike to turns. In addition, participants conceptualized angles to be different from turns through their appearance. For instance, both Sarah and Adi noted that angles are characterized by straight sides with a vertex, while turns have rounded sides. This finding suggests that although students may conceptualize angles as turns, they may also view them as different contexts because of students' conceptualization of angles as defined by straight sides meeting at a vertex. Thus, how do we support students' conceptualization of angles to incorporate contexts such as turns with no visible rays meeting at a common vertex.

Summary of Post-Interview Findings

In summary, the post-interview findings indicate that participants' conceptualization of an angle as a figure formed by two rays/lines meeting at a common point persisted. This was evident through participants' definitions, drawings, and identification of angles in solids. The findings also revealed that participants conceptualized 360 degrees as the largest angle, while a zero degree was conceptualized as the smallest angle. Some participants conceptualized that an angle of zero degree could be represented using a straight line. This poses a challenge to a straight angle representation, which is also a straight line.

Post-interviews also revealed that participants had broadened their angle conceptualization to include reflex angles. This was evidence when they identified angles in a circle as either acute and reflex, or obtuse and reflex. In addition, when participants were asked to name things, they would consider angles that previously they had not, they talked of straight angles. This suggests that the instructional unit supported students' conceptualization of straight and reflex angles.

The post-interview findings have also revealed that students conceptualized angle measure as preserved even with the change of the length of the rays of an angle. In addition, when participants were asked to state the difference between angle measure and side measure, they conceptualized the difference in terms of the units of measure. When asked the difference between angles and turns, students conceptualized angles as defined by two rays meeting at a common point and turns defined by rounded sides. This further confirms students' conceptualization of angles as defined by straight lines meeting at a common point. The post-interview findings also revealed that students conceptualized angle measure using their body turns or using 90° as a reference point. Participants were able to measure 110° and 25° correctly when they visualized the use of their protractors through their body turns. I next present the analysis of the research question two.

Research Question Two (RQ2)

Research question two investigated instructional supports that contributed to students' conceptualization of angle and angle measure in the course of the instructional sequence. As previously noted in Chapter three (pp. 60-64), I used Anghileri's (2006) levels of supports framework in the analysis of this question. According to Anghileri, levels of supports can be classified into those that require direct teacher interaction with the student(s)and those that do not. In reporting these findings, I identified three major cases of students' conceptualization of angle and angle measure that emerged or developed during the instructional unit. I then explained the instructional supports that contributed to the development or emergence of these conceptualization. The three cases are: (1) conceptualizing reflex angles and their measurement in relation to other angles in a complete turn, (2) conceptualizing the role of angles in approximating the measure of a curved side of a miniature golf hole, and (3) conceptualizing the measure of angles created when a ball hits a wall and bounces off. I next present the findings of the instructional supports that contributed to the emergence or development of the three aforementioned cases.

The First Case: Conceptualizing Reflex Angles and their Measurement in Relation to Other Angles in a Complete Turn

Before the instructional unit as reported in the pre-interview findings, students' conceptualization of angles revolved around acute, right, and obtuse angles. Conceptualization of reflex angles was not evident. However, during the instructional unit is when students' conceptualization of reflex angles emerged and developed. The following are Anghileri's instructional supports that contributed to the emergence and development of students' conceptualization of reflex angles.

Level 1 Supports: Environmental Affordances

According to Anghileri (2006), environmental affordances support students' mathematical learning without direct teacher interaction with the students. Some of the environmental affordances that supported students' conceptualization of reflex angles were: (1) provision of structured tasks, and (2) peer collaboration. I next discuss how each of these provisions is evident.

Provision of structured tasks. On Day 3 of the instructional sequence, the researcher gave students rough sketch drawings of a miniature golf hole on Worksheet 2 as shown in Figure 4.24.

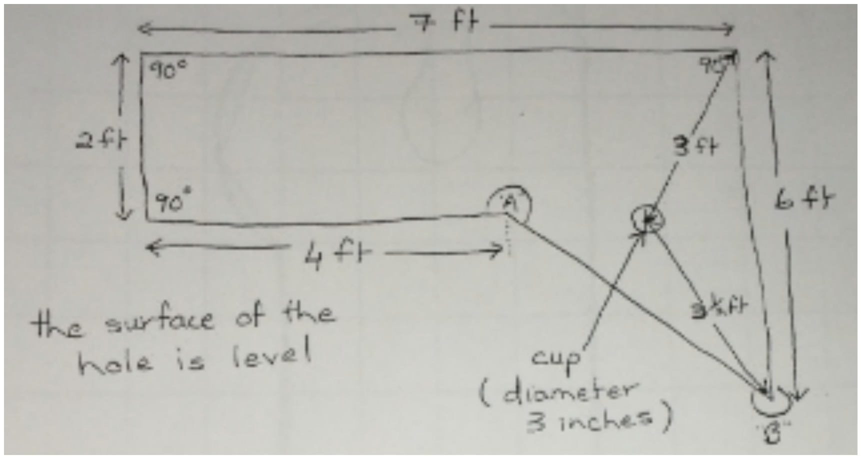


Figure 4.24: A sketch drawing of a miniature golf hole with straight edges.

As Figure 4.24 shows, some angle measures were provided, while others were not. Participants were required to identify and to measure the missing measurement on the sketch of the miniature golf hole. As had been noted under RQ1 findings, participants identified angles using their position on the drawing. For instance, Sarah and Emma identified the angles as “the two corner angles,” the angle at “the corner at the bottom right,” and the angle at “the middle corner.” Matt and Adi went further to label the angles as A and B on the drawing. In addition, participants conceptualized the measure of the reflex angles as the measure of obtuse angles as they did not understand reflex angles. Thus, this task supported students’ conceptualization of reflex angles in that it created an opportunity for students to learn how to name what they termed as “the angle at the corner at the bottom right” or “angles A and B,” which were the reflex angles (angles greater than 180°, but less than 360°), and how to measure these angles using a protractor.

Peer collaboration. During the task, students were asked to discuss in pairs before they could engage in a whole-class discussion. The excerpt following Figure 4.25 and Figure 4.26, show how one student supported another student's conceptualization of reflex angles.

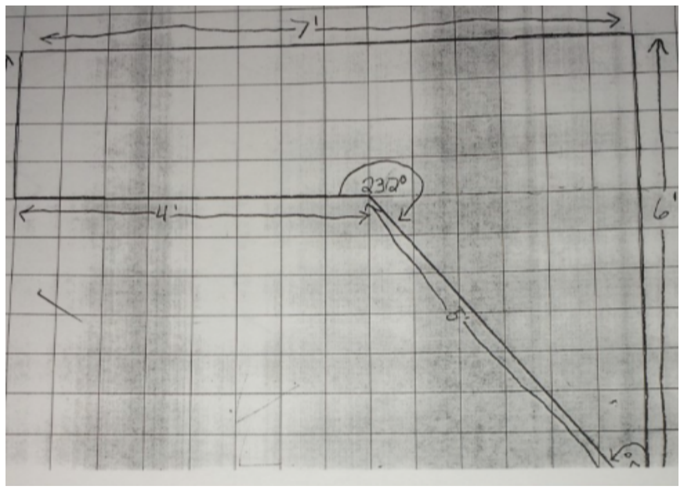


Figure 4.25: Matt's diagram indicating angle measures.

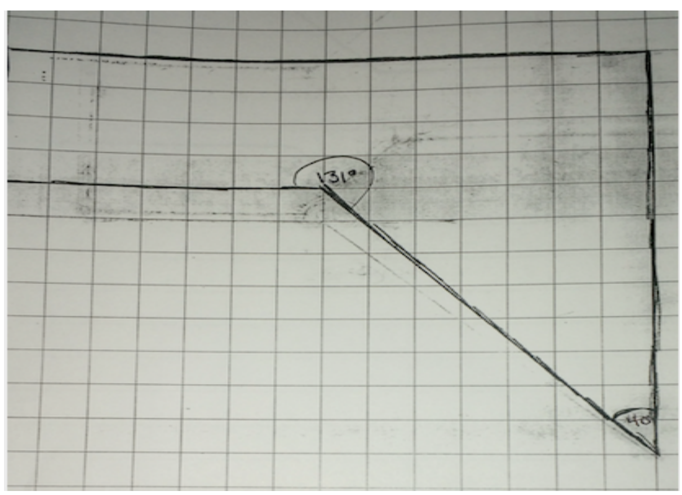


Figure 4.26: Adi's diagram indicating his angle measures.

Matt: I have 135. That is like right there. How did you like … this is a reflex angle.

Adi: Which one is?

Matt: This. It is not 135.

Adi: Yeah, it is a reflex because the hole is right there.

Matt: No, it is down there.

Adi: Right. But like the starting point, the starting point was right there. You had to come down to the cup, so that it could be like a reflex angle.

Figure 4.25 show the correct position and the measure of the reflex angle on Matt's drawing, while Figure 4.26 show Adi's measure of the same angle. The excerpt shows Matt trying to convince Adi of the position of the reflex angle on the drawing. It is evident that Adi conceptualized the measure of the reflex angle as the measure of an obtuse angle. This provide an example of how peer collaboration supported students' conceptualization of angle and angle measure, and in this case conceptualization of reflex angles as related to obtuse angles in a circle. I next present level 2 and level 3 supports. According to Anghileri (2006), level 2 and 3 supports are those that involve teacher-student(s) direct interactions.

Level 2 Supports: Reviewing and Restructuring through Focusing Students' Thinking

Students' conceptualization of angle and angle measure was supported through: (1) the use of prompts and probing questions, examples of "reviewing," (2) "negotiating meaning, an example of "restructuring" (Anghileri, 2006, p. 39). The following excerpt provides an example of how the researcher interacted with students in order to focus their thinking into conceptualizing reflex angles as well. This excerpt followed Figure 4.24 above.

Researcher:	How could you describe this instead of saying near the bottom? You might be looking at it this way. How could you describe it if you wanted to make sure that everybody knew exactly which angle you were talking about?
Matt:	The one on the bottom of the triangle?
Researcher:	Okay. Diana?
Diana:	I was going to say measure the acute.
Researcher:	Okay, is this the only acute angle here?
Adi:	No.
Matt:	Yeah.
Researcher:	There are a lot of 90 degree angles, three of them. There is an acute, and what is the other one called? Matt?
Matt:	Reflex.
Researcher:	Okay, which one is the reflex?
Matt:	Almost directly under the … where it says 7 feet.
Researcher:	Okay, this one right here?
Matt:	Yeah.

Use of probing questions. The above excerpt shows how the researcher used probing questions throughout as she interacted with students in order to focus them to conceptualize all kinds of angles. In particular, the researcher said, “there are a lot of 90°

angles, three of them … an acute, and what is the other one called?" This latter probing question led students to conceptualize the kind of angles that had not been mentioned, the reflex angles. Recall that students had termed the reflex angle as "angle at the middle bottom." Thus, students' conceptualizing the reflex angle was key as it was not evident from the beginning as acute, right, or obtuse angles.

Negotiating meanings. Anghileri (2006) noted that:

> The process of negotiating meanings involves a social process of developing a topic, by pooling and probing predicates and by selecting socially agreed-on predicates as classroom discussion becomes the collective learning of the classroom community, during which taken-as-shared mathematical meanings emerge as the teacher and students negotiate interpretations and solutions. (p. 46)

This means that making mathematical meanings is a process where both students and teacher should reach a shared understanding. The above excerpt provides an example where the researcher guided students into negotiating all kinds of angles that needed measurement, through the use of probing questions. In particular, when Matt responded that the other angle needed is reflex, the researcher followed with the question, "which one is reflex?" This probing question was seeking to find out which angle on the diagram was the reflex, before the response could be accepted-as-shared by the learning community.

The next excerpt provides another example of how negotiating meanings supported students' conceptualization of angles and how to measure reflex angles. The researcher paid close attention to students' words, used probing questions to guide students in making further interpretations and meanings of angles. This excerpt is also based on Figure 4.24 as students and the teacher negotiated how to measure the angle at the bottom middle of the sketch given, which was a reflex angle.

Researcher: How about this angle right here? First of all, how could we measure it? Adi?

Adi: I do know. Okay. 360 take away 130.

Researcher: Okay, how did you get the 130?

Adi: I don't know.

Researcher: Does anybody else understand what he is talking about? He said you need to take 360 minus some number, 130 degrees maybe.

Adi: Oh, I know how to do it.

Researcher: Okay, Adi.

Adi: Hold on.

Researcher: Okay. I would say this angle right here is more than 130 degrees. How do you think I know that? Sally?

Sally: You know what 130 degrees is, like how big it is.

Researcher: Well, from here all the way around to here. What do you know about this? If it was just straight like that, what would be that angle?

Many: 180.

Researcher: 180, right? If you turn this way, you turn completely around or turn facing the other way you have a 180. Right? From here to here is

180. So, I know that if I then go this much farther, it has to be more than 180. Where do you think the 130 comes from? Jessie?

Jessie: The number outside the scale.

Researcher: This part right here?

Jessie: Yeah.

Researcher: Okay. That is a good way to measure. Remember, your protractor only goes from 0 to 180. So, it is difficult to measure any kind of an angle like this that is more than 180. So, you can turn your paper around. I am not going to turn this around, but I am going to measure this angle, this one here. If I put my protractor there, now I am going to measure from here and use the inside scale. This looks like, mine is about 127. Close to 130. All right? So, this angle right here is about 127.

Researcher: Does anybody not understand how we got that? You would put it right wherever you have a little line, right in the middle there, right in the middle of the straight edge is where you put the vertex point.

As the above excerpt indicates, the researcher's attempt to have students explain their understandings of how to measure the reflex angle provided other learning opportunities about angles. I conjecture that the question "how do you think I know that," provided students an opportunity to conceptualize about: straight angles (180°), reflex angles being greater than 180° but less than 360°, how to measure obtuse angles (130 degrees), and angle measure as an amount of turning as they sought to provide explanations. For instance, when Sally responded that the researcher knew how big 130 degrees angle is, the researcher sought for more explanation on what students knew about the measure of the half turn in the amount of turning that the reflex angle made. Although, the researcher did not use the word half turn explicitly, students collectively responded "180." As a consequence, the researcher asserted that then the amount of reflex angle has to be greater than 180° and followed with the question "where do you think the 130 came from?" Jessie responded to the question by noting that 130 then has to be "the number outside the scale." At this moment, the researcher confirmed the obtuse angle was what Adi was referring to and went ahead to demonstrate to students how to use a protractor to measure it. The researcher got 127 degrees, which she confirmed to be close to 130 degrees, the measure of an angle that Adi had claimed. I conjecture that by students discussing where 130 degrees came from provided them opportunities to conceptualize an obtuse angle and its measure, as well as its relation to a reflex angle in a complete turn. These findings suggest that as students and the teacher continued negotiating how to measure the reflex angle, this opened an opportunity for students to conceptualize the straight angle (180°), the obtuse angle (127°), and their relation to the measure of the reflex angle. This led to Anghileri's (2006) level three support "making connections" which led to developing students' conceptual thinking about angles in a complete turn (p. 39).

Level 3 Supports: Developing Conceptual Thinking

Making Connections. The next episode shows how participants conceptualized the measure of the identified reflex angle in relation to other angles in a complete turn

(360 degrees), and how the ideas became taken-as-shared by the classroom community. Recall that Adi had claimed that the reflex angle is measured by "360 take away 130." The following episode shows how the whole-class discussion ensued.

Researcher: All right. If this angle is 127, how could I figure out what this part is? Steve?

Steve: Well, you add them together?

Researcher: Add what together?

Steve: Those two?

Researcher: 180 is just with the straight line, right? And this. Then there would be a part missing because the 180 is just from here to here. 127 is from here to here. What about this part? Suzie?

Suzie: Well, first I ....

Researcher: Just tell me what you got.

Suzie: I got 54 degrees.

Researcher: 54 degrees for this part in here?

Suzie: Yes.

Researcher: How did you get that?

Suzie: I put the protractor from 180 degrees and drew a line.

Researcher: Okay, so you drew a line here and measured that.

Suzie: Yes.

Researcher: Okay, so you figured out what this angle is and added it to 180. Okay. Jessie, do you have a different idea?

Jessie: Um, oh, no. That was mine. Never mind. Sorry.

Researcher: Okay … If we turn all the way around and face the same way you have gone 360 degrees. So, if this whole thing is 360, if we use Adi's idea and subtract this outside part and take 360 minus 127, we take the whole circle minus this part. That should give us this part, shouldn't it. Okay? So that would be 233 degrees.

The above episode shows how the researcher guided students in making connections of the relation of the measure of the reflex angle and other angles in a circle. Students presented two methods of figuring out the measure of the reflex angle after knowing that the obtuse part is 127 degrees. First, Steve stated that the two angles, that is 180 degrees and the 127 degrees should be added together. The researcher illustrated that the two angles could not be added together as that would not give the measure of the targeted reflex angle. The researcher demonstrated to the students where the measures of the two angles are on the drawing, indicating that there will remain a part of the reflex angle that will not be accounted for. Instead of the researcher asking for a different idea of measuring the reflex angle, she asked for the measure of the unaccounted part of the angle following Steve's idea. In response, Suzie claimed that she got 54 degrees. The researcher probed Suzie in attempt to have her explain how she obtained 54 degrees. Suzie explained that she drew a straight line from 180 degrees and measured the remaining part, which was 54 degrees. Following this, the researcher said, "… so, you figured out what this angle is, and you added it to 180." Here, the researcher clarified Suzie's explanation in order for other students to make connections. Suzie conceptualized the measure of the reflex angle as 180 degrees added to 54 degrees, which is an acute angle. These findings suggest that the researcher used probing questions in order to have students explain their claims, as well as build on students' ideas to help them make connections of how to measure reflex angles in relation to an acute angle and a straight angle in a complete turn.

Building on Adi's idea of 360 taking away a known obtuse angle measure, the researcher explained the second method of finding the measure of the reflex angle when the obtuse angle is known. The researcher said, "if a straight line has 180 degrees, what is the whole circle to go all the way around?" Emma made a claim that it is 360. The researcher said, "if we turn all the way around and face the same way you have gone 360 degrees. So, if this whole thing is 360, if we use Adi's idea and subtract this outside part and take 360 minus 127, we take the whole circle minus this part. That should give us this part, shouldn't it. Okay. So that would be 233 degrees." Thus, the researcher builds on a shared student's idea to help students see connection between a reflex angle, in relation to an obtuse angle in a complete turn.

Towards this end, it is evident that the three levels of Anghileri's (2006) instructional supports assisted students in developing their conceptualization of reflex angles in relation to acute, obtuse, and straight angles in a complete turn. While it appeared easy to discuss Anghileri's level one supports separately, it is seemed challenging to discuss level 2 and 3 supports separately. For instance, teacher's use of probing questions in order to have students explain their thinking, led to negotiating meaning, which consequently led to students making connections of the intended mathematics meanings. Hence, it appears that level 2 supports led directly to level 3 supports. I next discuss the emergence of the second case.

The Second Case: Conceptualizing the Role of Angles in Approximating the Measure of a Curved Side of a Miniature Golf Hole

The emergence of the second case took place on Day 5 of the instructional sequence. The following are Anghileri's (2006) instructional supports that contributed to the emergence of students' conceptualization of the role of angles in approximating the measure of a curved side of a miniature golf hole.

Level 1 Supports: Environmental Affordances

Provision of structured tasks. On Day 5 of the instructional sequence, students were given a sketched drawing of a miniature golf hole as shown in Figure 4.27 and were asked to discuss what measurements they needed to know in order to draw the hole to scale. As can be noted, Figure 4.27 consist of straight sides and curved sides.

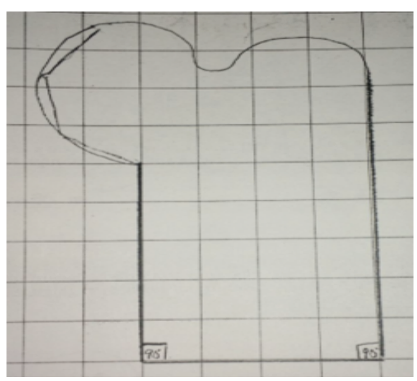


Figure 4.27: A sketch drawing of a miniature golf hole with a curved side.

This task provided students opportunities to conceptualize how they could use angles in finding the measurements of the curved sides.

Another evidence of the support that structured tasks provided to students was revealed in students' Journal 3 assignment and the corresponding midway interviews that took place in the course of the instructional sequence. Students were presented with the following diagrams as shown in Figure 4.28 and were asked what measurements they needed to take.

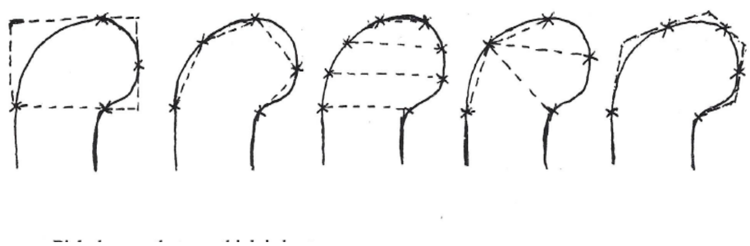


Figure 4.28: A diagram showing five strategies for measuring a curved side of a hole Journal 3.

Students' written responses and interviews showed that they all conceptualized angles as important measurements to consider. For instance, Sarah responded as shown in the following excerpt.

Researcher:	… Here, we had asked you specifically to say what measurements you would take.
Sarah:	Um, how long they are and like maybe the angle of it so you can try to make an angle with the circle. (*referring to the curve*)
Researcher:	Okay. So, the angle in those places.
Sarah:	Yes.

In the above excerpt, when Sarah was asked the measurements she would consider when measuring a curved side of a hole, she said, "um, how long they are and like maybe the angle of it so you can try to make an angle with the circle." This suggests that Sarah had taken-as-shared the idea of measuring angles for a better approximation of a curved side during the instructional unit. The next excerpt also illustrates that Emma had taken the idea as shared.

Researcher: Can you show me on this picture what are the angles that you need to measure? (*referring to strategy 5*)

Emma: Well, you have to like draw the points first. Then connect some of the lines and then the angles would be there, there, there, and there.

Researcher: Okay And could you tell me, how would you decide on the points?

Emma: Probably, I would pick one of the spaces where it was most curved, and I would put two points opposite each other so I could make an angle with the points. It would just go around the thing.

In this excerpt, when the researcher asked Emma what angles that she needed to measure, she claimed that she has to draw points first. The researcher followed by asking Emma, "how would you decide on the points?" Emma explained how she would place two points opposite each other in order to make an angle with the points all around the curve. Emma's response indicate that she conceptualized the need to create angles when measuring a curved side.

Level 2 Supports: Reviewing through Focusing Students' Thinking

The following excerpt illustrates the researcher's use of probing questions in order to have students conceptualize other measurements needed to approximate the measure of the curved side of a miniature golf hole as given in Figure 4.27, besides length measures.

Researcher: This is what you have. Jimmy said measure this (*referring to the maximum length from one end of the straight edge to the farthest point on the curve)* So, I said, okay, let's just say this is 35 in., so I could draw a line out here, but how would I know where it goes? It is 35 in., where? Maggy?

Maggy: You have to measure the degree there. (*referring to the angle the straight line would make with the curve)*

Researcher: This one right here?

Matt: The angle.

Maggy: Yeah.

Researcher: So, measure the angle?

Maggy: Yeah.

Researcher: Okay, let's say it is 140. (see Figure 4.29) Okay. What will that tell me then?

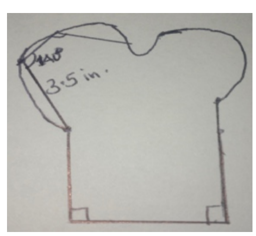


Figure 4.29: An illustration of an angle to locate the direction of a curved side.

Maggy: That will tell you like the direction that the line is going to go into.

Researcher: Okay. All right. So, if I was here, I could use my protractor, 140, I am measuring from here, this is 140, I would extend this out beyond and then I could measure off 35 in., which would be what—3 1/2 units?

Maggy: Yeah.

Researcher: So, I am just going to approximate on here. Okay. So, here it is. What does that point tell me then? Where is that point?

Emma: Like the peak of the angle.

Researcher: It is on the curve, right? So, I could draw something like that to approximate it. Okay.

Use of prompts and probing questions. Building on Jimmy's idea of measuring the full length from one corner of the hole to the curved side the researcher said, "so, I could draw a line out here, but how would I know where it goes? It is 35 in., where? This question led to the emergence of the need to measure angles to better approximate a curved side of a hole. In response to the researcher's question, Maggy said, "measure the degree there." Consistent with previous studies (e.g., Browning et al., 2008), students use the word degree to refer to an angle. As the dialogue progressed, Matt clarified what Maggy implied and said, "the angle." The researcher's use of probing questions led to a conceptual discourse.

Level 3 Supports: Developing Conceptual Thinking

As the dialogue continued as shown in the above excerpt, the focus now shifted from just the need to measure the angle to conceptualizing the reason to measure that angle. This illustrates how the researcher led students to generating conceptual discourse.

Generating conceptual discourse. According to Anghileri (2006), conceptual discourse begins when "what is said … becomes an explicit topic of discussion" (p. 49). In the above excerpt, the researcher guided students to conceptualizing the reason to measure angles, after stating the need to measure angles. The researcher said, "okay, let's say it is 140. Okay. What will that tell me then?" This question was the researcher's further attempt to have students explain why the angle measure was needed. In response to the question, Maggy explained that the angle measure would tell the direction the line who measure was suggested to be 35 inches would go to. The researcher elaborated further Maggy's response as she illustrated it as shown in Figure 32. As the researcher illustrated, she asked, "what does that point tell me then?" (*referring to the point created on the curve after measuring 140 degrees*). In response to this question, Emma said, "like the peak of the angle," as the researcher said, "it is on the curve, right? So, I could draw something like that to approximate it. Okay." According to Anghileri (2006), the researcher used probing questions to lead students conceptualizing the need to measure angles in approximating a better curve. The researcher had students explain their thinking of how the need to measure angles would help. As noted, measuring angles would tell the direction the segments locating the path of the curve would go.

To this end, the use of structured tasks (a level 1 support), the use of probing questions (a level 2 support), and generating conceptual discourse (a level 3 support) contributed to students' conceptualization of the need to create and measure angles in order to have a better approximation of a curved part of a miniature golf hole. Although the idea of approximating a curve is typically learned in calculus at the high school or undergraduate level, it is evident that the ideas of measuring a curve using angles emerged as the sixth-grade students engaged in a task that provided the opportunity. This suggests that well-structured tasks, coupled with teacher's guidance and conceptual discourse, can provide students learning opportunities of advanced ideas in mathematics.

The Third Case: Conceptualizing the Measure of Angles Created When a Ball Hits a Wall and Bounces Off

The emergence of case three was initiated on Day 10 of the instructional sequence. All three levels of Anghileri's (2006) instructional supports contributed to students' conceptualization of the measure of angles created by a path of a ball and its rebound once it hits a wall.

Level 1 Supports: Environmental Affordances

Provision of structured tasks. Students were given Worksheet 10 and the researcher explained that in the next three days they will explore the path of the ball and its rebound when it bounces off a wall. Using a constructed miniature golf hole in the classroom, the researcher led students in hitting the ball through different paths. Students were asked to make observations and see if they could make predictions of what was to happen to the ball once it hit the class wall. The researcher hit the ball through three different paths, where path number one was marked with a solid line and letter X, path two was marked a dashed line, and path three was marked with a dotted line. The researcher asked the teacher to draw the sketch of what was observed on the board and students on the space under number 1 on Worksheet 10. (*see Figure 4.30 for an example*)

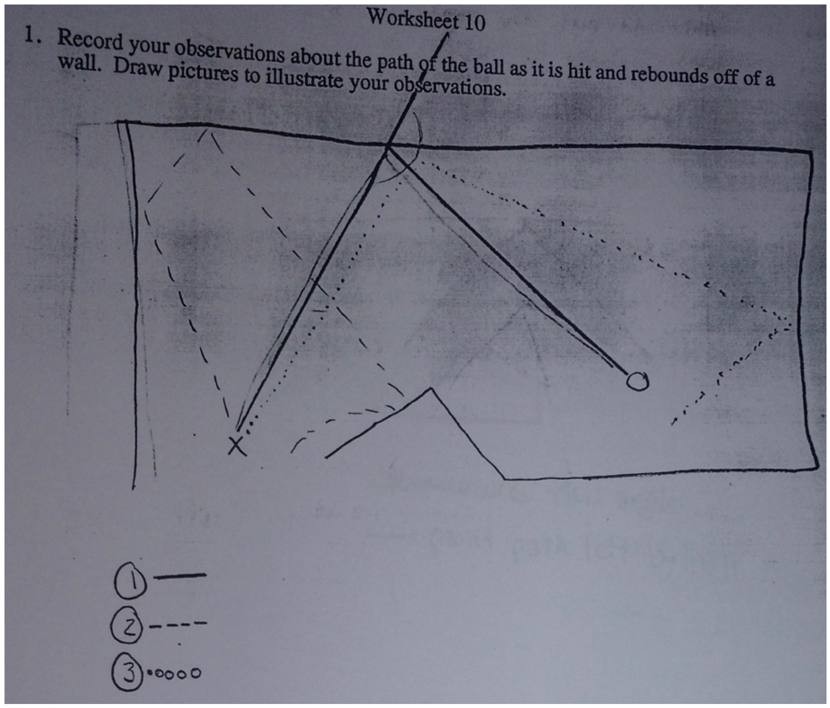


Figure 4.30: Matt's and Adi's sketch drawing illustrating three paths of a ball and its rebound.

As Figure 4.30 indicates, Matt and Adi drew a sketch following the researcher's model of what she expected the students to do. This task provided students an opportunity to conceptualize angles formed by an incoming path of a ball and its rebound once it hit the wall.

Peer collaboration. The following excerpt show the discussion that ensued between peers as they made predictions about the path of the ball and its rebound when it hits the wall.

Researcher: Okay, now you have 3 paths at least. So, let's stop for just a minute. #2 says to make a prediction about the path of the ball as it hits and rebounds off the wall. So, talk with your partner. Finish this sentence. When I hit the ball and it rebounds off the wall, I think it will.... You don't have to use that exact sentence, but that is the kind of thing you are aiming for in general. How can you predict where the ball is going to go after you hit it off a wall?

Teacher: When the ball bounces off the wall, you say where it is going to go. Predict the path it is going to take.

Researcher: Talk with your partner and write your predictions for #4.

Matt: It bounces back at the exact same angle that it was hit to the board.

Adi: The speed also affects.

Matt: The only thing that affects it is if it has spin.

Adi: This is English, if you turn it like that, then kind of cut through it.

Matt: What?

Adi: English. It is the spin on the ball. It is like in pool and basketball. It is the Englishman's spin.

Matt: But if it has no spin it will bounce exactly at the same angle that it hit.

Adi: Right. The same angle, but it is speed

Teacher: Okay, in tennis you intentionally put spin or top spin on the ball, and it does, top spin will not change the direction. It will change the speed. Back spin can change the direction. For right now, I would look at predicting if you didn't put spin on the ball. Then what would happen with spin. But first look at it with no spin.

Adi: Would speed affect? It is not going to affect the path, well, actually it could affect the path because it depends on how far it is going to go, like that one over there.

Teacher: That is right. So, it could make it go farther. Will it change the beginning pattern?

Adi: No. Okay.

Matt: A ball with no spin.

Adi: Yeah.

The above excerpt illustrates how peer collaboration supported students' conceptualization of the measure of angles created by an incoming path of a ball and its rebound once it hit a wall. Matt claimed that the ball will bounce off the wall at exactly same angle it hit, while Adi responded that the speed of the ball will affect the angle of rebound. Matt responded back that the only way the speed could affect is if the ball had a spin. At this point, the teacher who was listening to this discussion interjected. The teacher used an example of a tennis ball to explain how different types of spin could affect, the top spin affecting the speed and the back spin affecting the direction of the ball. However, the teacher asked both Matt and Adi to consider that the ball has no spin. As the discussion progressed, it becomes clear that Adi agreed with Matt that a ball with no spin will bounce off the wall at the same angle it hit. However, Adi still believed that speed could affect the path of the ball. Adi said, “would speed affect? It is not going to affect the path, well, actually it could affect the path because it depends on how far it is going to go, like that one over there.” This discussion led to level 2, where the teacher used probing questions to support Adi’s conceptualization that the speed of the ball will not affect the measure of the angle of rebound.

Level 2 Supports: Reviewing Through Focusing Students’ Thinking

The use of prompts and probing questions. The teacher agreed with Adi about the speed affecting how far the ball can go but asked “will it change the beginning pattern?” This question was the teacher’s attempt to have Adi see that the angle an incoming path create with the wall is not affected by the speed of the ball, and thus the rebound angle is also not affected. As the episode ends, it is evident that both Matt and Adi are in agreement that with no spin, the ball bounces off the wall at the same angle it hit. This illustrates how peer collaboration coupled with a teacher’s use of probing questions supported students’ conceptualization of the angles formed by the incoming path of a ball and its rebound once it hits a wall. The following excerpt also illustrates how a researcher’s and a teacher’s use of questions supported students in deepening further their conceptualization during a whole-class discussion.

Researcher: Okay, let’s hear about what predictions you have. Brent?

Brent: We think that if it is a wall like this and you hit it at this angle, then this angle right here (*referring to the angle created by the incoming path of the ball and the wall*) will be the same angle that it bounces off at.

Researcher: So, the angle it goes in will be the same angle it comes out at?

Brent: Yeah.

Researcher: Okay. Addy?

Addy: Me and Brian thought that when a ball comes and hits the wall straight on like that it will bounce back. But if it is at an angle, it will go, bounce back at a 90 degree angle from where it was hit. So, if it hits here like this, it will bounce back like that.

Teacher: We can test your theory of hitting it straight ahead. If you hit something straight ahead, you are actually hitting it at an angle against the wall. What angle is it that you are hitting against the wall?

Matt: 180.

Teacher: But you think every time it hits it is going to bounce off at 90 degrees. Okay. What are you guys thinking?

Emmy: When you hit a wall at a certain angle, it will bounce off at the same angle to the wall in the opposite direction.

Teacher: Is that the same or different than Brian's and Addy's. Does anybody have a different theory for either of those two?

Matt: I have a comment about Addy and Brian's.

Teacher: The one that was going to be 90 degrees.

Matt: Right. I think that it depends on the angle you hit the ball to. Like if you hit a ball and you just skim it, like it was going just a little bit like this and it won't hit back at a 90 degree angle. It would be more like a 130 degree angle or something like that. But it won't always hit on the, it won't always rebound on a 90 degree angle. It really depends on how you hit it toward the wall.

Researcher: Let's just test out the theory of hitting it straight into the wall and see what happens. First of all, that wasn't straight, but ...So, it seems to pretty much bounce straight back. Right? So that still doesn't, you had accounted for that in your theory, right? All right. Where should we try it now to test out, there are basically two theories out there. One is that it goes in at the same angle that it bounces off with and the other is Brian and Addy's which says they go out at 90 degree angles.

Brian: 90 degrees not from where it comes into the wall, but from the ... Like you see the dashed one, where it goes in and it comes out at a 90 degree angle?

Researcher: So, this right here?

Brian: Yeah.

Researcher: You thought this was a 90 degree angle.

Brian: Yeah. That is right.

Teacher: How about this one over here? It is kind of hard to see, where I put those orange dots, does that look 90 to you?

Many: No.

Teacher: Here over by the cup side.

Researcher: Over here. Does this one look like it is 90?

Many: No.

Teacher: So that one would negate your theory, right?

Brian: Yeah.

Researcher: Let's talk a little bit about, these people who, a number of you and I saw a lot of you on this side here, have the idea that the angle that it goes it at, the same that it goes out at. Which angles are we talking about? Let's just take this dashed one. It looks like there are 3 angles. This one, this one and this one. Which angles are we talking about that are the same? If you just look here, this is the path of the ball and it rebounds off one wall. It seems that there are 3 angles. This one, this one, and that one. So, which angles are we talking about that will be the same? Bethan?

Bethan: The two, the one [cannot hear]

Researcher: Okay, so you are saying maybe this one here and this one here are the same (*referring to the angles created by the incoming path of the ball and the wall and the rebounding path of the ball and the wall*).

Matt: Yeah, they are.

Researcher: Does everybody else who has their hand up, was that what you were going to say, also? Okay. Let's examine another one and see if this theory holds up. This one right here. Look at your own paper. Does it look like the two angles, where the ball goes in and rebounds off the wall, the two angles that have part of the board as their sides, do they look like they are congruent?

Many: Yes.

As the above excerpt shows, the researcher begins by calling students to share their predictions. She said, "okay, let's hear about what predictions you have." That prompt called on students during the whole-class discussion to share their ideas on how they conceptualized the measure of angles formed when an incoming path of a ball hit a wall and rebound. The researcher and the teacher also used questions throughout as students' predictions were tested. For instance, when Brian's and Addy's theory of the ball rebounding at 90 degrees was being tested, the researcher asked, "does this one look like it is 90?" The teacher followed with the prompt, "so, that would negate your theory, right?" Brian said, "yeah." The researcher's question and the teacher's prompt supported students in conceptualizing that the angle of rebound is not always 90 degrees. The above excerpt also illustrates how students explaining and listening to the thinking of others, a level 3 support, helped them make connections of the mathematical ideas they were discussing.

Level 3 Supports: Developing Conceptual Thinking

Students explaining and listening to the thinking of others. The above episode also illustrates how students shared and explained their predictions, as others listened and commented on the shared ideas. Brent began by making a claim that the ball will bounce off the wall at the same angle it hit the wall. However, Addy thought that the ball hitting the wall straight will bounce back, but if the ball hit at angle, it will bounce back at 90 degree. These were two claims in one. The teacher initiated testing the first claim that if the ball hits the wall straight (*at an angle of 180 degrees*), it bounces back straight, which many students agreed to be true. However, the teacher called students to discuss about the second claim that the ball that hits the wall at angle will bounce back at 90 degrees. The teacher asked other students what they thought about the claim. Emmy made the previous claim that the ball bounces off the wall at the same angle it hit but in opposite direction. The teacher asked whether there was anyone with a different theory about the claims made. Matt said that he had a comment about Addy and Brian's claim on angle bouncing at 90 degrees. Matt said, "it depends on the angle you hit the ball to" that the ball cannot always bounce off the wall at 90 degrees. Matt's reaction to Addy's and Brian's claim illustrates that students were listening to the thinking of others. According to Anghileri (2006), students develop their conceptual thinking when they can listen and respond to the thinking of others. This eventually led to making connections of the topic of discussion.

Generating conceptual discourse. As the discussion continued in the above episode, the researcher and the teacher led students in testing Addy's and Brian's theory of the ball always rebounding at 90 degrees. Here, although Matt had provided some explanation to show that the ball will bounce off the wall at the angle it comes in, the researcher and the teacher led students in testing Addy's and Brian's theory of a ball bouncing off at 90 degrees. This move by the researcher was a further attempt to lead students in conceptualizing the prediction that would count as accepted mathematically. According to Anghileri (2006), generating conceptual discourse is characterized by norms such as what counts as an acceptable mathematical explanation. Thus, this episode illustrated how generating conceptual discourse contributed to students' conceptualization that indeed the angles created by the incoming path of the ball and the wall, and the rebounding path of the ball and the wall have equal measures. This was the prediction that was accepted as mathematically correct.

Summary of the Anghileri's Instructional Supports

As RQ2 has revealed, the instructional unit provided students opportunities to conceptualize about angle and angle measure. In particular, students conceptualized about reflex angles, their measures and their relation to other angles such as acute or obtuse, or straight angles. Curriculum documents such as the CCSS for mathematics (NGA & CCSSO, 2010) do not emphasize reflex angles as they do for acute, right, and obtuse. Students also conceptualized the need for creating and measuring angles on a curved side of a miniature golf hole in order to tell the direction the segments would go for a better approximation of the curved side. Lastly, students conceptualized the measures of angles created by a path of a ball and its rebound once a ball hits a wall. The instructional supports that contributed to students' conceptualization of angle and angle measure during the miniature golf instructional unit ranged from those that did not require direct teacher-student interactions to those that required (Anghileri, 2006).

Using Anghileri's (2006) levels of supports framework, environmental affordances (level 1 supports) that contributed to students' conceptualization of angle and angle measure were: provision of structured tasks through worksheets and activities, and peer collaboration. During the instructional unit, students were provided with worksheet and activities that focused their thinking to conceptualizing about angles and angle measure. Anghileri's level 2 supports that contributed to students' conceptualization of angle and angle measure were reviewing and restructuring. These were through researcher's or teacher's use of prompts and probing questions that focused students' thinking towards angle and angle measure conceptualization. The use of probing questions also led to students and teacher or researcher to negotiate mathematical meanings and interpretations. Lastly, the Anghileri's level 3 supports that contributed to students' conceptualization of angle and angle measure were students' explaining their ideas and listening to the ideas of others, which helped them make connections of the ideas discussed. In addition, through generating conceptual discourse, the researcher guided students to negotiate the explanations that counted as mathematically acceptable.

Chapter Summary

In this chapter, I presented major findings of my study. For the RQ1, I presented findings of students' conceptualization of the angle concept and angle measure under three major categories: before, during, and after learning through a geometry unit of instruction set in a miniature golf context. Overall, the findings showed that students' conceptualization of the meaning of angle ranged from an angle as a point or a corner to a geometrical figure formed by the union of two rays across the three categories. Students also conceptualized angle measure in terms of measuring geometrical figures such as right, acute, or reflex. In addition, students conceptualized that changing the length of rays of an angle does not change the size of an angle, and that angle measures are preserved for a scale drawing. The post-interview findings also revealed that students conceptualized angle measure using their body turns or using 90° as a reference point.

The findings have revealed that the instructional unit set through a miniature golf context provided students with opportunities to conceptualize about: (1) reflex angles, their measures and how they are related to other angles such as acute, obtuse, and straight angles, (2) the need to create and measure angles on a curved side of a hole in order to get a better approximation of a curve, and (3) the measure of angles created by an incoming path of a ball and its rebound once a ball hit a wall. At the beginning of the instructional unit, students conceptualized reflex angles and their measures as obtuse angles and vice versa. However, by the end of the instructional sequence, students conceptualized reflex angles as angles greater than 180 degrees but less than 360 degrees. In addition, students learned about the relationship of the reflex angles with other angles such as acute, obtuse, and straight angles. During the course of the instructional unit, students conceptualized that they could use angles in order to get a better approximation of a curved side of miniature golf hole as the angles can help them know the direction the curve will take. Lastly, students conceptualized that the angles created by an incoming path of a ball and its rebound once the ball hit a wall have equal measures.

The findings of RQ2 revealed that various instructional supports contributed to the development of students' conceptualization of angle and angle measure during the instructional sequence. Using the Anghileri's (2006) levels of supports framework, use of structured tasks and peer collaboration were the main environmental affordances, level 1 supports that contributed to students' conceptualization of angle and angle measure ideas. For level 2 supports, the researcher and the teacher used prompts and probing questions, as they led students in negotiating the intended mathematical meanings of angle and angle measure ideas in order to focus students' thinking. Lastly, level 3 supports that contributed to students' conceptualization of angle and angle measure were: students explaining and listening to the thinking of others which helped them to make connections among angle and angle measure ideas under discussion. Through generating a conceptual discourse, also a level 3 support, students were supported to know the mathematical explanations that counted as acceptable. Anghileri's level 2 and 3 supports, such as having students explain and listen to the thinking of others as well as discussing what count as an acceptable mathematical explanation are consistent with the social norms and sociomathematical norms of the emergent perspective, respectively.

Chapter Five: Discussion

Research has shown that sixth-grade students struggle in understanding the multifaceted nature of the angle concept (Keiser, 2003, 2004). Thus, the aim of this study was to investigate sixth-grade students' conceptualization of angle and angle measure before, during, and after learning through an instructional unit set in a miniature golf context, an example of a real-world context. The study also sought to find out the instructional supports that contributed to the participants' conceptualization of angle and angle measure in that context. The study conducted a retrospective analysis of an existing data from a larger study that followed a design-based research (DBR) methodology, whose underlying philosophy relates to the adage "if you want to change something, you have to understand it, and if you want to understand something, you have to change it" (Gravemeijer & Cobb, 2006, p. 73).

This study drew data from a larger project that carried out 17 days of a collaborative teaching experiment through a miniature golf geometry unit of instruction. The instructional unit of the larger study was guided by the Realistic Mathematics Education (RME) theory, a domain specific instructional theory that posits that mathematics is invented by human and it is not a system of products already made (Freudenthal, 1971, 1973). In this sense, students are to be positioned as inventors of mathematics and be provided opportunities that can allow them invent math. This study data analysis was guided by an emergent interpretive framework that posits learning as both social and individual (Cobb & Yackel, 1996). To answer research question one (RQ1), I used emergent codes that I developed from 30 transcripts consisting of pre-interviews, observations of collaborative teaching experiment, and post-interviews with two pairs of participants situated in two sixth-grade classes. To answer research question two (RQ2), I used Anghileri's (2006) levels of supports framework.

In this chapter, I present an overview of the study findings, interpretation of findings in relation to the reviewed literature, limitations of the study, recommendations of the study, generalization of the analyses, trustworthiness of the analyses, and conclusions.

An Overview of the Study

In this section, I present a summary of the findings of the two research questions of this study. I begin with the sixth-grade students' conceptualization about angle and angle measure before, during, and after learning through a geometry unit of instruction set in a miniature golf context. I then move to the instructional supports that contributed to students' conceptualization of angle and angle measure during the instructional sequence.

Sixth-grade Students' Conceptualization of Angle and Angle Measure Before, During, and After the Instructional Unit

As pre-interview findings revealed, before the instructional unit students conceptualized an angle as: (1) a point or a corner related to straight lines, (2) a geometrical figure formed when two straight lines meet at a common point, (3) a thing related to the tool of measure – the protractor, or simply a right angle. For example, when participants were given two-dimensional shapes and were asked to describe them, for shape I shown in Figure 5.1, Adi said "… it has got four … it is kind of like a square, but the two sides like these are like that and there are 1, 2, 3, 4." When the researcher asked Adi what he meant with "1, 2, 3, 4" he said, "points." On the same question, Matt said, "… the bottom and the top are parallel to each other and so are the sides … and has got four sides and four corners." This suggest that Adi referred to the angles as points and Matt as corners. Browning et al. (2008) noted that students tend to think of an angle as a corner. In addition, the standard documents such as CCSS note that students at kindergarten can use corners as an informal language (NGA & CCSSO, 2010). Thus, if sixth-grade students can use "corners" for angles, then the question that arises is at what level do students transition from using the informal language "corners" to using the formal language "angles" considering that angles are formally introduced at fourth grade (NGA & CCSSO, 2010). Richardson and Koyunkaya (2017) noted that students' conceptualization of an angle as a point or a corner relates to the dominant definition of an angle as a figure formed when two rays meet at a common point.

Figure 5.1: Shape I, one of the shapes participants were required to describe.

Students conceptualized angle measure as related to the protractor and the unit of measure – degree. For example, when participants were asked how they could measure an angle, Adi said, "with a protractor … we use degrees." Browning et al. (2008) and Keiser (2003) have also documented students' tendency to conceptualize angle or angle measure as degrees. Students also conceptualized that angle measure is not affected by changing the length of the rays of an angle. For instance, both Matt and Adi said, "no change" on the measure of angle will occur by extending the rays of the angle. This indicates that participants challenged the side-length obstacle reported by Devichi and Munier (2013). Students also conceptualized an angle as a turn in the context of their body movements. For instance, when students were given the floorplan of a small shopping mall activity (see Figure 4.8, p. 88), and were asked to think how their bodies moved from the flower shop to the music store, Matt said, "I am taking all right turns." Matt used the word turn after encouraged to think of his body movement. The post-interview findings also revealed that students were able to conceptualize the measure of 110° and 25° using a protractor when they were encouraged to consider how their bodies were turning, as well as using 90° as a reference point. Smith et al. (2014) noted that tasks that provide students opportunities to conceptualize angles through their body motions support conceptualization of angle as a turn as well as students' use of a protractor when measuring angles.

The findings also revealed that students' conceptualization of an angle as a figure that is formed when two rays meet at a common point persisted even during and after the instructional unit. For example, during the instructional unit, students were asked to discuss the measurements they needed in order to reproduce a miniature golf hole. Adi said, "measuring of like angles. So, like the way to measure the angles like an obtuse, right and acute." This suggests that Adi conceptualized an angle as a figure defined by two rays meeting at a common point. In addition, during post-interviews, when students were asked to define an angle, Matt and Adi said, "when two rays meet at a point," while Sarah and Emma said, "when two lines meet at a point." Furthermore, when students were asked to state the difference between angles and turns, some students like Sarah said, "angles and turns are different because angles have two lines and turns have a rounded side. On the same question, Adi said, "angles and turns are different because angles have straight sides and turns have rounded sides with no straight sides. This suggest that students' tendency to conceptualize an angle as defined by straight lines. This conceptualization of angles is limited as it does not incorporate all angle contexts such as turns and slopes where both rays may not be visible (Mitchelmore, 1997, 1998).

The post-interview findings revealed that participants conceptualized 360 degrees as the largest angle, and zero degree as the smallest angle. They conceptualized that zero degree could be represented using a straight line. This poses a challenge to a straight angle representation, which is also a straight line. When students were asked things, they would consider angles that initially they did not, Matt said, "like a straight angle." These findings suggest the likelihood of the instructional unit supporting students to conceptualize 0°, 180°, and 360° as angles. Keiser (2004) reported sixth-grade students' difficulties to conceptualize 0°, 180°, and 360° as angles.

During the instructional unit, students engaged in activities that provided opportunities to conceptualize: (1) reflex angles, their measures, and relation to other angles such as acute, straight, and obtuse in a circle, (2) the need to create and measure angles in order to better approximate a curved side of a miniature golf hole, and (3) the measures of angles formed by an incoming path of a balk and its rebound when a ball hit a wall. When students were given a sketch of a miniature golf hole and were asked to identify the angles that needed measurements, at first students conceptualized obtuse angles as reflex angles and vice versa. For example, Adi conceptualized the measure of a reflex angle to be 131degrees, a measure of an obtuse angle as shown in Figure 5.2. However, by the end of the activity students developed their conceptualization of the right position and measure of reflex angles as angles greater than 180 degrees but less than 360 degrees. As a consequence, students developed their conceptualization of the relation of reflex angles and acute, straight, and obtuse angles in a complete turn. This finding suggests the importance of introducing angles in a circle, instead of introducing angles from one aspect of two rays meeting at a point, a partial geometrical figure. Tanguay and Venant (2016) suggested reflex angles to be introduced as angles are being introduced and defined in a circle.

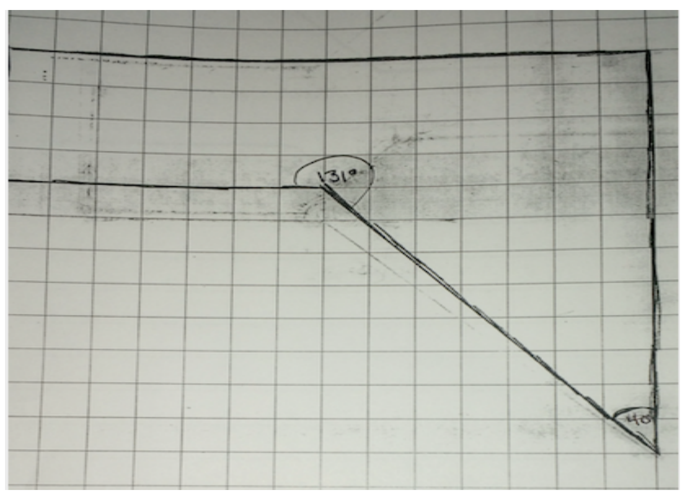


Figure 5.2: Adi's drawing indicating the measure of a reflex angle as 131 degree, which is a measure of an obtuse angle.

The instructional unit also provided students opportunities to conceptualize the need for angles in telling the direction the curved side of a miniature golf hole would take when drawing the hole to scale. When students were asked things, they would consider measuring when given another chance to visit an actual miniature golf hole, Sarah said, "… the curved sides," and Adi said, "inclines." This suggest the likelihood of the miniature golf context instructional unit supporting students in conceptualizing how to measure inclines and curved sides, which are slope and turn angle contexts, respectively (Mitchelmore, 1997, 1998). Lastly, the instructional unit provided students opportunities to conceptualize that the measure of an angle created by an incoming path of a ball when it hits a wall is of equal measure with the angle created by its rebound.

To this end, these findings raise the following needs: (1) to rethink the definition of an angle in the curriculum documents, and (2) to incorporate body motion activities when teaching angle and angle measure. In addition, the findings raise the following questions: (1) how do sixth-grade students represent a zero degree angle and a straight angle? (2) how can a miniature golf geometry unit of instruction support students' conceptualization of slope and turn as angle contexts? I next present a summary of the findings of RQ2.

Instructional Supports that Contributed to Students' Conceptualization of Angle and Angle Measure During the Instructional Unit

As noted in the latter summary of research question one findings, the instructional unit provided students opportunities to conceptualize: (1) reflex angles, their measures, and relation to other angles such as acute, straight, and obtuse, (2) the need to create and measure angles for a better approximation of a curved side of a miniature golf hole, and (2) the measure of angles created by an incoming path of a ball once it hit a wall, and its rebound. Using Anghileri's (2006) levels of supports framework, I identified the following supports as having contributed to students' conceptualization of the aforementioned angle and angle measure ideas. Anghileri has defined instructional supports to comprise of those that need non-direct teacher interaction with students such as environmental affordances (level one supports), as well as those that need direct teacher interactions with students such as explaining, reviewing, restructuring (level two supports), and developing students' conceptual thinking (level three supports). The findings revealed that provision of structured task through worksheets and activities and peer collaboration contributed to students' conceptualization of angle and angle measure as environmental affordances. For example, the following structured task illustrated in Figure 5.3 contributed to students' conceptualization of reflex angles, their measure in relation to obtuse and acute angles, in the sense that this task provided students with the opportunity to conceptualize different kinds of angles such as right angles and reflex angles labelled A and B in the context of an obtuse and an acute angle, respectively.

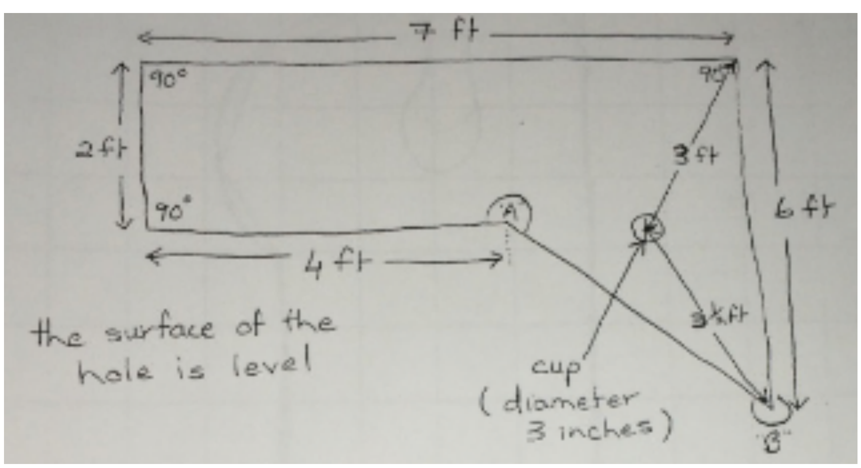


Figure 5.3: A sketch drawing of a miniature golf hole with different kinds of angles. Through peer collaboration, students in pairs engaged with the tasks and activities given before a whole class discussion followed. According to emergent perspective, learning is both individual and social (Cobb & Yackel, 1996). In this sense, peer collaboration is an example of a classroom social norm that contributed to students' conceptualization of angle and angle measure ideas.

In order to focus students' thinking to conceptualize angle and angle measure, the researcher and the teacher used prompts and probing questions (level two supports), as they led students through negotiating meanings of the intended conceptualization of the angle and angle measure. For example, when students were negotiating how they could measure a curved side of a miniature golf hole, building on Jimmy's ideas of measuring length from one corner of the hole to the curved side, the researcher posed the question, "so, I could draw a line out here, but how would I know where it goes? It is 35 in., where?" This question led students to conceptualize the need to create and measure angles in order to tell the direction the curve would take. For instance, Maggy responded, "measure the degree there" which implied measuring the angle as Matt noted. The researcher continued the use of probing questions, which culminated into a conceptual discourse as students conceptualized the importance of measuring angles in telling the direction the curve of a miniature golf hole would take.

Lastly, in order to develop students' conceptual thinking about angles and angle measure, the researcher and the teacher asked students to explain, listen and respond to each other's thinking (level three supports), which are also norms that students were expected to execute (Cobb & Yackel, 1996). Through generating a conceptual discourse (a level three support), the researcher led students in testing their predictions on what would count as an acceptable mathematical explanation of the measures of angles formed by an incoming path of a ball and its rebound once the ball hits a wall. For example, Addy and Brian prediction was that the ball will rebound at 90 degrees, while Matt commented on their prediction and noted that the angle of rebound would depend on the angle at which the ball comes in. This discussion moved further as the researcher and the teacher led students to test their predictions. Through a conceptual discourse, a consensus was reached that the angles created by an incoming path of a ball and its rebound once a ball hits a wall are of equal measures, and this angle measure idea became accepted as mathematically sound. According to Cobb and Yackel (1996), the accepted mathematical explanation is an example of a sociomathematical norm that contributed to students' conceptualization of angle and angle measure ideas.

To this end, these findings have revealed that in the context of a miniature golf hole, all the three levels of Anghileri's (2006) supports contributed to students' conceptualization of angle and angle measure ideas. For level one supports, the use of structured tasks and peer collaboration played a significant role. For level two supports, teacher's/researcher's use of probing question and negotiating meanings contributed through focusing students' thinking towards angle and angle measure ideas. And for level three, students explaining, listening and responding to others' ideas through conceptual discourse contributed to students making connections and developing their conceptual understanding of angle and angle measure ideas, such as identifying, measuring, and relating of reflex angles with other angles in a circle. In a nutshell, a miniature golf context as an example of a real-world context for teaching and learning angle ideas is not a standalone. The findings have revealed that other supports such as the use of worthwhile structured tasks, peer collaboration, use of prompts and probing questions, negotiating meanings, developing conceptual discourse (Anghileri, 2006) are key in supporting students' conceptualization of angle and angle measure. Research done on angles has mostly focused on supports provided by non-direct teacher-students interaction supports such as the use of technology (Clements et al., 1996), the use of body motions (Smith et al., 2014), the use of physical situations and contexts (Mitchelmore, 1997, 1998), but rare studies if any have focused on investigating the direct teacher-students interaction supports, such as use of probing questions and students' explaining their ideas. This study findings suggest the need to pay attention to Anghileri's instructional supports in the teaching and learning of angle and angle measure ideas holistically. I next discuss the interpretation of these findings in relation to the reviewed literature.

Interpretation of Findings

In this section, I discuss three key areas that emerged from the findings that are worth consideration. First, redefining the angle concept in the curriculum documents. Second, the use of body motion activities in the teaching and learning of angle and angle measure. Lastly, the use of Anghileri's (2006) instructional supports in the teaching and learning of angles in a real-world context.

Redefining the Angle Concept in the Curriculum Documents

As noted in Chapter one, the National Council of Teachers of Mathematics (NCTM) emphasizes the need to attend to precision particularly with meanings of mathematical concepts (2000). With this demand, curriculum documents provide the common definition of an angle as “the union of two rays that share a common endpoint” (Smith, 2017, p. 372). As the findings of this study have revealed, sixth-grade students’ conceptualization of the meaning of an angle aligned with the common definition. Participants’ definitions, drawings, and identification of angles suggested their tendency to conceptualize an angle as a geometrical figure formed by two rays meeting at a common point. In addition, participants identified the inside of an angle as the convex region defined by the two rays meeting at a common point. Participants also described angles in shapes as points or corners. These findings suggest participants emphasis on the point of intersection and the straight lines in their conceptualization of an angle, which aligns with the provided common definition. To some students, if there are no visible points and straight lines, then there are no angles. This is a conceptualization that does not include all angle contexts, such as slopes or turns that have one or zero visible rays, respectively (Mitchelmore, 1997). In addition, the common definition of two rays meeting at a common point tends to exclude some angles such as reflex angles (Tanguay and Venant, 2016).

This study is acknowledging Tanguay and Venant’s (2016) definition. Tanguay and Venant suggested that the definition of an angle should first acknowledge that the rays separate the plane into two regions, where each region is an angle. The convex region is the salient angle and the non-convex region is the reflex angle. Tanguay and

Venant definition is also recognized in some high school geometry textbooks, such as Lang and Murrow (1983). Lang and Murrow (1983) put it as a note in their textbook.

> **Note**. You may already be familiar with the definition of an angle as "the union of two rays having a common vertex." We have chosen a different convention for two reasons. First, people do tend to think of one or the other sides of the rays when they meet two rays as pictured (see Figure 5.4), they do not think neutrally. Second, and more importantly, when we want to measure angles later, and assign a number to an angle, as when we shall say that an angle has 30 degrees, or 270 degrees, adopting the definition of an angle as the union of two rays would not provide sufficient information for such purposes, and we would need to give additional information to determine the associated measure. Thus, it is just as well to incorporate this information in our definition of an angle. (p. 21)

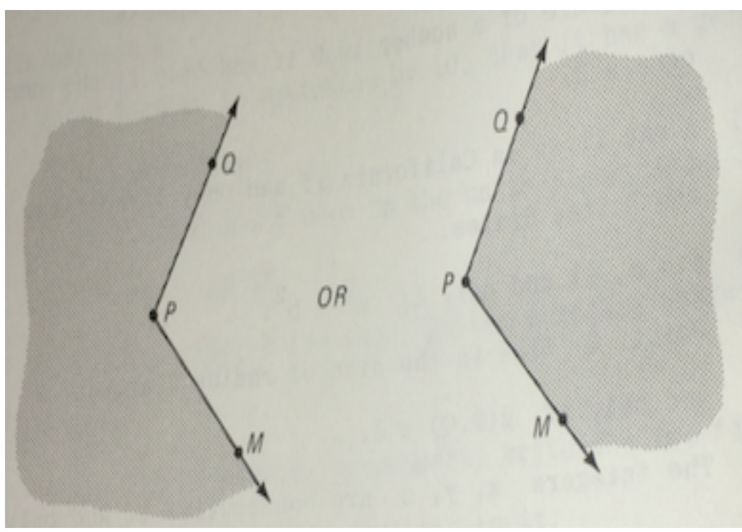


Figure 5.4: Two regions created when rays separate a plane, where the non-convex region is the reflex angle, and the convex region is the salient angle. (Lang & Murrow, 1983, p. 21)

Consistent with Lang and Murrow's note, the tendency of students to conceptualize the salient region as the only angle – excluding the reflex angle – has been confirmed in this study. Particularly, when students were asked to identify the inside and outside of an angle, following the common definition of two rays meeting at a point, all of them indicated the inside of the convex region. This study argues that the inside of an angle depends on the angle of interest as determined by the rays. In other words, if one is focusing on the reflex angle, then that would define the inside of that particular angle. As this study has shown, students can conceptualize other kinds of angles, such as acute, obtuse, straight, and reflex angles, as related to each other in a circle. This emphasizes the Tanguay and Venant suggested definition of an angle since two regions form a complete circle. When students conceptualize an angle as formed when two rays divide a region, this can help them make sense of angles in a circle. As students conceptualize about the measure of angles in a circle, they can develop their understanding of angle measure as an amount of turning, which can incorporate turn and slope contexts whose both rays may not be visible.

The Use of Body Motion Activities in the Teaching and Learning of Angle and Angle Measure

As noted in Chapter One, angle and angle measure are at the confluence of both geometry and measurement. In this sense, angle and angle measure need to be conceptualized together. This study has revealed that at the beginning of the instructional unit, participants conceptualized angle measure in terms of measuring geometrical figures such as acute, obtuse, and reflex angles. In addition, participants conceptualized angle and angle measure as related to the protractor, the tool of measure, as well as the degree, the unit of measure. Students' conceptualization of angle and angle measure as a geometrical figure or a degree has been documented elsewhere (Biber et al., 2013; Tanguay & Venant, 2016). Students' tendency to conceptualize angle and angle measure as a degree is attributed to the systematic use of a protractor, which is labelled in degrees (Tanguay & Venant, 2016).

This study has revealed that students were able to conceptualize the correct measurement of an acute angle 25 degrees and an obtuse angle 110 degrees using protractors when they visualized the amount of turning their bodies were making. By conceptualizing the measure of an angle through the amount of turn their bodies were making, participants were able to place and locate the angle measure correctly using their protractors. A protractor has two readings that corresponds to the same angle measure, the inner and the outer reading as Figure 38 shows. Thus, it requires a student to be able to identify the initial point and the direction the turn is taking when measuring an angle. For example, as Figure 5.5 shows, the angle marked with blue lines measure is 70 on the inner scale, but if one does not identify the initial point and make the correct turn, the angle measure might be wrongly read as 110 degrees on the outer scale.

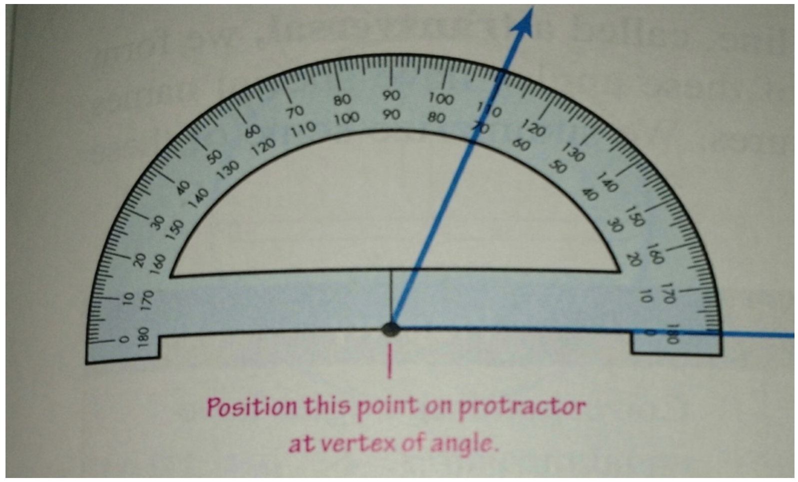


Figure 5.5. A picture of a protractor showing the inner and the outer scale (Pirnot, 2014, p. 439).

Students difficulty with using of protractors to measure angles has been documented even at high school level (Moore, 2013). Thus, this study confirms and emphasizes Smith et al. (2014) findings of the use of body movements activities in supporting students' conceptualization of angle and angle measure, particularly when using protractors as the tool of measure.

The Use of Anghileri's Supports in the Teaching and Learning of Angles in a Real-World Context

This study has revealed that structured tasks coupled with teacher's use of probing questions contributed to the participants' conceptualizing the need to create and measure angles in order to have a better approximation of a curved side of a miniature golf context. Although, there could be other ideas that could be used to approximate a curved side of a hole such as a freehand drawing bisecting the segments in order to locate the center that can serve to draw the curved part, conceptualizing how angles can be helpful in telling the direction the segment would go was key. This showed that these sixth graders were placed in a position to mathematize a realistic real-world situation such as measuring a curved side and asked to think of how angles could help. In this way, the curved hole became a model for conceptualizing about angles and their measure in solving a realistic problem (van den Heuvel-Panhuizen & Drijvers, 2014).

The study has also revealed that structured tasks coupled with peer collaboration and teacher's use of probing questions that lead to a conceptual discourse can support students in conceptualizing angle and angle measure. A sketch of a miniature golf hole with kinds of angles provided students an opportunity to conceptualize angles such as acute, right, obtuse, and reflex together. In particular, this study show that participants developed in their conceptualization of reflex angles and their measure in relation to other angles. Keiser (2003) had documented sixth-grade students with difficulty in measuring of reflex angles.

In addition, students developed in their conceptualization of the measures of angles created by an incoming path of a ball and its rebound when the ball hits a wall. While some students initially thought that the angles created will be of different measures, through a conceptual discourse, the researcher led all students into a consensus of an acceptable explanation. The angles created by incoming path of a ball and its rebound when a ball hits a wall having equal measures was accepted as a sound mathematical explanation (Cobb & Yackel, 1996). To this end, a real-world context is not standalone in supporting students' conceptualization of angle and angle measure. This study has revealed that other instructional supports, both those that need teacher-student direct interactions and those that do not need direct teacher-student interactions contribute to students' conceptualization of angle and angle measure. Well-structured mathematical tasks, teacher's use of prompts and probing questions to focus students' thinking through a conceptual discourse (Anghileri, 2006), contribute to students making connections and developing in their conceptual understandings of mathematical meanings, such as angle and angle measure.

Limitations and Suggestions for Further Studies

Students' understanding of angles in slope and turn contexts is of significance during their future understanding of trigonometry. If I were to design this study, I would consider specific activities that would provide students with more opportunities to conceptualize angles in slope and turn contexts. That was a limitation in the design of activities where future studies can focus on. One instance that was an opportune for students to conceptualize about angles in slope contexts was when they had identified the need to measure inclines on the miniature golf context. When students conceptualized of using tape measures, rulers, or pair of compasses in obtaining the length, follow up questions such as *why the incline or slope is not vertical or horizontal* would have been helpful. Such a question would have provided students with an opportunity to conceptualize angles and angle measure in slope contexts. More so, while there were questions in the pre-interview and post-interview that provided students with an opportunity to conceptualize angles in turn contexts, there lacked explicit activities during the unit of instruction to support that. Future studies should provide activities in a real-world context that can support students' conceptualization of slopes and turns as angle contexts.

The fact that I did not directly conduct this study, I did not have the opportunity to participate in preparing and designing activities of the larger study, as well as choosing of participants. I also did not have the opportunity to conduct the actual experiment during which testing and modifying of the instructional activities aimed at improving instruction on angles would have occurred. As a consequence, a further study specifically on angle and angle measure in a real-world context following a research-based methodology would illuminate more light in this area.

This study raised pertinent areas that future studies can focus on: (1) The role that definitions play in students' understanding of mathematical concepts, (2) the level when students transition from using the informal mathematical language, for instance "corners" to using the formal mathematical language, "angles." Lastly, how sixth-grade students represent a zero-degree angle and a straight angle could be areas of focus. Such studies may add to the knowledge base of students' mathematical learning.

Recommendations

Based on my reflection about the findings of this study, I have the following suggestions for the following sectors of mathematics education: (a) the school mathematics curriculum, (b) pedagogical implications, and (c) mathematics teacher educators.

For the School Mathematics Curriculum

In this section, suggestions are addressed to both curriculum developers and policy makers of the mathematics standards.

Need to emphasize the teaching of angle and angle measure in the standards. The mathematics standards greatly determine what goes into the school mathematics curriculum. For instance, in the United States, the Common Core State Standards (CCSS) are partly or wholly adopted by each state and local levels, and hence what is in them acts as a guide to a curriculum. The CCSS for mathematics (NGA & CCSSO, 2010) show that students are formally introduced to angle and its measure at fourth grade, but the topic is not emphasized compared to others such as length and its measure. As a consequence, less attention is paid to angle and angle measure compared to other measurements, such as length. As revealed in this study, students find it easier to estimate lengths than estimating angles because they have several instances to explore length compared to angles. This study suggests that *mathematics standards should emphasize the teaching of angle and its measure as it does for length measure*, as this will influence what goes into the curriculum.

Need to emphasize introduction of reflex angles as the angle concept is formally introduced. According to the CCSS (NGA & CCSSO, 2010), angles are formally introduced at fourth grade, the standards emphasizes acute, right and obtuse angles with no mention of reflex angles. This is likely the reason the participants in this study found it easier to conceptualize right, acute, and obtuse angles compared to reflex angles. On further evaluating K-8 standards, there was no mention of reflex angles at all (NGA & CCSSO, 2010). Thus, one wonders when reflex angles are introduced to students. This study suggests that *the standards need to note and emphasize introduction of reflex angles and their measure as the angle concept is being formally introduced*. This way, as the school curriculum is being developed with a goal of reaching the standards, reflex angles can also be considered and emphasized as other kinds of angles.

Need for the school curriculum to redefine the angle concept. As previously noted, most textbooks define an angle as a geometric figure that "consists of the union of two rays that have a common endpoint," which is called the vertex of the angle and each ray a side of the angle (e.g., Bassarear, 2001, p. 450) as Figure 2.1 (page 20) shows. In addition, an angle is said to partition a plane into three disjoint sets: the angle itself, the interior of the angle, and the exterior of the angle (Bassarear, 2001; Masingila, Lester, & Raymond, 2011). This definition of an angle tends to make people conceptualize one or the other side of the rays as the only angle excluding the other, and thus as a consequence reflex angles are excluded when conceptualizing the definition or the meaning of an angle (Lang & Murrow, 1983). Consistent with previous researchers (Tanguay & Venant, 2016), this study suggests *the need for the school mathematics curriculum to redefine the angle concept* in order to have sufficient information that incorporate all kinds of angles. This study emphasizes Tanguay and Venant (2016) definition of first acknowledging that two rays divide a region into two angles, the salient angle and the reflex angle, and that an angle is an amount of turning.

Need to emphasize a zero degree angle representation in the textbooks. Participants in this study conceptualized a representation of a zero degree angle as a straight line. By representing a zero degree with a straight line, that would make it difficult to differentiate from a straight angle representation. In most textbooks, a straight angle is represented as a single line having two rays on opposite directions as shown in Figure 5.6. Thus, for a zero degree angle, it would make sense to use a ray only, which is rarely shown in the textbooks. This study suggests *the need for the curriculum developers to emphasize a zero degree angle representation as it does for a straight angle in the textbooks*. This way, students can be able to differentiate the two representations.

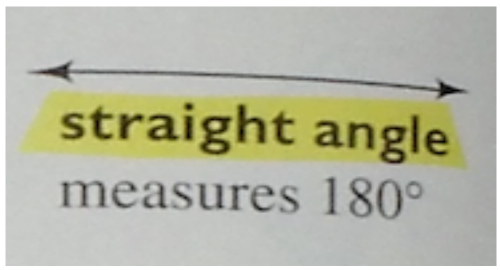


Figure 5.6: A diagram of a straight angle as represented in middle grades mathematics, an interactive approach course 2 textbook.

Need to increase real-world contexts for angle exploration. This study extends Masingila and de Silva (1997) findings that a real-world context such as a miniature golf context can support students in angle exploration. The study has shown that a miniature golf geometry unit of instruction, coupled with structured tasks and conceptual discourse supported students in conceptualizing: (1) reflex angles and their relationship with other kinds of angles, (2) the use of angles in telling the direction a segment would take when measuring a curved side of a hole for a better approximation, and (3) conceptualizing that the measure of angles created when an incoming path of a ball hits a wall and bounce off are of equal measure. Although, the use of a miniature golf hole in studying of angles may not be evident in the curriculum, this study has shown that provision of real-world contexts would support students in conceptualizing angle and angle measure. As noted by Bustang et al. (2013), angles are closely related to real-world contexts. Thus, this study suggests the need for the *curriculum to increase real-world contexts* for angle and angle measure exploration.

Need to increase body movement activities in the learning of angle and angle measure. Turn-as-body-motion is documented to support students' understanding of angle and angle measure. While most of the research studies in this area have focused on third- and fourth-grade students (e.g., Clements et al., 1996; Smith et al., 2014), this study focused on sixth-grade students. This study showed that the sixth-grade students conceptualized angle measure as an amount of turning or a turn, when they were encouraged to use their body movements in spotting the right direction, as well as getting the correct measure of an angle when using a protractor. The use of protractor is documented to pose difficulties to students even at high school level, particularly when selecting which scale to read (Moore, 2013). This study has shown that students at sixth grade can develop in their conceptualization of angle and angle measure while conceptualizing the amount of turning their bodies are making for given measures using protractors. Thus, the curriculum needs *to increase activities that will engage students in using their body motions when conceptualizing angles and their measure*. Students' conceptualization of angles as an amount of turning can be instilled as early as kindergarten as students turn themselves around during physical education games, music or dance (Kennedy & Tipps, 2000). This way, students as early as kindergarten can develop their spatial sense of turns and angles, well before they are even introduced to the angle measure using a standard protractor.

Pedagogical Implications

The suggestions for the school mathematics curriculum can be useless if no one is able to implement them. In addition, as much as textbooks are important guide for teachers, some lack instructional activities for helping students develop through their geometrical levels of thinking (Fuys, Geddes, & Tischler, 1988). Thus, the teacher is the key to effective implementation of the curriculum. The following are the suggestions from this study for teaching angles.

Need to introduce the term angle in place of “corner” as the angle concept is formally introduced. In early stages of learning geometry, students need angle knowledge in classifying geometric shapes. According to the Common Core State Standards, the expectation is at the fourth-grade level students are able to classify and/or describe shapes based on their side and angle properties. This expectation aligns with the analysis level, the second level of the van Hiele model of geometrical thinking. According to the van Hiele model, students at the second level should be able to conceptualize classes of shapes based on their properties (Crowley, 1987). More so, students at the informal deduction level, the third level of the van Hiele model, should be able to give meaningful definitions. This study has revealed that even at sixth grade, some students conceptualized angles as corners and/or points. This is a sympathizing situation given that this is the second stage in van Hiele model of geometrical thinking. Since students begin classifying shapes as early as kindergarten and can use informal language such as corners to describe shapes (NGA & CCSSO, 2010), this study suggests that t*eachers should emphasize the term angle in place of corners as the angle concept is formally introduced*. This way, students can adopt the right terminology to describe shapes based on their properties. As Crowley (1987) emphasized, instruction is the most significant factor in supporting students' through geometrical levels of thinking rather than maturation.

Need to support students in developing the cognitive skills required to conceptualize the measure of reflex angles. Most standard protractors have both an inner and an outer scale running from 0 to 180 degrees in opposite directions. Reading such a scale for angles above 180 degrees can be challenging. This study suggests that *teachers should support students in developing the cognitive skills required to conceptualize how to measure reflex angles that goes beyond the use of a protractor*. Students need to know which angle to measure in order to add it to 180 or subtract it from 360, if acute, or subtract it from 360 if obtuse in order to obtain the reflex angle. This study suggests that re-defining the definition of an angle and conceptualizing about angles in a circle can support students in developing the cognitive skills required to find the measure of reflex angles. In addition, where available, teachers should introduce to students a complete revolution protractor, when introducing reflex angles.

Need to incorporate Anghileri's levels of supports in real-world contexts during angle and angle measure exploration. This study has shown that real-world contexts are not standalone in supporting students' conceptualization of angle and angle measure. Structured tasks through worksheet and activities can help students tease out the expected angle and angle measure conceptualizations. In addition, teachers are a great resource in focusing students' thinking through use of prompts and probing questions that can lead to a conceptual discourse of the acceptable mathematical explanations (Cobb & Yackel, 1996). This study suggests *the need to incorporate the Anghileri's (2006) levels of supports in angle and angle measure exploration in a real-world context*.

For Mathematics Teacher Educators

Most novice teachers will often teach following what they know and how they know to do it. It is therefore important for mathematics teacher educators to prepare prospective teachers adequately in their subject areas. Research shows that many beginning teachers lack sufficient knowledge required to teach geometry topics (Hourigan & Leavy, 2017; Keith, 2000; Robichaux-Davis & Guarino, 2016). One of the areas of geometry that teachers need knowledge in order to be prepared to teach effectively is the "mastery of core concepts and principles of Euclidean geometry in the plane and space" (Keith, 2000, p. 111). Prospective teachers need to understand the angle concept and its measure well in order to effectively teach it. Hence, mathematics teacher educators have the obligation to prepare prospective teachers with the necessary repertoires for their future teaching. This study has the following suggestions for the mathematics teacher educators with respect to preparing prospective teachers for teaching of the angle concept and angle measure effectively.

(1) Engage prospective teachers (PTs) with rich activities that will support them to conceptualize angle and angle measure. For example, have PTs engage and reason with shapes, as this study has showed that shapes provide an opportunity to

conceptualize about angles as properties of shapes. Other activities, like identifying angles in given shapes and figures, can also be used with prospective teachers.

(2) Mentor PTs in how to teach the angle concept and its measure in ways that are accessible to learners. For instance, this study has showed that sixth-grade students were able to conceptualize about reflex angles and their measures in relation to acute, obtuse, or straight angles in a complete turn through structured tasks in a miniature golf context. Prospective teachers should be encouraged to teach angles beginning with students' informal angle experiences, such as the use of body movements (Smith et al., 2014), the use of physical angle situations like a turning door, an oven knob, a road bend (Mitchelmore, 1997, 1998), before introducing the formal abstract angle concept to students.

(3) Mentor PTs in choosing, adapting and using tasks that go beyond standards' stipulations to meet the needs of the students. This study has shown that reflex angles are rarely mentioned in the standards or emphasized in the K-8 curriculum. Thus, prospective teachers need to know how to modify and supplement curriculum materials on the teaching of geometry topics. One way to do this is to teach prospective teachers how to integrate research with practice. Current research should be a guide to prospective teachers' future teaching.

Generalization of the Analyses

According to Cobb (2003), generalization in a design research is "accomplished by means of an explanatory framework rather than by means of a representative sample, in that the theoretical insights and understandings developed during one or more experiments can feed forward to influence the analysis of events and thus pedagogical planning and decision making in other classrooms" (p. 4). Thus, it is not my claim that the activities used in this study are necessarily useful in new settings, but I propose that a framework explaining why specific means of support promoted students' conceptualization of angles can be useful for instructional design decisions and curriculum development. This way, the current study design research findings should be seen to go beyond the specific contexts in which they were developed and generalized to new contexts and situations (Steffe & Thompson, 2000).

Trustworthiness of the Analyses

The trustworthiness of the findings depends on the "extent to which they are reasonable and justifiable given the researcher's interests and concerns" (Cobb & Whitenack, 1996, p. 225). Cobb and Whitenack further noted that for analyses involving small groups, it is pertinent to acknowledge that "other plausible interpretations of the children's mathematical activity could be made for alternative purposes" (p. 225). Similar sentiments were made by Cobb, Stephan, and Gravemeijer (2001). In order to bolster trustworthiness, the current study documented all phases of analysis, including testing of conjectures (Cobb, Stephan, & Gravemeijer, 2001).

I justified final claims and assertions by backtracking through the various phases of the analysis (Cobb & Whitenack, 1996; Gravemeijer & van Eerde, 2009), and provided a limited number of critical episodes which were informed by the entire data under consideration (Cobb, Stephan, & Gravemeijer, 2001). In general, this study empirical grounding is based on a systematic and thorough analysis of the data set, and not on statistical analysis (Gravemeijer & Cobb, 2006).

According to Cobb and Whitenack (1996), another way of enhancing credibility of the analysis is by the researcher ensuring a prolonged engagement with the study participants. For the larger project on which the current study drew, the lead researcher was present throughout the whole school year, including during the design experiment, and also taught the lessons. This provided her with the opportunity of having a prolonged engagement with the students. Given that she provided feedback for the analysis of this study, this enhances the credibility of the analysis.

The analysis of the current study was also read and critiqued by others in order to ascertain its credibility (Cobb, Stephan, & Gravemeijer, 2001; Cobb & Whitenack, 1996). In particular, Cobb and Whitenack (1996) noted that peer debriefers who are familiar with the participants can ascertain the credibility of the analysis at a global level. For this study, the lead researcher of the larger project consistently read the analyses and ascertained their credibility.

As explained in Chapter Three, I also built some coding reliability by having an external coder code six transcripts, an equivalent of 20% of the total data that I analyzed. On calculating our rate of coding agreement, Cohen's kappa statistic yielded 0.9524 with a 98 % agreement, which is an excellent rate of agreement (Syed & Nelson, 2015). In addition, the credibility of the analyses was also ascertained by my dissertation committee members who read the final document.

Chapter Summary

In this chapter, I presented an overview and interpretation of the study findings in relation to the reviewed literature. I also highlighted limitations and recommendations of the study, particularly for the school mathematics curriculum, pedagogical implications, as well as for mathematics teacher educators. In addition, I discussed generalization and trustworthiness of the analyses. The findings emphasized three areas: (1) redefining the angle concept in the curriculum documents; (2) the use of body motions activities in the teaching and learning of angle and angle measure; and (3) the use of Anghileri's (2006) instructional supports in the teaching and learning of angles in a real-world context.

Laying a strong foundation for students' understanding of the multifaceted angle concept and angle measure at lower levels of schooling is of paramount significance. When students are limited to one definition of an angle, this can jeopardize their experience of angles as rotation, an experience needed in trigonometry and other advanced courses in mathematics later at high school levels. This study has confirmed the need to redefine the angle concept in order to first acknowledge that the two rays divide a plane into two angles, a convex angle and a reflex angle (Tanguay & Venant, 2016). This way the definition can include reflex angles, excluded from the static definition of an angle as a union of two rays meeting at a common vertex. This study also suggests the curriculum documents to emphasize the definition of an angle as an amount of turning as this include slope and turn angle contexts which are dynamic in nature. Modifying the definition of angles in the curriculum documents will provide more opportunity to support students in developing more sophisticated understandings of the multifaceted nature of the angle concept.

This study emphasizes the use of body motion activities in the teaching and learning of angle and angle measure with sixth-grade students, extending Smith et al.'s (2014) studies with third- and fourth-grade students. It is evident that students' difficulty with the use of protractors to measure angles can be solved when they learn to visualize an angle as an amount of turning through their body turns. In addition, real-world context coupled with structured tasks and use of conceptual discourse can support students in making connections of the mathematical meanings (Anghileri, 2006), such as conceptualization of angle and angle measure in this study. While other studies have focused on Anghileri's level one supports in angle exploration, this study added the knowledge base in presenting analyses of all Anghileri's levels of supports in students' conceptualization of angle and angle measure.

Besides this study confirming previous studies in several ways, it has also contributed to the knowledge base by providing a detailed analysis of sixth-grade students' conceptualization of angle and angle measure, as well as instructional supports through a miniature golf geometry unit of instruction. The philosophy of a design-based research methodology emphasizes the need to understand in order to bring change and vice versa (Gravemeijer & Cobb, 2006). Thus, this study effort to understand how sixth-grade students conceptualize angle and angle measure, emphasizes the need to redefine the concept in order to include all angle contexts. More so, this study has shown that teacher's supports such as use of prompts and probing questions that focus students' thinking towards making connections of mathematical meanings can lead to a conceptual discourse (Anghileri, 2006), and hence generation of acceptable mathematical explanations of concepts (Cobb & Yackel, 1996).

Standards stipulations determines what goes in the curriculum. It is my sense that students' struggle to understand angle and angle measure because of insufficient emphasis given to this topic, particularly for K-8 grades. According to the CCSS, students are introduced to length from kindergarten and its measure is emphasized from $1^{st} – 3^{rd}$ grades. In the same standards, students are formally introduced to angle and its measure at 4^{th} grade, and no emphasis continues in the grades after (NGA & CCSSO, 2010). This would explain why students struggle with estimating angle measures than estimating length measures as a student in this study lamented. To this end, I suggest the emphasis given to length measures; the same emphasis should be given to angle measures in the standard stipulations and hence in the school mathematics curriculum.

Appendix A 1: Day 1 Lesson Plan, Journal 1, Worksheet 1

Introduction to Unit and Preparation for Field Trip (March 10th)
Goals: Engage students in thinking about designing a miniature golf course and what is involved with that. Have students brainstorm about what they might want to observe and measure on the field trip tomorrow.

Materials 3 miniature golf hole(s) in center of room, rope, copies of Worksheet 1, overhead transparencies, overhead pens

Timetable

5 min Introduce unit to students by explaining that for this next unit we will be investigating some mathematical ideas while designing a miniature golf course. Ask who has played miniature golf. Say that during the class today the students will be working with their partner to come up with ideas of what they should take note of while on the field trip tomorrow. They can draw on their experience and also use ideas from the hole(s) set up in the class. Ask if there are any questions about what they are to do. Distribute Worksheet 1.

10 min Students should work with their partners to draw a rough sketch of a miniature golf hole they have played. Those who have not played miniature golf, can make a rough sketch of the class hole(s). Each student should complete his or her worksheet even though it may be exactly like his or her partner's. They should also describe the hole and note what measurements they think should be taken in order to reproduce the hole. We may need to ask them questions while circulating to start them thinking about what details are involved with a hole. While students are working, note mentally the different kinds of details in their drawings (point of view/perspective, details of obstacles, etc.). We can have overhead transparencies and pens available and ask some students to draw their sketch on the transparency so that they can share with the class.

10 min Facilitate a whole-class discussion by having students share their insights about #2 and #3 on Worksheet 1. Compile a list of things to measure and how each might be measured on the board. Recall some unusual details or ones not yet brought up from time of circulating and have those students share their ideas. Also discuss how accurate the measuring should be at Pinescape.

5 min Assign homework. Say that we will be keeping a journal during this unit and will have an assignment from time to time to write in there. Today is the first assignment. Try to write in complete sentences as much as possible. Also give any directions that are needed about the field trip tomorrow (e.g., meet in the cafeteria at 8:45 a.m.). Discuss that students will receive a folder tomorrow that they will use for the whole unit. Tomorrow each pair will receive a protractor and a tape measure (demonstrate how to use the protractor)

HW Journal #1: What mathematics do you think might be involved in designing a miniature golf course?

Note: We will compile the measurement ideas from all three classes and make a sheet for students to take on the field trip.

Worksheet 1

- Draw a rough sketch of a miniature golf hole you have played (or of a class hole).
- Describe in short phrases the details of the hole (its general shape, details of its sides and angles, any obstacles, etc.).
- What measurements should you take to be able to reproduce the hole at another time?

A miniature golf hole set at the center of the classroom

Appendix A 2: Day 2 Lesson Plan and Journal 2

Field Trip and Discussion (March 11th)

Goals: Engage students in making observations and measurements on an actual miniature golf course.

Materials brought by teacher tape measure and large protractor for each pair, unit folders for students, copies of observation sheet and measurement ideas, 5 Pinescape tickets for each student, directions to Pinescape

Materials brought by student pencil

Field Trip Instructions: Before we board the buses, give students directions while they are in the cafeteria. They should sit with their partner. Distribute the observation sheet, tape measure, protractor, and hole and course assignments. For the first 15 minutes, all of the pairs are assigned to a specific hole and make observations and measurements at that hole. At the end of 15 minutes, each pair moves to their second hole assignment. At the end of 30 minutes, all pairs may play for 30 minutes. Distribute folders, protractor, tape measure, and hole assignments. Discuss the hole and course assignments with the students and make sure they understand what they are to do when we arrive at Pinescape. As students get off the buses once we have arrived at Pinescape, we will give them the 5 tickets that they need to get in (or it may be that we can hand these in all at once).

Timetable for Discussion in Afternoon

10 min Collect Journal #1. Have two pairs work together to share their work and discuss what mathematics they did during the field trip.

15 min Facilitate a whole-class discussion about the field trip. Possible questions to ask might be: What was difficult/easy? What kinds of things did you measure? What kinds of things did you estimate? What kinds of things did you reason out in order to find the measure? How did you do the measuring/estimating/ reasoning? What kind of problem-solving strategies did you use? (almost anything is acceptable here) How did you work together as a team? What new ideas do you have now about the mathematics involved in designing a miniature golf course? What would you like to learn more about in order to design a course?

5 min Collect students' observation sheet from the field trip. We could possibly display their work somehow. Assign homework.

HW Journal #2: Describe how you went about drawing and measuring a hole. Describe what was easy to do. Describe what was difficult to do. Describe what strategies you used in playing two different holes.

Measurement, Drawing, and Observation Sheet

- Draw a rough sketch of your first miniature golf hole.

- Measure anything that you think will help you to reproduce the hole later. (You can look at the sheet with the ideas we came up with in class yesterday to help you.) Make note of the measurements on your drawing and/or below.

- Describe any details of the hole that you cannot draw.

- Describe your strategy for playing the hole.

- Draw a rough sketch of your second miniature golf hole.

- Measure anything that you think will help you to reproduce the hole later. (You can look at the sheet with the ideas we came up with in class yesterday to help you.) Make note of the measurements on your drawing and/or below.

- Describe any details of the hole that you cannot draw.

- Describe your strategy for playing the hole.

Appendix A 3: Days 3 - 5 Lesson Plan, Journal 3 and Worksheet 2

Scale Drawing (March 12th, 13th and 17th)

Goals: Engage students in thinking about differences between a rough sketch and a scale drawing. Engage students in making scale drawings of holes.

Materials brought by teacher: copies of Worksheets 2 and 3, sheets of blank paper, compasses, graph paper, copies of two sketches from Pinescape, overhead transparencies of how scale drawings are used in certain types of work and of Worksheets 2 and 3

Materials brought by student: unit folder, ruler, protractor, compasses (if they have them)

Timetable

<u>Day 3</u>
10 min Collect Journal #2. Distribute Worksheet 2 and have students spend a few minutes discussing the questions with their partners.

20 min Have the class discuss why it is a rough sketch and what might make it a better sketch. Ask who has heard of scale drawings. Ask questions like: What things do we need to consider when making a scale drawing? Will the scale drawing be as large as the actual hole? Why or why not? What might be a good scale to use? Why? Lead students in making a scale drawing of the hole.

HW Complete the scale drawing of Anne's hole if you did not finish in class.

<u>Day 4</u>
10 min Have students discuss with their partners for several minutes, comparing their measurements for the parts that Anne did not record, then discuss as a whole class. Some students may not remember how to use their protractor so we should demonstrate this. They have also not measured reflex angles before so we should have them discuss their strategies for finding this angle. Some students may measure the obtuse and subtract its measure from 360°, some may make the reflex into a straight angle (by extending the one ray of the angle) and an acute angle and add their measures, while many students will not know how to find the measure.

10 min Have students discuss with their partners how to use the compass to find the location of the cup. Then discuss as a class and demonstrate using the large wooden compass. Have students find the location of the cup on their drawing. Discuss if this would be the only place that the cup could be—is this the only location that is 3 ft from the one vertex and 3 1/2 ft from the other vertex?

5 min Ask the students if they can think of situations where the scale drawing would be bigger than the object it represents (in contrast to our situation where the scale drawing is much smaller than the actual miniature golf hole). Give examples of how scale drawings are used in electronics and in architecture. Use overheads to illustrate.

5 min Distribute graph paper and copy of the two holes from Pinescape. Discuss what "means and the scale (1 unit = 10 inches). Assign homework.

HW Make a scale drawing of hole #1 (hole with all straight sides) using the scale 1 unit = 10 inches.

<u>Day 5</u>
5 min Discuss that scale should be 1 unit = 10 inches (it is confusing to say 1 box = 10 inches since box is usually thought of as 3-dimensional—cube—and square is not correct because that denotes area). Discuss homework—have partners compare their drawings. Ask what difficulties the students encountered, if any. Have students place the cup if they didn't have a compass to do it at home.

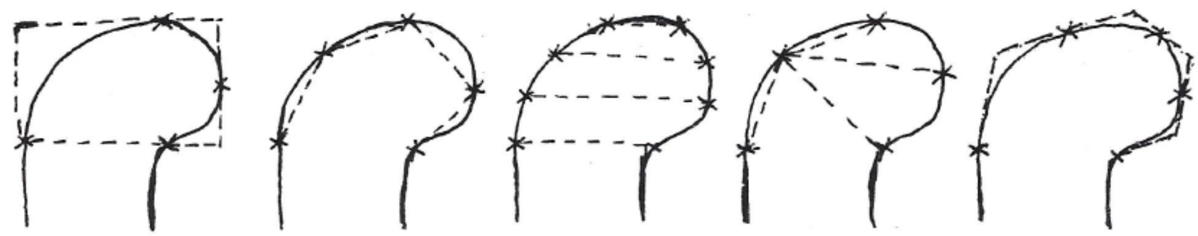
5 min Have partners compare and discuss similarities and differences between the scale drawing and the actual object. What measurements stayed the same from the original to the scale drawing? What measurements changed? How did they change? Is it possible to have a 30" side of a hole be represented by 3 units and have a 40" side of a hole be represented by 5 units? Will a 45° angle on the hole be represented by a 45° angle on the scale drawing or by another angle? Facilitate a whole-class discussion on the merits of a scale drawing vs. a rough sketch. Explain how scale drawings can be used to reduce or enlarge, but that different parts of the figure remain in the same "proportion" to each other. Introduce the terms "congruent" (same shape, same size) and "similar" (same shape, not necessarily the same size). Use overheads from electronics and architecture as needed.

10 min Distribute hole #2 and have students discuss what other measurements they would need to make a scale drawing of it. Focus the discussion on their strategies for measuring and drawing the curved side of the hole. Give information as needed.

5 min Distribute Journal #3.

HW Complete a scale drawing of hole #2.

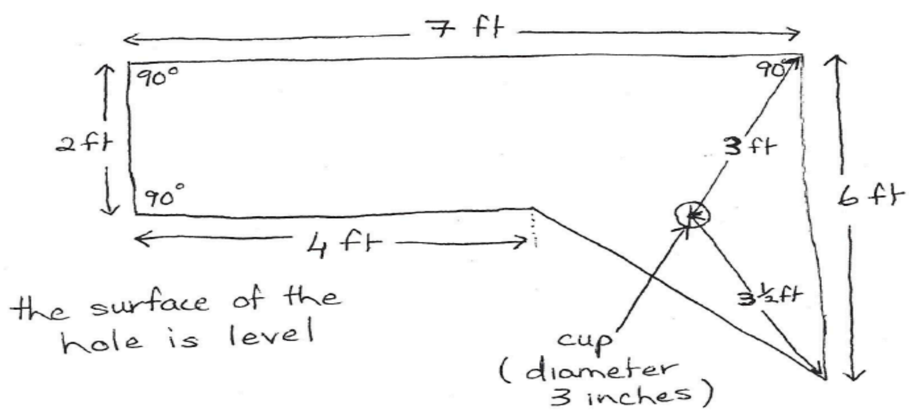
Journal #3: Five different strategies that students in Math 6, 7 and 9 have come up with for measuring a curved side of a hole are shown below.



- Pick the one that you think is best.
- Explain what measurements you need to take.
- Explain why you think this is the best way to measure a curved side.

Worksheet 2

Anne made the following rough sketch of a miniature golf hole she had played. The sketch includes the measurements and notes she made about the hole.



1. What geometrical shape is the hole?
2. What other measurements might Anne have recorded?
3. Why is this a rough sketch?

Appendix A 4: Day 10 Lesson Plan and Worksheet 10

Path of Rebound (March 24th)
Goals: Engage students in exploring the path of the ball when it hits a wall and rebounds.

Materials brought by teacher: 1 miniature golf hole in center of room, putters, golf balls, cups, copies of Worksheet 10, overhead transparency copies of hole and Worksheets 10 and 11

Materials brought by student: unit folder, pencil

Timetable

5 min Discuss and collect Worksheet 9.

20 min Distribute Worksheet 10. Have volunteers take turns to play shots on the class holes and observe the path of the ball after it rebounds off of a wall. Have students make a prediction about the path of a rebound. Have the students then experiment systematically to determine the path a ball will travel after it hits a spot on the wall by varying the path of approach. Have students record their observations, predictions, and results of testing their predictions.

5 min Wrap up the discussion. Assign homework.

Worksheet 10

1. Record your observations about the path of the ball as it is hit and rebounds off of a wall. Draw pictures to illustrate your observations.

2. Make a prediction about the path of the ball as it is hit and rebounds off of a wall.

3. Record what we did to test our prediction and what the results were. Draw pictures to illustrate what we did.

   Do you think your prediction is true? Why or why not?

Appendix A 5: Days 11-12 Lesson Plans, Journal 4 and Worksheets 12 and 13

Angles of Incidence and Reflection (March 25th)

Goals: Engage students in exploring angles of incidence and reflection.

Materials brought by teacher: copies of Worksheet 12 and Journal #4, overhead transparency copies of Worksheet 12, Miras (one per student), graph paper

Materials brought by student: unit folder, pencil, ruler, protractor

Timetable

5 min Discuss and collect Worksheet 11.

20 min Distribute Worksheet 12 and Miras to students and lead the class through the activity. Demonstrate how to use the Miras. Have them discuss their observations. Define angles of incidence and reflection.

5 min Wrap up discussion and assign homework.

HW **Journal #4**: What angle ideas are involved in the path of a ball and its rebound? Be specific and include a drawing.

Worksheet 12

In this activity, you will explore the idea of reflection and examine angles created by the incoming and rebounding paths of the ball.

- The diagrams below show different paths a ball might take when rebounding off a certain spot on a wall.
- The wall acts like a mirror in that **the path of rebound is a reflection of the path the ball would have taken had the wall not been there**.

Do the following with each diagram:

1. Sketch a line to predict the path of rebound.
2. Extend the incoming path beyond the wall using a dotted line to show the path the ball would have taken had the wall not been there.
3. Place the Mira along the wall as demonstrated by Dr. Masingila.

4. Draw the reflection of the extended path by placing your hand behind the Mira and sketching the line you see. This reflection is the path the ball will take.

5. Compare your predicted path with that the ball will take.

6. Mark any angles that are created by the incoming and outgoing paths of the ball. Compare these angles.

Diagram 1

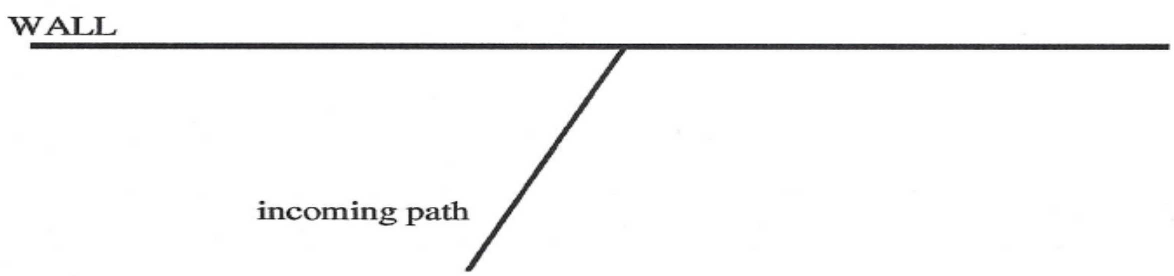


Diagram 2

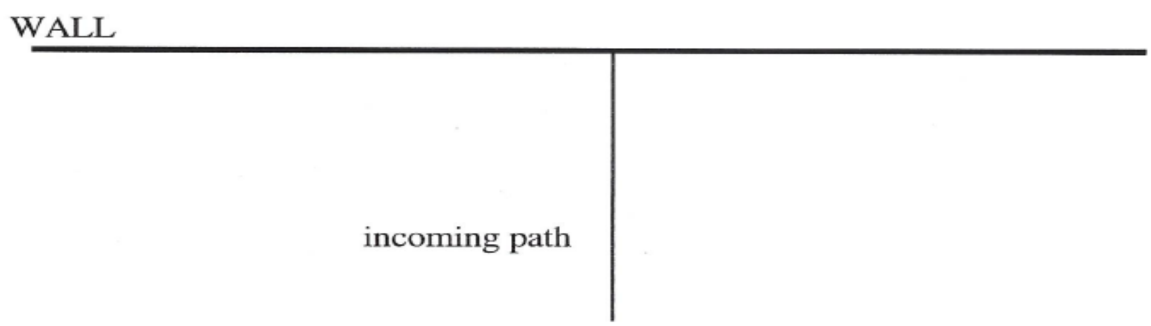
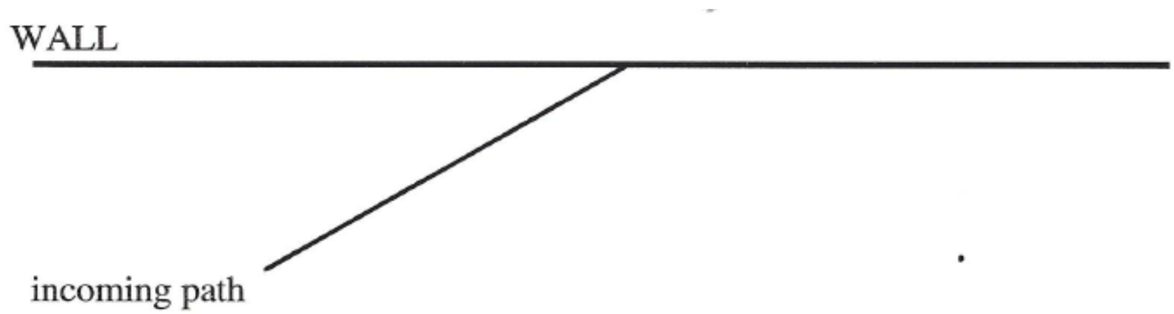


Diagram 3



Day 12: Angles of Incidence and Reflection (continued) (March 26th)
Goals: Engage students in exploring angles of incidence and reflection.

Materials brought by teacher: copies of Worksheet 13, overhead transparency copies of Worksheet 13, Miras (one per student), graph paper

Materials brought by student: unit folder, pencil, ruler, protractor

Timetable

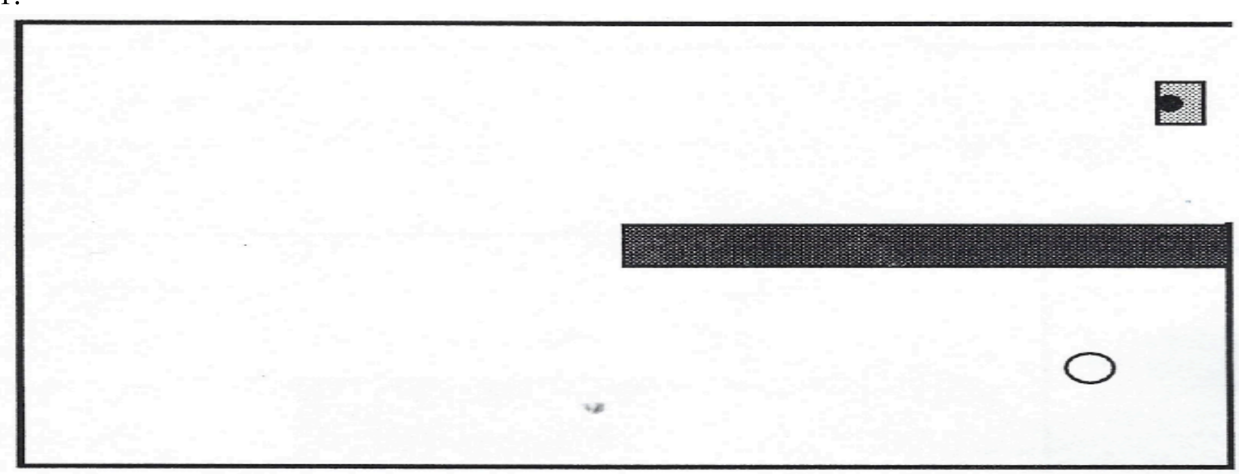
5 min Discuss and collect Worksheet 11 and Journal #4.

25 min Distribute Worksheet 13 and Miras to students and lead the class through the activity.

Worksheet 13

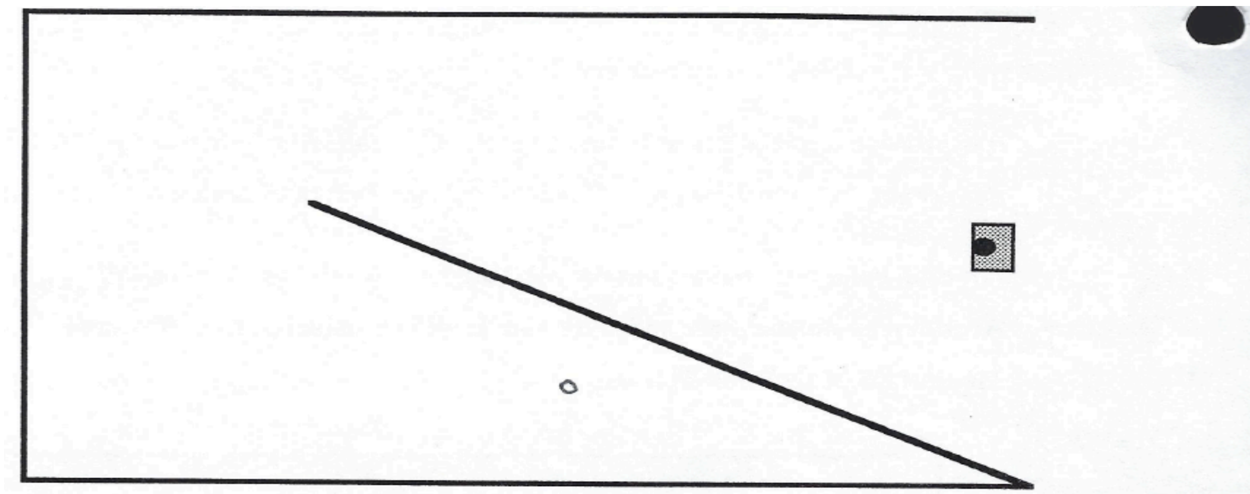
- It is possible to make a hole-in-one on each of the following miniature golf holes.
- For each hole, predict the number of rebounds the path of the ball would take to get a hole-in-one.
- For each hole, draw the path of the ball that could result in a hole-in-one.
- At each rebound, measure and record the angles of incidence and reflection.
- The surface of each hole is level

1.



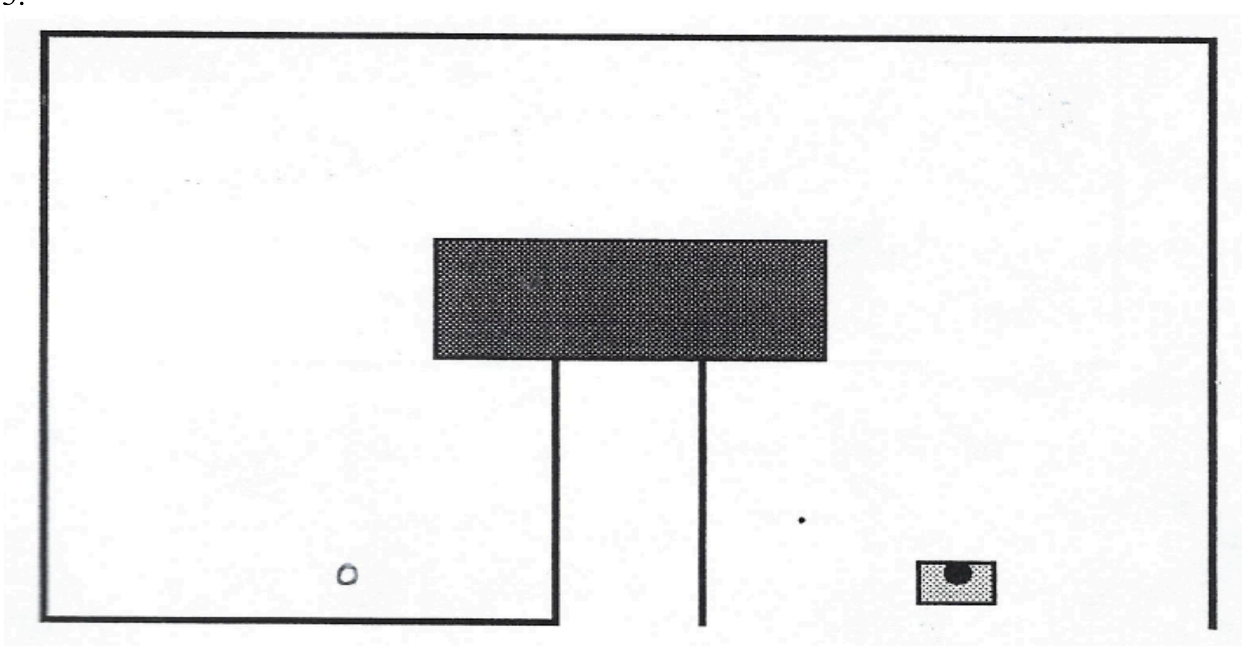
Prediction: _______

2.



Prediction: _______

3.



Prediction: _______

Appendix A6: Day 17 Lesson Plan and Worksheet 18

Culminating Activity (April 4th)

Goals: Engage students in designing a miniature golf hole (2-dimensionally) and applying some ideas of the unit.

Materials brought by teacher: copies of Worksheet 18, transparency copy of Worksheet 18, geometric shapes, grid paper

Materials brought by student: unit folder, pencil

Timetable
5 min Discuss and collect Worksheet 17.
20 min Distribute Worksheet 18, the geometric shapes and the grid paper.
5 min Collect folders.

Worksheet 18

1. Work with your partner to design a miniature golf hole by placing the three shapes together without overlapping. Your design must fit on the grid paper.

2. Place your design on grid paper, lining up the edges of the shapes with the grid lines as much as possible.

3. Trace the outline of each shape as placed on the grid paper. Used dashed lines to show where two shapes meet.

4. Label the shapes you drew A, B, and C. Below write the mathematical name of each shape.

   A is a ________________

   B is a ________________

   C is a ________________

5. Write "scale: 1 unit = 1 ft" at the top of your grid paper. Use your tape measure to determine the length of 1 unit in inches. Write this below.

   1 unit = ______ inches

6. Find the perimeter of your hole in feet. Keep a paper trail below.

7. Find the area of your hole in square feet. Keep a paper trail below.

8. On your drawing:

   a. mark the starting point with a "X".

   b. mark the cup with a small circle so that the ball will have to rebound off one or more walls in order to make a hole-in-one. On the full-size hole, the cup has a diameter of 6 inches. Draw your circle in the scale so it is the correct size.

   c. draw the top view of a cylinder of radius 6 inches and height of 2 ft that is placed on its side. You may place the cylinder in any reasonable location on your hole.

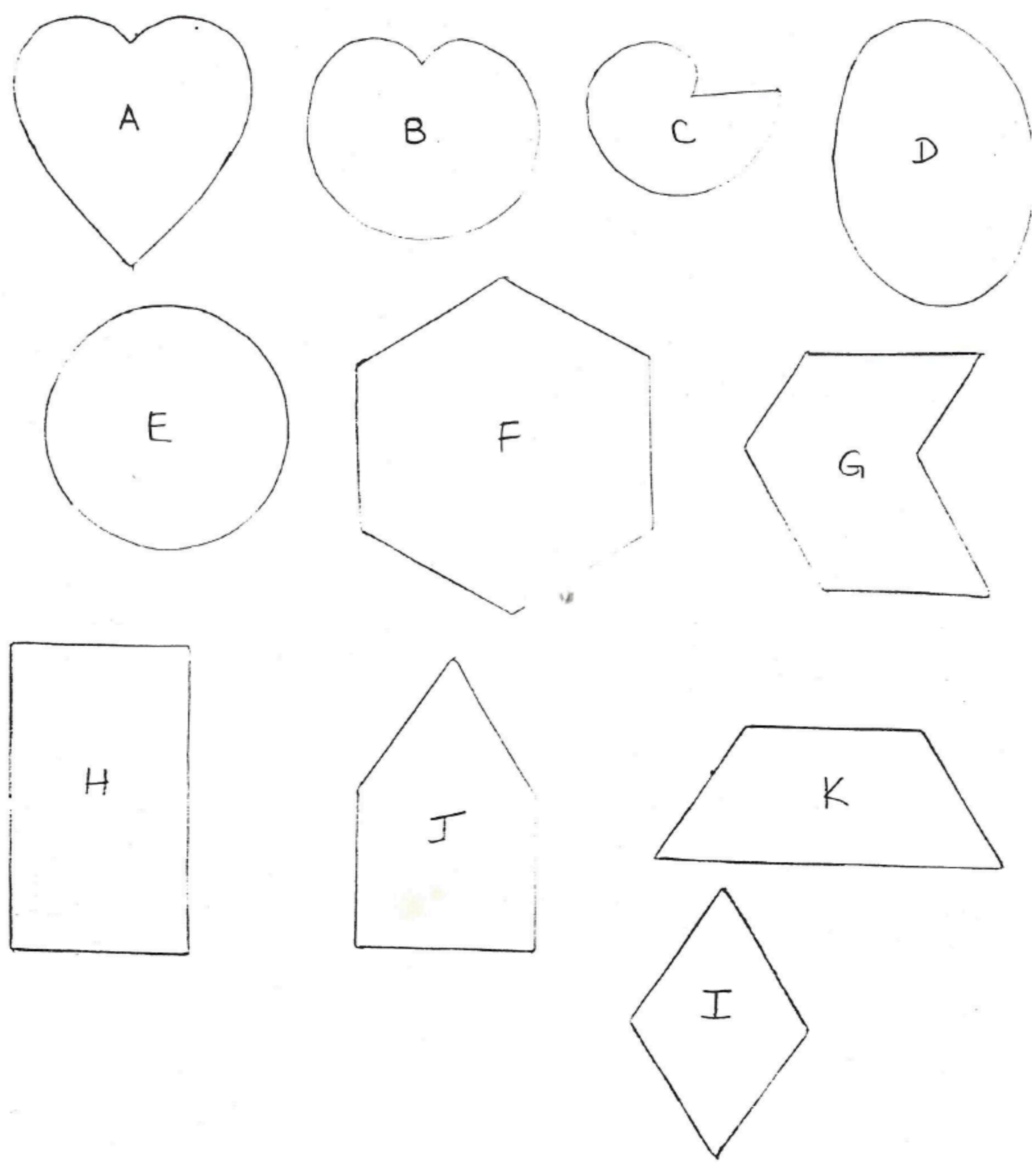
9. Predict the point where the ball would rebound first if you tried to make a hole-in-one. Label this point P. Draw the path the ball would take as it rebounds off of point P (and any other walls). Write down all angle measurements that you take.

Appendix B 1: Pre-interview A and B Protocol

Q.1 How would you describe these shapes? (give 2-D shapes)

- If student names them, have her/him elaborate -- how would you describe them to someone who doesn't know the names?
- Look out for words signifying angle -- use student's language throughout this interview

<u>Materials</u> polygons, circles, semicircles, ovals, other curved shapes



Q.2 a) Please sort the shapes into 2 groups

b) How have you sorted them?

- Have the student sort in 2 - 3 different ways if necessary, to see if "having/not having angles" is something s/he sees as an important attribute. (Can you sort them differently?)
- If sorting does occur in the above way, s/he may identify the principle as having straight sides vs. curved sides. If so, probe to see if s/he can identify other attributes that follow from each condition. Basically, see if s/he recognizes/connects with some concept of angle.

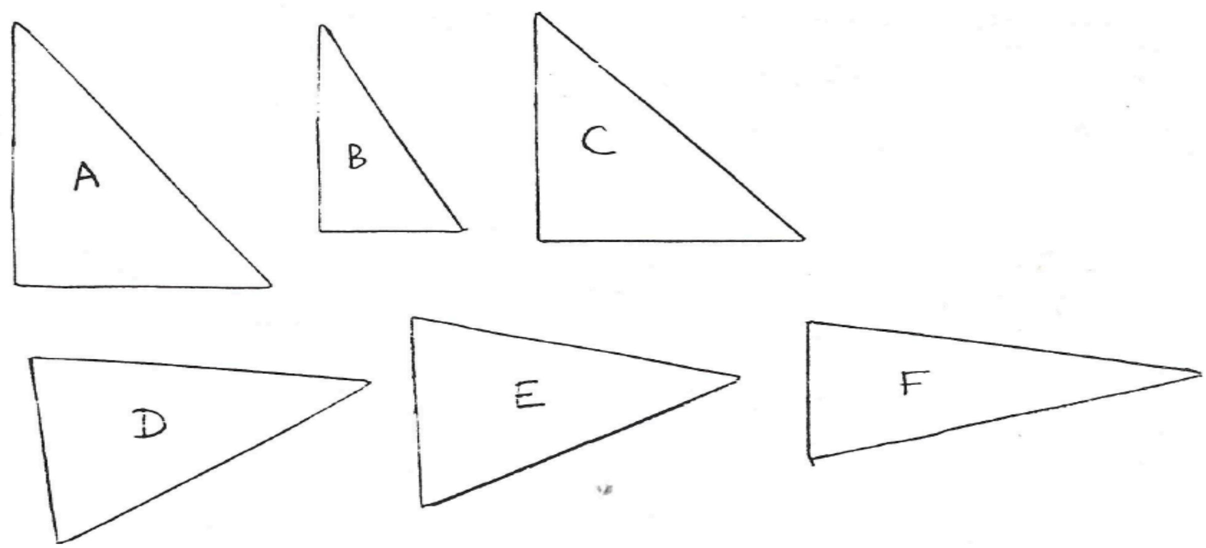
Materials - same as for Q.1

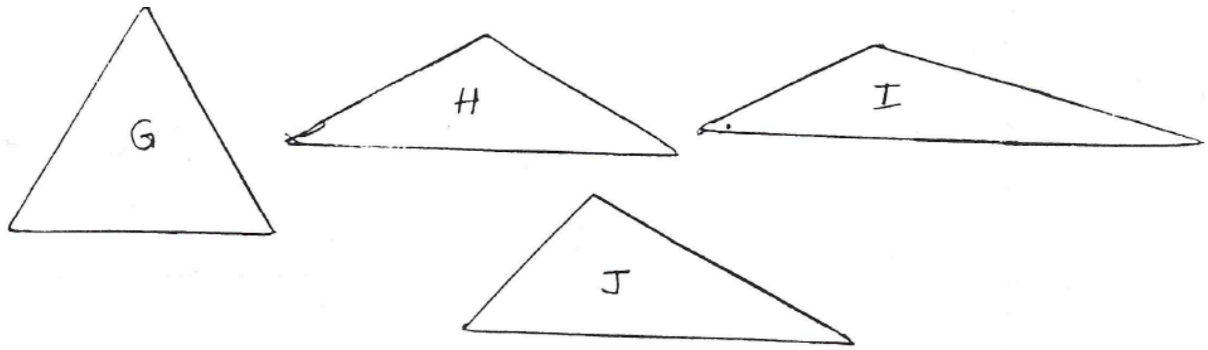
Q.3 a) Please sort these shapes into 2 or 3 groups (give the triangles)

b) What is your sorting principle?

- Have student sort in 2 - 3 different ways if necessary, to see if the kind of angle is something s/he sees as an important attribute, and what language s/he uses. (How else can you sort them?)

Materials acute, obtuse, and right triangles of different sizes

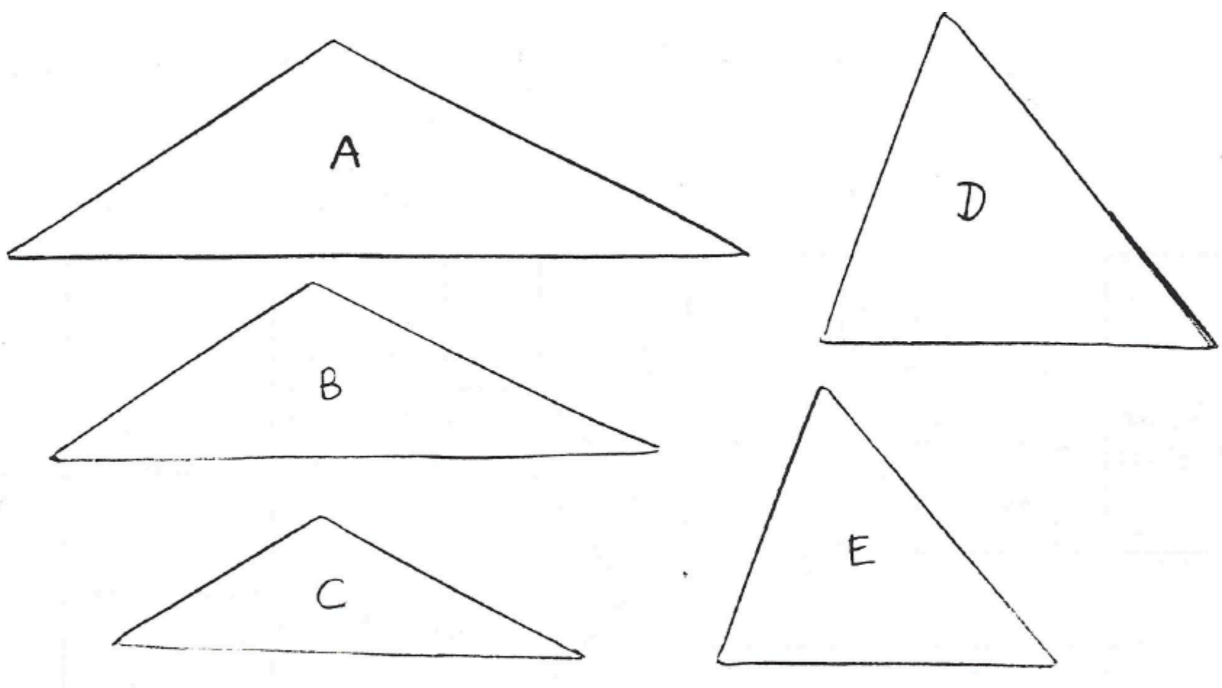


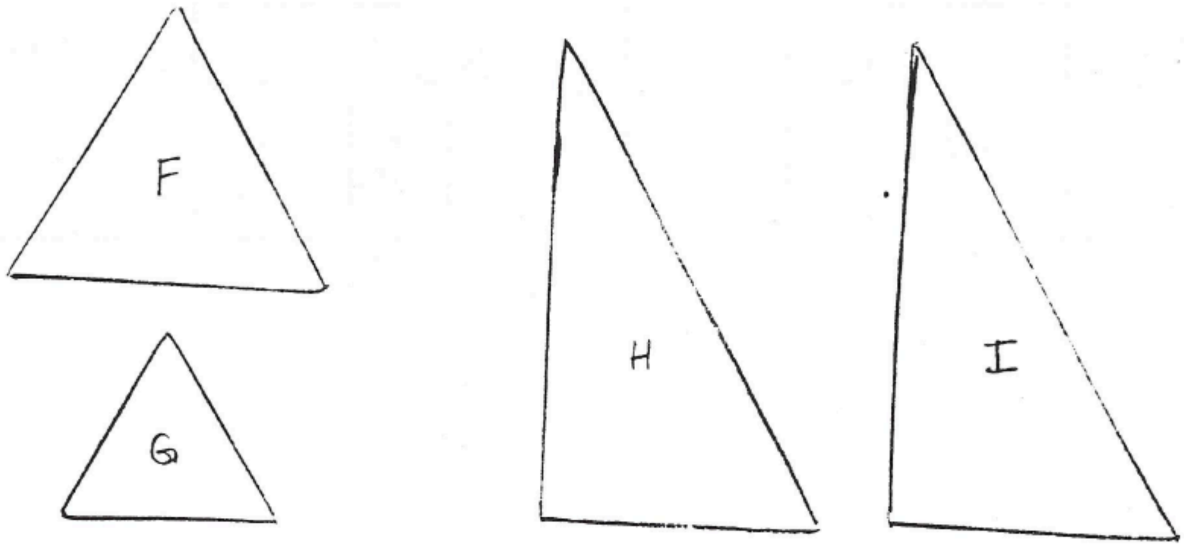


Q.4 a) How would you sort these shapes? (give the triangles)
b) Please explain your sorting principle

- Have student sort in 2 - 3 different ways if necessary, to see if equality of angles is something s/he sees as an important attribute, and what language s/he uses. (How else can you sort them?)

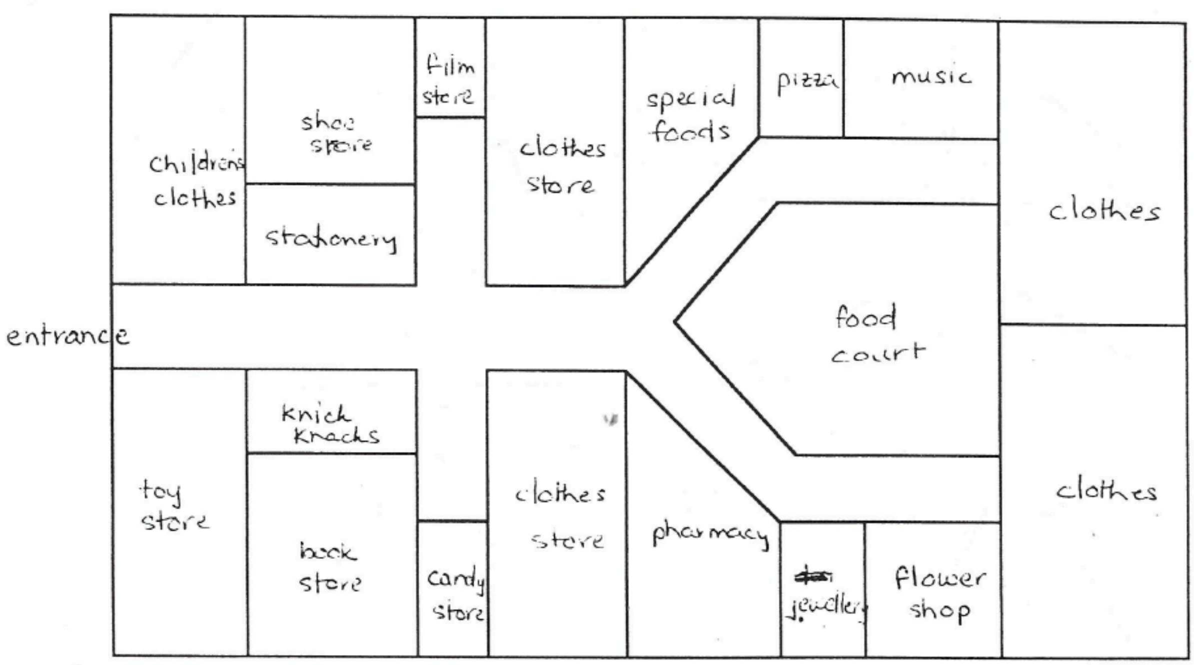
<u>Materials</u> set of triangles which can be sorted into 4 different groups of similar triangles





Q.5 a) Can you guess what this is? (show floor plan of school -- if they don't recognize it say
what it is)
b) Pretend you are explaining to a visitor how to get from (specific place) to (specific place). What would you say?
- Look out for language of turn and direction

<u>Materials</u> Floorplan of JD small shopping mall



Q.6 You described a few different turns (pick up on their language) in the directions you gave the "pretend" visitor. Can you tell me in what ways these turns are different?

- Look for sense of amount turned specially in comparing the 90˚ turns vs. those in the V corridor
- Look to see if they associate turn as measured from initial direction of motion.
- Look for use of body in addition to language

<u>Materials</u> same as for Q. 5

Q.7 Do you use any names for special turns? (use student's language)

- For example, "right", "180" may be common -- see if a direction is associated with each. If not, ask, have you ever heard of a "180"? a "right turn"?

Q.8 a) Have you ever watched a marching band on parade? What kinds of turns do they make?
(If they don't know explain that they usually make "right" turns.) Suppose you were in such a band and facing the band leader. How many right turns must you make to face the opposite direction? (build on their language to see their ability to combine turns and the way they do it)
b) In a certain computer treasure hunt you can move as much as you want in the direction you are facing, but you are only allowed to make right turns.
Suppose you are facing N when the game begins.

- Can you get to a pot of gold that is W of you? (If so how? If not, why not?)
- Can you get to a chest of jewels that is SE of you? (If so how? If not, why not?)

Q.9 a) I'm interested in what you think mathematics is...what comes to mind when you think of mathematics? (if responses to above are limited to numbers), Can you think of activities that you consider mathematical, but don't use numbers?
b) In what ways do you use mathematics when not in math class or doing math homework?
c) (for Pat's and Randy's students) What kind of activities do you do outside of school -- sports, hobbies etc.

Pre-interview. B

Q.1 a) When, if ever, do you use the word angle? Explain.
b) What do you think is an angle?

Q.2 a) Can you show me (draw?) an angle?
b) Explain why it is an angle.
c) Can you show/draw a different angle? How is it different?

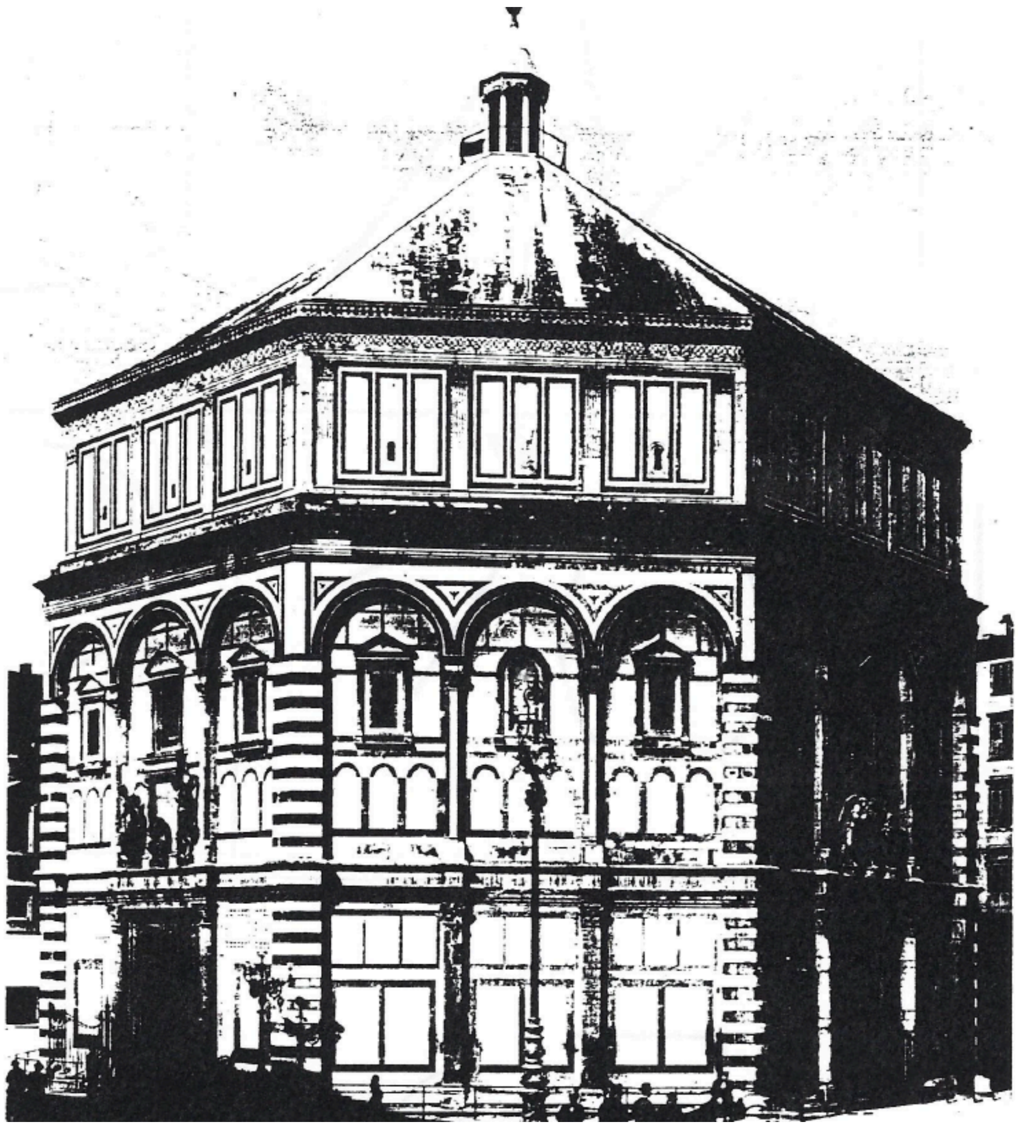
- Probe to see what constitutes the angle in what they show/draw (if students show/draw unexpected things, some of the later questions may have to be reframed)
- See if language of comparison is used. Probe further.

Q.3 a) Is this still an angle? (Using one of the student's angles extend the legs or in some way
change it so that the angle measure is preserved.)
b) How does it compare to your original angle?
c) Where is this X/Y/Z with respect to the angle? (draw an X on the paper)

- try to get at their understanding of what is the angle

Q.4 What angles does this photograph make you think of? (show photograph)

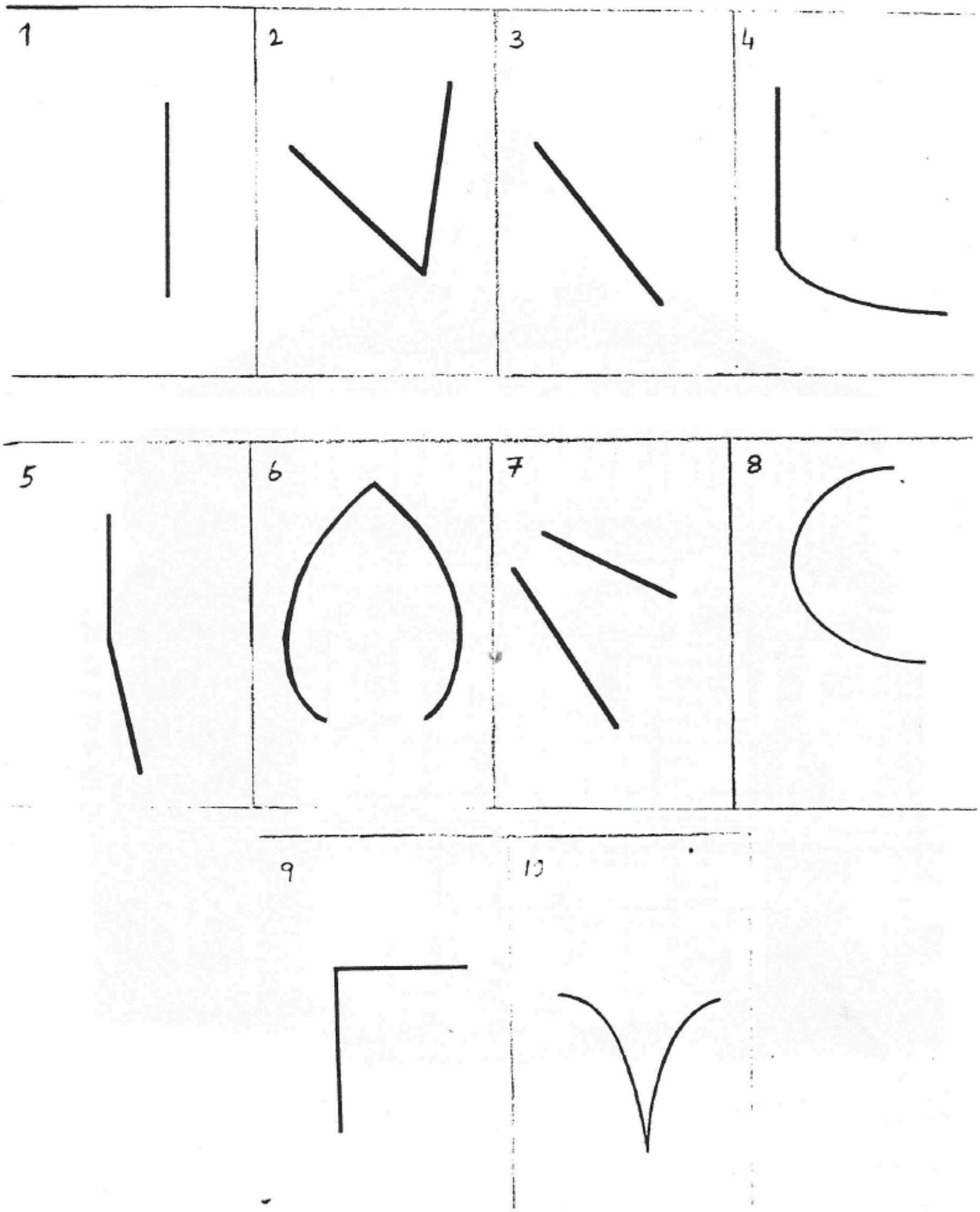
<u>Materials</u> photograph of the Baptistry, Florence



Q.5 I'm now going to show you some drawings. For each one, please tell me if you think it
could be an angle and explain why or why not.

- Have student show what makes/does not make a figure an angle

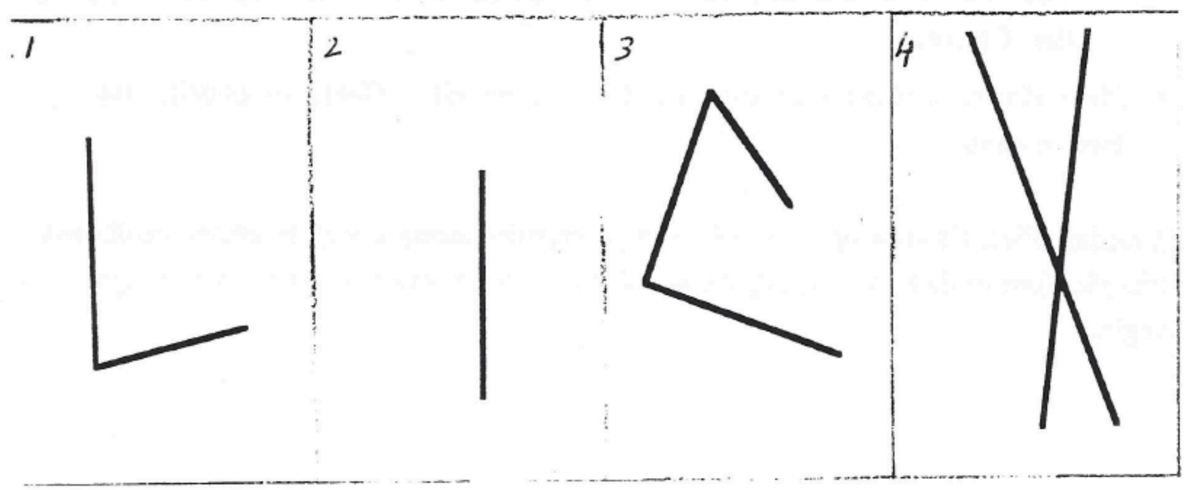
Materials different drawings of possible angles/non-angles


1
2
3
4
5
6
7
8
9
10

Q.6 How many angles can you find in each of the following figures?

- Have students indicate what they identify as an angle

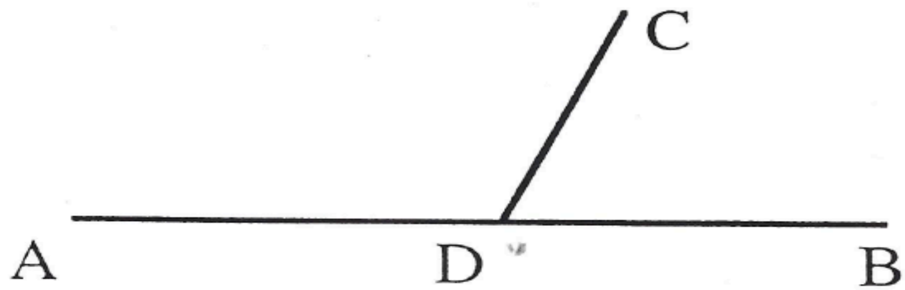
Materials diagrams

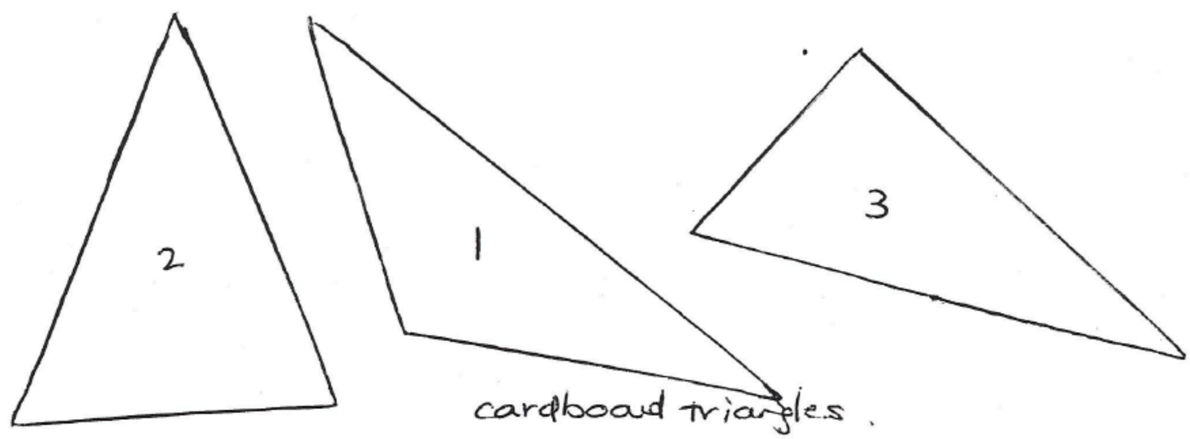


Q.7 Please look at the following figure carefully (give drawing and some time to look). I want you to try to copy the figure as exactly as you can on this sheet of paper (give paper). You can look at the figure as much as you want in-between drawing it, but you can't look at it while drawing. So, I'm going to put this figure behind your chair. You may also use any of these things (give stuff) to help you copy the figure.

- Note down student's actions in detail, specially efforts to coordinate two measures

Materials Piaget's drawing, sheet of clean paper, ruler, string, compass, eraser, cardboard triangles (one of them with an angle equal to one of those drawn, another having a right angle.

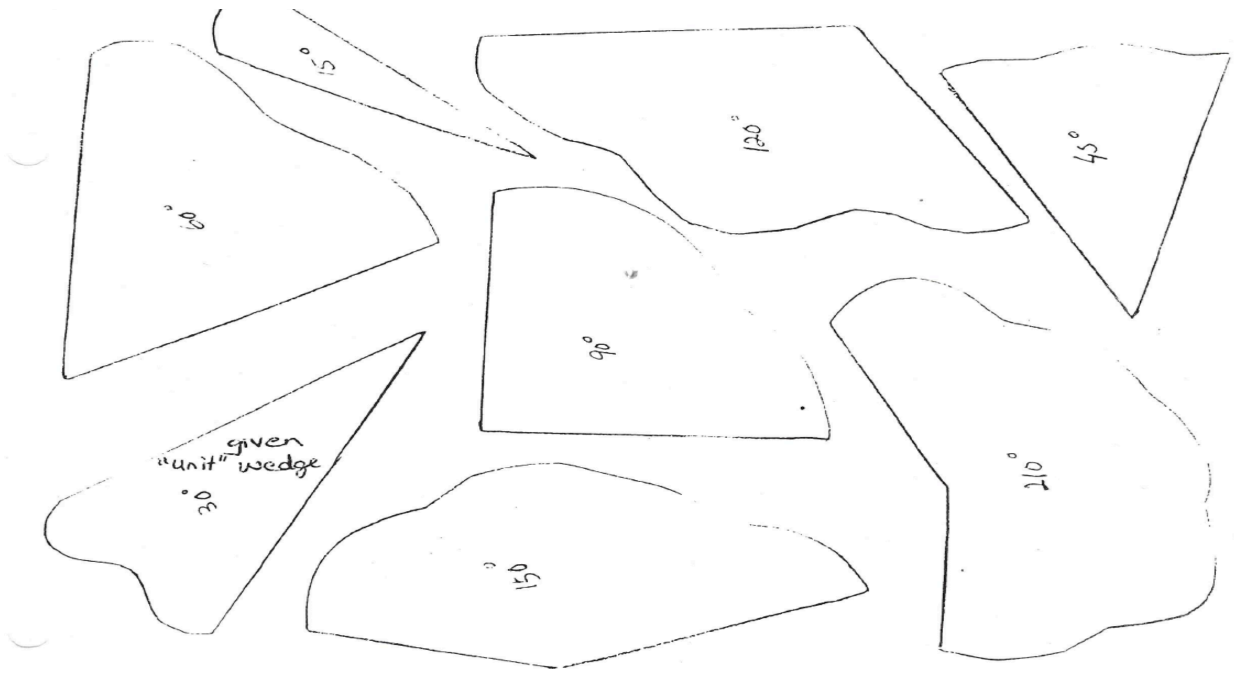




Q.8 a) What are some things we measure? How do we measure them?
b) How could one measure an angle?
c) Suppose we use this wedge shape to measure angle size (give a 30˚ paper wedge shape).
How many wedges would each of the following angles be? (give angles)

- try to get at informal measures as well as formal measures
- See if they can link up with any common measures they may have known -- for example to describe the measure of the wedge in terms of a 180.

<u>Materials</u> 30˚ white paper wedge, wedges of 60˚, 120º, 150˚, 90˚, 15˚, 45˚, 210º degree angles in blue



Q.9 Please complete this sentence (write on this sheet)

Measuring an angle of a shape is different from measuring a side of a shape because.......

Q.10 Can you give me some examples of things which you see or do that have angles or use angle ideas?

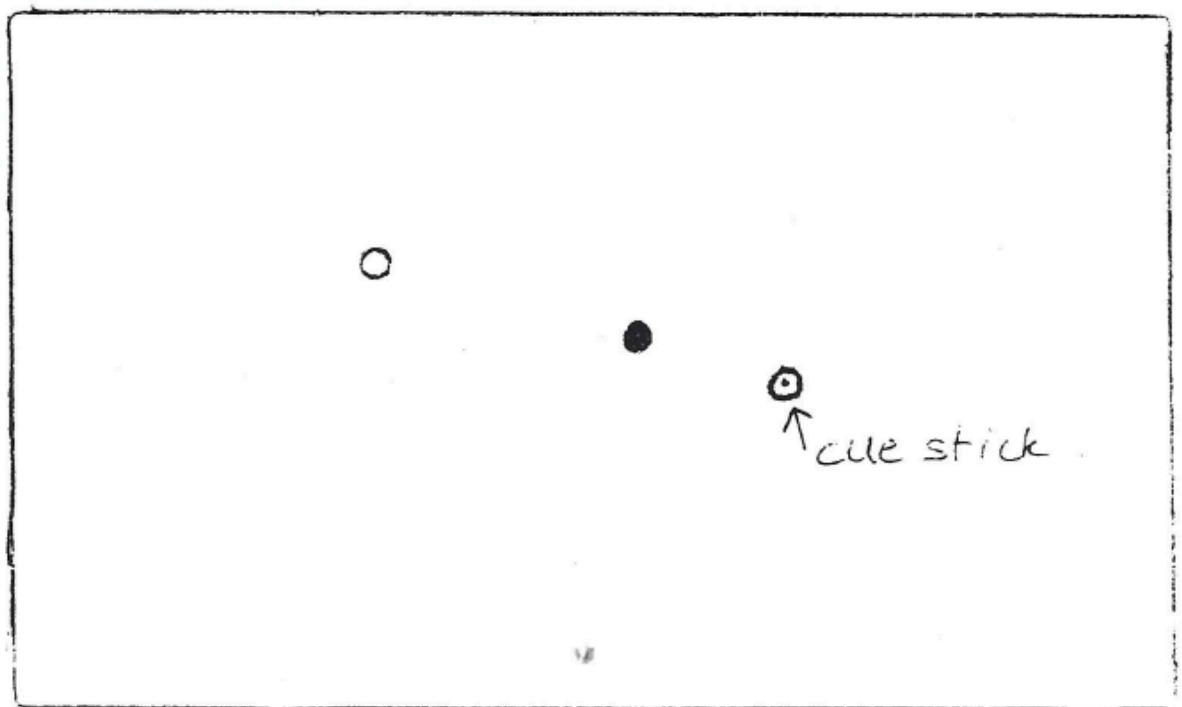
- Pick up on these and ask for elaborations. For example, if a triangle -- ask where the angles are, how many, can it have 4 angles etc.
- If sports or other activities which use a more dynamic idea of angle are mentioned ask for descriptions. In particular, link up with activities we know they do which may have angle ideas and probe further.

Q.11 a) Have you played billiards or any similar game? This is a computer simulation of the game of billiards (show game and how it works)

b) I'd like you to try to hit this ball with this one without hitting the one in the middle (3 balls are collinear). Please tell me what you are thinking as you do this.

- Let student try to hit the ball a few times.
- Note changes in direction of cue and emerging strategies

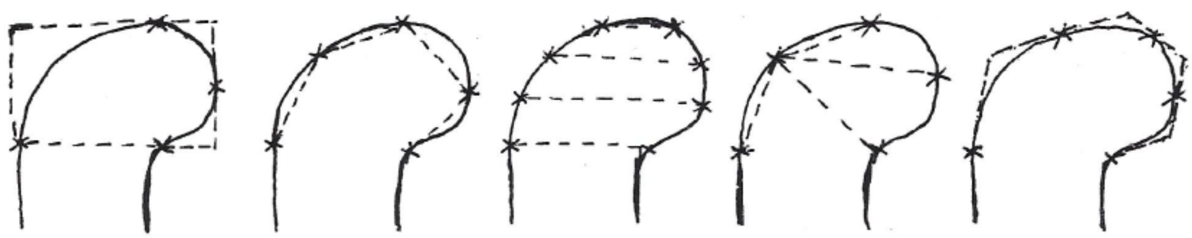
Initial set-up of billiard game.



Appendix B2: Midway Interview Protocol

This interview was based on students' responses to journal # 3 and journal # 4 assignments.

Journal #3 Five different strategies that students in Math 6, 7 and 9 have come up with for measuring a curved side of a hole are shown below.



- Pick the one that you think is best.
- Explain what measurements you need to take.
- Explain why you think this is the best way to measure a curved side.

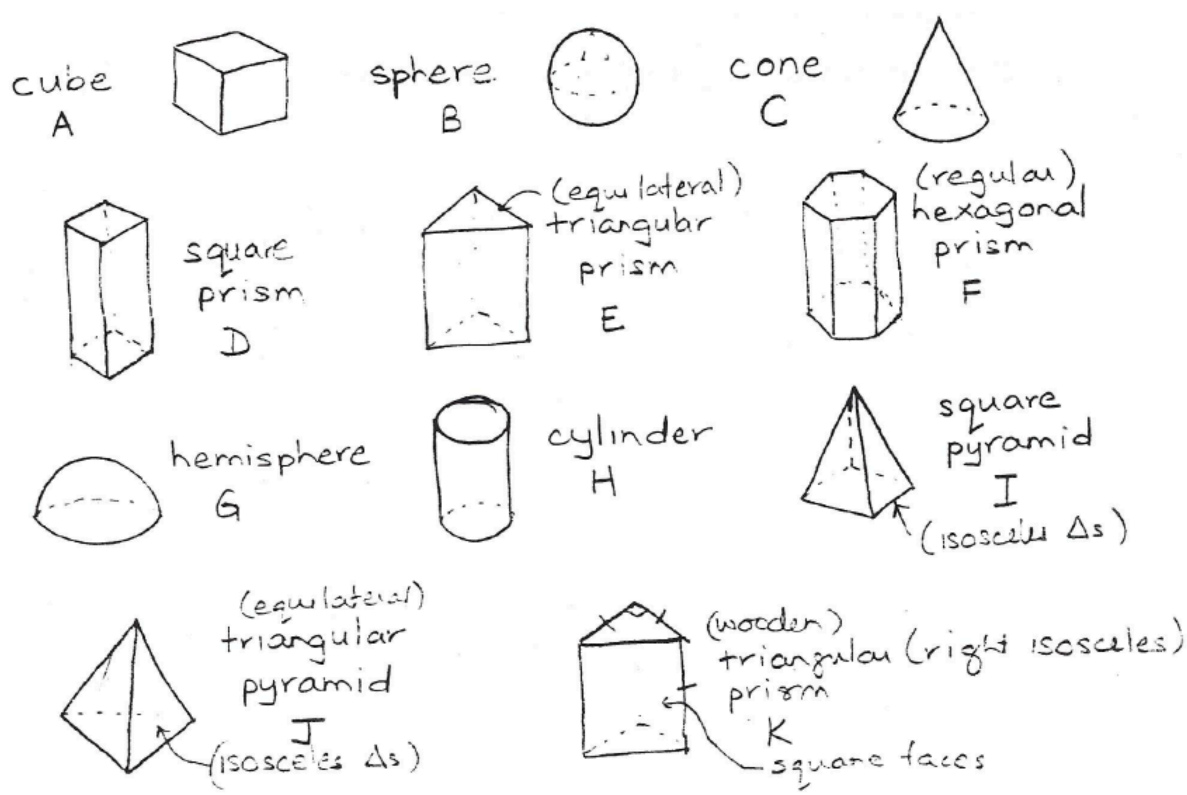
Journal #4 What angle ideas are involved in the path of a ball and its rebound? Be specific and include a drawing.

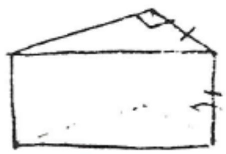
Appendix B3: Post-interview Protocol

Q. 1. a) What do you think is an angle?
b) Can you draw me an example of an angle? Explain why it is an angle.
c) Can you draw me a different angle? How is it different?
d) What is the largest/smallest angle you can think of/draw?
e) Can you draw me an example of something you used to think was an angle, but now
think is not an angle? (Also vice versa). Explain why you've changed your mind.

Q. 2 a) Is this still an angle? (Extends legs of an angle drawn by student)
b) How does it compare with your original angle?
c) What is the inside/outside of your angle?
d) How many angles do you see in the figure you drew?

Q. 3 What angles (if any) do you see when you look at these solids?
(power solids as well as some non-transparent solids)



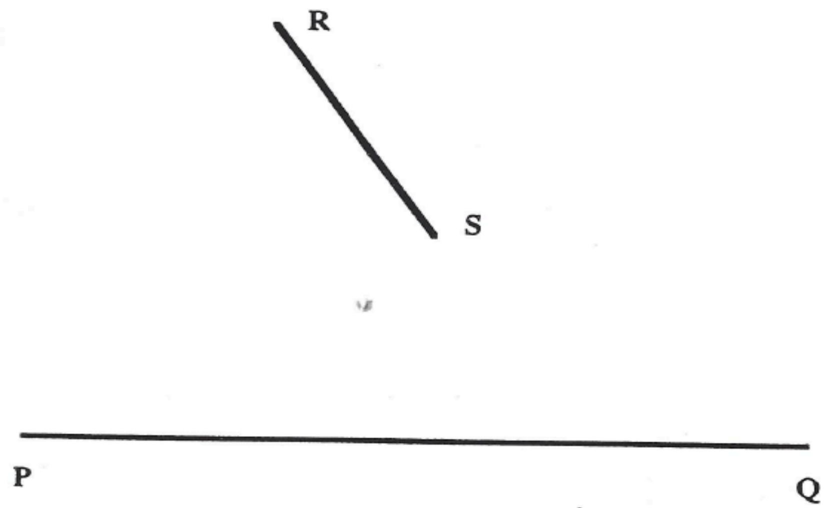


(wooden)
triangular (right, 25°, 65°) prism
L square face

Q. 4 Please look at the following figure carefully.

a) Tell me what you see.

b) Copy the figure as exactly as you can on this piece of paper. You may look at and measure the figure as much as you want in-between drawing.
Give student ruler and protractor.



Q. 5 I'm going to give you instructions that will help you to find the spot X where treasure is buried on the following map. Draw the path from point P as accurately as possible.

Go North 200 ft. Turn 110o left. Go forward 300 ft. Turn 25o right. Go forward 100 ft to get to the spot X.

Q. 6 Suppose you were to roll a ball. What kinds of things would make the ball go faster? go slower? stop?

Q. 7 Please respond to at least one of each of the statements in a) and b) in writing:

a) Measuring an angle of a shape is similar to measuring the side of a shape because …

   Measuring an angle of a shape is different to measuring the side of a shape because …

b) Angles and turns are similar because …

   Angles and turns are different because …

Appendix C: Research Approval Letter, IRB

SYRACUSE UNIVERSITY

INSTITUTIONAL REVIEW BOARD
MEMORANDUM

TO: Joanna Masingila
DATE: June 28, 2018
SUBJECT: **Determination of Exemption from Regulations**
IRB #: 18-220
TITLE: *Connecting In-School and Out-of-School Mathematics Practice*

The above referenced application, submitted for consideration as exempt from federal regulations as defined in 45 C.F.R. 46, has been evaluated by the Institutional Review Board (IRB) for the following:

1. determination that it falls within the one or more of the five exempt categories allowed by the organization;
2. determination that the research meets the organization's ethical standards.

It has been determined by the IRB this protocol qualifies for exemption and has been assigned to category **4**. This authorization will remain active for a period of five years from **June 28, 2018** until **June 27, 2023**.

CHANGES TO PROTOCOL: Proposed changes to this protocol during the period for which IRB authorization has already been given, cannot be initiated without additional IRB review. If there is a change in your research, you should notify the IRB immediately to determine whether your research protocol continues to qualify for exemption or if submission of an expedited or full board IRB protocol is required. Information about the University's human participants protection program can be found at: http://orip.syr.edu/human-research/human-research-irb.html Protocol changes are requested on an amendment application available on the IRB web site; please reference your IRB number and attach any documents that are being amended.

STUDY COMPLETION: Study completion is when all research activities are complete or when a study is closed to enrollment and only data analysis remains on data that have been de-identified. A Study Closure Form should be completed and submitted to the IRB for review (Study Closure Form).

Thank you for your cooperation in our shared efforts to assure that the rights and welfare of people participating in research are protected.

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