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Koszul Homology and Resolutions over Commutative Rings

Rachel Nicole Diethorn
Syracuse University

ABSTRACT

The Koszul homology algebra of a commutative local (or graded) ring R tends to reflect important information about the ring R and its properties. We examine Koszul homology and the relationship between Koszul homology and minimal free resolutions in various settings. We provide tools which allow one to view Koszul homology in a concrete way, and use these tools in a number of applications.

One setting in which we examine Koszul homology is provided by the notion of J-closed modules. We introduce this new class of modules and present explicit formulas for the generators of Koszul homology with coefficients in such modules. This generalizes work of Herzog and of Corso, Goto, Huneke, Polini, and Ulrich. We use these formulas to study connections between J-closed ideals and the ideals by which they were inspired, namely, weak complete intersection ideals.

As another application of such formulas, we study the Koszul homology algebra of quotients by edge ideals. We show that the Koszul homology algebra of a quotient by the edge ideal of a forest is generated by the lowest linear strand. This provides an answer, for such rings, to a question of Avramov about the Koszul homology algebra of a Koszul algebra. In order to obtain this result, we explicitly construct the minimal graded free resolution of the quotient by an edge ideal of a tree.

We also use these formulas to construct a self-dual complete resolution of a module defined by a pair of embedded complete intersection ideals in a local ring. The existence of such a complete resolution gives an isomorphism between certain stable homology and cohomology modules.

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Rachel N. Diethorn

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Chapter 1

Introduction

In this chapter we discuss the basic objects that we use throughout this thesis, and we summarize our main results.

1.1 The Basic Objects

In this section we recall some basic definitions and notations. Throughout this thesis, all rings are assumed to be commutative and Noetherian. Furthermore we often assume that a ring is local, or that it is graded. We say that a ring R is *graded* if $R = \bigoplus_{i \geq 0} R_i$ such that $R_i R_j \subseteq R_{i+j}$ and R_0 is a field. Throughout this section, we let $(R, \mathfrak{m}, k)$ be a local ring with unique maximal ideal $\mathfrak{m}$ and residue field k.

1.1.1 Commutative Rings

Here we recall some basic definitions from commutative ring theory. The following definition is one we use throughout this thesis.

Definition 1.1.1. Let M be an R-module. A sequence of elements $x_1, ..., x_n \in R$ is an *M-regular sequence* if x_1 is a nonzerodivisor on M and, for each $i \geq 2$, x_i is a nonzerodivisor on $M/(x_1, ..., x_{i-1})M$. We call $x_1, ..., x_n$ a *regular sequence* if it is an R-regular sequence. If an ideal I in R is generated by a regular sequence, we call I a *complete intersection ideal.*

We work with rings that are complete with respect to certain ideals in Chapters 2 and 3. This notion is defined as follows.

Definition 1.1.2. Let I be an ideal in R. The *I-adic completion* of an R-module M is the inverse limit

$$\widehat{M^I} := \varprojlim M/I^n M.$$

If the natural map $M \to \widehat{M^I}$ is an isomorphism, we say that M is *complete with respect to the I-adic topology.*

We will also use the following basic definitions.

Definition 1.1.3. Let I be an ideal and p a prime ideal in R. The *height* of p is

$$\operatorname{ht} p := \sup\{n \mid \text{there is a chain of prime ideals, } p = p_n \supset \ldots \supset p_1 \supset p_0\}.$$

The *height* of I is

$$\operatorname{ht} I := \inf\{\operatorname{ht} p \mid I \subset p\}.$$

Definition 1.1.4. The *dimension* of R is

$$\dim R := \sup\{n \mid \text{there is a chain of prime ideals, } p_n \supset \ldots \supset p_1 \supset p_0\}.$$

Definition 1.1.5. The *depth* of an R-module M is the length of the longest M-regular sequence in $\mathfrak{m}$, and the *depth* of the ring R is the depth of R as a module over itself. The ring R is *Cohen-Macaulay* or *CM* if $\operatorname{depth} R = \dim R$.

1.1.2 Homology

The main object we investigate in this thesis is Koszul homology. In this subsection we define the notion of homology and recall several related definitions that we use throughout this thesis. We begin with the following definition.

Definition 1.1.6. A *complex* (C, ∂) is a sequence

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$

of R-modules and homomorphisms, called *differentials*, satisfying $\partial_n \partial_{n+1} = 0$ for all n. We define $C[\,i\,]$ to be the shifted complex with modules $(C[\,i\,])_n = C_{n-i}$ and differentials as above.

Given this definition, the notion of homology is defined as follows.

Definition 1.1.7. The *n-th homology* of a complex (C, ∂) is the quotient

$$H_n(C) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}.$$

A complex C is said to be *exact* if $H_n(C) = 0$ for all n.

Next we define some basic objects that will appear often throughout this thesis.

Definition 1.1.8. A *chain map* f between two complexes (A, ∂^A) and (B, ∂^B) is a sequence of homomorphisms $f_n : A_n \to B_n$ such that $f_{n-1}\partial^A_n = \partial^B_n f_n$ for all n. Equivalently, the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^A_{n+1}} & A_n & \xrightarrow{\partial^A_n} & A_{n-1} & \xrightarrow{\partial^A_{n-1}} & \cdots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \xrightarrow{\partial^B_{n+1}} & B_n & \xrightarrow{\partial^B_n} & B_{n-1} & \xrightarrow{\partial^B_{n-1}} & \cdots \end{array}.$$

It is well-known that a chain map induces a map on homology. This brings us to the following definition which will be important in this thesis.

Definition 1.1.9. A chain map f is a *quasi-isomorphism* if its induced map on homology is an isomorphism.

We end this subsection with the following definition which we use in the proofs of several of our results.

Definition 1.1.10. An R-module M is *flat* if whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of R-modules, the sequence

$$0 \longrightarrow A \underset{R}{\otimes} M \longrightarrow B \underset{R}{\otimes} M \longrightarrow C \underset{R}{\otimes} M \longrightarrow 0$$

is also exact.

1.1.3 Free Resolutions

In this thesis, we investigate the connections between Koszul homology and free resolutions, which are defined as follows.

Definition 1.1.11. A *free resolution* of a finitely generated R-module M is a sequence

$$\ldots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

of free R-modules together with an augmentation map $F_0 \xrightarrow{\epsilon} M$ such that the sequence F

$$\ldots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is exact. We say that F is *minimal* if $\operatorname{Im} \partial_i \subseteq \mathfrak{m} F_{i-1}$ for all $i \geq 1$.

It is well-known that the minimal free resolution of a module is unique up to isomorphism; thus the following definition makes sense.

Definition 1.1.12. The *projective dimension* of an R-module M is the length of the minimal free resolution; that is, the largest integer n such that F_n is nonzero.

We end this section by defining the (co)homology modules, Ext and Tor. These objects will be important throughout this thesis.

Definition 1.1.13. Let M and N be finitely generated R-modules. Then we define

$$\operatorname{Tor}_n^R(M, N) := H_n(F \underset{R}{\otimes} N)$$

$$\operatorname{Ext}_R^n(M, N) := H^n(\operatorname{Hom}_R(F, N))$$

where F is a free resolution of M.

1.1.4 The Koszul Complex

The main object of study in this thesis is Koszul homology, which is the homology of a certain complex called the Koszul complex. We recall the definition of this complex in this subsection. We begin with the following definition, which we will need to define the Koszul complex.

Definition 1.1.14. The *tensor algebra* of an R-module M is

$$T(M) = \bigoplus_{k=0}^{\infty} T^k(M),$$

where $T^k(M) = M^{\otimes k}$ is the k-th tensor power. The *exterior algebra* of M is

$$\bigwedge M = T(M)/I,$$

where I is the ideal generated by all elements of the form $x \otimes x$ where $x \in M$. The *exterior product* of two elements $a, b \in \bigwedge M$ is the product $a \wedge b$ induced by the tensor product on $T(M)$. The *k-th exterior power* of M, denoted $\bigwedge^k(M)$, is the submodule of $\bigwedge M$ generated by elements of the form $a_1 \wedge ... \wedge a_k$ with $a_i \in M$ for all i.

Throughout this thesis, we suppress the wedge notation and denote by ab the exterior product $a \wedge b$. Now we are ready to define the Koszul complex. Let $\underline{x} = x_1, ..., x_n$ be a sequence of elements of R.

Definition 1.1.15. The *Koszul complex* $K(\underline{x}; R)$ is the complex with modules

$$K_i = \bigwedge^i (R^n)$$

for $0 \leq i \leq n$ and $K_i = 0$ for all $i > n$, where we fix a basis $dx_1, ..., dx_n$ of the free module R^n, and differentials

$$\partial_i^K(dx_{j_1} \cdots dx_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_k dx_{j_1} \cdots \widehat{dx_{j_k}} \cdots dx_{j_i}.$$

We note that $K_0 = R$, $K_1 = R^n$, and for $2 \leq i \leq n$, K_i is a free R-module of rank $\binom{n}{i}$ with basis

$$\{dx_{j_1} \cdots dx_{j_i} \,|\, 1 \leq j_1 < \cdots < j_i \leq n\}.$$

We note that one can also describe the Koszul complex as the exterior algebra of the free module with basis $dx_1, \ldots, dx_n$ equipped with differentials induced by the map sending dx_i to x_i, and we write

$$K(\underline{x}; R) = R\langle\, dx_1, \ldots, dx_n \,|\, \partial_K(dx_i) = x_i \,\rangle.$$

1.2 Summary of Results

This thesis consists of five chapters and an appendix. The main goal of Chapter 2 is to provide explicit formulas for the generators of Koszul homology modules with coefficients in what we call a J-closed module. This generalizes work of Herzog in [24] (see also [26]) and of Corso, Goto, Huneke, Polini, and Ulrich in [15], and allows one to view Koszul homology modules in a concrete way. We begin the chapter with some background on the perturbation lemma and the so called de Rham contraction, which are the main tools we use throughout the chapter. In Section 2.2, we introduce the notion of J-closed modules, which provide the setting for the main result of this chapter. We state and prove this result in Section 2.4.

The chapters that follow consist of various applications of the explicit generators of Koszul homology discussed in Chapter 2. In Chapter 3, we use these generators to investigate connections between J-closed ideals and weak complete intersection ideals, the ideals that inspired the notion of J-closed ideals. The main result of this chapter provides a general condition under which the quotient of J by a J-closed ideal is a weak complete intersection. This provides a new family of examples of weak complete intersections.

In Chapter 4, we study the Koszul homology algebra of quotients by edge ideals of forests. These rings are classical examples of Koszul algebras. The main result of this chapter, which we state and prove in Section 4.3, gives an answer, for such rings, to a question of Avramov about the structure of the Koszul homology algebra of a Koszul algebra. In Section 4.2 we construct the minimal graded free resolution of a quotient by the edge ideal of a tree, and use this construction along with generators of Koszul homology modules given by Herzog and Maleki in [26] in the proof of the main result.

The main goal of Chapter 5 is to construct the minimal complete resolution of a module defined by a pair of embedded complete intersection ideals, and show that it is self-dual. We begin this chapter with some background on complete resolutions and a survey of the known results on symmetric growth in such resolutions. In Section 5.2, we provide explicit generators of Koszul homology modules in the embedded complete intersection case, and we use them in Section 5.3 to prove the main result of this chapter.

The appendix consists of an alternate proof of the result in Chapter 5 that gives formulas for the generators of Koszul homology modules in the embedded complete intersection case. Although the proof we provide in Section 5.2 is a much simpler one, we include this alternate proof as it provides the intuition behind the methods used in Chapter 2 to provide generators of Koszul homology in a more general setting.

Chapter 2

Generators of Koszul Homology

Let Q be a commutative Noetherian ring and let J be a complete intersection ideal in Q. In this chapter, we introduce a class of ideals called J-closed ideals, that provides the setting in which we study Koszul homology throughout this chapter. This new notion is inspired by the weak complete intersection ideals studied by Rahmati, Striuli, and Yang in their recent paper [36]. In a local ring Q, they define weak complete intersection ideals to be the ideals I such that the differentials in the minimal free resolution F of the quotient Q/I land in IF. In any commutative Noetherian ring Q, we define J-closed ideals to be the ideals I such that there is a projective resolution P of Q/I whose differentials land in JP. We define the more general notion of a J-closed module in a similar way. Expanding the allowed range for the differentials allows additional flexibility, which as we show in Chapter 3, becomes useful in studying weak complete intersection ideals. Of course, when Q is a regular local ring and $\mathfrak{m}$ is its maximal ideal, then any ideal is an $\mathfrak{m}$-closed ideal, but more interesting examples are abundant. We show that a weak complete intersection ideal is a J-closed ideal if and only if it is a complete intersection ideal embedded in J. Although neither class of ideals is contained in the other, we find other connections between J-closed ideals and weak complete intersection ideals in Chapter 3. To accomplish this, we study the Koszul homology modules $H_i(\underline{g}; R)$, where $R = Q/I$ is the quotient by a J-closed ideal and $\underline{g} = g_1, \ldots, g_s$ is a regular sequence that generates J.

One approach to understanding Koszul homology is to describe the generators. In his 1991 paper [24] (see also [26]), Herzog gives explicit formulas for generators of the homology $H(\underline{x}; R)$ of the Koszul complex on the minimal generators of the of the irrelevant maximal ideal $m = (x_1, \ldots, x_n)$ of a finitely generated graded k-algebra R, where k is a field of characteristic zero. Recently, the authors of [15] provided explicit formulas for generators of Koszul homology in a more general setting. They studied the homology of the Koszul complex on a sequence $\underline{g} = x_1^{a_1}, \ldots, x_n^{a_n}$ with M-coefficients, where $J = (\underline{g})$ and M is what we call a J-closed module throughout this chapter. We further generalize to the setting where $\underline{g}$ is any full regular sequence generating J and M is a J-closed module. The formulas we obtain have new terms, which vanish in the case considered by Herzog.

One of the main tools we use to obtain these formulas is the classical perturbation lemma; see for example [16]. We also utilize the theory developed in [19] on the formulation of a sort of partial derivative with respect to a regular sequence $\underline{g}$ and the de Rham contraction built from it. The formulas we provide are given in terms of these partials.

2.1 The Main Tools

In this section we discuss the main tools used throughout this chapter, including the perturbation lemma and a version of the de Rham contraction developed in [19].

2.1.1 The Perturbation Lemma

In this section we discuss the classical perturbation lemma. We use it in Section 2.4 as the main tool for providing explicit formulas for the generators of Koszul homology. We begin with the definitions, which can be found in [16].

Definition 2.1.1. A *deformation retract datum*

$$\Big((F, \partial_F) \underset{i}{\overset{p}{\leftrightarrows}} (G, \partial_G), H\Big)$$

consists of the following:

(i) complexes (F, ∂_F) and (G, ∂_G)

(ii) quasi-isomorphisms p and i such that $pi = \mathrm{Id}_F$

(iii) a homotopy H between ip and Id_G (ie. $\partial_G H + H\partial_G = ip - \mathrm{Id}_G$).

We call F a *deformation retract* of G. A *special deformation retract datum* is a deformation retract datum that also satisfies the equalities

$$Hi = 0, \;\; pH = 0, \;\; H^2 = 0.$$

Given a deformation retract datum, one can define a (small) perturbation of the datum as follows.

Definition 2.1.2. A *perturbation* of a deformation retract datum is a map

$$G \xrightarrow{\epsilon} G$$

of degree -1, such that $(\partial_G + \epsilon)^2 = 0$. The perturbation is *small* if the map $\mathrm{Id}_G - \epsilon H$ is invertible.

Now we state the perturbation lemma; see for example [16, 2.4].

Theorem 2.1.3 (Perturbation Lemma). *If ϵ is a small perturbation of the deformation retract datum*

$$\Big((F, \partial_F) \underset{i}{\overset{p}{\leftrightarrows}} (G, \partial_G), H\Big),$$

then the perturbed datum

$$\Big((F, \widetilde{\partial_F}) \underset{\tilde{i}}{\overset{\tilde{p}}{\leftrightarrows}} (G, \partial_G + \epsilon), \tilde{H}\Big)$$

is a deformation retract datum, where

$$\widetilde{\partial_F} = \partial_F + pAi, \;\; \tilde{p} = p + pAH, \;\; \tilde{i} = i + HAi, \;\; \tilde{H} = H + HAH$$

and $A = (Id_G - \epsilon H)^{-1}\epsilon$. In particular, $\tilde{i}$ is a homotopy equivalence.

In Section 2.3, we modify a deformation retract datum constructed in [19], and apply the perturbation lemma to provide formulas for generators of Koszul homology.

2.1.2 The de Rham Contraction

In this section we use the theory in [19] on connections to formulate a sort of partial derivative with respect to the elements of a regular sequence $\underline{g}$. We use these partial derivatives in the formulas given in Section 2.4.

Throughout this section we assume that k is a field of characteristic zero and Q is a Noetherian k-algebra. We let $\underline{g}$ be a regular sequence in Q such that $Q/(\underline{g})$ is a finite dimensional k-vector space and Q is complete[1] in the $(\underline{g})$-adic topology.

We let $\Omega^1_{k[\underline{g}]/k}$ be the module of Kähler differentials, that is,

$$\Omega^1_{k[\underline{g}]/k} = \bigoplus_{i=1}^{s} k[\underline{g}]dg_i,$$

and we denote by $\Omega^\bullet_{k[\underline{g}]/k}$ the exterior algebra over $\Omega^1_{k[\underline{g}]/k}$ with differential induced by the map $\Omega^1_{k[\underline{g}]/k} \to k[\underline{g}]$ sending dg_i to g_i, where we note that Q is an algebra over the subring $k[\underline{g}]$, which is a polynomial ring since $\underline{g}$ is a regular sequence. The Koszul complex $K(\underline{g};Q)$ on $\underline{g}$ over Q is given by $Q \otimes_{k[\underline{g}]} \Omega^\bullet_{k[\underline{g}]/k}$. Indeed,

$$\begin{aligned} Q \underset{k[\underline{g}]}{\otimes} \Omega^\bullet_{k[\underline{g}]/k} &= Q \underset{k[\underline{g}]}{\otimes} \bigwedge \Omega^1_{k[\underline{g}]/k} \\ &= Q \underset{k[\underline{g}]}{\otimes} k[\underline{g}]\langle dg_1, \dots, dg_s \mid \delta(dg_i) = g_i \rangle \\ &= Q\langle dg_1, \dots, dg_s \mid \delta(dg_i) = g_i \rangle. \end{aligned}$$

Let $\pi : Q \to Q/(\underline{g})$ be the quotient map and fix a k-linear splitting

$$\sigma : Q/(\underline{g}) \to Q$$

of π. The following lemma is a well known; see for example [19, Appendix B] or [32, Lemma 3.1.1]. As a proof is not given in either source above, we include a proof here to clarify the constructions in this section.

[1]We use the term "complete" to mean "complete and separated".

Lemma 2.1.4. *For every element $q \in Q$, there exist unique residue classes $\overline{q_N} \in Q/(\underline{g})$ such that*

$$q = \sum_{N \in \mathbb{N}^s} \sigma(\overline{q_N}) g^N.$$

where $N = (n_1, \dots, n_s)$ and $g^N = g_1^{n_1} \dots g_s^{n_s}$.

Proof. Consider the quotient maps $\pi_i \colon (\underline{g})^i \to (\underline{g})^i/(\underline{g})^{i+1}$. We will show by induction on i that there exist elements $\overline{q_N} \in Q/(\underline{g})$ such that

$$q - \left(\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\dots+n_s<i}} \sigma(\overline{q_N}) g^N \right) \in (\underline{g})^i$$

for all $i \geq 1$.

For the base case, we note that $\pi(q - \sigma(\overline{q})) = \overline{q} - \pi(\sigma(\overline{q})) = \overline{q} - \overline{q} = 0$, where the second equality follows from the fact that σ is a splitting and where $\overline{q} = \pi(q)$. Hence, $q - \sigma(\overline{q}) \in (\underline{g})$.

Now we suppose that there exist elements $\overline{q_N} \in Q/(\underline{g})$ such that

$$q - \left(\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\dots+n_s<m-1}} \sigma(\overline{q_N}) g^N \right) \in (\underline{g})^{m-1}.$$

Then since $\underline{g}$ is a regular sequence, we have that $(\underline{g})^{m-1}/(\underline{g})^m$ is a free $Q/(\underline{g})$-module with basis $\{\overline{g_1}^{i_1} \dots \overline{g_s}^{i_s}\}_{i_1+\dots+i_s=m-1}$, so we have that

$$\pi_{m-1} \left(q - \left(\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\dots+n_s<m-1}} \sigma(\overline{q_N}) g^N \right) \right) = \sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\dots+n_s=m-1}} \overline{q_N g}^N$$

for some $\overline{q_N} \in Q/(\underline{g})$. But now, by the induction hypothesis and the fact that σ is a splitting, we have that

$$\pi_{m-1} \left(q - \left(\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\dots+n_s<m}} \sigma(\overline{q_N}) g^N \right) \right) = 0.$$

Thus, we have

$$q - \left(\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\cdots+n_s<m}} \sigma(\overline{q_N}) g^N \right) \in (\underline{g})^m$$

which completes induction.

Now we have that

$$q - \left(\sum_{N\in\mathbb{N}^s} \sigma(\overline{q_N}) g^N \right) \in \bigcap_{n\geq 0} (\underline{g})^n = (0)$$

since Q is complete (and thus separated) in $(\underline{g})$-adic topology. Thus,

$$q = \sum_N \sigma(\overline{q_N}) g^N$$

as desired.

To show uniqueness, we suppose that

$$\sum_N \sigma(\overline{q_N}) g^N = q = \sum_N \sigma(\overline{r_N}) g^N.$$

Applying π to both sides of the above equality, we get that $\overline{q_0} = \overline{r_0}$.

Now suppose that $\overline{q_N} = \overline{r_N}$ for all $N = (n_1, \dots, n_s)$ with $\sum_{i=1}^s n_i \leq m-1$. Then we have the equality

$$\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\cdots+n_s\geq m}} \sigma(\overline{q_N}) g^N = \sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\cdots+n_s\geq m}} \sigma(\overline{r_N}) g^N.$$

Applying π_m to both sides, we get that

$$\sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\cdots+n_s=m}} \overline{q_N g}^N = \sum_{\substack{N=(n_1,\dots,n_s) \\ n_1+\cdots+n_s=m}} \overline{r_N g}^N.$$

Since $\{\overline{g_1}^{i_1} \dots \overline{g_s}^{i_s}\}_{i_1+\cdots+i_s=m}$ is a basis of $(\underline{g})^m/(\underline{g})^{m+1}$, this implies that $\overline{q_N} = \overline{r_N}$ for $N = (n_1, \dots, n_s)$ with $\sum_{i=1}^s n_i = m$, which completes induction. □

By writing elements of Q in this way, Dyckerhoff and Murfet in [19] give an explicit k-linear connection on Q, that is, a map

$$\nabla^0 \colon Q \to Q \underset{k[\underline{g}]}{\otimes} \Omega^1_{k[\underline{g}]/k}$$

which satisfies the Leibniz rule. In this context, the Leibniz rule is

$$\nabla^0(aq) = a\nabla^0(q) + q \otimes d^0(a),$$

for $a \in k[\underline{g}]$ and $q \in Q$, where $d^0 : k[\underline{g}] \to \Omega^1_{k[\underline{g}]/k}$ is the Kähler differential; see for example [19, Def 2.7]. Explicitly, they define

$$\nabla^0(q) = \sum_{i=1}^{s} \sum_{N} N_i \sigma(\overline{q_N}) g^{N-e_i} \otimes dg_i$$

where e_i are the standard basis vectors of $\mathbb{Z}^s$ and g^N is defined to be zero whenever some component of N is negative. By means of this connection, one can define $\frac{\partial}{\partial g_i}$ to be the k-linear map given by the composition

$$\frac{\partial}{\partial g_i} : Q \xrightarrow{\nabla^0} Q \underset{k[\underline{g}]}{\otimes} \Omega^1_{k[\underline{g}]/k} \xrightarrow{(dg_i)^*} Q,$$

where

$$(dg_i)^*(q \otimes dg_j) = \begin{cases} q & i = j \\ 0 & i \neq j \end{cases}.$$

Remark 2.1.5. We note that in order for $\frac{\partial}{\partial g_j}$ to be well-defined, it is important to fix a splitting σ.

Using the map $\frac{\partial}{\partial g_j}$, one defines ∇ to be the k-linear map

$$\nabla \colon Q \underset{k[\underline{g}]}{\otimes} \Omega^\bullet_{k[\underline{g}]/k} \to Q \underset{k[\underline{g}]}{\otimes} \Omega^\bullet_{k[\underline{g}]/k} \tag{2.1}$$

given by

$$\nabla(q \otimes \omega) = \sum_{i=1}^{s} \frac{\partial}{\partial g_i}(q) \otimes dg_i \wedge \omega + q \otimes d\omega.$$

Recall that δ is the differential on $K(\underline{g}; Q)$. Since $\operatorname{char} k = 0$, we have that $\delta\nabla + \nabla\delta$ is invertible in nonzero degrees by [19, 8.1], so one can make the following definition; see [19, Definition 8.8].

Definition 2.1.6. Let H_∇ be the k-linear map

$$H_\nabla \colon Q \underset{k[\underline{g}]}{\otimes} \Omega^\bullet_{k[\underline{g}]/k} \to Q \underset{k[\underline{g}]}{\otimes} \Omega^\bullet_{k[\underline{g}]/k}$$
$$H_\nabla = (\delta\nabla + \nabla\delta)^{-1}\nabla.$$

This map is called the *de Rham contraction.*

The de Rham contraction is a homotopy on the Koszul complex, as we see in the following result of Dyckerhoff and Murfet in [19, 8.10].

Theorem 2.1.7. *The following is a special deformation retract datum*

$$\Big((Q/(\underline{g}), 0) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (Q \underset{k[\underline{g}]}{\otimes} \Omega^\bullet_{k[\underline{g}]/k}, \delta),\ H_\nabla\Big).$$

We modify this datum in Section 2.3 and apply the perturbation lemma to provide the formulas in Theorem 2.4.1.

2.2 J-Closed Modules

Let Q be a commutative Noetherian ring and let J be a complete intersection ideal. In this section we introduce a class of ideals called J-closed ideals, inspired by the weak complete intersection ideals defined in [36].

Definition 2.2.1. A finitely generated Q-module M is a *J-closed module* if there is a projective resolution (P, ∂) of M which satisfies the property

$$\operatorname{Im} \partial_i \subseteq JP_{i-1} \tag{2.2}$$

for every i. We call an ideal $I \subseteq Q$ a *J-closed ideal* if Q/I is a J-closed module.

If Q is local, we note that it suffices to consider the minimal free resolution. We also have the following characterization of J-closed modules in the local case.

Remark 2.2.2. If Q is local, then a module M is J-closed if and only if $\operatorname{Tor}_i^Q(Q/J, M)$ is a free Q/J-module for every i; see for example [36, Lemma 2.2].

We now give some examples of J-closed ideals.

Example 2.2.3. The complete intersection ideal $J \subseteq Q$ is a J-closed ideal. Indeed, Q/J is resolved by the Koszul complex on a minimal set of generators of J, whose differentials certainly land in the ideal J. Similarly, any embedded complete intersection ideal $I \subseteq J$ is also a J-closed ideal because Q/I is resolved by the Koszul complex on a minimal set of generators of I. In a regular local ring, every module is an $\mathfrak{m}$-closed module, where $\mathfrak{m}$ is the maximal ideal.

The next example gives a large class of J-closed ideals that are not complete intersection ideals.

Example 2.2.4. Let J be a complete intersection ideal in a local ring Q and take $\underline{g} = g_1, \ldots, g_s$ to be a regular sequence generating J. Then any ideal I generated by monomials in $\underline{g}$ is a J-closed ideal. Indeed, Q/I is resolved (possibly non-minimally) by the Taylor resolution for monomials in a regular sequence. The entries in the differentials are either monomials in $\underline{g}$ or units. After change of bases, the minimal free resolution splits off. The entries in the differentials of the minimal resolution are still contained in J as the appropriate row and column operations do not disturb this property.

The next example shows that there are non-monomial ideals that are J-closed.

Example 2.2.5. Let $Q = k[x, y, z]$ be a polynomial ring and let $I = (x^2y^4 + y^3z^7,\ y^6,\ x^4y^2)$. According to Macaulay2 [21], a free resolution of Q/I over Q is given by

$$0 \to Q \xrightarrow{\partial_3} Q^3 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \to Q/I \to 0$$

where the differentials are given by the following matrices:

$$\partial_3 = \begin{bmatrix} -z^7 - x^2y \\ -x^4 \\ y^3 \end{bmatrix},\ \partial_2 = \begin{bmatrix} -y^4 & 0 & -yz^7 - x^2y^2 \\ x^4 & -z^7 - x^2y & 0 \\ 0 & y^3 & x^4 \end{bmatrix},$$

$$\partial_1 = \begin{bmatrix} x^4y^2 & y^6 & x^2y^4 + y^3z^7 \end{bmatrix}.$$

It is easy to see that I is a J-closed ideal, where J is any complete intersection ideal containing (x^2, y^3, z^7).

Although the definition of J-closed ideals was inspired by the definition of weak complete intersection ideals, for a fixed complete intersection ideal J, the two classes of ideals are distinct and neither one is contained in the other. In fact, the intersection of the two classes is contained in the class of complete intersection ideals, as shown in the following proposition.

Proposition 2.2.6. *Let Q be a local (or graded) ring and let J be a (homogeneous) complete intersection ideal in Q.*

(1) Every finitely generated J-closed module has finite projective dimension via the resolution in Definition 2.2.1.

(2) A weak complete intersection ideal is a J-closed ideal if and only if it is a complete intersection ideal embedded in J.

Proof. We give a proof of the local case; the one in the graded case is similar.
(1) Let M be a finitely generated J-closed module and let F be a free resolution of M with $\operatorname{Im} \partial_i \subseteq JF_{i-1}$ for all i. Pick a regular sequence $\underline{g} = g_1, \ldots, g_s$ that generates J and let $H_\ell(\underline{g}; M) = H_\ell(K(\underline{g}; Q) \otimes_Q M)$, where $K(\underline{g}; Q)$ is the Koszul complex on $\underline{g}$ over Q. Then we have isomorphisms

$$H_\ell(\underline{g}; M) = H_\ell(K(\underline{g}; Q) \underset{Q}{\otimes} M) \cong \operatorname{Tor}_\ell^Q(Q/J, M) \cong H_\ell(Q/J \underset{Q}{\otimes} F) \cong Q/J \underset{Q}{\otimes} F_\ell$$

where the first isomorphism follows from the fact that $\underline{g}$ is a regular sequence and the last isomorphism follows directly from Definition 2.2.1. Note that $H_\ell(\underline{g}; M) = 0$ for all $\ell > s$. Then $Q/J \otimes_Q F_\ell = 0$ by above, and hence $F_\ell = 0$ for all $\ell > s$ by Nakayama's Lemma. Thus, $\mathrm{pd}_Q M < \infty$.
(2) Let I be a weak complete intersection ideal and suppose it is also J-closed. Note that

the minimal generators of I are contained in the ideal J since they are the entries in the first differential of the minimal free resolution F of Q/I over Q. By (1), Q/I has finite projective dimension; thus I is a complete intersection ideal by [40, Cor 1]. The other direction follows directly from [*loc. cit.*]. $\square$

In Chapter 3, we investigate further the connection between J-closed ideals and weak complete intersection ideals.

2.3 Some Technical Lemmas

Let Q a Noetherian k-algebra with k a field of characteristic zero and assume that Q is complete with respect to the J-adic topology. We fix a regular sequence $\underline{g}$ in Q and the ideal J generated by it such that Q/J is a finite dimensional k-vector space. Let M be a finitely generated J-closed Q-module. In this section we prove some technical lemmas that we will use in the following sections to study the homology of the Koszul complex on $\underline{g}$ with coefficients in M.

Now we fix some notation to be used throughout the section. Let F be a free resolution of M over Q such that $\operatorname{Im}\partial_F \subseteq JF$ as in Definition 2.2.1. Let $\pi : Q \to Q/J$ be the quotient map and fix a k-linear splitting $\sigma\colon Q/J \to Q$. Set $K = K(\underline{g}; Q) = Q \otimes_{k[\underline{g}]} \Omega^{\bullet}_{k[\underline{g}]/k}$ with differential δ. Let ∇ be the k-linear map (2.1) defined in Section 2.1.2, and let H_∇ be the de Rham contraction from Definition 2.1.6.

Consider the isomorphisms

$$H_\ell(\underline{g}; M) = H_\ell(M \underset{Q}{\otimes} K) \cong \operatorname{Tor}^Q_\ell(M, Q/J) \cong H_\ell(F \underset{Q}{\otimes} Q/J) \cong F_\ell \underset{Q}{\otimes} Q/J \qquad (2.3)$$

of Q/J-modules, which appeared in the proof of Proposition 2.2.6. Thus, the Koszul homology we are interested in is isomorphic to the homology of the complex $F \otimes_Q K$. We begin by giving a modification, involving a related complex, of the special deformation retract datum of Theorem 2.1.7. We apply the perturbation lemma to this datum to yield the desired formulas in Theorem 2.4.1. Before stating it, we make the following remark about the maps that appear in Lemma 2.3.2.

Remark 2.3.1. We note that σ and H_∇ are only k-linear maps, so to define maps $1 \otimes \sigma$ and $1 \otimes H_\nabla$, we first fix a basis $h_1^\ell, \dots, h_{b_\ell}^\ell$ for each module F_ℓ in the resolution F. Now we have the isomorphisms

$$F_\ell \underset{Q}{\otimes} K_i \cong Q^{b_\ell} \underset{Q}{\otimes} K_i \cong K_i^{b_\ell} \tag{2.4}$$

and

$$F_\ell \underset{Q}{\otimes} Q/J \cong Q^{b_\ell} \underset{Q}{\otimes} Q/J \cong (Q/J)^{b_\ell},$$

for $i \geq 0$. We define $1 \otimes \sigma : F_\ell \otimes Q/J \to F_\ell \otimes K_0$ by applying σ to each of the b_ℓ summands. We consider the composition

$$F \otimes Q/J \xrightarrow{1 \otimes \sigma} F \otimes K_0 \hookrightarrow F \otimes K$$

where the second map is the natural inclusion; abusing notation slightly, we call this map $1 \otimes \sigma$. Again using the isomorphisms in (2.4), we define $1 \otimes H_\nabla$ by applying H_∇ to each of the b_ℓ summands. Throughout the remainder of this section, we use the notation $1 \otimes \sigma$ and $1 \otimes H_\nabla$ with the understanding that the maps are defined with respect to the fixed bases above. Also, we note that $(1 \otimes \pi)(F \otimes K_i) = 0$ for all $i \geq 1$.

We will see that, in order to give a basis for the Koszul homology, it is enough to find a k-linear map which agrees with the Q-linear isomorphism in (2.3), and apply this map to our fixed bases above. We use the k-linear maps $1 \otimes \sigma$ and $1 \otimes H_\nabla$ to produce such a map.

Now we state the lemma.

Lemma 2.3.2. *The following is a special deformation retract datum*

$$\Big((F \underset{Q}{\otimes} Q/J, 0) \underset{1\otimes\sigma}{\overset{1\otimes\pi}{\leftrightarrows}} (F \underset{Q}{\otimes} K, (0, \delta)),\ 1 \otimes H_\nabla \Big).$$

Proof. We note that we have the equalities

$$(1 \otimes H_\nabla)(1 \otimes \sigma) = 0$$
$$(1 \otimes \pi)(1 \otimes H_\nabla) = 0$$
$$(1 \otimes H_\nabla)^2 = 0$$

since the deformation retract datum from Theorem 2.1.7 is special. It is also clear that $(1 \otimes \pi) \circ (1 \otimes \sigma) = \mathrm{Id}_{F \otimes Q/J}$. Thus we need only check that $1 \otimes H_\nabla$ is a homotopy between $(1 \otimes \sigma) \circ (1 \otimes \pi)$ and $\mathrm{Id}_{F \otimes K}$, and the fact that $1 \otimes \sigma$ and $1 \otimes \pi$ are quasi-isomorphisms will follow. But the fact that $1 \otimes H_\nabla$ is a homotopy follows directly from the definitions given in Remark 2.3.1 and from Theorem 2.1.7. □

We will need the following lemmas in the proof of Theorem 2.4.1 and its corollaries.

Lemma 2.3.3. $\frac{\partial^2}{\partial g_j \partial g_i} = \frac{\partial^2}{\partial g_i \partial g_j}$

Proof. Writing $q = \sum_N \sigma(\overline{q_N}) g^N$ as in Lemma 2.1.4, we have the equalities

$$\frac{\partial}{\partial g_i}(q) = \sum_N N_i \sigma(\overline{q_N}) g^{N-e_i} = \sum_M \sigma(\overline{r_M}) g^M$$

where $\overline{r_M} = N_i \overline{q_N}$ and $M = N - e_i$, and where the second equality follows from the fact that σ is k-linear. Thus we get that

$$\frac{\partial^2}{\partial g_j \partial g_i}(q) = \sum_M M_j \sigma(\overline{r_M}) g^{M-e_j} = \sum_N N_i N_j \sigma(\overline{q_N}) g^{N-e_i-e_j},$$

again by k-linearity. A similar argument for $\frac{\partial^2}{\partial g_i \partial g_j}(q)$, gives the desired equality. □

The next lemma is a version of the Leibniz rule for $\frac{\partial}{\partial g_j}$. In the lemma, we write elements $q, r \in Q$ as $q = \sum_N \sigma(\overline{q_N}) g^N$ and $r = \sum_M \sigma(\overline{r_M}) g^M$, respectively.

Lemma 2.3.4. *The map* $\frac{\partial}{\partial g_j}$ *satisfies the rule*

$$\frac{\partial}{\partial g_j}(qr) = \frac{\partial}{\partial g_j}(q) r + \frac{\partial}{\partial g_j}(r) q + \sum_{M,N} \left(\frac{\partial}{\partial g_j}\big(\sigma(\overline{q_N})\sigma(\overline{r_M})\big) \right) g^{M+N}.$$

Proof. We begin by noting that $qr = \sum_{M,N} \sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N}$ and writing $\sigma(\overline{q_N})\sigma(\overline{r_M}) = \sum_P \sigma(\overline{s_P})g^P$. Now we compute

$$\begin{aligned}
\frac{\partial}{\partial g_j}(qr) &= \frac{\partial}{\partial g_j}\left(\sum_{M,N,P} \sigma(\overline{s_P})g^{M+N+P}\right) = \sum_{M,N,P}(M_j + N_j + P_j)\sigma(\overline{s_P})g^{M+N+P-e_j} \\
&= \sum_{M,N,P}(M_j + N_j)\sigma(\overline{s_P})g^{M+N+P-e_j} + \sum_{M,N,P} P_j\sigma(\overline{s_P})g^{M+N+P-e_j} \\
&= \sum_{M,N}(M_j + N_j)\Big(\sum_P \sigma(\overline{s_P})g^P\Big)g^{M+N-e_j} + \sum_{M,N}\Big(\sum_P P_j\sigma(\overline{s_P})g^{P-e_j}\Big)g^{M+N} \\
&= \sum_{M,N}(M_j + N_j)\sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} + \sum_{M,N}\left(\frac{\partial}{\partial g_j}\big(\sigma(\overline{q_N})\sigma(\overline{r_M})\big)\right)g^{M+N}
\end{aligned}$$

and we see that

$$\begin{aligned}
&\sum_{M,N}(M_j + N_j)\sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} = \\
&\quad = \sum_{M,N} M_j\sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} + \sum_{M,N} N_j\sigma(\overline{q_N})\sigma(\overline{r_M})g^{M+N-e_j} \\
&\quad = \left(\sum_M M_j\sigma(\overline{r_M})g^{M-e_j}\right)\left(\sum_N \sigma(\overline{q_N})g^N\right) + \left(\sum_N N_j\sigma(\overline{q_N})g^{N-e_j}\right)\left(\sum_M \sigma(\overline{r_M})g^M\right) \\
&\quad = \frac{\partial}{\partial g_j}(q)r + \frac{\partial}{\partial g_j}(r)q
\end{aligned}$$

which completes the proof. □

Remark 2.3.5. We see from Lemma 2.3.4 that $\frac{\partial}{\partial g_j}$ satisfies the usual Leibniz rule,

$$\frac{\partial}{\partial g_j}(qr) = \frac{\partial}{\partial g_j}(q)r + q\frac{\partial}{\partial g_j}(r),$$

whenever for each pair (M, N), either $\sigma(\overline{q_N}) \in k$ or $\sigma(\overline{r_M}) \in k$. Indeed, in this case $\frac{\partial}{\partial g_j}(\sigma(\overline{q_N})\sigma(\overline{r_M})) = 0$. In particular, we have that

$$\frac{\partial}{\partial g_j}(qg_k) = \begin{cases} \frac{\partial}{\partial g_j}(q)g_k + q & j = k \\ \frac{\partial}{\partial g_j}(q)g_k & j \neq k \end{cases}.$$

However, examples which do not satisfy the usual product rule are plentiful. Consider the regular sequence $g_1 = x^2$, $g_2 = y^3$, $g_3 = z^5$ in $k[x, y, z]$. We have that

$$\frac{\partial}{\partial g_1}(xy^3 \cdot xz^5) = \frac{\partial}{\partial g_1}(x^2y^3z^5) = y^3z^5,$$

but

$$\frac{\partial}{\partial g_1}(xy^3)xz^5 + \frac{\partial}{\partial g_1}(xz^5)xy^3 = 0.$$

Note however that

$$\frac{\partial}{\partial g_1}(x \cdot x)y^3z^5 = y^3z^5$$

which illustrates Lemma 2.3.4.

Recall that the map $\delta\nabla + \nabla\delta$ is invertible in nonzero degrees by [19, 8.1]. The next lemma gives an explicit description of the map $(\delta\nabla + \nabla\delta)^{-1}$. This fact is known to experts, but to our knowledge, is not directly stated in the literature, so we state and prove it here.

Lemma 2.3.6. *The map $(\delta\nabla + \nabla\delta)^{-1} : K \to K$ is given by*

$$(\delta\nabla + \nabla\delta)^{-1}(q \otimes dg_{i_1} \dots dg_{i_r}) = \sum_N \frac{1}{|N| + r}\left(\sigma(\overline{q_N})g^N \otimes dg_{i_1} \dots dg_{i_r}\right)$$

where $q = \sum_N \sigma(\overline{q_N})g^N$ with $N = (n_1, \dots, n_s)$ and $|N| = n_1 + \cdots + n_s$.

Proof. We begin by computing

$$\begin{aligned}
&(\delta\nabla + \nabla\delta)(q \otimes dg_{i_1} \dots dg_{i_r}) = \\
&= \delta\left(\sum_{j=1}^{s}\frac{\partial}{\partial g_j}(q) \otimes dg_j dg_{i_1} \dots dg_{i_r}\right) + \nabla\left(q \otimes \left(\sum_{\ell=1}^{k}(-1)^{\ell+1} g_{i_\ell} dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r}\right)\right) \\
&= \sum_{j=1}^{s}\delta\left(\frac{\partial}{\partial g_j}(q) \otimes dg_j dg_{i_1} \dots dg_{i_r}\right) + \sum_{\ell=1}^{k}(-1)^{\ell+1}\nabla\left(q g_{i_\ell} \otimes \left(dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r}\right)\right) \\
&= \sum_{j=1}^{s}\left(\frac{\partial}{\partial g_j}(q) \otimes \left(g_j dg_{i_1} \dots dg_{i_r} + \sum_{m=1}^{k}(-1)^m g_{i_m} dg_j dg_{i_1} \dots \widehat{dg_{i_m}} \dots dg_{i_r}\right)\right) \\
&\quad + \sum_{\ell=1}^{k}(-1)^{\ell+1}\left(\sum_{p=1}^{s}\frac{\partial}{\partial g_p}(q g_{i_\ell}) \otimes \left(dg_p dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r}\right)\right).
\end{aligned}$$

Simplifying the first sum and applying Remark 2.3.5 to the second, we have

$(\delta\nabla + \nabla\delta)(q \otimes dg_{i_1} \dots dg_{i_r}) =$

$$= \sum_{j=1}^{s} \frac{\partial}{\partial g_j}(q) g_j \otimes dg_{i_1} \dots dg_{i_r} + \sum_{j=1}^{s} \sum_{m=1}^{k} (-1)^m \frac{\partial}{\partial g_j}(q) g_{i_m} \otimes dg_j dg_{i_1} \dots \widehat{dg_{i_m}} \dots dg_{i_r}$$

$$+ \sum_{\ell=1}^{k} \sum_{p=1}^{s} (-1)^{\ell+1} \frac{\partial}{\partial g_p}(q) g_{i_\ell} \otimes dg_p dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r} + \sum_{\ell=1}^{k} (-1)^{\ell+1} q \otimes dg_{i_\ell} dg_{i_1} \dots \widehat{dg_{i_\ell}} \dots dg_{i_r}.$$

We note that the middle two sums cancel with each other and we are left with the equality

$$(\delta\nabla + \nabla\delta)(q \otimes dg_{i_1} \dots dg_{i_r}) = \sum_{j=1}^{s} \frac{\partial}{\partial g_j}(q) g_j \otimes dg_{i_1} \dots dg_{i_r} + rq \otimes dg_{i_1} \dots dg_{i_r}. \tag{2.5}$$

But we have that

$$\begin{aligned} \sum_{j=1}^{s} \frac{\partial}{\partial g_j}(q) g_j &= \sum_{j=1}^{s} \sum_{N} N_j \sigma(\overline{q_N}) g^{N-e_j} g_j \\ &= \sum_{N} \sum_{j=1}^{s} N_j \sigma(\overline{q_N}) g^N \\ &= \sum_{N} |N| \sigma(\overline{q_N}) g^N, \end{aligned}$$

and thus, writing $q = \sum_N \sigma(\overline{q_N}) g^N$ in (2.5), we get

$$(\delta\nabla + \nabla\delta)(\sum_N \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r}) = \sum_N (|N| + r) \left(\sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r}\right). \tag{2.6}$$

Therefore, since σ is k-linear, we have

$$\begin{aligned} &(\delta\nabla + \nabla\delta)^{-1}(\sum_N \sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r}) = \\ &\quad = (\delta\nabla + \nabla\delta)^{-1} \left(\sum_N \frac{|N| + r}{|N| + r} \left(\sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r}\right) \right) \\ &\quad = (\delta\nabla + \nabla\delta)^{-1} \left(\sum_N (|N| + r) \left(\sigma\left(\overline{\frac{q_N}{|N| + r}}\right) g^N \otimes dg_{i_1} \dots dg_{i_r} \right) \right) \\ &\quad = \sum_N \sigma\left(\overline{\frac{q_N}{|N| + r}}\right) g^N \otimes dg_{i_1} \dots dg_{i_r} \\ &\quad = \sum_N \frac{1}{|N| + r} \left(\sigma(\overline{q_N}) g^N \otimes dg_{i_1} \dots dg_{i_r}\right), \end{aligned}$$

where the third equality follows from (2.6), and this completes the proof. □

We establish some notation which we use throughout the rest of the chapter.

Definition 2.3.7. Let $q = \sum_N \sigma(\overline{q_N}) g^N$ be an element of Q. We define $\frac{\partial^*}{\partial g_j}$ by

$$\frac{\partial^*}{\partial g_j}(q) = \sum_N \frac{N_j}{|N|} \sigma(\overline{q_N}) g^{N-e_j}.$$

In the next section, we give our main result of the chapter; namely, we provide explicit formulas for the generators of Koszul homology.

2.4 The Main Result: Explicit Generators

Throughout this section, we assume that Q is a Noetherian k-algebra where k is a field of characteristic zero. We fix a regular sequence $\underline{g}$ in Q and the ideal J generated by it such that Q/J is a finite dimensional k-vector space. For $f \in Q$, we denote by $\widehat{f}$ the image of f in $\widehat{Q^J}$. Under these assumptions, we have the following result.

Theorem 2.4.1. *Let M be a finitely generated J-closed module over Q with free resolution (F, ∂_F) satisfying (2.2) in Definition 2.2.1. Then a Q/J-basis of $H_\ell(\underline{g}; M)$ is given by the homology classes of the elements*

$$z_{j_1} = \sum_{1 \le k_1, \ldots, k_\ell \le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_{\ell+1}=1}^{b_0} m_{j_{\ell+1}} \frac{\partial^*}{\partial g_{k_\ell}} \left(\widehat{f^1_{j_{\ell+1}, j_\ell}} \frac{\partial^*}{\partial g_{k_{\ell-1}}} \left(\widehat{f^2_{j_\ell, j_{\ell-1}}} \cdots \frac{\partial^*}{\partial g_{k_1}} (\widehat{f^\ell_{j_2, j_1}}) \cdots \right) \right) dg_{k_1} \ldots dg_{k_\ell}$$

for $j_1 = 1, \ldots, b_\ell$, where $h^i_1, \ldots, h^i_{b_i}$ is a basis for F_i, where $\partial_F(h^i_p) = \sum_{m=1}^{b_{i-1}} f^i_{m,p} h^{i-1}_m$, where m_j is the image of h^0_j under the augmentation map $F \to M$, and where we identify $z_{j_1} \in H(\underline{g}; \widehat{M^J})$ with its image under the isomorphism $H(\underline{g}; \widehat{M^J}) \cong H(\underline{g}; M)$.

Proof. We first reduce to the case where Q is complete with respect to the J-adic topology. Suppose that we can find such a basis for $H(\underline{g}; \widehat{M^J})$ over the completion $\widehat{Q/J^J}$. Thus we have such a basis for

$$\widehat{H_\ell(\underline{g}; M)}^J \cong H_\ell(\underline{g}; M) \otimes \widehat{Q^J} \cong H_\ell(\underline{g}; M \otimes \widehat{Q^J}) \cong H_\ell(\underline{g}; \widehat{M^J})$$

by flatness. But

$$\widehat{H_\ell(\underline{g}; M)}^J = \varprojlim_t H_\ell(\underline{g}; M)/J^t H_\ell(\underline{g}; M) = H_\ell(\underline{g}; M)$$

where the last equality follows from the fact that $J \subseteq \operatorname{ann}_Q H_\ell(\underline{g}; M)$. Thus, it suffices to find such a basis in the complete case.

Applying Lemma 2.3.2, we get a special deformation retract datum

$$\Big((F \underset{Q}{\otimes} Q/J, 0) \underset{1\otimes\sigma}{\overset{1\otimes\pi}{\leftrightarrows}} (F \underset{Q}{\otimes} K, \delta),\, 1 \otimes H_\nabla\Big).$$

For ease of notation we write $\partial_F \otimes 1$ as ∂_F and $1 \otimes \delta$ as δ, and we note that ∂_F is a perturbation of the special deformation retract datum above. Indeed, the degree of ∂_F is -1 and $(\partial_F + \delta)^2 = 0$ since it is the differential on the total complex $F \otimes K$. By Proposition 2.2.6 (1), $F \otimes Q/J$ is a finite complex, say $F_i \otimes Q/J = 0$ for all $i > r$, so we have the following equalities

$$\begin{aligned}
&\Big(1 - \partial_F(1 \otimes H_\nabla)\Big)\Big(1 + \partial_F(1 \otimes H_\nabla) + \big(\partial_F(1 \otimes H_\nabla)\big)^2 + \cdots + \big(\partial_F(1 \otimes H_\nabla)\big)^r\Big) \\
&\quad = 1 + \partial_F(1 \otimes H_\nabla) + \big(\partial_F(1 \otimes H_\nabla)\big)^2 + \cdots + \big(\partial_F(1 \otimes H_\nabla)\big)^r \\
&\qquad - \Big(\partial_F(1 \otimes H_\nabla) + \big(\partial_F(1 \otimes H_\nabla)\big)^2 + \cdots + \big(\partial_F(1 \otimes H_\nabla)\big)^{r+1}\Big) \\
&\quad = 1 - \big(\partial_F(1 \otimes H_\nabla)\big)^{r+1} = 1.
\end{aligned}$$

Hence, we have that the map $1 - \partial_F(1 \otimes H_\nabla)$ is invertible, with inverse

$$\big(1 - \partial_F(1 \otimes H_\nabla)\big)^{-1} = 1 + \partial_F(1 \otimes H_\nabla) + \big(\partial_F(1 \otimes H_\nabla)\big)^2 + \cdots + \big(\partial_F(1 \otimes H_\nabla)\big)^r;$$

thus ∂_F is small. By the perturbation lemma, the perturbed datum is a deformation retract datum. In particular, we have the homotopy equivalence

$$(F \underset{Q}{\otimes} Q/J, \tilde{0}) \underset{\widetilde{1\otimes\sigma}}{\overset{\widetilde{1\otimes\pi}}{\leftrightarrows}} (F \underset{Q}{\otimes} K, \delta + \partial_F).$$

We note that $\delta + \partial_F$ is the usual differential on the total complex $F \otimes K$ and that the perturbed map $\widetilde{0}$ is given by

$$\begin{aligned}
\tilde{0} &= 0 + (1 \otimes \pi)A(1 \otimes \sigma) \\
&= (1 \otimes \pi)\Big(1 + \partial_F(1 \otimes H_\nabla) + \big(\partial_F(1 \otimes H_\nabla)\big)^2 + \cdots + \big(\partial_F(1 \otimes H_\nabla)\big)^r\Big)\partial_F(1 \otimes \sigma) \\
&= (1 \otimes \pi)\partial_F(1 \otimes \sigma)
\end{aligned}$$

where the last equality follows from the fact that $1 \otimes \pi$ composed with $(\partial_F(1 \otimes H_\nabla))^i$ for $i > 0$ is zero. Now since $\operatorname{Im} \partial_F \subseteq JF$, we get that π composed with ∂_F is zero, thus $\tilde{0} = 0$, which gives the homotopy equivalence

$$F \underset{Q}{\otimes} Q/J \xrightarrow{\widetilde{1 \otimes \sigma}} F \underset{Q}{\otimes} K$$

where

$$\begin{aligned}
\widetilde{1 \otimes \sigma} &= (1 \otimes \sigma) + (1 \otimes H_\nabla) A (1 \otimes \sigma) \\
&= (1 \otimes \sigma) + (1 \otimes H_\nabla) \left(1 + \partial_F(1 \otimes H_\nabla) + \cdots + \left(\partial_F(1 \otimes H_\nabla)\right)^r\right) \partial_F(1 \otimes \sigma) \\
&= \left(1 + (1 \otimes H_\nabla)\partial_F + ((1 \otimes H_\nabla)\partial_F)^2 + \cdots + ((1 \otimes H_\nabla)\partial_F)^r\right) (1 \otimes \sigma).
\end{aligned}$$

We also note that $\widetilde{1 \otimes \pi}$ is just the Q-linear map $1 \otimes \pi$ since it sends all elements to zero except ones lying in the first row of the double complex, $F \otimes_Q K$. Thus, the induced map on homology

$$F_\ell \underset{Q}{\otimes} Q/J \xrightarrow{\cong} \operatorname{Tor}_\ell^Q(M, Q/J) \cong H_\ell(M \underset{Q}{\otimes} K(\underline{g}; Q)) \cong H_\ell(\underline{g}; M)$$

is an isomorphism whose inverse is induced by the Q-linear map $1 \otimes \pi$, and hence agrees with the Q-linear isomorphism (2.3). As a result, a basis for $H_\ell(\underline{g}; M)$ is given by first applying $\widetilde{1 \otimes \sigma}$ to the basis elements of $F_\ell \otimes Q/J$ and then applying the augmentation map $F_0 \to M$ and taking homology classes of the results.

To this end, we compute

$$\begin{aligned}
((1 \otimes H_\nabla)\partial_F)^\ell \left(h^\ell_{j_1} \otimes 1 \otimes 1\right) &= ((1 \otimes H_\nabla)\partial_F)^{\ell-1} (1 \otimes H_\nabla) \left(\sum_{j_2=1}^{b_{\ell-1}} f^\ell_{j_2,j_1} h^{\ell-1}_{j_2} \otimes 1 \otimes 1 \right) \\
&= \sum_{j_2=1}^{b_{\ell-1}} ((1 \otimes H_\nabla)\partial_F)^{\ell-1} (1 \otimes H_\nabla) \left(h^{\ell-1}_{j_2} \otimes f^\ell_{j_2,j_1} \otimes 1\right) \\
&= \sum_{j_2=1}^{b_{\ell-1}} ((1 \otimes H_\nabla)\partial_F)^{\ell-1} \left(h^{\ell-1}_{j_2} \otimes \sum_{k_1=1}^{s} \frac{\partial^*}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \otimes dg_{k_1} \right) \\
&= \sum_{k_1=1}^{s} \sum_{j_2=1}^{b_{\ell-1}} ((1 \otimes H_\nabla)\partial_F)^{\ell-1} \left(h^{\ell-1}_{j_2} \otimes \frac{\partial^*}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \otimes dg_{k_1} \right)
\end{aligned}$$

Applying the procedure above $\ell - 1$ more times, we get that

$$((1 \otimes H_{\nabla})\partial_F)^{\ell} (h^{\ell}_{j_1} \otimes 1 \otimes 1) = \sum_{1 \le k_1, \ldots, k_\ell \le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_{\ell+1}=1}^{b_0} \left(h^0_{j_{\ell+1}} \otimes Z^{j_1,\ldots,j_{\ell+1}}_{k_1,\ldots,k_\ell} \otimes dg_{k_1} \ldots dg_{k_\ell} \right),$$

where

$$Z^{j_1,\ldots,j_{\ell+1}}_{k_1,\ldots,k_\ell} = \frac{\partial^*}{\partial g_{k_\ell}} \left(f^1_{j_{\ell+1},j_\ell} \frac{\partial^*}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial^*}{\partial g_{k_1}} \left(f^{\ell}_{j_2,j_1} \right) \ldots \right) \right).$$

Applying the augmentation map $F_0 \otimes_Q K_\ell \to M \otimes_Q K_\ell$, we obtain the desired formula. $\square$

We make the following remark regarding Theorem 2.4.1.

Remark 2.4.2. (1) We see that in the case that $M = Q/I$ and I is a J-closed ideal, the formulas in Theorem 2.4.1 are given by

$$z_{j_1} = \sum_{1 \le k_1, \ldots, k_\ell \le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} \overline{\frac{\partial^*}{\partial g_{k_\ell}} \left(\widehat{f^1_{1,j_\ell}} \frac{\partial^*}{\partial g_{k_{\ell-1}}} \left(\widehat{f^2_{j_\ell,j_{\ell-1}}} \cdots \frac{\partial^*}{\partial g_{k_1}} (\widehat{f^{\ell}_{j_2,j_1}}) \ldots \right) \right)} dg_{k_1} \ldots dg_{k_\ell}$$

where the bar denotes the image in the quotient.
(2) The proof of Theorem 2.4.1 actually yields the following more general result. Let F be any complex such that $\operatorname{Im} \partial_i \subseteq JF_{i-1}$ for all i. Define $K(\underline{g}; F) = K(\underline{g}; Q) \otimes_Q F$ and let $H(\underline{g}; F)$ be its homology. Then the homology classes of the elements given in Theorem 2.4.1 are a basis for $H(\underline{g}; F)$.

2.5 The Generators in Some Special Cases

We now give versions of these formulas for some special cases of interest. In the following corollaries, we consider the case where M is a cyclic module, but similar formulas can be given in the non-cyclic case as in Theorem 2.4.1. First we establish some notation. We denote

$$\frac{\partial(f_1, \ldots, f_i)}{\partial(g_{k_1}, \ldots, g_{k_i})} = \det \left(\frac{\partial}{\partial g_{k_j}} (f_\ell) \right)_{j,\ell}.$$

Definition 2.5.1. We call f *homogeneous in $\underline{g}$ of degree n* if there is an integer n such that

$$f = \sum_N \sigma(\overline{f_N}) g^N$$

for $N = (n_1, \ldots, n_s) \in \mathbb{N}^s$ satisfying $\sum_{\ell=1}^s n_\ell = n$. We call the $\sigma\left(\overline{f_N}\right)$ the *coefficients* of f.

In the following corollary, we provide more explicit versions of the formulas from Theorem 2.4.1. These formulas have new terms which vanish in the classical case, where $\underline{g} = x_1, \ldots, x_n$ are minimal generators of the maximal ideal.

Corollary 2.5.2. *Let I be a J-closed ideal in Q with $R = Q/I$ and (F, ∂_F) a Q-free resolution of R such that the entries in the matrices ∂_F are homogeneous in $\underline{g}$ with coefficients in Q. Then a Q/J-basis of $H_\ell(\underline{g}; R)$ is given by the homology classes of the elements*

$$z_{\ell_{j_1}} = \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1,\ldots,j_\ell} \left(\sum_{1 \le k_1 < \cdots < k_\ell \le s} \overline{\frac{\partial(f^1_{1,j_\ell}, f^2_{j_\ell,j_{\ell-1}}, \ldots, f^\ell_{j_2,j_1})}{\partial(g_{k_1}, \ldots, g_{k_\ell})}} + \mathcal{E} \right) dg_{k_1} \ldots dg_{k_\ell}$$

for $j_1 = 1, \ldots, b_\ell$, where

$$\mathcal{E} = \sum_{1 \le k_1, \ldots, k_\ell \le s} \sum_{m=2}^{\ell} \prod_{n=m+1}^{\ell} \overline{\frac{\partial}{\partial g_{k_n}} (f^{\ell-n+1}_{j_{n+1},j_n}) \sum_{M,N} \frac{\partial}{\partial g_{k_m}} \left(\sigma\left(\overline{f^{\ell-m+1}_{j_{m+1},j_m\, M}}\right) \sigma\left(\overline{y_{(m-1)_{j_1}}}_{N}\right) \right) g^{M+N}}$$

and the elements $y_{(m-1)_{j_1}}$ are defined inductively by

$$z_{(m-1)_{j_1}} = \sum_{1 \le k_1, \ldots, k_{m-1} \le s} \sum_{j_2=1}^{b_{m-2}} \cdots \sum_{j_{m-1}=1}^{b_1} D_{j_1,\ldots,j_{m-1}} y_{(m-1)_{j_1}} dg_{k_1} \ldots dg_{k_{m-1}}$$

and where

$$D_{j_1,\ldots,j_\ell} = \frac{1}{d^\ell_{j_2,j_1}} \cdot \frac{1}{d^{\ell-1}_{j_3,j_2} + d^\ell_{j_2,j_1} - 1} \cdot \cdots \cdot \frac{1}{d^1_{1,j_\ell} + d^2_{j_\ell,j_{\ell-1}} + \cdots + d^\ell_{j_2,j_1} - \ell + 1}$$

with $d^k_{i,j}$ the degree of $f^k_{i,j}$.

Proof. We first note that for a $\underline{g}$-homogeneous polynomial, $f^k_{i,j}$ we have that

$$\frac{\partial^*}{\partial g_j}(f^k_{i,j}) = \frac{1}{d^k_{i,j}} \frac{\partial}{\partial g_j}(f^k_{i,j}).$$

Thus, since $\frac{\partial}{\partial g_j}$ reduces the $\underline{g}$-degree of its argument by one whenever it is nonzero, the formulas from Theorem 2.4.1 become

$$z_{\ell_{j_1}} = \sum_{1\le k_1,\ldots,k_\ell\le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1,\ldots,j_\ell} \frac{\partial}{\partial g_{k_\ell}} \left(f^1_{1,j_\ell} \frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \cdots \right)\right) dg_{k_1} \ldots dg_{k_\ell}, \tag{2.7}$$

where we omit the bar for ease of exposition. It is now clear that we have the desired formula for $z_{1_{j_1}}$. To obtain the desired formula for $z_{\ell_{j_1}}$, we apply the Leibniz rule from Lemma 2.3.4 to (2.7), to get

$$\begin{aligned} z_{\ell_{j_1}} = \sum_{1\le k_1,\ldots,k_\ell\le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1,\ldots,j_\ell} \Bigg(& \frac{\partial}{\partial g_{k_\ell}} (f^1_{1,j_\ell}) \frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \cdots \right) \\ & + f^1_{1,j_\ell} \frac{\partial}{\partial g_{k_\ell}} \left(\frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \cdots \right)\right) \\ & + \sum_{M,N} \frac{\partial}{\partial g_{k_\ell}} \left(\sigma(\overline{f^1_{1,j_\ell}}_M) \sigma \left(\overline{\frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \cdots \right)}_N \right)\right) g^{M+N} \Bigg) dg_{k_1} \ldots dg_{k_\ell}. \end{aligned}$$

However, by Lemma 2.3.3, we have that

$$\sum_{1\le k_1,\ldots,k_\ell\le s} \frac{\partial}{\partial g_{k_\ell}} \left(\frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \right) \cdots \right) dg_{k_1} \ldots dg_{k_\ell} = 0,$$

which gives the following formula

$$\begin{aligned} z_{\ell_{j_1}} = \sum_{1\le k_1,\ldots,k_\ell\le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1,\ldots,j_\ell} \Bigg(& \frac{\partial}{\partial g_{k_\ell}} (f^1_{1,j_\ell}) \frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \cdots \right) \\ & + \sum_{M,N} \frac{\partial}{\partial g_{k_\ell}} \left(\sigma(\overline{f^1_{1,j_\ell}}_M) \sigma \left(\overline{\frac{\partial}{\partial g_{k_{\ell-1}}} \left(f^2_{j_\ell,j_{\ell-1}} \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \cdots \right)}_N \right)\right) g^{M+N} \Bigg) dg_{k_1} \ldots dg_{k_\ell}. \end{aligned}$$

Now we apply the Leibniz rule from Lemma 2.3.4 repeatedly, simplifying at each step as above, to obtain

$$\begin{aligned} z_{\ell_{j_1}} = \sum_{1\le k_1,\ldots,k_\ell\le s} \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1,\ldots,j_\ell} \Bigg(& \frac{\partial}{\partial g_{k_\ell}} (f^1_{1,j_\ell}) \frac{\partial}{\partial g_{k_{\ell-1}}} (f^1_{j_\ell,j_{\ell-1}}) \cdots \frac{\partial}{\partial g_{k_1}} (f^\ell_{j_2,j_1}) \\ & + \sum_{m=2}^{\ell} \prod_{n=m+1}^{\ell} \overline{\frac{\partial}{\partial g_{k_n}} (f^{\ell-n+1}_{j_{n+1},j_n}) \sum_{M,N} \frac{\partial}{\partial g_{k_m}} \left(\sigma(\overline{f^{\ell-m+1}_{j_{m+1},j_m}}_M) \sigma(\overline{y_{(m-1)_{j_1}}}_N) \right) g^{M+N}} \Bigg) dg_{k_1} \ldots dg_{k_\ell} \end{aligned}$$

where

$$y_{(m-1)_{j_1}} = \frac{\partial}{\partial g_{k_{m-1}}} \left(f^{\ell-m+2}_{j_m, j_{m-1}} \cdots \frac{\partial}{\partial g_{k_1}} \big(f^{\ell}_{j_2, j_1} \big) \cdots \right).$$

We observe that the first part of the above formula can be written as a determinant, thus giving the desired formulas. □

We now give formulas in the case that the entries in the matrices given by ∂_F are homogeneous in $\underline{g}$ with coefficients in k rather than Q. This happens for example when $M = Q \otimes_{k[\underline{g}]} \widetilde{M}$, where $\widetilde{M}$ is a $k[\underline{g}]$-module.

Corollary 2.5.3. *Let I be a J-closed ideal in Q with $R = Q/I$ and (F, ∂_F) a Q-free resolution of R such that the entries in the matrices ∂_F are homogeneous in $\underline{g}$ with coefficients in k. Then a Q/J-basis of $H_\ell(\underline{g}; R)$ is given by the homology classes of the elements*

$$z_{\ell_{j_1}} = \sum_{j_2=1}^{b_{\ell-1}} \cdots \sum_{j_\ell=1}^{b_1} D_{j_1,\ldots,j_\ell} \sum_{1 \le k_1 < \cdots < k_\ell \le s} \overline{\frac{\partial(f^1_{1,j_\ell}, f^2_{j_\ell, j_{\ell-1}}, \ldots, f^\ell_{j_2,j_1})}{\partial(g_{k_1}, \ldots, g_{k_\ell})}} dg_{k_1} \ldots dg_{k_\ell}$$

for $j_1 = 1, \ldots, b_\ell$.

Proof. This follows immediately from Corollary 2.5.2 and the fact that $\frac{\partial}{\partial g_j}$ satisfies the usual Leibniz rule in the case that the coefficients involved in the product are elements of k, as discussed in Remark 2.3.5. □

Remark 2.5.4. Taking $\underline{g} = x_1, \ldots, x_n$ to be the minimal generators of the maximal ideal, the corollary recovers the formulas given by Herzog in [24].

2.6 An Example

We finish this chapter with an example which illustrates our results. We note that for this example, the new terms in the formulas given in Corollary 2.5.2, which vanish in the case where $\underline{g} = x_1, \ldots, x_n$ are minimal generators of the maximal ideal, do not vanish.

Example 2.6.1. Let $Q = k[x, y, z]$ be a polynomial ring with k a field of characteristic zero and let J be a complete intersection ideal generated by the regular sequence $g_1 = x^2 + yz$, $g_2 = y^3$, $g_3 = z^5$. Consider the ideal $I = (x^2y^4 + y^5z + xz^{10},\ y^6,\ x^4y^2 + x^2y^3z)$. Let us compute the generators of $H_\ell(\underline{g}; R)$, where $R = Q/I$. We begin by fixing a k-linear splitting of $\pi : Q \to Q/J$. One can find the following basis for Q/J as a k-vector space either by hand or using Macaulay2 [21]

$$\bar{1},\ \overline{x},\ \overline{xy},\ \overline{xy^2},\ \overline{xy^2z},\ \overline{xy^2z^2},\ \overline{xy^2z^3},\ \overline{xy^2z^4},\ \overline{xyz},\ \overline{xyz^2},\ \overline{xyz^3},\ \overline{xyz^4},\ \overline{xz},\ \overline{xz^2},$$
$$\overline{xz^3},\ \overline{xz^4},\ \overline{y},\ \overline{y^2},\ \overline{y^2z},\ \overline{y^2z^2},\ \overline{y^2z^3},\ \overline{y^2z^4},\ \overline{yz},\ \overline{yz^2},\ \overline{yz^3},\ \overline{yz^4},\ \overline{z},\ \overline{z^2},\ \overline{z^3},\ \overline{z^4}.$$

We choose the splitting $\sigma(\bar{a}) = a$ for every basis element $\bar{a}$. According to Macaulay2, a free resolution of R over Q is given by

$$0 \to Q \xrightarrow{\partial_3} Q^3 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \to R \to 0$$

where the differentials are given by the following matrices:

$$\partial_3 = \begin{bmatrix} z^{10} + xy^4 \\ -y^4 \\ x^3 + xyz \end{bmatrix},\ \partial_2 = \begin{bmatrix} -y^4 & -z^{10} - xy^4 & 0 \\ x^4 + x^2yz & -x^3yz - xy^2z^2 & -xz^{10} - x^2y^4 - y^5z \\ 0 & x^3y^2 + xy^3z & y^6 \end{bmatrix},$$

$$\partial_1 = \begin{bmatrix} x^4y^2 + x^2y^3z & y^6 & x^2y^4 + y^5z + xz^{10} \end{bmatrix}.$$

We now write the entries in the differentials as in Lemma 2.1.4. Note for example that the first entry in ∂_1 can be written as

$$x^4y^2 + x^2y^3z = x^2y^2g_1$$

but that x^2y^2 is not in the image of our splitting map σ, so we write

$$\begin{aligned} x^4y^2 + x^2y^3z &= x^2y^2g_1 \\ &= (g_1 - yz)y^2g_1 \end{aligned}$$

$$= y^2 g_1^2 - y^3 z g_1$$
$$= y^2 g_1^2 - z g_1 g_2$$

and we can see that the coefficients are now in the image of σ. Using a similar procedure on the other entries, we obtain the following matrices

$$\partial_3 = \begin{bmatrix} xyg_2 + g_3^2 \\ -yg_2 \\ xg_1 \end{bmatrix}, \ \partial_2 = \begin{bmatrix} -yg_2 & -xyg_2 - g_3^2 & 0 \\ g_1^2 - yzg_1 & -xyzg_1 & -yg_1g_2 - xg_3^2 \\ 0 & xy^2 g_1 & g_2^2 \end{bmatrix},$$

$$\partial_1 = \begin{bmatrix} y^2 g_1^2 - z g_1 g_2 & g_2^2 & y g_1 g_2 + x g_3^2 \end{bmatrix}.$$

Applying Theorem 2.4.1, we get the following set of elements whose homology classes generate $H_1(\underline{g}; R)$,

$$\widetilde{h_1^1} = \overline{y^2 g_1 - z g_2} dg_1 - \frac{1}{2}\overline{z g_1} dg_2 = \overline{x^2 y} dg_1 - \frac{1}{2}\overline{x^2 z + y z^2} dg_2$$
$$\widetilde{h_2^1} = \overline{g_2} dg_2 = \overline{y^3} dg_2$$
$$\widetilde{h_3^1} = \frac{1}{2}\overline{y g_2} dg_1 + \frac{1}{2}\overline{y g_1} dg_2 + \overline{x g_3} dg_3 = \frac{1}{2}\overline{y^4} dg_1 - \frac{1}{2}\overline{x^2 y + y^2 z} dg_2 + \overline{x z^5} dg_3$$

where $\widetilde{h_i^j}$ is the generator which corresponds to the basis element h_i^j of F_j. We get the following generators for $H_2(\underline{g}; R)$

$$\widetilde{h_1^2} = -\frac{1}{2}\overline{y^4 z} dg_1 dg_2$$
$$\widetilde{h_2^2} = \big(-\frac{1}{2}\overline{x y^4 z} - \frac{1}{3}\overline{z^{11}}\big) dg_1 dg_2 - \frac{1}{3}\overline{y^3 z^6} dg_1 dg_3 + \frac{1}{3}\overline{x^2 z^6 + y z^7} dg_2 dg_3$$
$$\widetilde{h_3^2} = -\frac{1}{3}\overline{y^7} dg_1 dg_2$$

and for $H_3(\underline{g}; R)$

$$\widetilde{h_1^3} = \big(\frac{1}{6}\overline{x^2 y^3 z^5} + \frac{2}{3}\overline{y^4 z^6} + \frac{1}{9}\overline{x^2 z^5} - \frac{1}{18}\overline{y z^6}\big) dg_1 dg_2 dg_3.$$

Chapter 3

Weak Complete Intersections

In this chapter, we discuss an application of the explicit generators of Koszul homology given in the previous chapter. The generators become useful in studying the connections between J-closed ideals and weak complete intersection ideals. In particular, we study the ideal J/I in the quotient Q/I, where I is a J-closed ideal, and in Proposition 3.1.4 we give a general sufficient condition under which J/I is a weak complete intersection. The condition we give involves the partials with respect to a regular sequence that generates J, discussed in the previous chapter. This result provides a new family of examples of weak complete intersections. We show that the condition in Proposition 3.1.4 is not a necessary one in Example 3.2.2.

3.1 A Sufficient Condition for WCI's

In this section we use the formulas for generators of Koszul homology given in the previous chapter to study the ideal $\bar{J} = J/I$ of the quotient $R = Q/I$, where Q is local and I is a J-closed ideal of Q.

In Proposition 3.1.4, we provide a general condition under which $\bar{J}$ is a weak complete intersection ideal in R, which expands the class of known examples of weak complete intersection ideals given in [36]. However, there are examples of ideals $\bar{J}$ which are not weak complete intersection ideals in quotients by J-closed ideals. We discuss one example in the following remark.

Remark 3.1.1. Taking $\underline{g} = x^2, y^3, z^5$ in Example 2.2.5, the ideal $\bar{J}$ of $R = Q/I$ is not a weak complete intersection ideal. One can verify this by looking at the beginning of the minimal free resolution of $R/\bar{J}$ over R on Macaulay2 [21]. There are entries in the differentials which are not elements of $\bar{J}$, for example, y^2 is one such entry.

We need the following definition in the proof of Proposition 3.1.4; see for example [36, Definition 2.7] or [2, Remark 5.2.1].

Definition 3.1.2. Let R be a local ring and assume $H_\ell(\underline{g}; R)$ is a free $R/(g)$-module for all ℓ. We say that $K(\underline{g}; R)$ admits a *trivial Massey operation* if for some basis $\mathcal{B} = \{h_\lambda\}_{\lambda\in\Lambda}$ of $H_{\geq 1}(\underline{g}; R)$, there is a function

$$\mu : \bigsqcup_{n=1}^{\infty} \mathcal{B}^n \to K(\underline{g}; R)$$

such that $\mu(h_\lambda) = z_\lambda$ is a cycle with $\operatorname{cls}(z_\lambda) = h_\lambda$ and

$$\partial^K \mu(h_{\lambda_1}, \dots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \dots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \dots, h_{\lambda_p}) \tag{3.1}$$

where $\overline{a} = (-1)^{|a|+1}a$.

In [36, Theorem 2.9], the authors prove that if J/J^2 and $H_i(\underline{g}; R)$ are free R/J-modules for all i, and if $K(\underline{g}; R)$ admits a trivial Massey operation, then J is a weak complete intersection ideal. We will use this fact in the proof of Proposition 3.1.4, but first we establish some notation.

Definition 3.1.3. Let I be an ideal in Q. We define $\frac{\partial}{\partial g}(I)$ to be the ideal of $\widehat{Q^J}$ generated by the elements $\{\frac{\partial}{\partial g_j}(\widehat{f}) | f \in I, j = 1, \dots, s\}$.

The following result gives a condition under which $\bar{J}$ is a weak complete intersection ideal in R.

Proposition 3.1.4. *If I is a J-closed ideal of Q and $\left(\frac{\partial}{\partial g}(I)\right)^2 \subseteq \widehat{I^J}$, then $\bar{J} = J/I$ is a weak complete intersection ideal in $R = Q/I$.*

Proof. First we note that the weak complete intersection property descends along the J-adic completion, so we may assume that Q is complete in the J-adic topology.

Since J/J^2 is a free Q/J-module and $H_\ell(\underline{g}; R)$ is a free Q/J-module for every ℓ, it suffices to show that $K(\underline{g}; R)$ admits a trivial Massey operation by [36, Theorem 2.9].

We take the z_λ to be the elements given in Theorem 2.4.1 and lift them to form a basis $\mathcal{B}$ of $H(\underline{g}; R)$. We define $\mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) = 0$ for all $p \geq 2$ and $h_{\lambda_i} \in \mathcal{B}$. By Theorem 2.4.1, we see that every z_λ has coefficients in $\frac{\partial}{\partial g}(I)$, since the elements f^1_{1,j_ℓ} are the entries in the first differential in the minimal free resolution of R, and hence are elements of I. Thus, the coefficients of $\mu(h_{\lambda_i})\mu(h_{\lambda_j})$ are elements of $\left(\frac{\partial}{\partial g}(I)\right)^2 \subseteq I$ for all i and j, so the products are zero in $K(\underline{g}; R)$. It is now easy to see that our definition of μ satisfies

$$\partial^K \mu(h_{\lambda_1}, \ldots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \ldots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \ldots, h_{\lambda_p}).$$

Thus, μ is a trivial Massey operation on $K(\underline{g}; R)$, as desired. □

We note that taking $\underline{g} = x_1, \ldots, x_n$ to be minimal generators of the maximal ideal in the proof above, we get that R is Golod, and this is precisely the result [25, Theorem 1.1] of Herzog and Huneke; see also [22, Theorem 3.5] for a similar result regarding Golod modules. The proof of Proposition 3.1.4 shows that the condition on $\frac{\partial}{\partial g}(I)$ cannot hold in the case that $I \subseteq J$ is an embedded complete intersection ideal, as $H(\underline{g}; R)$ cannot have trivial products in this case.

We see in the next section that Proposition 3.1.4 provides new examples of weak complete intersection ideals. We also see that the sufficient condition given in the proposition is not a necessary condition for weak complete intersections.

3.2 Examples

In this section, we give several examples which highlight the main ideas of this chapter. The following example illustrates Proposition 3.1.4 and shows that it produces new examples of weak complete intersection ideals.

Example 3.2.1. Let $Q = k[x, y]$ with $\operatorname{char} k = 0$ and let J be the complete intersection ideal in Q generated by the regular sequence $g_1 = x^2 + y^2$, $g_2 = y^3$. We consider the ideal $I = (g_1^2 g_2,\ g_1^4,\ g_2^3) \subseteq Q$. A free resolution of Q/I is

$$0 \to Q^2 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \to R \to 0$$

with differentials

$$\partial_2 = \begin{bmatrix} -g_1^2 & -g_2^2 \\ g_2 & 0 \\ 0 & g_1^2 \end{bmatrix}, \quad \partial_1 = \begin{bmatrix} g_1^2 g_2 & g_1^4 & g_2^3 \end{bmatrix},$$

so I is a J-closed ideal. The ideal $\frac{\partial}{\partial g}(I)$ is generated by the elements $g_1 g_2, g_1^2, g_2^2$. Indeed, elements of I are of the form $f = a g_1^2 g_2 + b g_1^4 + c g_2^3$ for $a, b, c \in Q$, and

$$\frac{\partial}{\partial g_1}(f) = 2a g_1 g_2 + 4b g_1^3 + \frac{\partial}{\partial g_1}(c) g_2^3$$
$$\frac{\partial}{\partial g_2}(f) = a g_1^2 + \frac{\partial}{\partial g_2}(b) g_1^4 + 3c g_2^2$$

which are both elements of the ideal $(g_1 g_2,\ g_1^2,\ g_2^2)$. It is easy to check that $\left(\frac{\partial}{\partial g}(I)\right)^2 \subseteq I$. Thus, by Proposition 3.1.4, $\bar{J}$ is a weak complete intersection ideal in $R = Q/I$.

The converse of Proposition 3.1.4 is not true in general, as shown by the following example.

Example 3.2.2. Let $Q = k[x, y, z]$ with $\operatorname{char} k{=}0$ and let J be a complete intersection ideal in Q generated by the regular sequence $g_1 = x^2$, $g_2 = y^3$, $g_3 = z^5$. The ideal $I = (x^2y^8,\ y^8z^9,\ x^3z^{14} + x^5y^5) \subseteq Q$ is a J-closed ideal; a free resolution of $R = Q/I$ being

$$0 \to Q^2 \xrightarrow{\partial_2} Q^3 \xrightarrow{\partial_1} Q \to R \to 0$$

with differentials

$$\partial_2 = \begin{bmatrix} -z^9 & -x^3y^5 \\ x^2 & -x^3z^5 \\ 0 & y^8 \end{bmatrix}, \quad \partial_1 = \begin{bmatrix} x^2y^8 & y^8z^9 & x^3z^{14}+x^5y^5 \end{bmatrix}.$$

By Theorem 2.4.1, the homology classes of the elements $\{\tilde{h}_1^1, \tilde{h}_2^1, \tilde{h}_3^1, \tilde{h}_1^2, \tilde{h}_2^2\}$, where

$$\begin{aligned}
\tilde{h}_1^1 &= \frac{1}{3}\overline{y^8}dg_1 + \frac{2}{3}\overline{x^2y^5}dg_2 \\
\tilde{h}_2^1 &= \frac{2}{3}\overline{y^5z^9}dg_2 + \frac{1}{3}\overline{y^8z^4}dg_3 \\
\tilde{h}_3^1 &= \frac{1}{3}\overline{(xz^{14}+2x^3y^5)}dg_1 + \frac{1}{3}\overline{x^5y^2}dg_2 + \frac{2}{3}\overline{x^3z^9}dg_3 \\
\tilde{h}_1^2 &= \frac{2}{3}\overline{y^5z^9}dg_1dg_2 - \frac{2}{3}\overline{x^2y^5z^4}dg_2dg_3 \\
\tilde{h}_2^2 &= \frac{1}{6}\overline{x^3y^{10}}dg_1dg_2 - \frac{1}{6}\overline{x^2y^8z^4}dg_1dg_3 + \frac{1}{3}\overline{x^3y^5z^9}dg_2dg_3
\end{aligned}$$

is a basis for $H_{\geq 1}(\underline{g}; R)$. To obtain a trivial Massey operation on $K(\underline{g}; R)$, we first multiply the cycles

$$\begin{aligned}
\tilde{h}_1^1 \cdot \tilde{h}_2^1 &= \frac{1}{9}\overline{y^{16}z^4}dg_1dg_3 \\
\tilde{h}_1^1 \cdot \tilde{h}_3^1 &= \frac{4}{9}\overline{x^5y^5z^9}dg_2dg_3 \\
\tilde{h}_2^1 \cdot \tilde{h}_3^1 &= \frac{2}{9}\overline{xy^5z^{23}}dg_1dg_2
\end{aligned}$$

and we observe that all multiplications involving $\tilde{h}_1^2$ and $\tilde{h}_2^2$ are zero. So we define μ as follows

$$\begin{aligned}
\mu([\tilde{h}_1^1], [\tilde{h}_2^1]) &= -\frac{1}{9}y^{13}z^4dg_1dg_2d_3 \\
\mu([\tilde{h}_1^1], [\tilde{h}_3^1]) &= \frac{4}{9}x^3y^5z^9dg_1dg_2d_3 \\
\mu([\tilde{h}_2^1], [\tilde{h}_2^1]) &= \frac{2}{9}xy^5z^{18}dg_1dg_2d_3
\end{aligned}$$

where $[\cdot]$ denotes the homology class, and otherwise we define μ to be zero. It is straightforward to check that μ satisfies (3.1); thus it is a trivial Massey operation. Therefore, by [36, Theorem 2.9], $\bar{J}$ is a weak complete intersection ideal in R. However, $y^8 \in \frac{\partial}{\partial g}(I)$, so y^{16} is in $\left(\frac{\partial}{\partial g}(I)\right)^2$, but not in I. This shows that $\left(\frac{\partial}{\partial g}(I)\right)^2 \subseteq I$ is not a necessary condition for $\bar{J}$ to be a weak complete intersection ideal in R.

Chapter 4

Quotients by Edge Ideals

Let k be a field and let $R = \bigoplus_{i \geq 0} R_i$ be a standard graded k-algebra; that is, R is Noetherian and finitely generated by elements of R_1 as an $R_0 = k$ algebra. Let $K(R)$ be the Koszul complex on a minimal set of generators of R_1. It is well-known that the differential graded algebra structure on $K(R)$ induces a k-algebra structure on its homology, $H(R)$; see for example [2, 1.3]. This algebra structure on Koszul homology holds important information about the ring R. For example, R is a complete intersection if and only if $H(R)$ is generated by $H_1(R)$ as a k-algebra [1, Thm 2.7], R is Gorenstein if and only if $H(R)$ satisfies Poincare duality [6], and R is Golod if and only if $K(R)$ admits a trivial Massey operation [2, Thm 5.2.2].

Another property of R that has strong connections to the structure of $H(R)$ is the Koszul property. R is said to be *Koszul* if k has a linear resolution over R, and a classical example of a Koszul algebra is the quotient by a quadratic monomial ideal (e.g. an edge ideal). To discuss the connections between R and $H(R)$ when R is Koszul, one views the Koszul homology algebra $H(R) = \bigoplus_{i,j} H_i(R)_j$ as a bigraded algebra, where i is the homological degree and j is the internal degree given by the grading on R. If R is Koszul, then it is known that $H_i(R)_j = 0$ for all $j > 2i$ [4, Thm 3.1], that $H_i(R)_{2i} = (H_1(R)_2)^i$ [5, Thm 5.1], and that $H_i(R)_{2i-1} = (H_1(R)_2)^{i-2} H_2(R)_3$ [9, Thm 3.1]. These results show that when R

is Koszul certain parts of $H(R)$ are generated by what we will call the lowest linear strand throughout this chapter. Avramov asked the following question regarding this behavior.

Question 4.0.1. If R is Koszul, is the Koszul homology algebra of R generated as a k-algebra by the lowest linear strand? That is, is $H(R)$ generated by $\bigoplus_i H_i(R)_{i+1}$?

The answer to this question is negative in general. The authors of [9] show that the Koszul homology algebra of the quotient by the edge ideal of an n-cycle where $n \equiv 1 (\mathrm{mod}\, 3)$ is not generated by the lowest linear strand. However, interest remains in determining for which Koszul algebras this question has a positive answer. The answer is positive for the Koszul homology algebra of the quotient by the edge ideal of an n-path [9, Thm 3.15] and for the Koszul homology algebra of the second Veronese algebra [14, Cor 2.4]. Still the question remains open for many classes of algebras known to be Koszul.

In this chapter, we give a positive answer to this question for a large class of edge ideals. Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over any field k and let I be an edge ideal associated to a tree. We show that the Koszul homology algebra of the quotient $R = Q/I$ is generated by the lowest linear strand. This result extends easily to edge ideals of forests, and our result recovers [9, Thm 3.15]. To obtain this result, we use the so-called iterated mapping cone construction to produce the minimal graded free resolution of R and apply the explicit k-bases of each $H_i(R)$ given by Herzog and Maleki in [26, Thm 1.3].

We now outline the contents of this chapter. In Section 4.1, we recall some important terminology which we use throughout the chapter and we discuss the main tools we use in our results. In Section 4.2, we construct the minimal graded free resolution of Q/I over Q which we use in the proof of our main result. We also recover a result of Roth and Van Tuyl in [37] on the Betti numbers of such quotients Q/I. In Section 4.3, we state and prove the main result.

4.1 Preliminaries

In this section, we set up the basic terminology which we use throughout the chapter and discuss the main tools we use to obtain our results. Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k.

We begin by recalling the notion of graded Betti numbers. We consider the minimal graded free resolution F

$$\cdots \longrightarrow \bigoplus_j Q(-j)^{\beta_{i,j}} \longrightarrow \bigoplus_j Q(-j)^{\beta_{i-1,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j Q(-j)^{\beta_{0,j}} \longrightarrow M \longrightarrow 0$$

of a finitely generated Q-module M. The *i-th graded Betti number of internal degree j* is $\beta_{i,j}$. The graded Betti numbers $\beta_{i,j}$ are commonly presented in what is called the *Betti table* of M, which is given by

$j-i$ \ i	0	1	2	3	$\ldots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\beta_{3,3}$	$\ldots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$	$\ldots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$	$\ldots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,6}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Now we recall the following basic isomorphism, which we use throughout this chapter. Let I be a homogeneous ideal in Q and let $R = Q/I$. Throughout this chapter, we denote the homology of the Koszul complex $K(x_1, \ldots, x_n; R)$ by $H(R)$. If F is the minimal graded free resolution of R over Q, then there is an isomorphism of bigraded k-algebras

$$\Phi : F \otimes k \to H(R). \tag{4.1}$$

Thus, given a basis $e_1^i, \ldots, e_{b_i}^i$ of F_i, we have that the elements $\Phi(e_j^i \otimes \bar{1})$ for $j = 1, \ldots, b_i$, form a basis for $H_i(R)$. Furthermore, if $\deg e_j^i = k$, then $\Phi(e_j^i \otimes \bar{1}) \in H_i(R)_k$. Given this isomorphism we can present the bigraded pieces $H_{i,j}$ of the Koszul homology algebra of R in the table

$j-i$ \ i	0	1	2	3	$\dots$
0	$H_{0,0}$	$H_{1,1}$	$H_{2,2}$	$H_{3,3}$	$\dots$
1	$H_{0,1}$	$H_{1,2}$	$H_{2,3}$	$H_{3,4}$	$\dots$
2	$H_{0,2}$	$H_{1,3}$	$H_{2,4}$	$H_{3,5}$	$\dots$
3	$H_{0,3}$	$H_{1,4}$	$H_{2,5}$	$H_{3,6}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

where $H_{i,j} = H_i(R)_j$. In this chapter we often discuss the *lowest linear strand* of $H(R)$, which is the second row (i.e. row 1) in the table above.

4.1.1 Edge Ideals

Let $Q = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k. We begin this subsection by recalling the notion of an edge ideal.

Definition 4.1.1. Let G be a simple graph (that is, with no loops nor multiple edges) on vertices $x_1, \dots, x_n$. The *edge ideal* associated to G is the ideal

$$I_G = (x_i x_j \mid x_i x_j \text{ is an edge in } G).$$

If G is a graph on the variables of Q and G' is a subgraph of G, we write $I_{G'}$ for the edge ideal associated to G' in Q. The class of edge ideals we focus on in this chapter is that of trees.

Definition 4.1.2. Let G be a simple graph. The graph G is a *tree* if G is connected and contains no cycle. Equivalently, G is a *tree* if every pair of vertices in G is connected by exactly one path. A *leaf* is a vertex in G of degree 1. A *forest* is a disjoint union of trees.

We illustrate the above definitions with the following example.

Example 4.1.3. Let $Q = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ be a polynomial ring. The edge ideal associated to the tree G shown in Figure 4.1 is $I_G = (x_1x_2,\ x_2x_3,\ x_2x_4,\ x_2x_5,\ x_3x_6,\ x_4x_7)$.

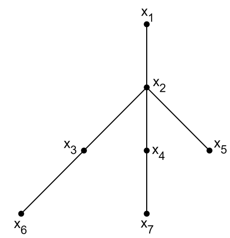

Figure 4.1: A tree G on 7 vertices

We make the following easy remarks about trees that will be useful throughout this chapter.

Remark 4.1.4.

(i) By definition, a tree G must have a leaf, otherwise G would contain a cycle.

(ii) It is easy to see that any subgraph of a tree is a forest.

In the following subsection, we discuss a method by which one can obtain the minimal graded free resolution of a quotient by the edge ideal of a tree.

4.1.2 Iterated Mapping Cones

In this subsection, we discuss one of the main tools we use to obtain the results in this chapter, namely, the iterated mapping cone construction. We begin by recalling the notion of a mapping cone.

Definition 4.1.5. Let (F, ∂^F) and (G, ∂^G) be two complexes of finitely generated Q-modules and let $\phi : F \to G$ be a map of complexes. The *mapping cone* of ϕ, denoted $\text{cone}\,(\phi)$, is the complex $(\text{cone}\,(\phi), \partial)$ with

$$(\text{cone}\,(\phi))_i = G_i \oplus F_{i-1}$$

$$\partial_i = \begin{bmatrix} \partial_i^G & \phi_{i-1} \\ 0 & -\partial_{i-1}^F \end{bmatrix}.$$

It is easy to see the following fact.

Remark 4.1.6. If $\phi : F \to G$, then there is a short exact sequence of complexes

$$0 \longrightarrow G \longrightarrow \operatorname{cone}(\phi) \longrightarrow F[-1] \longrightarrow 0.$$

Thus, G is a subcomplex of cone (ϕ).

Mapping cones can be used to build free resolutions of quotients by monomial ideals in the following way; see for example [35, Constr 27.3].

Construction 4.1.7. Let Q be a graded polynomial ring, and let I be the ideal minimally generated by monomials $m_1, \ldots, m_r$. Denote by d_i the degree of the monomial m_i and by I_i the ideal generated by $m_1, \ldots, m_i$. For each $i \geq 1$, we have the following graded short exact sequence

$$0 \longrightarrow Q/(I_i : m_{i+1})(-d_{i+1}) \xrightarrow{m_{i+1}} Q/I_i \longrightarrow Q/I_{i+1} \longrightarrow 0.$$

Note that we have shifted the first term by the degree of the monomial m_{i+1} to make multiplication by m_{i+1} a degree zero map. Thus, given graded Q-free resolutions G^i of Q/I_i and F^i of $Q/(I_i : m_{i+1})$, there is a map of complexes $\phi_i : F^i \to G^i$ induced by multiplication by m_{i+1}, which we call the *comparison map*. The mapping cone of the comparison map cone $(\phi_i) = F^{i+1}$ is a graded free resolution of Q/I_{i+1}. Applying this construction for each $i = 1, \ldots, r-1$ to obtain a graded free resolution of $Q/I = Q/I_r$ is called the *iterated mapping cone construction.*

We make the following important remarks about the iterated mapping cone construction.

Remark 4.1.8. Using the notation from Construction 4.1.7, we note the following.

(i) The resolution of Q/I produced by the mapping cone construction depends on the given order of the monomial generators $m_1, \ldots, m_r$ of I. We illustrate this remark in Example 4.1.12 below.

(ii) For any $i \geq 1$, cone(ϕ_i) need not be minimal, even if the given free resolutions F^i and G^i are minimal. Thus, the resolution of Q/I produced by the iterated mapping cone construction need not be minimal. We illustrate this remark in Example 4.1.10 below.

We now recall the following theorem that follows from results of Hà and Van Tuyl in [23] and was proved independently by Bouchat in [10, Thm 3.0.16]. It will be useful in the proofs of our results.

Theorem 4.1.9. *Let $Q = k[x_1, \ldots, x_n]$ and let G be a simple graph on vertices $x_1, \ldots, x_n$ such that x_n is a vertex of degree 1 and is connected by an edge to the vertex x_{n-1}. Then the mapping cone construction applied to the map*

$$Q/(I_{G\backslash x_n} : x_{n-1}x_n)(-2) \xrightarrow{x_{n-1}x_n} Q/I_{G\backslash x_n}$$

is a minimal graded free resolution of Q/I_G.

The following example shows that the conclusion of Theorem 4.1.9 need not hold if the graph G has no vertex of degree 1.

Example 4.1.10. Let G be the 5-cycle shown in Figure 4.2 and consider its associated edge ideal $I_G = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_1x_5)$.

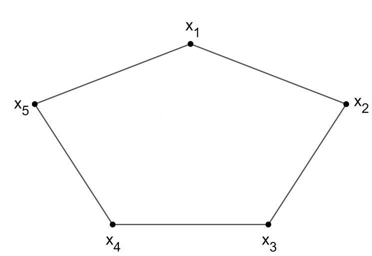

Figure 4.2: The 5-cycle

Applying the iterated mapping cone construction to resolve Q/I_G, we get the following comparison map in the last iteration.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Q(-4) & \xrightarrow{\begin{bmatrix} -x_4 \\ x_2 \end{bmatrix}} & Q(-3)^2 & \xrightarrow{\begin{bmatrix} x_2 & x_4 \end{bmatrix}} & Q(-2) \\
 & & \Big\downarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} & & \Big\downarrow \begin{bmatrix} x_5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & x_1 \end{bmatrix} & & \Big\downarrow x_1x_5 \\
Q(-5) & \xrightarrow[\begin{bmatrix} x_4x_5 \\ x_1x_5 \\ x_1x_2 \\ -x_3 \end{bmatrix}]{} & Q(-3)^3 \oplus Q(-4) & \xrightarrow[\begin{bmatrix} x_3 & 0 & 0 & x_4x_5 \\ -x_1 & x_4 & 0 & 0 \\ 0 & -x_2 & x_5 & 0 \\ 0 & 0 & -x_3 & -x_1x_2 \end{bmatrix}]{} & Q(-2)^4 & \xrightarrow[\begin{bmatrix} x_1x_2 & x_2x_3 & x_3x_4 & x_4x_5 \end{bmatrix}]{} & Q
\end{array}
$$

We see that the cone of this comparison map will produce a non-minimal resolution.

By Remark 4.1.4, Theorem 4.1.9 provides an inductive method for finding the minimal graded free resolution of Q/I_G, where G is any tree. We state this as a corollary.

Corollary 4.1.11. *If G is a tree, then, in some order, the iterated mapping cone construction gives the minimal graded free resolution of Q/I_G over Q.*

The following example illustrates the importance of the order in which the iterated mapping cone construction is applied.

Example 4.1.12. Let G be the tree shown in Figure 4.3 and consider its associated edge ideal $I_G = (x_1x_3,\ x_2x_3,\ x_3x_4,\ x_4x_5)$. Applying the iterated mapping cone construction to resolve Q/I_G, we get the following comparison map in the last iteration.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Q(-3) & \xrightarrow{x_3} & Q(-2) \\
 & & & & & & \Big\downarrow \begin{bmatrix} 0 \\ 0 \\ x_5 \end{bmatrix} & & \Big\downarrow x_4x_5 \\
0 & \longrightarrow & Q(-4) & \xrightarrow{\begin{bmatrix} -x_4 \\ x_2 \\ -x_1 \end{bmatrix}} & Q(-3)^3 & \xrightarrow{\begin{bmatrix} x_2 & x_4 & 0 \\ -x_1 & 0 & x_4 \\ 0 & -x_1 & -x_2 \end{bmatrix}} & Q(-2)^3 & \xrightarrow{\begin{bmatrix} x_1x_3 & x_2x_3 & x_3x_4 \end{bmatrix}} & Q
\end{array}
$$

However, if we instead order the minimal generators of the edge ideal as $I_G = (x_1x_3,\ x_2x_3,\ x_4x_5,\ x_3x_4)$ and apply the iterated mapping cone construction, we get the following comparison map in the last iteration.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q(-5) & \xrightarrow{\begin{bmatrix} x_5 \\ -x_2 \\ x_1 \end{bmatrix}} & Q(-4)^3 & \xrightarrow{\begin{bmatrix} -x_2 & -x_5 & 0 \\ x_1 & 0 & -x_5 \\ 0 & x_1 & x_2 \end{bmatrix}} & Q(-3)^3 & \xrightarrow{\begin{bmatrix} x_1 & x_2 & x_5 \end{bmatrix}} & Q(-2) \\
 & & \Big\downarrow -1 & & \Big\downarrow \begin{bmatrix} -x_4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & & \Big\downarrow \begin{bmatrix} x_4 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_3 \end{bmatrix} & & \Big\downarrow x_3x_4 \\
0 & \longrightarrow & Q(-5) & \xrightarrow{\begin{bmatrix} x_4x_5 \\ -x_2 \\ x_1 \end{bmatrix}} & Q(-3) \oplus Q(-4)^2 & \xrightarrow{\begin{bmatrix} x_2 & x_4x_5 & 0 \\ -x_1 & 0 & x_4x_5 \\ 0 & -x_1x_3 & -x_2x_3 \end{bmatrix}} & Q(-2)^3 & \xrightarrow{\begin{bmatrix} x_1x_3 & x_2x_3 & x_4x_5 \end{bmatrix}} & Q
\end{array}
$$

It is clear that applying the mapping cone construction in these two orders produce different resolutions. We note that the second resolution is not minimal. If it was, it would have to be isomorphic to the first one.

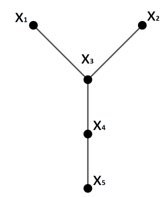

Figure 4.3: A tree G on 5 vertices

We use the iterated mapping cone construction in Section 4.2 to explicitly build the minimal graded free resolution of Q/I_G, where G is a tree. This resolution is an important ingredient in our proof of the main result in Section 4.3.

4.1.3 Multiplicative Structures on Resolutions

Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over any field k and let I be a monomial ideal of Q. Let F be the minimal graded free resolution of Q/I over Q. In this section we recall the notion of a multiplicative structure on F; see for example [30].

Definition 4.1.13. A Q-linear map $F \otimes_Q F \to F$ sending $a \otimes b$ to $a \cdot b$ is a *multiplication* if it satisfies the following conditions for $a, b \in F$

(i) it extends the usual multiplication on $F_0 = Q$

(ii) it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$

(iii) it is homogeneous with respect to the homological grading: $|a \cdot b| = |a| + |b|$

(iv) it is graded commutative: $a \cdot b = (-1)^{|a||b|} b \cdot a$

Notice we do not require that a multiplication is associative. The following fact is a result due to Buchsbaum and Eisenbud in [12, Prop 1.1].

Proposition 4.1.14. *The resolution F admits a multiplication.*

This fact will be useful in the proofs of our results.

4.1.4 Explicit Bases for Koszul Homology

Let $Q = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over any field k, let I be a homogeneous ideal of Q, and let $R = Q/I$. In this section, we discuss explicit bases of the Koszul homology modules $H_i(R)$ given by Herzog and Maleki in [26]. In order to describe these bases explicitly, we first set up some notation.

Herzog and Maleki define operators on Q as follows. For $f \in (x_1, \ldots, x_n)$ and for $r = 1, \ldots, n$, let

$$d^r(f) = \frac{f(0, \ldots, 0, x_r, \ldots, x_n) - f(0, \ldots, 0, x_{r+1}, \ldots, x_n)}{x_r}.$$

It is clear that the operators $d^r : Q \to Q$ are k-linear maps and that they depend on the order of the variables. In this chapter, we apply these operators to monomials. The following basic lemma describes how d^r behaves in this context.

Lemma 4.1.15. *Let f be the monomial $x_{k_1} \ldots x_{k_i}$ with $k_1 \le \cdots \le k_i$. Then*

$$d^r(f) = \begin{cases} x_{k_2} \ldots x_{k_i} & r = k_1 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. If $r = k_1$, then by definition $d^r(f) = \frac{x_{k_1} \ldots x_{k_i} - 0}{x_{k_1}} = x_{k_2} \ldots x_{k_i}$. If $r < k_1$, then $d^r(f) = \frac{f-f}{x_r} = 0$. If $r > k_1$, then $d^r(f) = \frac{0-0}{x_r} = 0$. □

This simple fact will be useful in the proof of our main result. The following theorem due to Herzog and Maleki in [26, Thm 1.3] describes explicit bases for Koszul homology modules. To set notation for the theorem, let F be the minimal graded free resolution of Q/I over Q and let b_i be the rank of F_i for each i. For each i, fix a homogeneous basis $e_1^i, \ldots, e_{b_i}^i$ of F_i and let $\partial(e_j^i) = \sum_{k=1}^{b_{i-1}} f_{k,j}^i e_k^{i-1}$. Let $dx_1, \ldots, dx_n$ be the standard generators of $K_1(\underline{x}; Q)$.

Theorem 4.1.16. *For each $i = 1, \ldots, n$, a k-basis of $H_i(R)$ is given by $[\bar{z}_{i,j}]$ for $j = 1, \ldots, b_i$, where*

$$z_{i,j} = \sum_{1 \le k_1 < \cdots < k_i \le n} \sum_{j_1=1}^{b_1} \cdots \sum_{j_{i-1}=1}^{b_{i-1}} d^{k_i}(f_{j_{i-1},j}^i) d^{k_{i-1}}(f_{j_{i-2},j_{i-1}}^{i-1}) \ldots d^{k_2}(f_{j_1,j_2}^2) d^{k_1}(f_{1,j_1}^1) dx_{k_1} \ldots dx_{k_i}.$$

In the proof of Theorem 4.1.16, Herzog and Maleki show that the isomorphism (4.1) is given explicitly by

$$\Phi : F \underset{Q}{\otimes} k \to H(R)$$

$$\Phi(e^i_j \otimes \bar{1}) = [\bar{z}_{i,j}].$$

We conclude this section with the following remark.

Remark 4.1.17. We note that in [24], Herzog gives a different description of bases for $H_i(R)$ under the assumption that k is a field of characteristic zero. In Chapter 2, we give explicit bases in a more general case, namely, the case of the Koszul complex on any full regular sequence, which recover Herzog's bases in characteristic zero. In this chapter, we use the description of bases given by Herzog and Maleki as they hold in any characteristic, and also because, by Lemma 4.1.15, many terms vanish in the case of a monomial ideal. This vanishing plays an important role in the proof of our main result in this chapter.

4.2 Construction of the Resolution

Let $\mathbb{G}$ be a tree and let Q be a standard graded polynomial ring over any field k with variables given by the vertices in $\mathbb{G}$. In this section we construct the minimal graded free resolution of $Q/I_{\mathbb{G}}$ over Q. We begin by setting the notation to be used throughout the section.

We name the vertices in $\mathbb{G}$ as follows. Since $\mathbb{G}$ is a tree, it has at least one vertex of degree 1; call it x_1 and call the vertex it is connected to by an edge x_2. Call the other vertices to which x_2 is connected to by an edge, $x_{2,1}, \ldots, x_{2,r}$. For each $\ell = 1, \ldots, r$, call the other vertices to which $x_{2,\ell}$ is connected to by an edge, $x_{2,\ell,1}, \ldots, x_{2,\ell,m_\ell}$. Note that $\mathbb{G} \smallsetminus \{x_1, x_2, x_{2,1}, \ldots, x_{2,r}\}$ is a forest. In particular, it is the disjoint union of $M := \sum_{\ell=1}^{r} m_\ell$ trees, call them $T_1, \ldots, T_M$. With this notation in mind, we view $\mathbb{G}$ as the diagram in Figure 4.4.

We aim to resolve $Q/I_{\mathbb{G}}$ minimally. By Theorem 4.1.9, this can be done by applying the iterated mapping cone construction as long as at each iteration we add a vertex of degree one. We choose the following order to apply the iterated mapping cone construction:

$$I_{\mathbb{G}} = (x_2x_{2,1}, \ldots, x_2x_{2,r}, \{x_{2,\ell}x_{2,\ell,p}\}_{\substack{\ell=1,\ldots,r \\ p=1,\ldots,m_\ell}}, e(T_1), \ldots, e(T_M), x_1x_2) \tag{4.2}$$

where, abusing notation, we write $e(T_i)$ to mean the set of relations coming from the edges of the tree T_i, taking the edge connecting T_i to the corresponding $x_{2,\ell,p}$ to be the first one. By Corollary 4.1.11 there is an ordering for each T_i which will preserve minimality in the iterated mapping cone construction; we choose such an ordering for each one. In this way, we obtain the minimal graded free resolution of $Q/I_{\mathbb{G}}$. Throughout the remainder of this section, we will write this resolution more explicitly in order to obtain our main result in the next section.

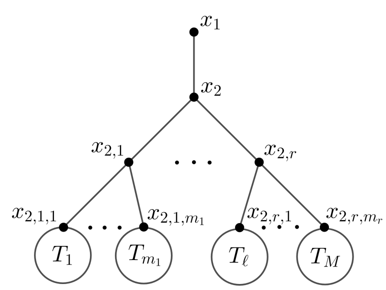

Figure 4.4: The tree $\mathbb{G}$ with $\ell = M - r_m - 1$

Denote by $\mathbb{G}_1$ the graph $\mathbb{G} \smallsetminus x_1$ and by C the colon ideal $(I_{\mathbb{G}_1} : x_1x_2)$. It is easy to see that

$$\begin{aligned} C &= (x_{2,1}, \ldots, x_{2,r}, e(T_1), \ldots, e(T_M)) \\ &= (x_{2,1}, \ldots, x_{2,r}) + \sum_{i=1}^{M} I_{T_i}. \end{aligned}$$

For the remainder of this chapter we denote by $F^{\mathbb{G}_1}$ and F^C, the minimal graded Q-free resolutions of $Q/I_{\mathbb{G}_1}$ and Q/C, respectively. The following fact is a key ingredient in our results.

Lemma 4.2.1. *The minimal graded free resolution of Q/C over Q is*

$$F^C = K(x_{2,1}, \ldots, x_{2,r}; Q) \underset{Q}{\otimes} F^1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} F^M$$

where F^i is the minimal graded free resolution of Q/I_{T_i} for each i.

Proof. Let A denote the subring $k[x_{2,1}, \ldots, x_{2,r}]$ of Q and let B denote the polynomial subring on all other variables in Q so that $Q = A \otimes_k B$. Let $\bar{K}$ be the minimal graded free resolution of A/J over A, where $J = (x_{2,1}, \ldots, x_{2,r})$, and let $\bar{F}$ be the minimal graded free resolution of B/L over B, where $L = \sum_{i=1}^{M} I_{T_i}$. Then we have that $K = \bar{K} \otimes_k B$ and $F = A \otimes_k \bar{F}$ are minimal graded free resolutions of Q/J and Q/L, respectively, over Q. We note that K is precisely the Koszul complex $K(x_{2,1}, \ldots, x_{2,r}; Q)$.

Now we have that

$$K \underset{Q}{\otimes} F = (\bar{K} \underset{k}{\otimes} B) \underset{Q}{\otimes} (A \underset{k}{\otimes} \bar{F}) = (\bar{K} \underset{k}{\otimes} B) \underset{A \underset{k}{\otimes} B}{\otimes} (A \underset{k}{\otimes} \bar{F}) = (\bar{K} \underset{A}{\otimes} A \underset{k}{\otimes} B) \underset{A \underset{k}{\otimes} B}{\otimes} (A \underset{k}{\otimes} B \underset{B}{\otimes} \bar{F})$$
$$= \bar{K} \underset{A}{\otimes} (A \underset{k}{\otimes} B) \underset{B}{\otimes} \bar{F} = \bar{K} \underset{k}{\otimes} \bar{F}.$$

Thus, taking homology, we see that

$$H_n(K \underset{Q}{\otimes} F) = H_n(\bar{K} \underset{k}{\otimes} \bar{F}) = \bigoplus_{i+j=n} \Big(H_i(\bar{K}) \underset{k}{\otimes} H_j(\bar{F}) \Big),$$

where the last equality follows from the Künneth Formula over k; see for example [38, Cor 10.84]. Thus $K \otimes_Q F$ is exact in all positive degrees and $H_0(K \otimes_Q F) = A/J \otimes_k B/L = Q/(J + L) = Q/C$. Minimality is clear, so we have that

$$K(x_{2,1}, \ldots, x_{2,r}; Q) \underset{Q}{\otimes} F$$

is the minimal graded free resolution of Q/C. Noticing that $T_1, \ldots, T_M$ involve disjoint sets of variables, we can apply a similar argument repeatedly to conclude that $F \cong F^1 \otimes_Q \cdots \otimes_Q F^M$, thus giving the desired result. □

Now we have that the minimal graded free resolution $\mathbb{F}$ of $Q/I_{\mathbb{G}}$ over Q is the cone of ϕ, where ϕ is a comparison map given by

$$\begin{array}{ccc} F^{C}(-2) & \longrightarrow & Q/C(-2) \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle x_1x_2} \\ F^{\mathbb{G}_1} & \longrightarrow & Q/I_{G_1} \end{array} \tag{4.3}$$

Thus, we have that $\mathbb{F}$ is the complex with modules $\mathbb{F}_i = F_i^{\mathbb{G}_1} \oplus F_{i-1}^{C}(-2)$ and differentials

$$\partial_i = \begin{bmatrix} \partial_i^{\mathbb{G}_1} & \phi_{i-1} \\ 0 & -\partial_{i-1}^{C} \end{bmatrix}. \tag{4.4}$$

We make the following remark about the resolutions $F^1, \ldots, F^M$ from Lemma 4.2.1.

Remark 4.2.2. The resolutions $F^1, \ldots, F^M$ are isomorphic to subcomplexes of $F^{\mathbb{G}_1}$. Indeed, a minimal resolution of $Q/I_{\mathbb{G}_1}$ can be obtained from each F^q by the iterated mapping cone construction; thus by Remark 4.1.6, each F^q is a isomorphic to a subcomplex of $F^{\mathbb{G}_1}$. We denote these isomorphisms by $\psi^i : F^i \to S^i \hookrightarrow F^{\mathbb{G}_1}$.

We now aim to give an explicit description of the map ϕ. To obtain such a description, we first observe that it is enough to define ϕ on elements of the form $\alpha \otimes 1 \otimes \cdots \otimes 1$.

Lemma 4.2.3. *If $\widetilde{\phi}$ is a comparison map*

$$\begin{array}{ccc} K(x_{2,1}, \ldots, x_{2,r}; Q)(-2) & \longrightarrow & Q/(x_{2,1}, \ldots, x_{2,r})(-2) \\ \big\downarrow{\scriptstyle \widetilde{\phi}} & & \big\downarrow{\scriptstyle x_1x_2} \\ F^{\mathbb{G}_1} & \longrightarrow & Q/I_{\mathbb{G}_1} \end{array}$$

then $\phi(\alpha \otimes \beta_1 \otimes \cdots \otimes \beta_M) = \widetilde{\phi}(\alpha) \cdot (\psi(\beta_1) \cdot (\cdots (\psi(\beta_{M-1}) \cdot \psi(\beta_M)) \ldots))$ defines a comparison map in (4.3).

Before giving a proof, we note that the multiplication appearing in the definition of ϕ in the lemma is a multiplication on the resolution $F^{\mathbb{G}_1}$; it has one by Proposition 4.1.14.

Proof. We must check that ϕ is a chain map. Thus we compute

$\phi(\partial^{C}(\alpha \otimes \beta_1 \otimes \cdots \otimes \beta_M)) =$

$$= \phi\left(\partial^K(\alpha)\otimes\beta_1\otimes\cdots\otimes\beta_M + \sum_{i=1}^{M}(-1)^{|\alpha|+\cdots+|\beta_i|}\alpha\otimes\beta_1\otimes\cdots\otimes\partial^{F^i}(\beta_i)\otimes\cdots\otimes\beta_M\right)$$

$$= \widetilde{\phi}(\partial^K(\alpha))\cdot\Big(\psi(\beta_1)\cdot(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots)\Big)$$

$$+ \sum_{i=1}^{M}(-1)^{|\alpha|+\cdots+|\beta_i|}\,\widetilde{\phi}(\alpha)\cdot\Big(\psi(\beta_1)\cdot\Big(\cdots\Big(\psi(\partial^{F^i}(\beta_i))\cdot(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots)\Big)\dots\Big)\Big).$$

On the other hand, we have that

$$\partial^{\mathbb{G}_1}(\phi(\alpha\otimes\beta_1\otimes\cdots\otimes\beta_M)) = \partial^{\mathbb{G}_1}\Big(\widetilde{\phi}(\alpha)\cdot(\psi(\beta_1)\cdot(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots))\Big)$$

$$= \partial^{\mathbb{G}_1}(\widetilde{\phi}(\alpha))\cdot\Big(\psi(\beta_1)\cdot(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots)\Big)$$

$$+ (-1)^{|\widetilde{\phi}(\alpha)|}\,\widetilde{\phi}(\alpha)\cdot\partial^{\mathbb{G}_1}\Big(\psi(\beta_1)\cdot(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots)\Big).$$

Applying the Leibniz rule repeatedly, and by the fact that $\widetilde{\phi}$ and ψ are chain maps, we see that $\phi(\partial^C(\alpha\otimes\beta_1\otimes\cdots\otimes\beta_M)) = \partial^{\mathbb{G}_1}(\phi(\alpha\otimes\beta_1\otimes\cdots\otimes\beta_M))$, which completes the proof. □

Now we work towards defining a comparison map $\widetilde{\phi}$. To accomplish this, we need to examine $F^{\mathbb{G}_1}$ more closely. We apply the iterated mapping cone construction in the order given in (4.2), and we observe that the resolution of $Q/(x_2x_{2,1},\dots,x_2x_{2,r})$ produced by the iterated mapping cone procedure is precisely the Taylor resolution; see for example [35, Constr 26.5], which we write as follows. Let E be the exterior algebra over k on basis elements $e_1,\dots,e_r$. Then the minimal graded free resolution of $Q/(x_2x_{2,1},\dots,x_2x_{2,r})$ over Q is F, where $F_i = Q\otimes E_i$ and the differentials are given by

$$\partial^F(e_{j_1}\dots e_{j_i}) = \sum_{\ell=1}^{i}(-1)^{\ell-1}\frac{\operatorname{lcm}(x_2x_{2,j_1},\dots,x_2x_{2,j_i})}{\operatorname{lcm}(x_2x_{2,j_1},\dots,\widehat{x_2x_{2,j_\ell}},\dots,x_2x_{2,j_i})}e_{j_1}\dots\widehat{e_{j_\ell}}\dots e_{j_i}$$

$$= \begin{cases} \sum_{\ell=1}^{i}(-1)^{\ell-1}x_{2,j_\ell}e_{j_1}\dots\widehat{e_{j_\ell}}\dots e_{j_i} & i\geq 2\\ x_2x_{2,j_1} & i=1\end{cases}.$$

Recall that, by Remark 4.1.6, F is a subcomplex of $F^{\mathbb{G}_1}$.

We are now ready to define ϕ. To set up notation, we write the Koszul complex $K(x_{2,1},\dots,x_{2,r};Q)$ as $Q\langle a_1,\dots,a_r|\partial^K(a_j) = x_{2,j}\rangle$. We note that if $F^{\mathbb{G}_1}$ is a DG algebra, the result below follows from standard DG algebra results; see [2, Prop 2.1.9]. Otherwise, a different proof is needed.

Proposition 4.2.4. *Define* $\phi : F^C(-2) \to F^{\mathbb{G}_1}$ *by*

$$\phi(a_{j_1}\dots a_{j_i}\otimes\beta_1\otimes\cdots\otimes\beta_M) = x_1e_{j_1}\dots e_{j_i}\cdot\Big(\psi(\beta_1)\cdot\big(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots\big)\Big).$$

Then ϕ *is a comparison map for*

$$\begin{array}{ccc} F^C(-2) & \longrightarrow & Q/C(-2) \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle x_1x_2} \\ F^{\mathbb{G}_1} & \longrightarrow & Q/I_{\mathbb{G}_1} \end{array}.$$

Proof. We define $\widetilde{\phi}(a_{j_1}\dots a_{j_i}) = x_1e_{j_1}\dots e_{j_i}$ so that

$$\phi(a_{j_1}\dots a_{j_i}\otimes\beta_1\otimes\cdots\otimes\beta_M) = \widetilde{\phi}(a_{j_1}\dots a_{j_i})\cdot\Big(\psi(\beta_1)\cdot\big(\cdots(\psi(\beta_{M-1})\cdot\psi(\beta_M))\dots\big)\Big).$$

Then by Lemma 4.2.3, it suffices to check that $\partial^{\mathbb{G}_1}\widetilde{\phi} = \widetilde{\phi}\,\partial^K$. For $i \geq 2$, we compute

$$\begin{aligned}\phi(\partial^C(a_{j_1}\dots a_{j_i}\otimes 1\otimes\cdots\otimes 1)) &= \phi\left(\sum_{\ell=1}^{i}(-1)^{\ell-1}x_{2,j_\ell}a_{j_1}\dots\widehat{a_{j_\ell}}\dots a_{j_i}\otimes 1\otimes\cdots\otimes 1\right)\\ &= \sum_{\ell=1}^{i}(-1)^{\ell-1}x_1x_{2,j_\ell}e_{j_1}\dots\widehat{e_{j_\ell}}\dots e_{j_i}\\ &= \partial^{\mathbb{G}_1}(x_1e_{j_1}\dots e_{j_i})\\ &= \partial^{\mathbb{G}_1}(\phi(a_{j_1}\dots a_{j_i}\otimes 1\otimes\cdots\otimes 1)).\end{aligned}$$

And, for any j, we have

$$\widetilde{\phi}(\partial^K(a_j)) = \widetilde{\phi}(x_{2,j}) = x_1x_2x_{2,j} = \partial^{\mathbb{G}_1}(x_1e_j) = \partial^{\mathbb{G}_1}(\widetilde{\phi}(a_j))$$

which completes the proof. □

To summarize the discussions in this section, we decompose $F^C(-2)$ and $F^{\mathbb{G}_1}$ as

$$\begin{aligned}F_i^C(-2) &= K_i(x_{2,1},\dots,x_{2,r};Q)(-2)\oplus\widetilde{F_i^C}\\ F_i^{\mathbb{G}_1} &= F_i\oplus\widetilde{F_i^{\mathbb{G}_1}}\end{aligned}$$

and think of the minimal graded free resolution $\mathbb{F}$ of $Q/I_{\mathbb{G}}$ as the cone of the diagram,

$$
\begin{array}{ccccccccc}
\longrightarrow Q(-(r+2)) \oplus \widetilde{F_r^C} & \longrightarrow & \dots & \longrightarrow & Q(-4)^{\binom{r}{2}} \oplus \widetilde{F_2^C} & \longrightarrow & Q(-3)^r \oplus \widetilde{F_1^C} & \longrightarrow & Q(-2) \\
\Big\downarrow \begin{bmatrix} x_1 & * \\ 0 & * \end{bmatrix} & & & & \Big\downarrow \begin{bmatrix} x_1 & * \\ 0 & * \end{bmatrix} & & \Big\downarrow \begin{bmatrix} x_1 & * \\ 0 & * \end{bmatrix} & & \Big\downarrow x_1x_2 \\
\longrightarrow Q(-(r+1)) \oplus \widetilde{F_r^{\mathbb{G}_1}} & \longrightarrow & \dots & \longrightarrow & Q(-3)^{\binom{r}{2}} \oplus \widetilde{F_2^{\mathbb{G}_1}} & \longrightarrow & Q(-2)^r \oplus \widetilde{F_1^{\mathbb{G}_1}} & \longrightarrow & Q
\end{array}
\tag{4.5}
$$

The following corollary is an immediate consequence of the construction and diagram above.

Corollary 4.2.5. *For $\alpha \in K(x_{2,1}, \dots, x_{2,r}; Q)$ with $\ell = |\alpha|+1$, the elements $\Phi(\alpha \otimes 1 \otimes \cdots \otimes 1)$ are elements of $H_\ell(R)$ of internal degree $\ell + 1$; thus they lie on the lowest linear strand.* □

We conclude this section by noting that the constructions above provide a way of counting the Betti numbers on the lowest linear strand of $\mathbb{F}$, and equivalently the generators on the lowest linear strand of $H(R)$. In particular, we recover the following result of Roth and Van Tuyl [37, Cor 2.6].

Corollary 4.2.6. *Let $\mathbb{G}$ be a tree. Then $\beta_{1,2}(Q/I_{\mathbb{G}}) = |e(\mathbb{G})|$ and*

$$\beta_{i,i+1}(Q/I_{\mathbb{G}}) = \sum_{v \in \mathbb{G}} \binom{\deg(v)}{i}$$

for all $i \geq 2$.

Proof. We use induction on the number of edges in $\mathbb{G}$. For the base case we consider the tree with one edge. In this case, $I_{\mathbb{G}} = (x_1x_2)$ and the minimal graded free resolution F of R is

$$0 \longrightarrow Q(-2) \xrightarrow{x_1x_2} Q \longrightarrow 0.$$

Thus we see that $\beta_{1,2}(Q/I_{\mathbb{G}}) = 1 = |e(\mathbb{G})|$.

Now take $\mathbb{G}$ to be any tree and assume that the result is true for every tree with strictly fewer edges. We obtain the minimal graded free resolution $\mathbb{F}$ of $Q/I_{\mathbb{G}}$ as the cone of the diagram (4.5) constructed in this section. We count the Betti numbers on the lowest linear strand as follows. From (4.5) and Corollary 4.2.5, we see that for $i \geq 2$ we have

$$\beta_{i,i+1}(Q/I_{\mathbb{G}}) \geq \beta_{i,i+1}(Q/I_{\mathbb{G}_1}) + \binom{r}{i-1} = \sum_{v \in \mathbb{G}_1} \binom{\deg_{\mathbb{G}_1}(v)}{i} + \binom{r}{i-1}$$

where the equality follows from induction. Separating the summand corresponding to x_2 from the rest of the sum, we get that

$$\beta_{i,i+1}(Q/I_{\mathbb{G}}) \geq \sum_{x_2 \neq v \in \mathbb{G}_1} \binom{\deg(v)}{i} + \binom{r}{i} + \binom{r}{i-1} = \sum_{x_2 \neq v \in \mathbb{G}_1} \binom{\deg(v)}{i} + \binom{r+1}{i}$$

where the equality follows from the identity called Pascal's Rule. We note that $r+1$ is precisely the degree of x_2 in $\mathbb{G}$. Thus we have the inequality

$$\beta_{i,i+1}(Q/I_{\mathbb{G}}) \geq \sum_{v \in \mathbb{G}} \binom{\deg(v)}{i}.$$

To show equality, it suffices to take e to be any basis element of $\widetilde{F^C_{i-1}}$ and show that it cannot be on the lowest linear strand. By Lemma 4.2.1, we have that $e = \alpha \otimes \beta_1 \otimes \cdots \otimes \beta_M$, for some basis elements α of $K_\ell(x_{2,1}, \ldots, x_{2,r})$ and β_p of $(F^p)_{i_p}$, where $\ell + i_1 + \cdots + i_M = i-1$ and at least one $\beta_p \neq 1$. In the following computations, we denote by $|\cdot|$ the homological degree and by $\deg(\cdot)$ the internal degree in $\mathbb{F}$. We denote by $\deg_C(\cdot)$ the internal degree in F^C. We have that

$$\begin{aligned}
&\deg(e) = \deg_C(e) + 2 \\
&= \deg_C(\alpha \otimes 1 \otimes \cdots \otimes 1) + \deg_C(1 \otimes \beta_1 \otimes \cdots \otimes 1) + \cdots + \deg_C(1 \otimes 1 \otimes \cdots \otimes \beta_M) + 2 \\
&= \deg(\alpha \otimes 1 \otimes \cdots \otimes 1) + \deg(\psi(\beta_1)) + \cdots + \deg(\psi(\beta_M))
\end{aligned}$$

where the first equality follows from the fact that $\mathbb{F} = \operatorname{cone}(F^C(-2) \to F^{\mathbb{G}_1})$, and the last equality follows from this same fact and also from Remark 4.2.2. Now by minimality we have that

$$\deg(e) \geq |\alpha \otimes 1 \otimes \cdots \otimes 1| + |\psi(\beta_1)| + \cdots + |\psi(\beta_M)| + M + 1$$

$$= \ell + i_1 + \cdots + i_M + M + 2$$

$$= i + M + 1$$

$$> i + 1.$$

Therefore, since $|e| = i$, e cannot possibly be on the lowest linear strand, and we have the desired equality. It is clear that the desired formula holds for $\beta_{1,2}(Q/I_{\mathbb{G}})$. □

4.3 The Main Result

In this section, we show that Question 4.0.1 has a positive answer for quotients by edge ideals of trees, and thus also of forests. Throughout this section we let $\mathbb{G}$ be a tree and let Q be a standard graded polynomial ring over any field k with variables given by the vertices in $\mathbb{G}$. Denote by N the set of indices for the vertices, so that $Q = k[\{x_n\}_{n \in N}]$. Following the notation established in the previous section, let $\Phi_{\mathbb{G}_1}$ and Φ_{F^i} be the isomorphisms $F^{\mathbb{G}_1} \otimes_Q k \to H(Q/I_{\mathbb{G}_1})$ and $F^i \otimes_Q k \to H(Q/I_{T_i})$ as in (4.1), where F^i is the minimal graded free resolution of Q/I_{T_i} as in Lemma 4.2.1. Before proving the main result of this chapter, we need the following lemma.

Lemma 4.3.1. *The canonical map of k-algebras*

$$\theta_{\mathbb{G}_1} : H(Q/I_{\mathbb{G}_1}) \longrightarrow H(R)$$

induced by the surjection $Q/I_{\mathbb{G}_1} \to Q/I_{\mathbb{G}} = R$ satisfies the equality $\theta_{\mathbb{G}_1}(\Phi_{\mathbb{G}_1}(e \otimes \bar{1})) = \Phi(e \otimes \bar{1})$ for any $e \in F^{\mathbb{G}_1}$.

Proof. First recall that the quotient map

$$Q/I_{\mathbb{G}_1} \longrightarrow Q/I = R$$

induces the map of DG algebras

$$K(Q/I_{\mathbb{G}_1}) \longrightarrow K(R)$$

that sends dx_i to dx_i for all i, and the induced map

$$\theta_{\mathbb{G}_1} : H(Q/I_{\mathbb{G}_1}) \longrightarrow H(R)$$

on homology is a map of k-algebras. Since, by Remark 4.1.6, $F^{\mathbb{G}_1}$ is a subcomplex of $\mathbb{F}$, it is clear by Theorem 4.1.16 that the equality $\theta_{\mathbb{G}_1}(\Phi_{\mathbb{G}_1}(e \otimes \bar{1})) = \Phi(e \otimes \bar{1})$ holds. □

Now we show that the Koszul homology algebra of the quotient by $I_{\mathbb{G}}$ is generated by the lowest linear strand.

Theorem 4.3.2. *If $R = Q/I_{\mathbb{G}}$, then $H(R)$ is generated by $\bigoplus_j H_j(R)_{j+1}$ as a k-algebra.*

Proof. We use induction on the number of edges in $\mathbb{G}$. For the base case, we consider the tree with one edge. In this case, $I_{\mathbb{G}} = (x_1x_2)$ and the minimal graded free resolution F of R is

$$0 \longrightarrow Q(-2) \xrightarrow{x_1x_2} Q \longrightarrow 0.$$

Thus applying the isomorphism Φ from (4.1) to $F \otimes k$, we see that the only basis element of $H_1(R)$ lies in $H_1(R)_2$. Hence $H(R)$ is trivially generated by the lowest linear strand.

Now take $\mathbb{G}$ to be any tree and assume that the result is true for every tree with strictly fewer edges. Let $\mathbb{F}$ be the minimal graded resolution of R over Q constructed in Section 3 and fix the basis of each $H_i(R)$ given in Theorem 4.1.16. It is enough to show that each basis element of $H_i(R)$ is in the subalgebra generated by $\bigoplus_j H_j(R)_{j+1}$. As such, we take h to be any basis element of $H_i(R)$. Then $h = \Phi(e \otimes \bar{1})$, for some basis element e of $\mathbb{F}_i$. We have that $\mathbb{F}$ is the cone of the map

$$F^C \xrightarrow{\phi} F^{\mathbb{G}_1}$$

defined in Proposition 4.2.4. Thus $\mathbb{F}_i = F_i^{\mathbb{G}_1} \oplus F_{i-1}^C$ and e must either be a basis element of $F_i^{\mathbb{G}_1}$ or of F_{i-1}^C.

We first consider the case where e is a basis element of $F_i^{\mathbb{G}_1}$. By Lemma 4.3.1,

$h = \theta_{\mathbb{G}_1}(\Phi_{\mathbb{G}_1}(e \otimes \bar{1}))$, but by the induction hypothesis, $H(Q/I_{\mathbb{G}_1})$ is generated by the lowest linear strand. So we have that

$$\Phi_{\mathbb{G}_1}(e \otimes \bar{1}) = \sum_{\lambda \in \Lambda} c_\lambda \prod_{\ell, m} \Phi_{\mathbb{G}_1}(e_\ell^{m,m+1} \otimes \bar{1})^{\lambda_{\ell,m}}$$

where Λ is a finite set of tuples $\lambda = (\lambda_{\ell,m})$ and for $\ell = 1, \dots, b_{m,m+1}$ the elements $e_\ell^{m,m+1}$ are basis elements of $F_m^{\mathbb{G}_1}$ of internal degree $m+1$, and where $c_\lambda \in k$. Now we have that

$$\begin{aligned} h &= \theta_{\mathbb{G}_1}\left(\sum_{\lambda \in \Lambda} c_\lambda \prod_{\ell,m} \Phi_{\mathbb{G}_1}(e_\ell^{m,m+1} \otimes \bar{1})^{\lambda_{\ell,m}} \right) = \sum_{\lambda \in \Lambda} c_\lambda \prod_{\ell,m} \theta_{\mathbb{G}_1}(\Phi_{\mathbb{G}_1}(e_\ell^{m,m+1} \otimes \bar{1}))^{\lambda_{\ell,m}} \\ &= \sum_{\lambda \in \Lambda} c_\lambda \prod_{\ell,m} \Phi(e_\ell^{m,m+1} \otimes \bar{1})^{\lambda_{\ell,m}} \end{aligned}$$

by Lemma 4.3.1. The elements $\Phi(e_\ell^{m,m+1} \otimes \bar{1})$ are basis elements of $H_m(R)$ that are in $H_m(R)_{m+1}$; thus h is in the subalgebra generated by the lowest linear strand.

Now we assume that e is a basis element of F_{i-1}^C. Then, by Lemma 4.2.1, $e = \alpha \otimes \beta_1 \otimes \cdots \otimes \beta_M$, for some basis elements α of $K_\ell(x_{2,1}, \dots, x_{2,r})$ and β_p of $(F^p)_{i_p}$, where $\ell + i_1 + \cdots + i_M = i - 1$. By Theorem 4.1.16, $h = [\bar{g}]$, where

$$g = \sum_{\{k_1 < \cdots < k_i\} \subseteq N} \sum_{j_1=1}^{b_1} \cdots \sum_{j_{i-1}=1}^{b_{i-1}} d^{k_i}(f_{j_{i-1},j_i}^i) \dots d^{k_2}(f_{j_1,j_2}^2) d^{k_1}(f_{1,j_1}^1) dx_{k_1} \dots dx_{k_i} \tag{4.6}$$

and each $f_{i,j}^k$ is the (i,j)-th entry in the kth differential of $\mathbb{F}$ when viewing the differentials as matrices with respect some fixed bases.

Recall that, the operators d^k depend on the order of the variables, so we fix an ordering on the variables in Q as follows

$$x_1 < x_2 < x_{2,\ell} < v(T_1) < \cdots < v(T_M)$$

for all ℓ, where by $v(T_q)$ we mean the variables given by the vertices in the tree T_q listed in some fixed order.

Now we analyze the terms in (4.6) more carefully in order to remove the initial sum. We note that since the differentials of $\mathbb{F}$ are given by (4.4), we have that for each set $\{j_1, \dots, j_{i-1}\}$, there is some m such that $f_{j_{i-1},j_i}^i, \dots, f_{j_m,j_{m+1}}^{m+1}$ are entries of ∂^C, f_{j_{m-1},j_m}^m is an entry of ϕ_{m-1}, and $f^{m-1}_{j_{m-2},j_{m-1}}, \ldots, f^1_{1,j_1}$ are entries of $\partial^{\mathbb{G}_1}$; that is, for each $p = 1, \ldots, m-1$, we have that $1 \leq j_p \leq \beta_p(Q/I_{\mathbb{G}_1})$, for $p = m, \ldots, i$, we have $\beta_p(Q/I_{\mathbb{G}_1}) + 1 \leq j_p \leq b_p$, and we have that $f^m_{j_{m-1},j_m} \in (x_1)$ by Proposition 4.2.4. Thus, by Lemma 4.1.15, we have that $d^{k_m}(f^m_{j_{m-1},j_m}) = 0$ unless $k_m = 1$. Also since $k_1 < \cdots < k_m < \cdots < k_i$, our fixed ordering on the variables implies that $m = 1$. So, every term in the sum with $k_m \neq 1$ or $m \neq 1$ vanishes, giving the equality

$$g = \sum_{\{1<k_2<\cdots<k_i\}\subseteq N} \sum_{j_1=a_1}^{b_1} \cdots \sum_{j_{i-1}=a_{i-1}}^{b_{i-1}} d^{k_i}(f^i_{j_{i-1},j_i}) \ldots d^{k_2}(f^2_{j_1,j_2}) d^1(f^1_{1,j_1}) dx_1 dx_{k_2} \ldots x_{k_i} \tag{4.7}$$

where $a_p = \beta_p(Q/I_{\mathbb{G}_1}) + 1$, and with $f^i_{j_{i-1},j_i}, \ldots, f^2_{j_1,j_2}$ entries of ∂^C, hence monomials, and f^1_{1,j_1} an entry of ϕ_0. We note that since $F^C_0 = Q$, we have that $b_1 = a_1$. Thus there is only one possible index j_1, and the sum over j_1 can be removed. Also, ϕ_0 is given by multiplication by x_1x_2, so $f^1_{1,j_1} = x_1x_2$.

By Lemma 4.1.15, for each $f^p_{j_{p-1},j_p}$ with $p > 1$, there is only one value of k_p such that $d^{k_p}(f^p_{j_{p-1},j_p})$ is nonzero. Thus, we see that for each set $\{j_1, \ldots, j_{i-1}\}$, there exist unique $k_p = k_p(j_1, \ldots, j_{i-1})$, for $p = 2, \ldots, i$, such that the terms in the sum (4.7) do not vanish. Therefore, we can indeed remove the initial sum, and the equality in (4.7) simplifies to

$$g = \sum_{j_2=a_2}^{b_2} \cdots \sum_{j_{i-1}=a_{i-1}}^{b_{i-1}} d^{k_i}(f^i_{j_{i-1},j_i}) \ldots d^{k_2}(f^2_{j_1,j_2}) d^1(x_1x_2) dx_1 dx_{k_2} \ldots dx_{k_i} \tag{4.8}$$

with $f^i_{j_{i-1},j_i}, \ldots, f^2_{j_1,j_2}$ entries of ∂^C. For ease of exposition in the rest of the proof, we drop the bounds on the sums in (4.8), and we denote by $\sum_{j_p}$ the sum from $j_p = a_p$ to $j_p = b_p$, for each p.

Now we aim to factor g. We note that since

$$\partial^C(\alpha \otimes \beta_1 \otimes \cdots \otimes \beta_M) = \partial^K(\alpha) \otimes \beta_1 \otimes \cdots \otimes \beta_M + \sum_{i=1}^{M} (-1)^s \alpha \otimes \beta_1 \otimes \cdots \otimes \partial^{F^i}(\beta_i) \otimes \cdots \otimes \beta_M,$$

where $s = |\alpha| + |\beta_1| + \cdots + |\beta_i|$, our chosen order of variables and Lemma 4.1.15 imply that the only nonzero terms in (4.8) are the ones such that

$$f^i_{j_{i-1},j_i}, \ldots, f^{i-i_M+1}_{j_{i-i_M},j_{i-i_M+1}} \text{ are entries of } \partial^{F^M}$$

$$f^{i-i_M}_{j_{i-i_M-1},j_{i-i_M}},\dots,f^{i-i_M-i_{M-1}+1}_{j_{i-i_M-i_{M-1}},j_{i-i_M-i_{M-1}+1}} \text{ are entries of } \partial^{F^{M-1}}$$
$$\vdots$$
$$f^{\ell+i_1+1}_{j_{\ell+i_1},j_{\ell+i_1+1}},\dots,f^{\ell+2}_{j_{\ell+1},j_{\ell+2}} \text{ are entries of } \partial^{F^1}$$
$$f^{\ell+1}_{j_\ell,j_{\ell+1}},\dots,f^{2}_{j_1,j_2} \text{ are entries of } \partial^{K}.$$

where here we identify each ∂^{F^p} with a matrix with respect to our fixed bases. Notice that (4.8) can be written as

$$g=\sum_{j_{\ell+1},\dots,j_{i-1}} d^{k_i}(f^i_{j_{i-1},j_i})\dots d^{k_{\ell+2}}(f^{\ell+2}_{j_{\ell+1},j_{\ell+2}})\sum_{j_2,\dots,j_\ell} d^{k_{\ell+1}}(f^{\ell+1}_{j_\ell,j_{\ell+1}})\dots d^1(x_1x_2)dx_1\dots dx_{k_i}. \tag{4.9}$$

Next note that the sum over $j_{\ell+1}$ can be removed. Indeed $f^{\ell+2}_{j_{\ell+1},j_{\ell+2}}$ is an entry of $\partial_1^{F^1}$ and since F^1 is the minimal graded free resolution of a cyclic module only one index $j_{\ell+1}$ is possible. The last group of sums in (4.9) can be pulled out to yield $g=g_1\cdot g_2$, where

$$g_1=\sum_{j_2,\dots,j_\ell} d^{k_{\ell+1}}(f^{\ell+1}_{j_\ell,j_{\ell+1}})\dots d^1(x_1x_2)dx_1\dots dx_{k_{\ell+1}}$$

and

$$g_2=\sum_{j_{\ell+2},\dots,j_{i-1}} d^{k_i}(f^i_{j_{i-1},j_i})\dots d^{k_{\ell+2}}(f^{\ell+2}_{j_{\ell+1},j_{\ell+2}})dx_{k_{\ell+2}}\dots dx_{k_i}.$$

We observe that the homology class of g_1 is precisely $\Phi(\alpha\otimes\bar{1})$. Repeating the procedure above with the sums in g_2, and then taking homology classes, we find that

$$h=\Phi(\alpha\otimes\bar{1})\cdot\theta_{T_1}\left(\Phi_{T_1}(\beta_1\otimes\bar{1})\right)\cdot\dots\cdot\theta_{T_M}\left(\Phi_{T_M}(\beta_M\otimes\bar{1})\right),$$

where

$$\theta_{T_i}: H(Q/I_{T_i})\longrightarrow H(R)$$

is the canonical map of k-algebras induced by the surjection $Q/I_{T_i}\to R$ as in Lemma 4.3.1.

By our induction hypothesis, $H(Q/I_{T_i})$ is generated by the lowest linear strand; thus similarly to the first case we have

$$h=\Phi(\alpha\otimes\bar{1})\cdot\Gamma,$$

where Γ is given by the product

$$\left(\sum_{\lambda\in\Lambda} c_\lambda \prod_{\ell,m} \theta_{T_1}\left(\Phi_{T_1}((\beta_1)_\ell^{m,m+1}\otimes\bar{1})\right)^{\lambda_{\ell,m}}\right)\cdot\cdots\cdot\left(\sum_{\lambda\in\Lambda} c_\lambda \prod_{\ell,m} \theta_{T_M}\left(\Phi_{T_M}((\beta_M)_\ell^{m,m+1}\otimes\bar{1})\right)^{\lambda_{\ell,m}}\right)$$

and where each $(\beta_q)_\ell^{m,m+1}$ is a basis element of F_m^q of internal degree $m+1$. By Corollary 4.2.5, $\Phi(\alpha\otimes\bar{1})$ is a generator of $H_{\ell+1}(R)$ that is in $H_{\ell+1}(R)_{\ell+2}$, and therefore, since the maps θ_{T_j} and Φ_{T_j} preserve bidegree for $j=1,\ldots,M$, h is in the subalgebra generated by the lowest linear strand. $\square$

Since paths are trees, this recovers [9, Thm 3.15]. Now let G be a forest, which by definition is a disjoint union of trees, $T_1,\ldots,T_m$. Thus, the quotient of the edge ideal of G is of the form

$$Q/I_G = Q_1/I_{T_1} \underset{k}{\otimes} \ldots \underset{k}{\otimes} Q_m/I_{T_m}$$

where $Q_1,\ldots,Q_m$ are polynomial rings on disjoint sets of variables such that $Q = Q_1\otimes_k\cdots\otimes_k Q_m$. This induces an isomorphism on the Koszul homology algebras

$$H(Q/I_G) \cong H(Q_1/I_{T_1}) \underset{k}{\otimes} \ldots \underset{k}{\otimes} H(Q_m/I_{T_m}),$$

thus yielding the following corollary as a direct consequence of Theorem 4.3.2.

Corollary 4.3.3. *If $R = Q/I_G$ and G is a forest, then $H(R)$ is generated by $\bigoplus_i H_i(R)_{i+1}$ as a k-algebra.* $\square$

Chapter 5

Embedded Complete Intersections

In this chapter, we construct a self-dual complete resolution of a module defined by a pair of embedded complete intersection ideals in a local ring. We find that the existence of such a complete resolution has nice (co)homological consequences for the module.

The motivation for studying self-duality of complete resolutions is the following classical result of Buchsbaum and Eisenbud in [12, Theorem 1.5] on self-duality of finite free resolutions.

Theorem 5.0.1. *If R is a local ring and $I \subseteq R$ is a Gorenstein ideal, then the minimal free resolution of R/I over R is self-dual.*

When one moves from finite free resolutions to infinite complete resolutions, the idea of self-duality is not so well understood. However, a related, but weaker notion of symmetric growth in complete resolutions has been a topic of interest in recent years. In 2000, Avramov and Buchweitz proved that over a complete intersection ring, the growth of the Betti sequence of a finitely generated module is the same as the growth of its Bass sequence [3, Theorem 5.6]. In particular every minimal complete resolution over a complete intersection has symmetric growth. Jorgensen and Şega constructed an example in 2005 that showed this is not at all the case over an arbitrary Gorenstein ring [29, Theorem 3.2]. However in the same paper they show that minimal complete resolutions over a Gorenstein ring of minimal multiplicity or low codimension have symmetric growth. In 2014, Bergh and Jorgensen studied minimal totally acyclic complexes and gave a criterion under which these complexes have symmetric growth. In particular, they proved that in a local ring, the minimal complete resolution of a module of finite complete intersection dimension has symmetric growth [8, Corollary 4.3].

Given these results, it seems natural to ask which modules have complete resolutions that not only exhibit symmetric growth, but are actually self-dual. In this chapter, we give a class of modules over a local ring which have self-dual complete resolutions. In particular, we construct a self-dual complete resolution of a module defined by a pair of embedded complete intersection ideals over a local ring. As a consequence of the existence of such a resolution we establish an isomorphism between certain stable homology and cohomology modules. We note that such a resolution was simultaneously and independently given in [31]. Although the resolutions are isomorphic, the constructions are rather different. A similar resolution, from a different perspective, also appeared in the Oberwolfach report [13], but we were not aware of this report during our work on this project.

We now outline the contents of this chapter. In Section 5.1 we recall some basic notions that we use throughout the chapter. We also introduce a precise notion of self-duality of complete resolutions and give a result that describes the (co)homological consequences when such a resolutions exists. In Section 5.2, we develop a key ingredient we will need in our main construction of the resolution; namely, we provide explicit generators of Koszul homology modules in the embedded complete intersection case. In Section 5.3, we construct the self-dual complete resolution.

5.1 Preliminaries

Throughout this chapter, we let Q be a commutative, Noetherian, local ring, and we use the notation $(\_\_)^* := \operatorname{Hom}_Q(\_\_, Q)$. We begin this section by recalling some basic definitions and facts that we use throughout the chapter. They can also be found in [3].

Definition 5.1.1. A *complete resolution* of a finitely generated Q-module M is a complex

$$T\colon\ \cdots \to T_2 \to T_1 \to T_0 \to T_{-1} \to T_{-2} \to \cdots$$

of finitely generated free Q-modules which satisfies the following conditions:

(1) both T and T^* are exact

(2) for some $r \geq 0$, the truncation $T_{\geq r}$ is isomorphic to $F_{\geq r}$, where F is a free resolution of M over Q.

Next we discuss stable (co)homology.

Definition 5.1.2. The *stable (co)homology modules* of M and N are defined as

$$\widehat{\operatorname{Tor}}_n^Q(M,N) := H_n(T \underset{Q}{\otimes} N)$$
$$\widehat{\operatorname{Ext}}_Q^n(M,N) := H^n(\operatorname{Hom}_Q(T,N))$$

where T is a complete resolution of M.

The following fact is well-known.

Fact 5.1.3. *If a module M has complete resolutions, then any two complete resolutions of M are homotopy equivalent. Thus, for all integers n, the stable (co)homology modules are well-defined.*

On the other hand, modules do not always have complete resolutions. The following well-known fact gives a necessary and sufficient condition for a module to have a complete resolution.

Fact 5.1.4. *A module has a complete resolution if and only if the module has finite Gorenstein dimension. Thus in a Gorenstein ring, every module has a complete resolution.*

We now introduce a precise definition for self-duality of complete resolutions, which we use throughout this chapter.

Definition 5.1.5. A complete resolution T is *self-dual* if T^* is isomorphic to $T[\,i\,]$ for some integer i.

In other words, we consider a complete resolution that is isomorphic to its dual, up to some shift, to be self-dual. Modules that have self-dual complete resolutions satisfy nice (co)homological properties, as shown in the next proposition.

Proposition 5.1.6. *If a Q-module M has a self-dual complete resolution then for any Q-module N there is an isomorphism*

$$\widehat{\operatorname{Tor}}_n^Q(M,N) \cong \widehat{\operatorname{Ext}}_Q^{n+i}(M,N)$$

for some integer i and for all integers n.

Proof. If T is a self-dual resolution of M and $T^* \cong T[\,i\,]$, then we have the following isomorphisms

$$\begin{aligned}\widehat{\operatorname{Tor}}_n^Q(M,N) = H_n(T \underset{Q}{\otimes} N) &\cong H_n(\operatorname{Hom}_Q(T^*,N)) \cong H_n(\operatorname{Hom}_Q(T[\,i\,],N)) \\ &\cong H_{n+i}(\operatorname{Hom}_Q(T,N)) = \widehat{\operatorname{Ext}}_Q^{n+i}(M,N)\end{aligned}$$

where the first isomorphism follows from the standard isomorphism $T \otimes_Q N \cong \operatorname{Hom}_Q(T^*,N)$. □

5.2 Generators of Koszul Homology

We begin this section by establishing some notation that is used throughout the remainder of this chapter. Let Q be a commutative Noetherian Cohen-Macaulay local ring and fix regular sequences $\underline{f} = f_1,\ldots,f_r$ and $\underline{g} = g_1,\ldots,g_s$ in Q, such that $I := (\underline{f}) \subseteq (\underline{g})$. In this case, we call I an *embedded complete intersection ideal* in $(\underline{g})$. Let $R = Q/I$ and J be the ideal $(\underline{g})/I$ in R, with g the *grade* of J, that is, the length of the longest R-regular sequence contained in J. Denote by

$$K(\underline{g};Q) = Q\langle dg_1,\ldots,dg_s | \partial_K(dg_i) = g_i\rangle$$

and

$$K(\underline{f};Q) = Q\langle df_1, \ldots, df_r | \partial_K(df_i) = f_i \rangle$$

the Koszul complexes on the sequences $\underline{g}$ and $\underline{f}$, respectively, and let $H_i(\underline{g};R)$ be the Koszul homology module $H_i(R \otimes_Q K(\underline{g};Q))$.

Since $(\underline{f}) \subseteq (\underline{g})$, for each $j = 1, \ldots, r$, we can write

$$f_j = \sum_{i=1}^{s} a_{ji} g_i$$

where $a_{ji} \in Q$. Denote by A the $s \times r$ matrix of coefficients (a_{ji}) and by $A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ the $k \times k$ minor of A corresponding to the rows $j_1, \ldots, j_k$ and the columns $i_1, \ldots, i_k$.

In this section we give explicit generators of each Koszul homology module $H_i(\underline{g};R)$, which we use in the next section to construct a self-dual complete resolution of R/J over R. We first note that in the case of embedded complete intersection ideals, the structure of $H(\underline{g};R)$ is well-understood in the following sense. We have that J is a quasi-complete intersection ideal of R. Indeed, $(\underline{g})$ is a quasi-complete intersection ideal of Q, and thus J is a quasi-complete intersection ideal of R by [7, Lemma 1.4]. Therefore, $H_\ell(\underline{g};R) = \bigwedge^\ell H_1(\underline{g};R)$ is an exterior power of the first Koszul homology. We use this structure to describe the generators of Koszul homology. The generators we provide in Lemma 5.2.1 are given in terms of the minors of the matrix A defined above, and this plays a crucial role in our constructions in the next section.

We also note that one could instead find formulas for the generators of Koszul homology in the embedded complete intersection case using Corollary 2.5.3. However, due to the extra structure on Koszul homology in this case, the general formulas obtained in Chapter 2 are not necessary. See Appendix A for an alternative proof of Lemma 5.2.1 which follows the spirit of Chapter 2.

Lemma 5.2.1. *An R/J-basis for $H_\ell(\underline{g};R)$ is given by homology classes of the elements*

$$\sum_{1 \le i_1 < \cdots < i_\ell \le s} \overline{A^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell}} dg_{i_1} \ldots dg_{i_\ell}$$

for $1 \leq j_1 < \cdots < j_\ell \leq r$*, where* $\overline{(\ \cdot\ )}$ *denotes the image in* R*.*

Proof. By the discussion above, it suffices to find a basis for $H_1(\underline{g}; R)$ and show that the product of each set of ℓ such basis elements yields the desired formulas.

Consider the following isomorphisms

$$H_\ell(\underline{g}; R) \cong H_\ell(R \underset{Q}{\otimes} K(\underline{g}; Q)) \cong \operatorname{Tor}_\ell^Q(R, R/J) \cong H_\ell(K(\underline{f}; Q) \underset{Q}{\otimes} R/J) \cong K_\ell(\underline{f}; Q) \underset{Q}{\otimes} R/J,$$

where the second and third isomorphisms follow from the fact that $\underline{f}$ and $\underline{g}$ are regular sequences and the last follows from the fact that $I \subseteq (\underline{g})$. The isomorphism above is defined by sending an element $b \otimes 1$ in $K_\ell(\underline{f}; Q) \otimes_Q R/J$ to $[\bar{z}_\ell]$ where $z = (z_0, \ldots, z_\ell)$ is a cycle in $K(\underline{f}; Q) \otimes_Q K(\underline{g}; Q)$ such that z_0 is sent to $b \otimes 1$ in the surjection

$$K_\ell(\underline{f}; Q) \otimes_Q K_0(\underline{g}; Q) \twoheadrightarrow K_\ell(\underline{f}; Q) \otimes_Q R/J,$$

z_ℓ is sent to $\bar{z}_\ell$ in the surjection

$$K_0(\underline{f}; Q) \otimes_Q K_\ell(\underline{g}; Q) \twoheadrightarrow R \otimes_Q K_\ell(\underline{g}; Q),$$

and $[\_\_]$ denotes the homology class.

Take $z_0 = df_j \otimes 1$ and $z_1 = \sum_{i=1}^s a_{ji} \otimes dg_i$. Then we see that

$$(1 \otimes \partial_K)(z_1) = f_j \otimes 1 = (\partial_k \otimes 1)(z_0)$$

for each $i = 1, \ldots, r$. Thus, (z_0, z_1) is a cycle, and since the elements $df_j \otimes \bar{1}$ for $j = 1, \ldots, r$ form a basis of $K_1(\underline{f}; Q) \otimes_Q R/J$, the homology classes of the elements

$$\sum_{i=1}^s \overline{a_{ji}} dg_i$$

for $j = 1, \ldots, r$ form a basis of $H_1(\underline{g}; R)$.

Multiplying any distinct two of these generators together yields

$$\left(\sum_{i_1=1}^s \overline{a_{j_1,i_1}} dg_{i_1} \right) \cdot \left(\sum_{i_2=1}^s \overline{a_{j_2,i_2}} dg_{i_2} \right) = \sum_{i_1=1}^s \sum_{i_2=1}^s \overline{a_{j_1,i_1} a_{j_2,i_2}} dg_{i_1} dg_{i_2} = \sum_{1 \leq i_1 < i_2 \leq s} \overline{A_{j_1,j_2}^{i_1,i_2}} dg_{i_1} dg_{i_2}.$$

Inductively, we obtain the desired formulas. □

We end this section by giving a formula for the generator of the first nonvanishing Koszul cohomology module, which we use in the next section.

Corollary 5.2.2. *The homology class of the element*

$$\sum_{1\leq i_1<\cdots<i_r\leq s} (-1)^{rg+k_1+\cdots+k_g} A^{i_1,\ldots,i_r}_{1,..,r}\, dg^*_{k_1}\ldots dg^*_{k_g}$$

where $\{k_1,\ldots,k_g\}$ *is the complement of the set* $\{i_1,\ldots,i_r\}$ *in* $\{1,\ldots,s\}$, *generates the Koszul cohomology module* $H^g(\underline{g};R)$.

Proof. We first note that since Q is Cohen-Macaulay, we have equalities

$$g = \mathrm{ht}J = \dim R - \dim R/J = \dim Q - r - (\dim Q - s) = s - r.$$

Now the result follows directly from Lemma 5.2.1 and the self-duality isomorphism of the Koszul complex,

$$K_j(\underline{g};R) \xrightarrow{\cong} K^{s-j}(\underline{g};R)$$
$$dg_{i_1}\ldots dg_{i_j} \longmapsto (-1)^p dg^*_{k_1}\ldots dg^*_{k_{s-j}}$$

where $\{k_1,\ldots,k_{s-j}\}$ is the complement of $\{i_1,\ldots,i_j\}$ in $\{1,\ldots,s\}$ and p is the sign of the permutation $dg_{i_1}\ldots dg_{i_j} dg_{k_1}\ldots dg_{k_{s-j}}$; see for example [20, Proposition 17.15]. It is easy to see that $p = j(s-j) + k_1 + \cdots + k_{s-j}$. □

5.3 The Self-Dual Complete Resolution

In this section we construct a self-dual complete resolution of R/J over R under the assumptions introduced at the beginning of the previous section.

We begin our construction by considering the Tate resolution of R/J over R, which is known to be a DG-algebra resolution. By [39, Theorem 4], since $\underline{f}$ and $\underline{g}$ are both regular sequences on Q, this resolution is precisely

$$\mathbb{F} = R\langle dg_1,\ldots,dg_s,T_1,\ldots,T_r|\ \partial(dg_i) = g_i,\ \partial(T_j) = \sum_{i=1}^{s} a_{ji}dg_i\rangle.$$

We view this resolution as the total complex of the double complex

$$\mathbb{F}_{i,j} = \bigwedge^i G \otimes D_j F$$

where G is the free R-module with basis $dg_1, \ldots, dg_s$, $\bigwedge G$ denotes the exterior algebra on that basis, F is the free R-module with basis $T_1, \ldots, T_r$, and D denotes the divided power algebra on that basis. Thus we see that the vertical maps are given by the differentials of the Koszul complex $K(\underline{g}; R)$, and the horizontal maps send T_j to $\sum_{i=1}^{s} a_{ji} dg_i$. For ease of notation we set $\bigwedge^i \otimes D_j := \bigwedge^i G \otimes D_j F$.

We now dualize the double complex $\mathbb{F}$, and we note that

$$(\mathbb{F}_{i,j})^* = \left(\bigwedge^i G \otimes D_j F\right)^* = \left(\bigwedge^i G\right)^* \otimes (D_j F)^* = \bigwedge^i G^* \otimes S_j F^*$$

where S denotes the symmetric algebra on the dual basis of F. We denote by $\bigwedge^i \otimes S_j := \bigwedge^i G^* \otimes S_j F^*$, and we note that in $\mathbb{F}^*$ the vertical maps μ_x are given by right multiplication by $x = \sum_{j=i}^{s} g_i dg_i^*$ and the horizontal maps d send dg_i^* to $\sum_{j=1}^{r} a_{ji} T_j^*$.

We also see that by [7, Theorem 2.5], since J is a quasi-complete intersection ideal, it is a *quasi-Gorenstein ideal*; that is, there are isomorphisms

$$\operatorname{Ext}_R^i(R/J, R) \cong \begin{cases} R/J & i = g \\ 0 & i \neq g \end{cases}. \tag{5.1}$$

Following work of Herzog and Martsinkovsky in [27, Section 3] — see also [7] for another gluing construction — we seek to glue the double complex $\mathbb{F}$ to its dual. That is, we aim to define a map $v : \mathbb{F} \to \mathbb{F}^*[-g]$ as shown in diagram (5.3) below that identifies the homologies $H_0(\mathbb{F}) = R/J$ and

$$H^g(\mathbb{F}^*) = H^g(\operatorname{Hom}_R(\mathbb{F}, R)) = \operatorname{Ext}_R^g(R/J, R) \cong R/J, \tag{5.2}$$

where the isomorphism follows from (5.1).

$$
\begin{array}{cccccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \bigwedge^1 \otimes D_2 & \xrightarrow{d^T} & \bigwedge^2 \otimes D_1 & \xrightarrow{d^T} & \bigwedge^3 \otimes D_0 & \xrightarrow{v_3} & \bigwedge^{g-3} \otimes S_0 & \xrightarrow{d} & \bigwedge^{g-4} \otimes S_1 & \xrightarrow{d} & \bigwedge^{g-5} \otimes S_2 & \longrightarrow \cdots \\
 & \downarrow\mu_x^T & & \downarrow\mu_x^T & & \downarrow\mu_x^T & & \downarrow\mu_x & & \downarrow\mu_x & & \downarrow\mu_x & & \\
0 \longrightarrow & \bigwedge^0 \otimes D_2 & \xrightarrow{d^T} & \bigwedge^1 \otimes D_1 & \xrightarrow{d^T} & \bigwedge^2 \otimes D_0 & \xrightarrow{v_2} & \bigwedge^{g-2} \otimes S_0 & \xrightarrow{d} & \bigwedge^{g-3} \otimes S_1 & \xrightarrow{d} & \bigwedge^{g-4} \otimes S_2 & \longrightarrow \cdots \\
 & \downarrow & & \downarrow\mu_x^T & & \downarrow\mu_x^T & & \downarrow\mu_x & & \downarrow\mu_x & & \downarrow\mu_x & & \\
 & 0 & \longrightarrow & \bigwedge^0 \otimes D_1 & \xrightarrow{d^T} & \bigwedge^1 \otimes D_0 & \xrightarrow{v_1} & \bigwedge^{g-1} \otimes S_0 & \xrightarrow{d} & \bigwedge^{g-2} \otimes S_1 & \xrightarrow{d} & \bigwedge^{g-3} \otimes S_2 & \longrightarrow \cdots \\
 & & & \downarrow & & \downarrow\mu_x^T & & \downarrow\mu_x & & \downarrow\mu_x & & \downarrow\mu_x & & \\
 & & & 0 & \longrightarrow & \bigwedge^0 \otimes D_0 & \xrightarrow{v_0} & \bigwedge^{g} \otimes S_0 & \xrightarrow{d} & \bigwedge^{g-1} \otimes S_1 & \xrightarrow{d} & \bigwedge^{g-2} \otimes S_2 & \longrightarrow \cdots \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{5.3}
$$

The diagram shows the resolution $\mathbb{F}$ on the left, its dual on the right, and a gluing map v between the two. Apriori, the gluing map v need not land in the first column as the diagram shows; however the gluing map we define in this section does indeed satisfy this property.

We note that since $\mathbb{F}^*$ is a module over $\mathbb{F}$, to define the gluing map v, it suffices to define v_0 and take v to be multiplication by the element $v_0(1)$. We will define v_0 carefully in order to construct a complete resolution of R/J over R which is self-dual.

To this end, we denote by β the image of $1 + J$ under the isomorphism

$$R/J \to \operatorname{Ext}_R^g(R/J, R). \tag{5.4}$$

from (5.1). Given the equality $\operatorname{Ext}_R^g(R/J, R) = H^g(\mathbb{F}^*)$ from (5.2) we have that

$$\beta = \alpha + \operatorname{Im}\partial$$

where $\alpha = (\alpha_0, \ldots, \alpha_{\lceil \frac{g+1}{2} \rceil - 1}) \in \operatorname{Ker}\partial \setminus \operatorname{Im}\partial$, and where ∂ denotes the differential on $\mathbb{F}^*$. With this notation and the diagram (5.3) in mind, we get the following technical lemma, which says that the element α_0 is not a boundary in the dual Koszul complex.

Lemma 5.3.1. *The element α_0 satisfies the property that $\alpha_0 \neq \mu_x(\delta_0)$ for any $\delta_0 \in \bigwedge^{g-1} \otimes S_0$.*

Proof. Suppose that $\alpha_0 = \mu_x(\delta_0)$ for some $\delta_0 \in \wedge^{g-1} \otimes S_0$. Since diagram (5.3) commutes, we have that $d(\alpha_0) = \mu_x(d(\delta_0))$. On the other hand, since $\alpha \in \mathrm{Ker}\partial$, we have that $\mu_x(\alpha_1) = d(\alpha_0)$. Thus $d(\delta_0) - \alpha_1 \in \mathrm{Ker}\mu_x$.

We note that each column on the right half of the diagram (5.3) is the dual Koszul complex on $g_1, \ldots, g_s$. By self-duality of the Koszul complex, since the last nonvanishing Koszul homology module is $H_{s-g}(\underline{g}; R)$, the first nonvanishing Koszul cohomology module is $H^g(\underline{g}; R)$. Thus the columns on the right are exact in degrees lower than g, and we have that $d(\delta_0) - \alpha_1 = \mu_x(\delta_1)$, for some $\delta_1 \in \wedge^{g-3} \otimes S_1$.

Inductively, we have elements δ_i, for $i = 0, \ldots, \lceil \frac{g+1}{2} \rceil - 1$, such that each $d(\delta_i) - \alpha_{i+1} = \mu_x(\delta_{i+1})$. Thus we have

$$\alpha = \partial(\delta_0, \ldots, \delta_{\lceil \frac{g+1}{2} \rceil - 1}),$$

which is a contradiction. $\square$

Continuing our construction, we note that since α is a cycle, we must have that α_0 is a cycle in the dual Koszul complex; however, it is not a boundary by Lemma 5.3.1. Thus by Corollary 5.2.2, we see that

$$\alpha_0 = t\left(\sum_{1 \le i_1 < \cdots < i_r \le s} (-1)^{k_1 + \cdots + k_g} A^{i_1, \ldots, i_r}_{1, .., r} dg^*_{k_1} \ldots dg^*_{k_g} \right) + \mu_x(\gamma), \tag{5.5}$$

for some nonzero $t \in R$ and some $\gamma \in \wedge^{g-1} \otimes S_0$.

Now we define the gluing map v as follows. Define the map $v_0 : \mathbb{F}_0 \to \mathbb{F}^*_g$ such that $v_0(1) = \epsilon$, where

$$\epsilon = \left(t\left(\sum_{1 \le i_1 < \cdots < i_r \le s} (-1)^{k_1 + \cdots + k_g} A^{i_1, \ldots, i_r}_{1, \ldots, r} dg^*_{k_1} \ldots dg^*_{k_g} \right), 0, \ldots, 0 \right), \tag{5.6}$$

and extend this to the chain map

$$v : \mathbb{F} \longrightarrow \mathbb{F}^*[-g] \tag{5.7}$$

$$\eta \longmapsto \epsilon\eta.$$

Thus the gluing map does indeed land in the first column on the right side of the diagram (5.3).

We aim to show that the map v is a quasi-isomorphism. To accomplish this, we need some technical lemmas.

Lemma 5.3.2. *The element*

$$\sum_{1\le i_1<\cdots<i_r\le s}(-1)^{k_1+\cdots+k_g}A^{i_1,\ldots,i_r}_{1,\ldots,r}\,dg^*_{k_1}\ldots dg^*_{k_g}$$

is in the kernel of the map d.

Proof. For ease of notation, we set $K = k_1 + \cdots + k_g$, and we denote by $\sum_I$ the sum over $1 \le i_1 < \cdots < i_r \le s$. Applying d to the given element, we see that it suffices to show that the coefficient of T_m in the sum

$$\sum_I(-1)^K A^{i_1,\ldots,i_r}_{1,..,r}d(dg^*_{k_1}\ldots dg^*_{k_g}) = \sum_I(-1)^K A^{i_1,\ldots,i_r}_{1,..,r}\sum_{n=1}^{g}dg^*_{k_1}\ldots\widehat{dg^*_{k_n}}\ldots dg^*_{k_g}\sum_{\ell=1}^{r}a_{\ell,k_n}T_\ell$$

is zero for all $m = 1,\ldots,r$. We observe that its coefficient is precisely

$$\sum_L(-1)^K A^{i_1,\ldots,i_r}_{1,..,r}\sum_{n=1}^{g}(-1)^{n+1}dg^*_{k_1}\ldots\widehat{dg^*_{k_n}}\ldots dg^*_{k_g}a_{m,k_n},$$

so it suffices to show that the coefficient of any $dg^*_{p_1}\ldots dg^*_{p_{g-1}}$ in this sum is zero.

Thus the terms of the sum we are interested in are the ones where we sum over all sets $\{i_1,\ldots,i_r\}$ whose complement in $\{1,..,s\}$ contains the set $\{p_1,\ldots,p_{g-1}\}$. There are $s-(g-1) = r+1$ such sets, namely,

$$\{q_2,\ldots,q_{r+1}\},\ldots,\{q_1,\ldots,\widehat{q_i},\ldots,q_{r+1}\},\ldots,\{q_1,\ldots,q_r\},$$

and their complements are

$$\{p_1,\ldots,p_{g-1},q_1\},\ldots,\{p_1,\ldots,p_{g-1},q_{r+1}\}$$

respectively, where $\{q_1,\ldots,q_{r+1}\}$ is the complement of $\{p_1,\ldots,p_{g-1}\}$ in $\{1,\ldots,s\}$.

Let s_i denote the spot in which q_i sits when $\{p_1,\ldots,p_{g-1},q_i\}$ is ordered. Thus the coefficient of interest is

$$\sum_{i=1}^{r+1}(-1)^{p_1+\cdots+p_{g-1}+q_i+s_i+1}A_{1,\ldots,r}^{q_1,\ldots,\widehat{q_i},\ldots,q_{r+1}}a_{m,q_i}.$$

We note that if the signs in the sum above are alternating, then this coefficient is zero. Indeed, it would equal the determinant $A_{m,1,\ldots,m,\ldots,r}^{q_1,\ldots,q_{r+1}}$, which is zero as a row is repeated.

Thus it suffices to show that $(-1)^{q_i+s_i}$ is the opposite of $(-1)^{q_{i+1}+s_{i+1}}$ for any $1 \leq i \leq r+1$. We assume that q_i and q_{i+1} have the same parity, as the argument for the case in which they have different parity is essentially the same. Note that in this case, there is an odd number of indices p_j between q_i and q_{i+1} in the set $\{1, \ldots, s\}$. Thus s_{i+1} is the sum of s_i and an odd number. Hence s_i and s_{i+1} have different parity, and $(-1)^{q_i+s_i}$ and $(-1)^{q_{i+1}+s_{i+1}}$ are opposites. Therefore the signs in the sum are alternating as desired. □

We also need the following lemma of Miller from [34] in the proof of our next proposition.

Lemma 5.3.3. *If* $\gamma = (\gamma_0, \ldots, \gamma_{\lceil \frac{g+1}{2}\rceil -1}) \in \operatorname{Ker}\partial$ *and* $d(\gamma_0) = 0$*, then there is an element* $\gamma' = (\gamma_0, 0, \ldots, 0) \in \operatorname{Ker}\partial$ *such that* $[\gamma] = [\gamma']$.

Given our construction above, we obtain the following result.

Proposition 5.3.4. *The map* $v : \mathbb{F} \to \mathbb{F}^*[-g]$ *defined in (5.7) is a quasi-isomorphism.*

Proof. Since $\operatorname{Ext}_R^i(R/J, R) = 0$ for all $i \neq g$, it suffices to show that $v_0 \colon \mathbb{F}_0 \to \mathbb{F}_g^*$ induces an isomorphism on homology. However, we observe that the map induced by v_0 on homology is precisely the isomorphism $R/J \to \operatorname{Ext}_R^g(R/J, R)$ in (5.4). Indeed, since $\beta = \alpha + \operatorname{Im}\partial$, we have the following equalities

$$\begin{aligned}
\beta &= \left(t\left(\sum_I (-1)^K A_{1,..,r}^{i_1,\ldots,i_r} dg_{k_1}^* \ldots e_{k_g}^*\right) + \mu_x(\gamma), \alpha_1, \ldots, \alpha_{\lceil \frac{g+1}{2}\rceil -1}\right) + \operatorname{Im}\partial \\
&= \left(t\left(\sum_I (-1)^K A_{1,\ldots,r}^{i_1,\ldots,i_r} dg_{k_1}^* \ldots dg_{k_g}^*\right), \alpha_1 - d(\gamma), \ldots, \alpha_{\lceil \frac{g+1}{2}\rceil -1}\right) + \partial(\gamma, 0, \ldots, 0) + \operatorname{Im}\partial \\
&= \left(t\left(\sum_I (-1)^K A_{1,\ldots,r}^{i_1,\ldots,i_r} dg_{k_1}^* \ldots dg_{k_g}^*\right), \alpha_1 - d(\gamma), \ldots, \alpha_{\lceil \frac{g+1}{2}\rceil -1}\right) + \operatorname{Im}\partial
\end{aligned}$$

$$= \left(t \left(\sum_I (-1)^K A^{i_1,\dots,i_r}_{1,\dots,r} dg^*_{k_1} \dots dg^*_{k_g} \right), 0, \dots, 0 \right) + \operatorname{Im} \partial$$

$$= \epsilon + \operatorname{Im} \partial$$

where $K = k_1 + \cdots + k_g$ and $\sum_I$ denotes the sum over $1 \leq i_1 < \cdots < i_r \leq s$, and where the first equality follows from (5.5), the second equality follows from the fact that

$$\partial(\gamma, 0, \dots, 0) = (\mu_x(\gamma), d(\gamma), 0, \dots, 0),$$

the fourth equality follows from Lemma 5.3.2 and Lemma 5.3.3, and the last follows from (5.6). Therefore $\beta = v_0(1) + \operatorname{Im} \partial$, and v is a quasi-isomorphism as desired.

□

Now we aim to show that the cone of the map constructed in (5.7), is self-dual. The next proposition gives conditions on any gluing map $\omega : \mathbb{F} \to \mathbb{F}^*[-g]$, which force its cone to be self-dual.

Proposition 5.3.5. *Let $\omega : \mathbb{F} \to \mathbb{F}^*[-g]$ be the map such that*

$$\omega_0(1) = \sum_{\substack{1 \leq k_1 < \cdots < k_n \leq s \\ 1 \leq \ell_1 < \cdots < \ell_m \leq r \\ n+2m=g}} c^{\ell_1,\dots,\ell_m}_{k_1,\dots,k_n} dg^*_{k_1} \dots dg^*_{k_n} T^*_{\ell_1} \dots T^*_{\ell_m}$$

and ω_i is defined by multiplication by $\omega_0(1)$. If either $c^{\ell_1,\dots,\ell_m}_{k_1,\dots,k_n} = 0$ for all $n \equiv 0, 1 \bmod 4$ or for all $n \equiv 2, 3 \bmod 4$, then cone(ω) is self-dual.

Proof. We first show that $\omega_j^T = \pm \omega_{g-j}$ for all j, where we view each ω_j as a matrix and $(\_\_)^T$ denotes the transpose. Note that for $t + 2u = j$, we have

$$\omega_j \left(dg_{p_1} \dots dg_{p_t} T_{q_1} \dots T_{q_u} \right) =$$

$$= \left(dg_{p_1} \dots dg_{p_t} T_{q_1} \dots T_{q_u} \right) \cdot \left(\sum_{\substack{1 \leq k_1 < \cdots < k_n \leq s \\ 1 \leq \ell_1 < \cdots < \ell_m \leq r \\ n+2m=g}} c^{\ell_1,\dots,\ell_m}_{k_1,\dots,k_n} dg^*_{k_1} \dots e^*_{k_n} T^*_{\ell_1} \dots T^*_{\ell_m} \right)$$

$$= \sum_{\substack{1\le k_1<\dots<k_n\le s\\ 1\le \ell_1<\dots<\ell_m\le r\\ n+2m=g}} c^{\ell_1,\dots,\ell_m}_{k_1,\dots,k_n} dg_{p_1}\dots dg_{p_t} T_{q_1}\dots T_{q_u} dg^*_{k_1}\dots dg^*_{k_n} T^*_{\ell_1}\dots T^*_{\ell_m},$$

and we observe that $dg_{p_1}\dots dg_{p_t} T_{q_1}\dots T_{q_u} dg^*_{k_1}\dots dg^*_{k_n} T^*_{\ell_1}\dots T^*_{\ell_m}$ is given by

$$\begin{cases} (-1)^z dg^*_{v_1}\dots dg^*_{v_{n-t}} T^*_{x_1}\dots T^*_{x_{m-u}} & dg_{p_1}\dots dg_{p_t} T_{q_1}\dots T_{q_u} | dg_{k_1}\dots dg_{k_n} T_{\ell_1}\dots T_{\ell_m} \\ 0 & \text{otherwise} \end{cases} \tag{5.8}$$

for some $z \in \mathbb{Z}$, where $\{v_1,\dots,v_{n-t}\}$ is the complement of $\{p_1,\dots,p_t\}$ in $\{k_1,\dots,k_n\}$ and $\{x_1,\dots,x_{m-u}\}$ is the complement of $\{q_1,\dots,q_u\}$ in $\{\ell_1,\dots,\ell_m\}$.

Similarly we have that

$$\omega_{g-j}\big(dg_{v_1}\dots dg_{v_{n-t}} T_{x_1}\dots T_{x_{m-u}}\big) =$$
$$= \sum_{\substack{1\le k_1<\dots<k_n\le s\\ 1\le \ell_1<\dots<\ell_m\le r\\ n+2m=g}} c^{\ell_1,\dots,\ell_m}_{k_1,\dots,k_n} dg_{v_1}\dots dg_{v_{n-t}} T_{x_1}\dots T_{x_{m-u}} dg^*_{k_1}\dots dg^*_{k_n} T^*_{\ell_1}\dots T^*_{\ell_m}.$$

From (5.8), in the case that $dg_{p_1}\dots dg_{p_t} T_{q_1}\dots T_{q_u} | dg_{k_1}\dots dg_{k_n} T_{\ell_1}\dots T_{\ell_m}$, we see that

$$dg^*_{k_1}\dots dg^*_{k_n} T^*_{\ell_1}\dots T^*_{\ell_m} = (-1)^z T^*_{q_u}\dots T^*_{q_1} dg^*_{p_t}\dots dg^*_{p_1} dg^*_{v_1}\dots dg^*_{v_{n-t}} T^*_{x_1}\dots T^*_{x_{m-u}}.$$

Thus we have the following equalities

$$\begin{aligned}
&dg_{v_1}\dots dg_{v_{n-t}} T_{x_1}\dots T_{x_{m-u}} dg^*_{k_1}\dots dg^*_{k_n} T^*_{\ell_1}\dots T^*_{\ell_m} = \\
&\quad = (-1)^z dg_{v_1}\dots dg_{v_{n-t}} T_{x_1}\dots T_{x_{m-u}} T^*_{q_u}\dots T^*_{q_1} dg^*_{p_t}\dots dg^*_{p_1} dg^*_{v_1}\dots dg^*_{v_{n-t}} T^*_{x_1}\dots T^*_{x_{m-u}} \\
&\quad = (-1)^z dg_{v_1}\dots dg_{v_{n-t}} dg^*_{p_t}\dots dg^*_{p_1} T^*_{q_1}\dots T^*_{q_u} dg^*_{v_1}\dots dg^*_{v_{n-t}} \\
&\quad = (-1)^z (-1)^{(n-t)(\frac{t+n-1}{2})} dg^*_{p_t}\dots dg^*_{p_1} T^*_{q_1}\dots T^*_{q_u} \\
&\quad = (-1)^z (-1)^{t(n-t)+\frac{(n-t)(n-t-1)}{2}+\frac{t(t-1)}{2}} dg^*_{p_1}\dots dg^*_{p_t} T^*_{q_1}\dots T^*_{q_u} \\
&\quad = (-1)^z (-1)^{\frac{n(n-1)}{2}} dg^*_{p_1}\dots dg^*_{p_t} T^*_{q_1}\dots T^*_{q_u} \\
&\quad = \begin{cases} (-1)^z dg^*_{p_1}\dots dg^*_{p_t} T^*_{q_1}\dots T^*_{q_u} & n \equiv 0,1 \bmod 4 \\ (-1)^{z+1} dg^*_{p_1}\dots dg^*_{p_t} T^*_{q_1}\dots T^*_{q_u} & n \equiv 2,3 \bmod 4 \end{cases},
\end{aligned}$$

and by our assumption on $c^{\ell_1,\ldots,\ell_m}_{k_1,\ldots,k_n}$, we see that either $\omega_{g-j} = \omega_j^T$ for all j or $\omega_{g-j} = -\omega_j^T$ for all j.

Now $T :=$cone(ω) is the complex

$$\cdots \longrightarrow \mathbb{F}_{g+1} \xrightarrow{D_{g+1}} \mathbb{F}_g \xrightarrow{D_g} \mathbb{F}_0^* \oplus \mathbb{F}_{g-1} \longrightarrow \cdots \longrightarrow \mathbb{F}_{g-2}^* \oplus \mathbb{F}_1 \xrightarrow{D_1} \mathbb{F}_{g-1}^* \oplus \mathbb{F}_0 \xrightarrow{D_0} \mathbb{F}_g^* \longrightarrow \ldots \tag{5.9}$$

with differentials given by the block matrices

$$D_j = \left[\begin{array}{c|c} \partial_{g-j}^T & \omega_j \\ \hline 0 & -\partial_j \end{array}\right].$$

We define $\phi \colon T \longrightarrow T^*[g-1]$, shown in the diagram below, as follows.

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & F_{g+1} & \xrightarrow{D_{g+1}} & F_g & \xrightarrow{D_g} & F_0^* \oplus F_{g-1} & \xrightarrow{D_{g-1}} \cdots \xrightarrow{D_2} & F_{g-2}^* \oplus F_1 & \xrightarrow{D_1} & F_{g-1}^* \oplus F_0 & \xrightarrow{D_0} & F_g^* & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \phi_{g+1}} & & \downarrow{\scriptstyle \phi_g} & & \downarrow{\scriptstyle \phi_{g-1}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_{-1}} & \\
\cdots \longrightarrow & F_{g+1} & \xrightarrow{D_{-1}^T} & F_g & \xrightarrow{D_0^T} & F_{g-1} \oplus F_0^* & \xrightarrow{D_1^T} \cdots \xrightarrow{D_{g-2}^T} & F_1 \oplus F_{g-2}^* & \xrightarrow{D_{g-1}^T} & F_0 \oplus F_{g-1}^* & \xrightarrow{D_g^T} & F_g^* & \longrightarrow \cdots
\end{array}$$

In the case that $\omega_{g-j} = \omega_j^T$ for all j, we define ϕ by

$$\phi_j = \begin{cases} \left[\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right] & \text{for } j \text{ even} \\ \left[\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline -\mathrm{Id} & 0 \end{array}\right] & \text{for } j \text{ odd} \end{cases},$$

and in the case that $\omega_{g-j} = -\omega_j^T$ for all j, we define ϕ by

$$\phi_j = \begin{cases} \left[\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right] & \text{for } j \text{ even} \\ \left[\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline -\mathrm{Id} & 0 \end{array}\right] & \text{for } j \text{ odd} \end{cases}.$$

In either case, it is easy to see that the diagram commutes, and that ϕ is an isomorphism between the cone(ω) and its shifted dual. □

To complete our construction, we let

$$\mathbb{T} := \operatorname{cone}(v), \tag{5.10}$$

where $v : \mathbb{F} \longrightarrow \mathbb{F}^*[-g]$ is the map defined in (5.7). We note that by Proposition 5.3.4, $\mathbb{T}$ is exact, and its dual is also exact. Indeed, by Proposition 5.3.5, it is self-dual. The fact that its truncation $\mathbb{T}_{\geq g}$ is the tail of a free resolution of R/J is clear from (5.9). Thus we have proved the following result.

Theorem 5.3.6. $\mathbb{T}$ *is a self-dual complete resolution of* R/J *over* R. □

In fact, the resolution we have constructed is the minimal complete resolution of R/J, as we see in the following remark.

Remark 5.3.7. We note that the generators of I and J can be chosen so that the a_{ji} are in the maximal ideal. Indeed, if

$$f_j = \sum_{i \in \Gamma^c} u_i g_i + \sum_{i \in \Gamma} a_{ji} g_i$$

where $\Gamma \subseteq \{1, \ldots, s\}$, each u_i is a unit, and each $a_{ji} \in \mathfrak{m}$, then

$$f_j + \mathfrak{m}(\underline{g}) = \sum_{i \in \Gamma^c} u_i g_i + \mathfrak{m}(\underline{g});$$

thus by Nakayama's Lemma, we can replace a generator g_k, for some $k \in \Gamma^c$, by f_j in our minimal generating set for $(\underline{g})$. However,

$$R/(g_1, \ldots, f_j, \ldots, g_s) = R/(g_1, \ldots, \widehat{f_j}, \ldots, g_s),$$

so we may remove the generator f_j from the ideal $(\underline{g})$. Repeating this argument as necessary, we see that we may remove all of the generators g_i with $i \in \Gamma^c$, and thus we may assume that all of the a_{ji} are in the maximal ideal in our construction.

By the discussion above, we see from (5.6) that the map v defined in (5.7) is minimal, and thus its cone $\mathbb{T}$ is also minimal.

Applying Proposition 5.1.6 and Theorem 5.3.6, we arrive at the following Corollary.

Corollary 5.3.8. *Let Q be a Noetherian local ring and let $I = (\underline{f}) \subseteq (\underline{g})$ be embedded complete intersection ideals in Q. If $R = Q/I$ and $J = (\underline{g})/I \subseteq R$, then for any R-module N, there are isomorphisms*

$$\widehat{\operatorname{Tor}}_n^R(R/J, N) \cong \widehat{\operatorname{Ext}}_R^{n+g-1}(R/J, N)$$

for all $n \in \mathbb{Z}$. □

Appendix A

Koszul Homology: Another Approach

In this appendix, we give an alternative proof of Lemma 5.2.1. We recall that Koszul homology in the embedded complete intersection case is well-understood; see the discussion preceding Lemma 5.2.1, and this fact is reflected in the proof of the lemma given in Chapter 5. However, we include a different proof here which does not utilize the known structure of Koszul homology over an embedded complete intersection, as this proof was a stepping stone to the formulas given in Chapter 2 for the generators of Koszul homology in a much more general case. We state the lemma again here for the convenience of the reader.

Lemma. Let Q be a Noetherian local ring and let $I = (\underline{f}) \subseteq (\underline{g})$ be embedded complete intersection ideals in Q. If $R = Q/I$ and $J = (\underline{g})/I \subseteq R$, then an R/J-basis for $H_\ell(\underline{g}; R)$ is given by homology classes of the elements

$$\sum_{1 \leq i_1 < \cdots < i_\ell \leq s} \overline{A^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell}} dg_{i_1} \ldots dg_{i_\ell}$$

for $1 \leq j_1 < \cdots < j_\ell \leq r$, where $\overline{(\ \cdot\ )}$ denotes the image in R.

We break the lemma into two separate statements; the first is that the proposed candidates for Koszul homology generators are cycles, and the second is that they are indeed generators.

Proposition A.0.1. *The elements*

$$\sum_{1\le i_1<\cdots<i_\ell\le s} \overline{A^{i_1,\dots,i_\ell}_{j_1,\dots,j_\ell}} dg_{i_1}\dots dg_{i_\ell},$$

for $1 \le j_1 < \cdots < j_l \le r$ *and where* $\overline{(\,\cdot\,)}$ *denotes the image in* R*, are cycles in the Koszul complex* $K(\underline{g}; R)$*.*

Proof. Applying the Koszul differential ∂_K to the given elements, we have

$$\partial_K\left(\sum_{1\le i_1<\cdots<i_\ell\le s} A^{i_1,\dots,i_\ell}_{j_1,\dots,j_\ell} dg_{i_1}\dots dg_{i_\ell}\right) = \sum_{1\le i_1<\cdots<i_\ell\le s} A^{i_1,\dots,i_\ell}_{j_1,\dots,j_\ell}\sum_{k=1}^{\ell}(-1)^{k+1} g_{i_k} dg_{i_1}\dots\widehat{dg_{i_k}}\dots dg_{i_\ell}.$$

It suffices to show that the coefficient of each $dg_{m_1}\dots dg_{m_{\ell-1}}$ is zero.

Thus, the terms of the sum we are interested in are the ones where we sum over the sets $\{i_1,\dots,i_\ell\}$ containing the set $\{m_1,\dots,m_{\ell-1}\}$. There are $s-\ell+1$ such sets, namely,

$$\{m_1,\dots,m_{\ell-1},p_1\},\dots,\{m_1,\dots,m_{\ell-1},p_{s-\ell+1}\},$$

where $\{p_1,\dots,p_{s-\ell+1}\}$ is the complement of the set $\{m_1,\dots,m_{\ell-1}\}$ in $\{1,\dots,s\}$. Denote by s_k, the spot in which p_k sits when the set $\{m_1,\dots,m_{\ell-1},p_k\}$ is ordered.

Then the coefficient of interest is

$$\sum_{k=1}^{\ell}(-1)^{s_k+1} A^{m_1,\dots,p_k,\dots,m_{\ell-1}}_{j_1,\dots,j_\ell} g_{p_k}. \tag{A.1}$$

Expanding each determinant in (A.1) along the s_k-th column, we get

$$\begin{aligned}\sum_{k=1}^{\ell}(-1)^{s_k+1} g_{p_k}\sum_{n=1}^{\ell}(-1)^{s_k+n} a_{j_n,p_k} A^{m_1,\dots,m_{\ell-1}}_{j_1,\dots,\widehat{j_n},\dots,j_\ell} &= \sum_{n=1}^{\ell}(-1)^{n+1} A^{m_1,\dots,m_{\ell-1}}_{j_1,\dots,\widehat{j_n},\dots,j_\ell}\sum_{k=1}^{\ell} a_{j_n,p_k} g_{p_k}\\ &= \sum_{n=1}^{\ell}(-1)^{n+1} A^{m_1,\dots,m_{\ell-1}}_{j_1,\dots,\widehat{j_n},\dots,j_\ell}\left(f_{j_n} - \sum_{k=\ell+1}^{s} a_{j_n,p_k} g_{p_k}\right).\end{aligned}$$

However,

$$\sum_{n=1}^{\ell}(-1)^{n+1} A^{m_1,\dots,m_{\ell-1}}_{j_1,\dots,\widehat{j_n},\dots,j_\ell} a_{j_n,p_k}$$

is precisely the determinant $A^{m_k,m_1,\dots,m_k,\dots,m_{\ell-1}}_{j_1,\dots,j_\ell}$ expanded along the first column, which is zero since a column is repeated. Thus, we have that the coefficient in (A.1) is given by

$$\sum_{k=1}^{\ell}(-1)^{s_k+1}A^{m_1,\dots,p_k,\dots,m_{\ell-1}}_{j_1,\dots,j_\ell}g_{p_k} = \sum_{n=1}^{\ell}(-1)^{n+1}A^{m_1,\dots,m_{\ell-1}}_{j_1,\dots,\widehat{j_n},\dots,j_\ell}f_{j_n}$$

which is zero in $R = Q/I$, and thus the given elements are cycles as desired. □

These cycles generate $H_\ell(\underline{g};R)$ as an R/J-module, as shown in the following proposition.

Proposition A.0.2. *An R/J-basis of $H_\ell(\underline{g};R)$ is given by the homology classes of the elements*

$$\sum_{1\le i_1<\cdots<i_\ell\le s}\overline{A^{i_1,\dots,i_\ell}_{j_1,\dots,j_\ell}}dg_{i_1}\dots dg_{i_\ell}$$

for $1 \le j_1 < \cdots < j_\ell \le r$.

Proof. We begin by considering the following isomorphisms

$$H_\ell(\underline{g};R) \cong H_\ell(R \underset{Q}{\otimes} K(\underline{g};Q)) \cong \operatorname{Tor}^Q_\ell(R, R/J) \cong H_\ell(K(\underline{f};Q) \underset{Q}{\otimes} R/J) \cong K_\ell(\underline{f};Q) \underset{Q}{\otimes} R/J,$$

where the second and third isomorphisms follow from the fact that $\underline{f}$ and $\underline{g}$ are regular sequences and the last follows from the fact that $I \subseteq (\underline{g})$.

As we stated in Chapter 5, the isomorphism above is defined by sending an element $b\otimes 1$ in $K_\ell(\underline{f};Q)\otimes_Q R/J$ to $[\bar{z}_\ell]$ where $z = (z_0,\dots,z_\ell)$ is a cycle in $K(\underline{f};Q)\otimes_Q K(\underline{g};Q)$ such that z_0 is sent to $b\otimes 1$ in the surjection

$$K_\ell(\underline{f};Q)\otimes_Q K_0(\underline{g};Q) \twoheadrightarrow K_\ell(\underline{f};Q)\otimes_Q R/J,$$

z_ℓ is sent to $\bar{z}_\ell$ in the surjection

$$K_0(\underline{f};Q)\otimes_Q K_\ell(\underline{g};Q) \twoheadrightarrow R\otimes_Q K_\ell(\underline{g};Q),$$

and [__] denotes the homology class.

Thus for each set $1 \le j_1 < \cdots < j_\ell \le r$ it suffices to find a cycle $z = (z_0,\dots,z_\ell)$ such that

$$z_\ell = 1\otimes\left(\sum_{1\le i_1<\cdots<i_\ell\le s}A^{i_1,\dots,i_\ell}_{j_1,\dots,j_\ell}dg_{i_1}\dots dg_{i_\ell}\right)$$

and z_0 is sent to the basis element $df_{j_1}\dots df_{j_\ell}\otimes 1$ of $K_\ell(\underline{f};Q)\otimes R/J$.

We define $z=(z_0,\dots,z_\ell)$, where

$$z_p=\sum_N(-1)^{b_{\ell,p}}\left[df_{j_{n_1}}\dots df_{j_{n_{\ell-p}}}\otimes\left(\sum_I A^{i_1,\dots,i_p}_{j_{n'_1},\dots,j_{n'_p}}dg_{i_1}\dots dg_{i_p}\right)\right],$$

where $\{n'_1,\dots,n'_p\}$ is the complement of the set $\{n_1,\dots,n_{\ell-p}\}$ in $\{1,\dots,\ell\}$, and where for ease of notation, we denote by $b_{\ell,p}$ the exponent $n_1+\dots+n_{\ell-p}+\ell-p$, by $\sum_N$ the sum over $1\le n_1<\dots<n_{\ell-p}\le\ell$, and by $\sum_I$ the sum over $1\le i_1<\dots<i_p\le s$.

To show that z is a cycle, we must show that $(1\otimes\partial_K)(z_p)=(\partial_K\otimes 1)(z_{p-1})$ for all $p=1,\dots,\ell$. As in the proof of Proposition A.0.1, we have that $(1\otimes\partial_K)(z_p)$ is given by

$$\sum_N(-1)^{b_{\ell,p}}\left[df_{j_{n_1}}\dots df_{j_{n_{\ell-p}}}\otimes\left(\sum_I\sum_{q=1}^{p}(-1)^{q+1}A^{i_1,\dots,i_{p-1}}_{j_{n'_1},\dots,\widehat{j_{n'_q}},\dots,j_{n'_p}}f_{j_{n'_q}}dg_{i_1}\dots dg_{i_{p-1}}\right)\right],\tag{A.2}$$

and on the other hand $(\partial_K\otimes 1)(z_{p-1})$ is given by

$$\sum_N(-1)^{b_{\ell,p-1}}\left[\left(\sum_{t=1}^{\ell-p+1}(-1)^{t+1}f_{j_{n_t}}df_{j_{n_1}}\dots\widehat{df_{j_{n_t}}}\dots df_{j_{n_{\ell-p+1}}}\right)\right.$$
$$\left.\otimes\left(\sum_I A^{i_1,\dots,i_{p-1}}_{j_{n'_1},\dots,j_{n'_{p-1}}}dg_{i_1}\dots dg_{i_{p-1}}\right)\right].\tag{A.3}$$

We consider in each sum (A.2) and (A.3), the term which is tensored with $df_{j_{m_1}}\dots df_{j_{m_{\ell-p}}}$ and show that they are equal. We first consider this term in (A.2) by looking at the terms where we sum over the sets $\{n_1,\dots,n_{\ell-p+1}\}$ containing the set $\{m_1,\dots,m_{\ell-p}\}$. There are p such sets, namely,

$$\{m_1,\dots,m_{\ell-p},u_1\},\dots,\{m_1,\dots,m_{\ell-p},u_p\}$$

where $\{u_1,\dots,u_p\}$ is the complement of the set $\{m_1,\dots,m_{\ell-p}\}$ in $\{j_1,\dots,j_\ell\}$. Note that once the set $\{u_1,\dots,u_p\}$ is ordered, it is the same set as $\{m'_1,\dots,m'_p\}$ according to our earlier notation. Let s_i denote the spot in which u_i sits in the set $\{m_1,\dots,u_i,\dots,m_{\ell-p}\}$ when it is ordered.

Picking out the appropriate terms from this part of the sum, we get

$$\sum_{k=1}^{p}(-1)^{b_{\ell,p}+u_k+1}\left[\left((-1)^{s_k+1}f_{j_{u_k}}df_{j_{m_1}}\dots df_{j_{m_{\ell-p}}}\right)\otimes\left(\sum_I A^{i_1,\dots,i_{p-1}}_{j_{u_1},\dots,\widehat{j_{u_k}},\dots,j_{u_p}}dg_{i_1}\dots dg_{i_{p-1}}\right)\right]$$

$$=df_{j_{m_1}}\dots df_{j_{m_{\ell-p}}}\otimes\left(\sum_{k=1}^{p}(-1)^{b_{\ell,p}+u_k+s_k+2}\sum_I A^{i_1,\dots,i_{p-1}}_{j_{u_1},\dots,\widehat{j_{u_k}},\dots,j_{u_p}} f_{j_{u_k}} dg_{i_1}\dots dg_{i_{p-1}}\right),$$

where, abusing notation, $b_{\ell,p}=m_1+\dots+m_{\ell-p}+\ell-p$.

Now we look at the corresponding terms in (A.3), namely,

$$df_{j_{m_1}}\dots df_{j_{m_{\ell-p}}}\otimes\left((-1)^{b_{\ell,p}}\sum_{k=1}^{p}(-1)^{s'_k+1}\sum_I A^{i_1,\dots,i_{p-1}}_{j_{u_1},\dots,\widehat{j_{u_k}},\dots,j_{u_p}} f_{j_{u_k}} dg_{i_1}\dots dg_{i_{p-1}}\right),$$

where s'_k denotes the spot in which u_i sits in the set $\{u_1,\dots,u_p\}$ once the set is ordered.

It suffices to verify that $(-1)^{u_k+s_k}=(-1)^{s'_k+1}$ for all $k=1,\dots,p$. We verify this fact in the case that s'_k is even; the other case is similar. Thus we must show that u_k and s_k have different parity. Since s'_k is even, there is an odd number of elements preceding u_k in the set $\{u_1,\dots,u_p\}$. If u_k is even, then there is an odd number of elements preceding u_k in the set $\{1,\dots,\ell\}$, and an odd number of those elements are contained in the set $\{u_1,\dots,u_p\}$, hence an even number of them must be contained in the set $\{m_1,\dots,m_{\ell-p}\}$. Therefore, s_k is odd as desired. The argument is similar if u_k is odd. Thus z is a cycle, and the result follows. □

The method we use here becomes more complicated in settings as general as in Chapter 2, so we streamline the proof with the Perturbation Lemma.

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VITA

NAME OF AUTHOR: Rachel N. Diethorn

PLACE OF BIRTH: Natrona Heights, PA   DATE OF BIRTH: March, 1991

DEGREES AWARDED:

Master of Science in Mathematics, Syracuse University, 2015

Bachelor of Science in Mathematics, Saint Vincent College, 2013

PUBLICATION: Rachel N. Diethorn, *Generators of Koszul homology with coefficients in a J-closed module*, J. Pure Appl. Algebra (2020), To appear.

AWARDS:

Kibbey Prize, For Excellence in the Mathematics Graduate Program, 2020

Outstanding Teaching Assistant Award, 2017

INVITED LECTURES:

AMS Special Session on Homological Methods in Algebra, University Park, PA, 2020

AMS Special Session on DG Methods in Commutative Algebra, Fresno, CA, 2020

AMS Special Session on Commutative Algebra, Charlottesville, VA, 2020

JMM Special Session on Commutative Algebra, Denver, CO, 2020

CMS Special Session on Commutative Algebra, Toronto, ON, 2019

Commutative Algebra Seminar, University of Nebraska - Lincoln, 2019

AMS Special Session on Homological Methods in Algebra, Gainesville, FL, 2019

AMS Special Session on Commutative Algebra, Binghamton, NY, 2019

PROFESSIONAL ACTIVITIES:

Instructor of Record, 2015-2020

Co-organizer: New York Regional Graduate Mathematics Conference, 2016-2019

Mentor: Directed Reading Program, 2018

Teaching Mentor: TA Orientation Program, 2017

President: Mathematics Graduate Organization, 2016-2017