Chapter

Survey of Some Exact and Approximate Analytical Solutions for Heat Transfer in Extended Surfaces

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Abstract

In this chapter we provide the review and a narrative of some obtained results for steady and transient heat transfer though extended surfaces (fins). A particular attention is given to exact and approximate analytical solutions of models describing heat transfer under various conditions, for example, when thermal conductivity and heat transfer are temperature dependent. We also consider fins of different profiles and shapes. The dependence of thermal properties render the considered models nonlinear, and this adds a complication and difficulty to solve these model exactly. However, the nonlinear problems are more realistic and physically sound. The approximate analytical solutions give insight into heat transfer in fins and as such assist in the designs for better efficiencies and effectiveness.

Keywords: exact solutions, approximate solutions, lie symmetry methods, approximate methods, heat transfer, fins

1. Introduction

In the study of heat transfer, a fin may be a solid or porous and stationary or moving that extends from an attached body to rapidly cool off heat of that surface. Cooling fins find application in a large real world phenomena particularly in engineering devices. Fins increase the surface area of heat transfer particularly for cooling of hot bodies. These come in different shapes, geometries and profiles. These differences provide variety of effectiveness and efficiencies. The literature with regard to the study of heat transfer in fins is well documented (see e.g. [1]). The solutions either exact, numerical or approximate analytical continue to be of immerse interest and this is due to continued use of fins in engineering devices.

Much attention has been given to linear one dimensional models [2–4] whereby Homotopy Analysis Method (HAM) was used to determine series solutions for heat transfer in straight fins of trapezoidal and rectangular profiles given temperature dependent thermal properties; nonlinear one dimensional models [5] wherein preliminary group classification methods were utilised to contract invariant (symmetry) solutions; heat transfer in linear two dimensional trapezoidal fins [6]; heat transfer in two dimensional straight nonlinear fins were considered [7] wherein Lie point symmetries and other standard methods were invoked and recently nonlinear three dimensional models [8] were considered wherein three dimensional Differential Transform Methods (DTM) were employed to construct approximate analytical solutions. The dependence of thermal properties on the temperature renders the equations highly nonlinear. The non-linearity brings an added complication or difficulty in the construction of solutions and particularly exact solutions.

Few exact solutions are recorded in the literature, for example for one dimensional problems [2–5, 9–15], two dimensions [6, 7, 16, 17]. An attempt to construct exact solutions for the three dimensional problems is found in [8], however these were general solutions. For this reason, either approximate analytical or numerical solutions are sought. However, the accuracy of numerical schemes is obtained by comparison with he exact solutions.

This chapter summaries the work of Moitsheki and collaborators in the area of heat transfer through fin. In their work, they employed Lie symmetry methods to construct exact solutions. These methods include, the preliminary group classification, the Lie point symmetries, conservation laws and associate Lie point symmetries, non-classical symmetry methods and recently non classical potential symmetries. It appeared that most of the constructed exact solutions do not satisfy the prescribed boundary conditions. The idea then becomes, start with the simple model that satisfy the boundary conditions and compare it with the approximate solutions to establish confidence in the approximate methods, then extend analysis to problems that are difficult to solve exactly.

We acknowledge that some scholars employed many other approximate methods to solve boundary value problems (BVPs); for example the Homotopy Analysis Method [18], Collocation Methods (CM) [19], Homotopy Perturbation Methods (HPM) [20], Haar Wavelet Collation Methods (HWCM) [21], Collocation Spectral Methods (CSM) [22], modified Homotopy Analysis Method (mHAM) [23], Spectral Homotopy Analysis Methods (SHAM) and the Optimal Homotopy Analysis Methods [24]. In this chapter we restrict discussions to Lie symmetry methods for exact solutions, and DTM and VIM for approximate analytical methods.

2. Mathematical descriptions

Mathematical descriptions represent some physical phenomena in terms of deterministic models given in terms of partial differential equations (PDEs). These differential equations become non-linear when heat transfer coefficient and thermal conductivity depend on the temperature (see e.g. [5]). This non-linearity was introduced as a significant modifications of the usually assumed models see e.g. [2].

In this chapter we present a few models for various heat transfer phenomena.

2.1 2 + 1 dimensional transient state models

Mathematical modelling for heat transfer in fins may be three dimensional models.

2.1.1 Cylindrical pin fins

We consider a two-dimensional pin fin with length L and radius R. The fin is attached to a base surface of temperature T_b and extended into the fluid of temperature T_s. The tip of the fin is insulated (i.e., heat transfer at the tip is negligibly small). The fin is measured from the tip to the base. A schematic representation of a pin fin is given in **Figure 1**. We assume that the heat transfer coefficient along the

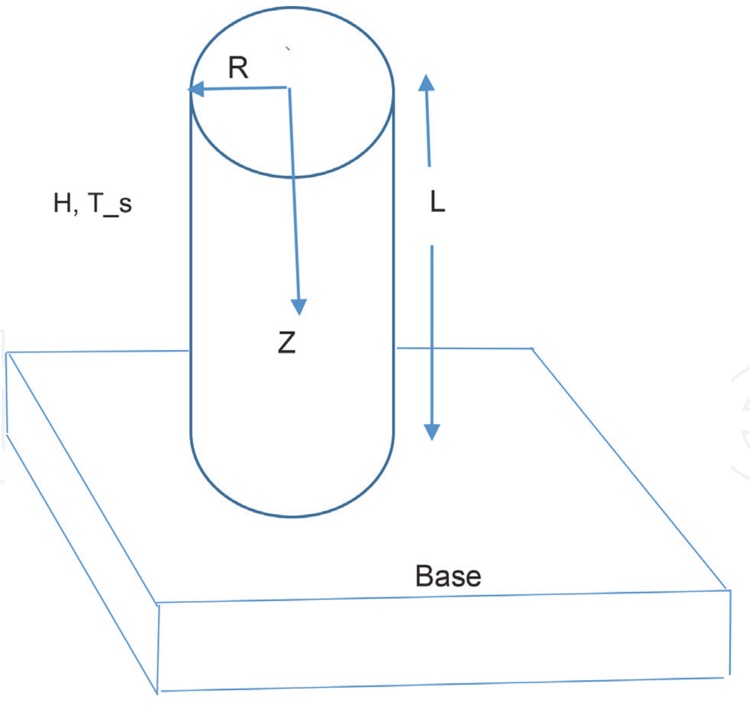

Figure 1.
Schematic representation of a pin fin.

fin is nonuniform and temperature dependent and that the internal heat source or sink is neglected. Furthermore, the temperature-dependent thermal conductivity is assumed to be the same in both radial and axial directions. The model describing the heat transfer in pin fins is given by the BVP (see e.g. [17])

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial Z}\left[K(T)\frac{\partial T}{\partial Z}\right] + \frac{1}{R}\frac{\partial}{\partial R}\left[K(T)R\frac{\partial T}{\partial R}\right]. \tag{1}$$

The initial condition is given by

$$T(0,R,Z) = T_s, \quad 0 \le R \le R_a, \quad 0 \le Z \le L,$$

here, T_s is the temperature of the surrounding fluid.

Boundary conditions are given by

$$\begin{aligned} T(t,R,L) &= T_b, \quad 0 \le R \le R_a, \quad t > 0, \\ \frac{\partial T}{\partial Z} &= 0, \quad Z = 0, \quad 0 \le R \le R_a, \quad t > 0. \\ K(T)\frac{\partial T}{\partial R} &= -H(T)[T - T_s], \quad R = 0, \quad 0 \le Z \le L, \quad t > 0, \\ \frac{\partial T}{\partial R} &= 0, \quad R = R_a, \quad 0 \le Z \le L, \quad t > 0, \end{aligned}$$

In non-dimensionalized variables and parameters we have,

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial z}\left[k(\theta)\frac{\partial \theta}{\partial z}\right] + E^2 \frac{1}{r}\frac{\partial}{\partial r}\left[k(\theta) r \frac{\partial \theta}{\partial r}\right], \tag{2}$$

subject to the initial condition

$$\theta(0,r,z) = 0, \quad 0 \le z \le 1, \quad 0 \le r \le 1,$$

and boundary conditions

$$\theta(\tau,1,r)=1, \quad 0, \leq r\leq 1, \quad \tau>0,$$

$$\frac{\partial\theta}{\partial z}=0, \quad z=0, \quad 0\leq r\leq 1, \quad \tau>0,$$

$$k(\theta)\frac{\partial\theta}{\partial z}=-Bih(\theta)\theta, \quad z=0, \quad 0\leq z\leq 1, \quad \tau>0,$$

$$\frac{\partial\theta}{\partial r}=0, \quad r=1, \quad 0\leq z\leq 1, \quad \tau>0,$$

where the non-dimensional quantities $E=\frac{L}{\delta}$, and $Bi=\frac{H_b\delta}{K_a}$, are the fin extension factor and the Biot number respectively. Also,

$$t=\frac{L^2\rho c_p}{K_a}\tau, \quad Z=Lz, \quad R=R_a r,$$
$$K=K_a k, \quad H=H_b h, \quad T=(T_b-T_s)\theta+T_s.$$

where τ, z, r, k, h and θ are all dimensionless variables. K_a and H_b are the ambient thermal conductivity and the fin base heat transfer coefficient respectively.

Notice that other terms may be added, for example internal heat generation (source term) and fin profile.

2.1.2 Rectangular straight fins

Following the similar pattern, in dimensionless variables we have (see e.g. [8])

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial x}\left[k(\theta)\frac{\partial\theta}{\partial x}\right]+E^2\frac{\partial}{\partial y}\left[k(\theta)\frac{\partial\theta}{\partial y}\right], \tag{3}$$

subject to the initial condition

$$\theta(0,x,y)=0, \quad 0\leq x\leq 1, \quad 0\leq y\leq 1,$$

and boundary conditions

$$\theta(\tau,1,y)=1, \quad 0, \leq y\leq 1, \quad \tau>0,$$

$$\frac{\partial\theta}{\partial x}=0, \quad x=0, \quad 0\leq y\leq 1, \quad \tau>0,$$

$$k(\theta)\frac{\partial\theta}{\partial y}=-Bih(\theta)\theta, \quad y=0, \quad 0\leq x\leq 1, \quad \tau>0,$$

$$\frac{\partial\theta}{\partial y}=0, \quad y=1, \quad 0\leq x\leq 1, \quad \tau>0,$$

2.2 Two-dimensional steady state models

In this section we consider the two dimensional steady state models. The symmetry analysis of these models have proven to be challenging. In some cases standard method such as separations of variables have been employed to determine exact solutions.

2.2.1 Cylindrical pin fins

For steady state problem, the heat transfer is independent of the time variable. For example, the time derivative in Eq. (2) vanish (see e.g. [16]).

2.2.2 Rectangular straight fins

For steady state problem, the heat transfer is independent of the time variable. For example, the time derivative in Eq. (3) is zero (see e.g. [7]).

2.3 1 + 1 dimensional transient model for straight fins

2.3.1 Solid stationary fins

For solid stationary straight fins the model is given by (see e.g. [25, 26])

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial x}\left[f(x)k(\theta)\frac{\partial\theta}{\partial x}\right]-M^2\theta h(\theta),\ \ 0\leq x\leq 1. \tag{4}$$

subject to initial and boundary conditions

$$\theta(0,x)=0,\quad 0\leq x\leq 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \tau\geq 0.$$

2.3.2 Solid moving fins

It appear, as far as we know, this is still an open problem and in preparation.

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial x}\left[f(x)k(\theta)\frac{\partial\theta}{\partial x}\right]-M^2\theta h(\theta)-Pef(x)\frac{\partial\theta}{\partial x},\ \ 0\leq x\leq 1. \tag{5}$$

subject to initial and boundary conditions

$$\theta(0,x)=0,\quad 0\leq x\leq 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \tau\geq 0.$$

2.3.3 Porous stationary fins

The model was considered in [27].

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial x}\left[f(x)k(\theta)\frac{\partial\theta}{\partial x}\right]-N_c(\theta-\theta_a)^{n+1}-N_r\left(\theta^4-\theta_a^4\right),\ \ 0\leq x\leq 1. \tag{6}$$

subject to initial and boundary conditions

$$\theta(0,x)=0,\quad 0\leq x\leq 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \tau\geq 0.$$

2.3.4 Porous moving fins

The model describing heat transfer in porous moving fin is considered in [28] and is given by

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left[f(x)k(\theta)\frac{\partial\theta}{\partial r}\right] - N_c(\theta-\theta_a)^{n+1} - N_p(\theta-\theta_a)^2 - N_r\left(\theta^4-\theta_a^4\right) - Pef(x)\frac{\partial\theta}{\partial x},\ 0\le x\le 1. \tag{7}$$

subject to initial and boundary conditions

$$\theta(0,x)=0,\quad 0\le x\le 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \tau\ge 0.$$

2.4 1 + 1 dimensional transient model for radial fins

2.4.1 Solid stationary fins

For solid stationary radial fins thge model is given by

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left[rf(r)k(\theta)\frac{\partial\theta}{\partial r}\right] - M^2\theta h(\theta),\ 0\le r\le 1. \tag{8}$$

subject to initial and boundary conditions

$$\theta(0,r)=0,\quad 0\le r\le 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial r}\right|_{r=0}=0,\quad \tau\ge 0.$$

2.4.2 Solid moving fins

For solid moving radial fins the model is given by (see e.g. [29]),

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left[rf(r)k(\theta)\frac{\partial\theta}{\partial r}\right] - M^2\theta h(\theta) - f(r)Pe\frac{\partial\theta}{\partial r},\ 0\le r\le 1. \tag{9}$$

subject to initial and boundary conditions

$$\theta(0,r)=0,\quad 0\le r\le 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial r}\right|_{r=0}=0,\quad \tau\ge 0.$$

2.4.3 Porous stationary fins

For solid stationary radial fins the model is given by

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left[rf(r)k(\theta)\frac{\partial\theta}{\partial r}\right] - N_p(\theta-\theta_a)^2 - N_r\left(\theta^4-\theta_a^4\right),\ 0\le r\le 1. \tag{10}$$

subject to initial and boundary conditions

$$\theta(0,r)=0,\quad 0\le r\le 1,\quad \theta(\tau,1)=1;\quad \left.\frac{\partial\theta}{\partial r}\right|_{r=0}=0,\quad \tau\ge 0.$$

2.4.4 Porous moving fins

For porous moving radial fins the model is given by

$$\frac{\partial\theta}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left[rf(r)k(\theta)\frac{\partial\theta}{\partial r}\right] - N_p(\theta-\theta_a)^2 - N_r\left(\theta^4-\theta_a^4\right) - f(r)Pe\frac{\partial\theta}{\partial r},\ 0\le r\le 1. \tag{11}$$

subject to initial and boundary conditions

$$\theta(0,r) = 0, \quad 0 \le r \le 1, \quad \theta(\tau,1) = 1; \quad \left.\frac{\partial\theta}{\partial r}\right|_{r=0} = 0, \quad \tau \ge 0.$$

2.5 One-dimensional steady state model for straight fins

Considering heat transfer in a one dimensional longitudinal fin of cross area A_c with various profiles. The perimeter of the fin is denoted by P and length by L. The fin is attached to a fixed prime surface of temperature T_b and extends to the fluid of temperature T_∞. in non-dimensional variables, one obtains

$$\frac{d}{dx}\left[f(x)k(\theta)\frac{d\theta}{dx}\right] - M^2\theta h(\theta) = 0, \; 0 \le x \le 1. \tag{12}$$

subject to

$$\theta(1) = 1, \quad \frac{d\theta(0)}{dx} = 0.$$

In case of a moving radial fin the term

$$f(x)Pe\frac{d\theta}{dx}$$

is added to Eq. (12).

2.6 One-dimensional steady state model for radial fins

Considering heat transfer in a one dimensional stationary radial fin of cross area A_c with various profiles. The perimeter of the fin is denoted by P and length by $Lr_b - r_t$ The fin is attached to a fixed prime surface of temperature T_b and extends to the fluid of temperature T_∞. One may assume that at the tip of the fin $r_t = 0$. In non-dimensional variables, one obtains

$$\frac{1}{r}\frac{d}{dr}\left[rf(r)k(\theta)\frac{d\theta}{dx}\right] - M^2\theta h(\theta) = 0, \; 0 \le x \le 1. \tag{13}$$

subject to

$$\theta(1) = 1, \quad \frac{d\theta(0)}{dr} = 0.$$

In case of a moving radial fin the term

$$f(r)Pe\frac{d\theta}{dr}$$

is added to Eq. (13) (see e.g. [30]).

3. Methods of solutions

3.1 Brief account on lie symmetry methods

In this subsection we provide a brief theory of Lie point symmetries. This discussion and further account can be found in the book of Bluman and Anco [31].

3.1.1 *m dependent and n independent variables*

m dependent variables $u = (u^1, u^2, \dots, u^m)$ and n independent variables $x = (x^1, x^2, \dots, x^n)$, $u = u(x)$ with $m \geq 2$, arise in studying systems of differential equations. We consider extended transformations from $(x, u)-$space to $\left(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}\right)-$ space. Here $u_{(k)}$ denotes the components of all kth-order partial derivatives of u wrt x..

definition *Total derivative.* The total differentiation operator wrt x^i is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_i^\alpha} + \dots + u_{ii_1 i_2 \dots i_k}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_n}^\alpha} + \dots, \qquad i = 1, 2, \dots, n.$$

where

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \quad u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}, \quad \text{etc.}$$

We seek the one-parameter Lie group of transformations

$$\begin{aligned} \overline{x}^i &= x^i + \varepsilon \xi^i(x, u) + O\left(\varepsilon^2\right), \\ \overline{u}^\alpha &= u^\alpha + \varepsilon \eta^\alpha(x, u) + O\left(\varepsilon^2\right), \end{aligned} \tag{14}$$

which leave the system of equation in question invariant. These transformations are generated by the base vector

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

The kth-extended transformation of (14) are given by

$$\left.\begin{aligned} \overline{u}_i^\alpha &= u_i^\alpha + \varepsilon \zeta_i^\alpha\left(x, u, u_{(1)}\right) + O(\varepsilon^2), \\ \overline{u}_{ij}^\alpha &= u_{ij}^\alpha + \varepsilon \zeta_{ij}^\alpha\left(x, u, u_{(1)}, u_{(2)}\right) + O(\varepsilon^2), \\ &\;. \\ &\;. \\ &\;. \\ \overline{u}_{i_1, i_2, \dots, i_k}^\alpha &= u_{i_1, i_2, \dots, i_k}^\alpha + \varepsilon \zeta_{i_1, i_2, \dots, i_k}^\alpha\left(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}\right) + O(\varepsilon^2), \end{aligned}\right\} \tag{15}$$

Theorem 1.1 The extended infinitesimals satisfy the recursion relations

$$\left.\begin{array}{ll} \zeta_i^\alpha & = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\ \zeta_{ij}^\alpha & = D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l), \\ . & . \\ . & . \\ . & . \\ \zeta_{i_1,i_2,\dots,i_k}^\alpha & = D_{i_k}\left(\zeta_{i_1\dots i_{k-1}}^\alpha\right) - u_{i_1,i_2\dots i_{k-1}l}^\alpha D_{i_k}(\xi^l), \end{array}\right\} \tag{16}$$

Introducing the *Lie Characteristic function* defined by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha,$$

then

$$\left.\begin{array}{ll} \zeta_i^\alpha & = D_i(W^\alpha) + \xi^j u_{ji}^\alpha, \\ \zeta_{ij}^\alpha & = D_i D_j(W^\alpha) + \xi^k u_{kij}, \\ . & . \\ . & . \\ . & . \\ \zeta_{i_1,i_2,\dots,i_k}^\alpha & = D_{i1}\dots D_{i_k}(W^\alpha) + \xi^j u_{ji_1,i_2\dots i_k}. \end{array}\right\} \tag{17}$$

The corresponding (kth extended) infinitesimal generator is given by

$$X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta_i^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha\frac{\partial}{\partial u_i} + \dots + \zeta_{i_1,i_2,\dots,i_k}^\alpha\frac{\partial}{\partial u_{i_1,i_2,\dots,i_k}}, \qquad k \ge 1.$$

Theorem 1.2 A differential function $F(x,u,u_1,\dots,u_{(p)})$ $p \ge 0$, is a pth-order differential invariant of a group G if

$$F(x,u,u_1,\dots,u_{(p)}) = F(\overline{x},\overline{u},\overline{u}_{\overline{1}},\dots,\overline{u}_{(\overline{p})}).$$

Theorem 1.3 A differential function $F(x,u,u_1,\dots,u_{(p)})$ $p \ge 0$, is a pth-order differential invariant of a group G if

$$X^{[p]}F = 0,$$

where $X^{[p]}$ is the pth prolongation of X..

3.2 Approximate methods

3.2.1 p-dimensional differential transform methods

For an analytic multivariable function $f(x_1,x_2,\dots,x_p)$, we have the p-dimensional transform given by

$$F(k_1,k_2,\dots,k_p) = \frac{1}{k_1!k_2!\dots k_p!}\left[\frac{\partial^{k_1+k_2+\dots+k_p}f(x_1,x_2,\dots,x_p)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_p^{k_p}}\right]\Bigg|_{(x_1,x_2,\dots,x_p)=(0,0,\dots,0)}. \tag{18}$$

The upper and lower case letters are for the transformed and the original functions respectively. The transformed function is also referred to as the T-function, the differential inverse transform is given by

$$f(x_1, x_2, \dots, x_p) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_p=0}^{\infty} F(k_1, k_2, \dots, k_p) \prod_{l=1}^{p} x_l^{k_l}. \tag{19}$$

It can easily be deduced that the substitution of (18) into (19) gives the Taylor series expansion of the function $f(x_1, x_2, \dots, x_p)$ about the point $(x_1, x_2, \dots, x_p) =$. $(0, 0, \dots, 0)$. This is given by

$$f(x_1, x_2, \dots, x_p) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_p=0}^{\infty} \frac{\prod_{l=1}^{p} x_l^{k_l}}{k_1! k_2! \dots k_p!} \left[\frac{\partial^{k_1+k_2+\dots+k_p} f(x_1, x_2, \dots, x_p)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_p^{k_p}} \right] \Bigg|_{x_1=0, \dots, x_p=0}. \tag{20}$$

For real world applications the function $f(x_1, x_2, \dots, x_p)$ is given in terms of a finite series for some $q, r, s \in \mathbb{Z}$. Then (19)becomes

$$f(x_1, x_2, \dots, x_p) = \sum_{k_1=0}^{q} \sum_{k_2=0}^{r} \dots \sum_{k_p=0}^{s} F(k_1, k_2, \dots, k_p) \prod_{l=1}^{p} x_l^{k_l}. \tag{21}$$

We now give some important operations and theorems performed in the p-dimensional DTM in **Table 1**. Those have been derived using the definition in (18) together with previously obtained results [32].

In the table

$$\delta(k_1 - e_1, k_2 - e_2, \dots, k_p - e_p) = \begin{cases} 1 & \text{if } k_i = e_i \text{ for } i = 1, 2, .., p. \\ 0 & \text{otherwise.} \end{cases}$$

We now provide one result of the p-dimensional DTM without proof.

Theorem. Proof in [32].

If

$$f(x_1, x_2, \dots, x_p) = g(x_1, x_2, \dots, x_p) h(x_1, x_2, \dots, x_p),$$

then

$$F(k_1, k_2, \dots, k_p) = \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} \dots \sum_{i_p=0}^{k_p} G(k_1, k_2, \dots, k_p + i_p) H(k_1 + i_1, \dots, k_{n-1} + i_{p-1}, k_p).$$

Original function $f(x_1, x_2, \dots, x_p)$	**T-function** $F(k_1, k_2, \dots, k_p)$
$f(x_1, x_2, \dots, x_p) = \lambda g(x_1, x_2, \dots, x_p)$	$F(k_1, k_2, \dots, k_p) = \lambda G(k_1, k_2, \dots, k_p)$
$f(x_1, x_2, \dots, x_p) = g(x_1, x_2, \dots, x_p) \pm p(x_1, x_2, \dots, x_p)$	$F(k_1, k_2, \dots, k_p) = G(k_1, k_2, \dots, k_p) \pm P(k_1, k_2, \dots, k_p)$
$f(x_1, x_2, \dots, x_p) = \frac{\partial^{r_1+r_2+\dots+r_p} g(x_1, x_2, \dots, x_p)}{\partial x_1^{r_1} \partial x_2^{r_2} \dots \partial x_p^{r_p}}$	$F(k_1, k_2, \dots, k_p) = \frac{(k_1+r_1)! \dots (k_p+r_p)!}{k_1! \dots k_p!} (k_1 + r_1, \dots, k_p + r_p)$
$f(x_1, x_2, \dots, x_p) = \prod_{l=1}^{p} x_l^{e_l}$	$F(k_1, k_2, \dots, k_p) = \delta(k_1 - e_1, k_2 - e_2, \dots, k_p - e_p)$

Table 1.
Theorems and operations performed in p-dimensional DTM.

3.2.2 Variational iteration methods

$$L\theta + N\theta = g(x), \tag{22}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the source inhomogeneous term. He [33], proposed the VIM where a correctional functional for Eq. (22) can be written as

$$\theta_{j+1}(x) = \theta_j(x) + \int_0^x \lambda(t)\left(L\theta_j(t) + N\tilde{\theta}_\theta(t) - g(t)\right)dt, \tag{23}$$

where λ is the general Lagrange multiplier, which can be be identified optimally via the variation theory, and $\tilde{\theta}_n$ is a restricted variation, which means $\delta\tilde{\theta}_n = 0$ [34]. The Lagrange multiplier can be a constant or a function depending on the order of the deferential equation under consideration. The VIM should be employed by following two essential steps. First we determine the Lagrange multiplier by considering the following second order differential equation,

$$\theta''(x) + a\theta'(x) + b\theta(x) = g(x), \theta(0) = \alpha, \theta'(0) = \beta, \tag{24}$$

where a and b are constants. The VIM admits the use of a correctional function for this equation as follows,

$$\theta_{j+1}(x) = \theta_j(x) + \int_0^x \lambda(t)\left(\theta''_j(t) + a\tilde{\theta}'_j(t) + b\tilde{\theta}_j(t) - g(t)\right)dt. \tag{25}$$

Taking the variation on both sides of Eq. (25) with respect to the independent variable θ_j gives,

$$\frac{\delta\theta_{j+1}}{\delta\theta_j} = 1 + \frac{\delta}{\delta\theta_j}\left(\int_0^x \lambda(t)\left(\theta''_j(t) + a\tilde{\theta}'_j(t) + b\tilde{\theta}_j(t) - g(t)\right)dt\right), \tag{26}$$

or equivalently

$$\delta\theta_{j+1}(x) = \delta\theta_j(x) + \delta\left(\int_0^x \lambda(t)\left(\theta''_j(t) + a\tilde{\theta}'_j(t) + b\tilde{\theta}_j(t) - g(t)\right)dt\right), \tag{27}$$

which gives

$$\delta\theta_{j+1}(x) = \delta\theta_j(x) + \delta\left(\int_0^x \lambda(t)\left(\theta''_j(t)dt\right)\right), \tag{28}$$

obtained upon using $\delta\tilde{\theta}_j = 0$ and $\delta\tilde{\theta}'_j = 0$. Evaluating the integral of Eq. (28) by parts gives,

$$\delta\theta_{j+1} = \delta\theta_j + \delta\lambda\theta'_j - \delta\lambda'\theta_j + \delta\int_0^x \lambda''\theta_j dt, \tag{29}$$

or equivalently

$$\delta\theta_{j+1} = \delta(1 - \lambda'|_{t=x})\theta_j + \delta\lambda\theta'_j + \delta\int_0^x \lambda''\theta_j dt. \tag{30}$$

The extreme condition of θ_{j+1} requires that $\delta\theta_{j+1} = 0$. Equating both sides of Eq. (30) to 0, yields the following stationary conditions

$$1 - \lambda'|_{t=x} = 0, \tag{31}$$

$$\lambda|_{t=x} = 0, \tag{32}$$

$$\lambda''|_{t=x} = 0. \tag{33}$$

This in turn gives

$$\lambda = t - x. \tag{34}$$

In general, for the n^{th} order ordinary differential equation, the Lagrange multiplier is given by,

$$\lambda = \frac{(-1)^j}{(j-1)!}(t-x)^{j-1}. \tag{35}$$

Having determined λ and substituting its value into (23) gives the iteration formula

$$\theta_{j+1}(x) = \theta_j(x) + \int_0^x (t-x)\Big(\theta''_j(t) + a\theta'_j(t) + b\theta_j(t) - g(t)\Big)dt, \tag{36}$$

The iteration formula Eq. (36), without restricted variation, should be used for the determination of the successive approximations $\theta_{j+1}(x), j \geqslant 0$, of the solution $\theta(x)$. Consequently, the solution is given by

$$\theta(x) = \lim_{j\to\infty} \theta_j(x). \tag{37}$$

4. Survey of some solutions

In this section we demonstrate the challenge in the construction of exact solution for heat transfer in pin fin. Also, we consider the work in [5].

4.1 Some exact solutions

4.1.1 Example 1

Given the power law thermal conductivity in heat transfer through pin fins, that is in Eq. (3) $k(\theta) = \theta^n$. The model admits four finite symmetry generators. Amongst the others, the two dimensional Lie subalgebra is given by

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \frac{2\theta}{n}\frac{\partial}{\partial \theta}$$

Notice that

$$[X_1, X_2] = X_1,$$

and hence we start the double reduction first with X_1 which implies (τ, r, θ) are invariants and leads to a steady state problem. Hence writing $\theta = F(\tau, r)$ and substitute in the original equation, one obtains

$$F_\tau = E^2 \frac{1}{r} \frac{\partial}{\partial r} (r F^n F_r).$$

X_2 becomes

$$X_2^* = r \frac{\partial}{\partial r} + \frac{2F}{n} \frac{\partial}{\partial F}.$$

This symmetry generator leads to the first order ODE

$$g'(\gamma) = 4E^2 \left(\frac{n+1}{n^2}\right) g(\gamma)^{n+1}.$$

In terms of original variables one obtains the general exact solution

$$\theta(\tau, r) = r^{n/2} \left[-4E^2 \left(\frac{n+1}{n^2}\right) \tau + c_1 \right]^{-1/n}.$$

The difficulty for group-invariant solutions is the satisfaction of the imposed or prescribed boundary conditions. This has been seen in two dimensional steady state problems [7, 16], and 1 + 1 D transient problems [25]. Perhaps the most successful attempt in in [26]. For nonlinear steady state problems, some transformation such as Kirchoff [7, 16], may linearise the two dimensional problems which then becomes easier to solve using standard methods. Linearisation of nonlinear steady state one dimensional problems is possible when thermal conductivity is a differential consequence of heat transfer coefficient [5].

4.1.2 Example 2

In [5], preliminary group classification is invoked to determine the thermal conductivity which lead to exact solutions. It turned out that given a power law heat transfer coefficient, thermal conductivity also takes the power law form. Given Eq. (12) with both $k(\theta)$ and $h(\theta)$ given by θ^n then one obtains the solution

$$\theta(x) = \left[\frac{cosh\ \left(\sqrt{n+1}\ Mx\right)}{cosh\ \left(\sqrt{n+1}\ M\right)} \right]^{1/(n+1)}. \tag{38}$$

The expressions for fin efficiency and effectiveness can be explicit in this case. Furthermore, this solution led to the benchmarking of the approximate analytical solutions [35]. With established confidence in approximate methods, then one may solve other problems that are challenging to solve exactly.

4.2 Some approximate solutions

4.2.1 Three dimensional DTM

In this subsection we consider heat transfer in a cylindrical pin fin. We consider thermal conductivity given as a linear function of temperature $1 + \beta\theta$ and a power law heat transfer coefficient. The three dimensional DTM solution of Eq. (2) is given by

$$\begin{aligned}\theta(\tau,r,z) &= c\tau + c\tau r + c\tau r^2 + c\tau r^3 + c\tau r^4 + c\tau r^5 + c\tau r^6 + c\tau r^7 + \ldots \ldots\\ &+c\tau z^2 - \frac{Bic^{m+1}}{(1+\beta c)}\tau r z^2 - \frac{5c}{2E^2}\tau r^2 z^2 + \frac{10Bic^{m+1}}{9E^2(1+\beta c)}\tau r^3 z^2 + \ldots \ldots\\ &+c\tau z^3 - \frac{Bic^{m+1}}{(1+\beta c)}\tau r z^3 - \frac{9c}{2E^2}\tau r^2 z^3 + \frac{2Bic^{m+1}}{E^2(1+\beta c)}\tau r^3 z^3 + \ldots \ldots\\ &\vdots\end{aligned} \tag{39}$$

One may determine the value of c by invoking the boundary at the base of the fin, as such

$$c\tau + c\tau r + c\tau r^2 + \ldots + c\tau - \frac{Bic^{m+1}}{(1+\beta c)}\tau r - \frac{5c}{2E^2}\tau r^2 \ldots + c\tau - \frac{Bic^{m+1}}{(1+\beta c)}\tau r - \frac{9c}{2E^2}\tau r^2 + \ldots = 1.$$

To plot a three dimensional figure for this solution one may fix temperature, say at $\tau = 0.4$ The results are shown in **Figure 2**.

4.2.2 Two dimensional DTM

The two dimensional DTM solution for a steady heat transfer through the cylindrical fin is given by

$$\theta(r,z) = \quad c - \frac{Bic^{m+1}}{1+\beta c}r + \ldots + cz^2 - \frac{Bic^{m+1}}{1+\beta c}z^2 r + \ldots + cz^3 - \frac{Bic^{m+1}}{1+\beta c}z^3 r + \ldots \tag{40}$$

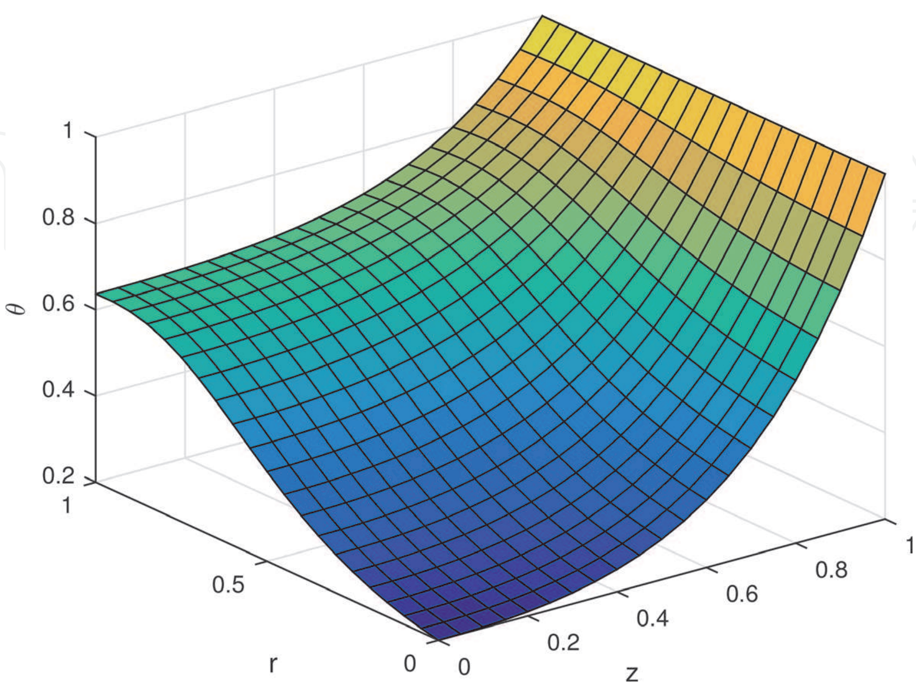

Figure 2.
Approximate analytical solutions for a two-dimensional cylindrical spine fin with a constant thermal conductivity ($\beta = 0$) for $\tau = 0.4$. The parameters are set such that $E = 2$, $Bi = 0.2$, and $m = 3$. (see also, [8]).

and c is obtained from

$$c - \frac{Bic^{m+1}}{1+\beta c} r + \ldots + c - \frac{Bic^{m+1}}{1+\beta c} r - \frac{12c + 18\beta c^2 + 4E^2\beta c^2 + \frac{4E^2\beta Bi^2 c^{2m+2}}{(1+\beta c)^2}}{4E^2(1+\beta c)} r^2 + \ldots = 1.$$

This solution is plotted in **Figure 3**

4.2.3 Comparison of one dimensional exact, DTM and VIM solutions

Here the solutions for the one dimensional heat transfer problems are compared, namely the exact solution given in Eq. (38). The VIM solutions is given by

$$\theta(x) = c + \frac{3c^2M^2x^2}{2} - \frac{3c^5M^2x^2}{2} + \frac{c^7M^2x^2}{2} + \frac{c^5M^4x^4}{8} - \frac{3c^7M^4x^4}{2} + \frac{5c^9M^4x^4}{4} - \frac{c^{11}M^4x^4}{4} + \cdots \tag{41}$$

The constant c may be obtained using the boundary condition at the fin base. The DTM solution is given by (see [35])

$$\theta(x) = c + \frac{3c^2M^2x^2}{2} - \frac{c^2M^2x^2}{6} + \frac{c^2M^2(3-4M^2c)x^4}{48} + \frac{c^2M^2(1-16M^2c)x^5}{240} - \cdots \tag{42}$$

Likewise, the constant c is obtained using the boundary conditions.
These solutions are depicted in **Figure 4**.

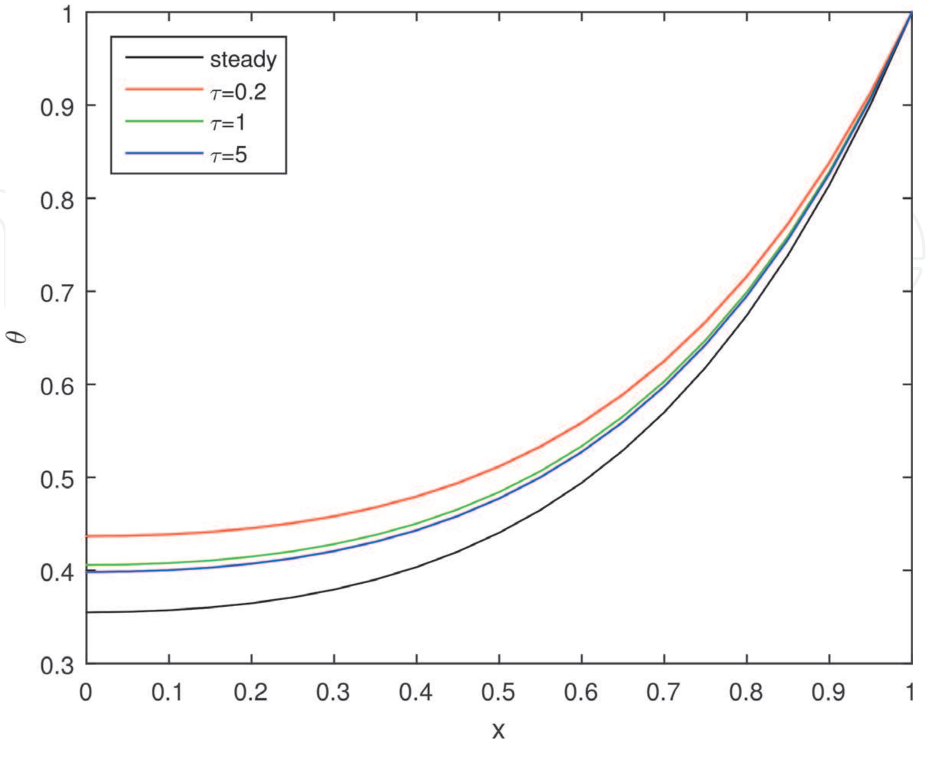

Figure 3.
Approximate analytical solutions for a two-dimensional cylindrical spine fin with a constant thermal conductivity ($\beta = 0$) for $\tau = 0.4$. The parameters are set such that $E = 2$, $Bi = 0.2$, and $m = 3$. (see also, [8]).

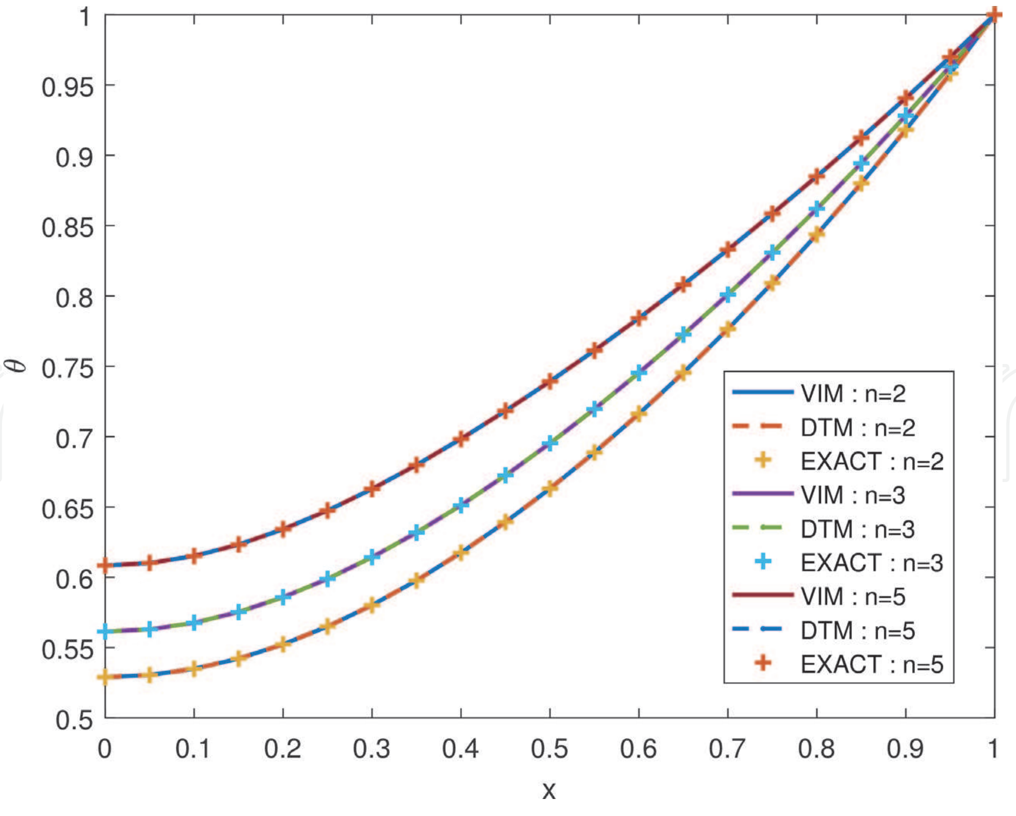

Figure 4.
A temperature distribution in a rectangular fin for varying values of n, $M = 1.7$.(see also [36]).

5. Outlook and some concluding remarks

The interest in heat transfer through fins will continue unabated. This is brought about by applications of fins in engineering appliances. The solutions to the problems give insight into effectiveness and efficiency of different fins. In this chapter we provided a summary of some of the work in recent times. In particular, we reviewed the exact and approximate analytical solutions. We demonstrated that although the models describing hear transfer seem to be simple, they are in fact challenging to solve exactly. When constructed, the exact solutions are used as benchmarks for the approximate solutions. It appears that some models including contracting or expanding have attracted some attention. The analysis of these problems provide insight into heat transfer phenomena and assist in the design of fins. The challenge is the construction of exact solutions, however one may construct approximate analytical solutions. The problems discussed here are not exhaustive.

Conflict of interest

The authors declare no conflict of interest.

Nomenclature

A_c	Cross-sectional area
Bi	Biot number
E	Aspect ratio

h	dimensionless heat transfer coefficient
H	Heat transfer coefficient
H_b	Heat transfer coefficient at the base of the fin
h	Dimensionless thermal conductivity
K_a	Thermal conductivity of the fluid
K	Thermal conductivity of the fin
L	Length of the fin
R	Radius
R_a	Radius
t	time
T_b	Base temperature
T_s	Fluid temperature
x	Dimensionless fin length
X	Fin length
y	Dimensionless fin length
Y	Fin length
Z	Length of a cylindrical pin fin. Greek letters
τ	Dimentionless time
θ	dimensionless temperature

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