Chapter

Deformed Sine-Gordon Models, Solitons and Anomalous Charges

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Abstract

We study certain deformations of the integrable sine-Gordon model (DSG). It is found analytically and numerically several towers of infinite number of anomalous charges for soliton solutions possessing a special space–time symmetry. Moreover, it is uncovered exact conserved charges associated to two-solitons with a definite parity under space-reflection symmetry, i.e. kink-kink (odd parity) and kink-antikink (even parity) scatterings with equal and opposite velocities. Moreover, we provide a linear formulation of the modified SG model and a related tower of infinite number of exact non-local conservation laws. We back up our results with extensive numerical simulations for kink-kink, kink-antikink and breather configurations of the Bazeia *et al.* potential $V_q(w) = \frac{64}{q^2}\tan^2\frac{w}{2}\left(1 - \left|\sin\frac{w}{2}\right|^q\right)^2$, $(q \in R)$, which contains the usual SG potential $V_2(w) = 2[1 - \cos(2w)]$.

Keywords: quasi-integrability, solitons, deformed sine-Gordon, anomalous charges, non-local charges

1. Introduction

Solitons can be regarded as isolated waves that travel without loss of energy. The solitons emerge with their velocities and shapes completely unchanged after collision to each other, the only outcome being their phase shifts. The soliton solution is the main feature of the integrable models [1–3]. However, certain non-linear models in physics, with solitary wave solutions, are not integrable. Recently, certain deformations of integrable models such as the sine-Gordon (SG), nonlinear Schrödinger (NLS), Korteweg-de Vries (KdV) and Toda models have been introduced, such that they exhibit soliton-type solutions with some properties resembling to their counterparts of the truly integrable ones. In this context the so-called quasi-integrability concept has been put forward [4]. These properties have been examined in the frameworks of the anomalous zero-curvature [4–7] and the Riccati-type pseudo-potential approaches [8–10], respectively.

The main developments have been focused on the construction of infinite number of quasi-conservation laws which give rise to asymptotically conserved charges, i.e. conserved charges, such that their values vary during the scattering of the solitons only. The main observation in the both approaches to quasi-integrability is that, in general, the conserved charges of the standard integrable systems turn out to be the so-called asymptotically conserved charges in the deformed models. In fact, the exact conservation laws of the usual integrable systems become quasi-conservation laws of the deformed integrable models. The non-homogeneous terms of the quasi-conservation laws are dubbed as anomalies such that they vanish when integrated on the space–time plane, provided that the fields satisfy a special space–time symmetry.

The properties of the soliton-like configurations in the quasi-integrable models are, so far, largely unknown. We summarize the main results. First, the one-soliton sectors exhibit infinite conserved charges. Second, the space–time integration of the anomalies vanish when one-soliton like solutions are located far away from each other. The anomalies are significant around the space–time regions of their interaction. Third, a sufficient condition for the vanishing of the space–time integrated anomalies is that the $N-$soliton possesses definite parity under a shifted parity and delayed time reversion ($\mathcal{P}_s\mathcal{T}_d$) symmetry. When the anomaly densities possess odd parities the space–time integration of them vanish, which imply the existence of anomalous charges. Fourth, the conserved charges of the usual integrable systems turn out to be the anomalous charges upon deformation. Fifth, there exist infinite towers of infinitely many anomalous charges, different in form from the ones of the usual integrable models. New towers of anomalous charges have been uncovered in [8–10]. Remarkably, even the usual integrable models possess quasi-conservation laws with anomalous charges for analytical $N-$ soliton with $C\mathcal{P}_s\mathcal{T}_d$ symmetry [9, 10]. For the standard SG theory it has been discussed for the 2-soliton sector of the theory [8]. Sixth, there is a subset of exact conserved charges for soliton eigenstates simply of the shifted space-reflection $\mathcal{P}_s$. The deformed NLS model for two-soliton solutions [6, 7] and the deformed sine-Gordon model [11] for two-kink and breather solutions exhibit this property.

In the context of the Riccati-type method there have been shown that the deformed SG, KdV and NLS models [8–10], respectively, possess linear system formulations and that they exhibit infinite towers of exact non-local conservation laws. The NLS-type, KdV-type and SG-type models share the same importance due to their potential applications, since they are ubiquitous in all areas of nonlinear physics, such as Bose-Einsten condensation and superconductivity [12–14], soliton gas and soliton turbulence in fluid dynamics [15–20], the Alice-Bob physics [21, 22] and the understanding of a kind of triality among the gauge theories, integrable models and gravity theories [23].

Here, we discuss the previous results in the field by utilizing a deformed sine-Gordon model. We will introduce the relationship between the space–time parity and asymptotically conserved charges. Next, we clarified on the space-reflection parity related to the linear combination of the dual sets of anomalous quantities. In addition, it is focused on the space-reflection symmetry of some two-soliton solutions of deformed sine-Gordon models. Then one proceeds to construct a tower of exactly conserved charges for each solution possessing a definite space-reflection parity. Lastly, by considering linear combinations of the anomalous conserved charges it is showed, through analytical and numerical methods, that there is a subset of exactly conserved charges.

A modified SG model and the space–time symmetries are presented in the next section. In Section 3, the towers of quasi-conservation laws are presented. In Section 4 our numerical simulations are described. The linear formulation and the non-local conservation laws are discussed in the Riccati-type pseudo-potential approach in Section 5. Finally, in Section 6 we present some conclusions.

2. A deformation of the sine-Gordon model

Let us consider the relativistic field theories in $(1+1)$-dimensions with equation of motion[1]

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[1] In the x and t laboratory coordinates: $\eta = \frac{t+x}{2}, \xi = \frac{t-x}{2}, \partial_\eta = \partial_t + \partial_x, \partial_\xi = \partial_t - \partial_x, \partial_\eta\partial_\xi = \partial_t^2 - \partial_x^2$

$$\partial_\xi \partial_\eta w + V^{(1)}(w) = 0, \tag{1}$$

where w is a real scalar field w, $V(w)$ is the scalar potential and $V^{(1)}(w) \equiv \frac{d}{dw} V(w)$. The family of potentials $V(w)$ will represent certain deformations of the usual SG model. The theory (1) has been studied using the techniques of integrable field theories, such as the anomalous zero-curvature [4, 11] and deformed Riccati-type pseudo-potential formulations [8], respectively. In our simulations we will consider [4, 24].

$$V(w,q) = \frac{2}{q^2} \tan^2 w [1 - |\sin w|^q]^2, \tag{2}$$

where q is a real parameter such that for $q = 2$ the potential reduces to the SG potential

$$V(w,2) = \frac{1}{16}[1 - \cos(4w)]. \tag{3}$$

So, we introduce the deformation parameter ε as $q = 2 + \varepsilon$, such that in the limit $\varepsilon = 0$ one reproduces the SG model.

The model (1) possesses several towers of anomalous charges associated to quasi-conservation laws [4, 8, 11]. In [11] it has been introduced a subset of exactly conserved charges associated to space-reflection eigenstates as kink-antikink, kink-kink and breather configurations, respectively. New types of two sets of dual towers of asymptotically conserved charges have been uncovered [8]. Remarkably, even the usual sine-Gordon models possesses anomalous charges. So far, it is attributed to the space–time symmetry properties of the solitons. Those charges can be relevant in the study of soliton gases and formation of certain structures, such as soliton turbulence, soliton gas dynamics and rogue waves [16].

The quasi-integrability has been introduced for deformed sine-Gordon models such that the field w and the potential V satisfy the symmetry [4, 8, 11].

$$\mathcal{P} : w \to -w + \text{const.}; \quad V(w) \to V(w), \tag{4}$$

under the special space–time reflection

$$\mathcal{P} \equiv \mathcal{P}_s \mathcal{T}_d, \ \mathcal{P}_s : \tilde{x} \to -\tilde{x}, \ \mathcal{T}_d : \tilde{t} \to -\tilde{\tilde{t}}, \ \tilde{x} \equiv x - x_\Delta, \ \tilde{t} = t - t_\Delta, \tag{5}$$

defined around a given point (x_Δ, t_Δ). Moreover, let us consider the space-reflection transformation

$$\mathcal{P}_x : x \leftrightarrow -x, \tag{6}$$

and assume that the scalar field is an eigenstate of the operator P_x

$$\mathcal{P}_x : w \to \varrho\, w, \ \ \varrho = \pm 1. \tag{7}$$

In addition, consider an even potential V under $\mathcal{P}_x$

$$\mathcal{P}_x(V) = V. \tag{8}$$

Several towers of quasi-conservation laws, with anomaly terms possessing odd parities under (6)–(8), have been found [8, 11]. Next, we consider those quasi-conservation laws and examine their anomalies in view of the symmetries (4)–(5) and (6)–(8), respectively.

3. Quasi-conservation laws of the deformed SG model

We will discuss some of the infinite towers of quasi-conservation laws of the deformed SG model (1).

3.1 First type of towers: The SG-type quasi-conservation laws

The usual SG charges turn out to be the anomalous charges of the DSG. So, one has the infinite set of quasi-conservation laws [4, 11].

$$\frac{d}{dt} q_a^{(2n+1)} = \int dx \, \beta^{(2n+1)}, \;\; n = 1, 2, 3, \ldots \tag{9}$$

where the quantities $q_a^{(2n+1)}$ define the anomalous charges, provided that the time-integrated anomalies $\int dt \int dx \, \beta^{(2n+1)}$ vanish for solitons satisfying (4) and (5). This condition, when combined with Eq. (9), implies $q_a^{(2n+1)}(t \to +\infty) = q_a^{(2n+1)}(t \to -\infty)$. So, we have that $q_a^{(2n+1)}$ are anomalous for $n = 1, 2, 3, \ldots$. The charges $q_a^{(2n+1)}$ maintain the same form as the ones of the usual SG.

In $(1+1)$-dimensional Lorentz invariant integrable field theories one has dual integrability conditions or Lax equations. Analogously, for the deformations of the SG model there exist a dual formulation for each equation as in (9) by interchanging $\xi \leftrightarrow \eta$ in the procedure to obtain the relevant quasi-conservation laws. So, one can get

$$\frac{d}{dt} \tilde{q}_a^{(2n+1)} = \int dx \, \tilde{\beta}^{(2n+1)}, \;\; n = 1, 2, 3, \ldots \tag{10}$$

where the quantities $\tilde{q}_a^{(2n+1)}$ define the dual asymptotically conserved charges, provided that the time-integrated anomalies $\int dt \int dx \, \tilde{\beta}^{(2n+1)}$ vanish. Likewise, this result implies $\tilde{q}_a^{(2n+1)}(t \to +\infty) = \tilde{q}_a^{(2n+1)}(t \to -\infty)$.

These towers of quasi-conservation laws reproduce the same polynomial form as in the usual sine-Gordon charge densities. In fact, the anomalies $\beta^{(2n+1)}$ and $\tilde{\beta}^{(2n+1)}$ vanish identically provided that the deformed potential $V(w)$ recovers the form of the standard SG potential.

The importance and the relevance of such a dual construction will become clear below when the linear combinations of the charges in (9) and (10) give rise to infinite towers of exactly conserved charges, provided that the space-integral of the linear combination of the anomaly densities $\beta^{(2n+1)}$ and $\tilde{\beta}^{(2n+1)}$ vanish for special two-soliton solutions.

3.1.1 Space-reflection parity and conserved charges

The above dual sets of quasi-conservation laws are used to construct a sequence of conserved charges and vanishing anomalies. The space-reflection symmetry of some soliton solutions of the deformed SG model will imply the existence of an infinite tower of conserved charges. So, let us examine a linear combination, at each order $n = 1, 2, \ldots$, of the above two sets of quasi-conserved charges $q_a^{(2n+1)}$ (9) and $\tilde{q}_a^{(2n+1)}$ (10). Consider the new quasi-conservation laws

$$\frac{d}{dt} q_{a,\pm}^{(2n+1)} = -\int dx \, \beta_{\pm}^{(2n+1)}, \;\; n = 1, 2, \ldots, \tag{11}$$

with the charges $q_{a,\pm}^{(2n+1)}$ and anomalies $\beta_{\pm}^{(2n+1)}$, respectively, defined as

$$q_{a,\pm}^{(2n+1)} \equiv \mp \frac{1}{16}\left(q_a^{(2n+1)} \pm \tilde{q}_a^{(2n+1)}\right), \tag{12}$$

$$\beta_{a,\pm}^{(2n+1)} \equiv \mp \frac{1}{16}\left(\beta^{(2n+1)} \pm \tilde{\beta}^{(2n+1)}\right) \tag{13}$$

in which the quantities $q^{(2n+1)}$ and $\beta^{(2n+1)}$ defined in (9) and the quantities $\tilde{q}_a^{(2n+1)}$ and $\tilde{\beta}^{(2n+1)}$ in (10) have been used, respectively.

Since the theory (1) is invariant under space–time translations one has that the energy momentum tensor is conserved. In fact, one has $\beta^{(1)} = \tilde{\beta}^{(1)} = 0$ at the zero'th order $n = 0$, and the linear combinations of the charges $q_a^{(1)}$ and $\tilde{q}_a^{(1)}$ leads to the energy and momentum, respectively [11].

$$q_+^{(1)} = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}(\partial_t w)^2 + \frac{1}{2}(\partial_x w)^2 + V\right], \tag{14}$$

$$q_-^{(1)} = \int_{-\infty}^{+\infty} dx\, \partial_x w \partial_t w, \tag{15}$$

where $E = q_+^{(1)}$ is the energy and $P = q_-^{(1)}$ is the momentum.

The first non-trivial anomalies become [11].

$$\beta_{\pm}^{(3)} = \pm\frac{1}{2} Z\left\{\partial_\xi\left[(\partial_\xi w)^2\right] \mp \partial_\eta\left[(\partial_\eta w)^2\right]\right\}, \quad Z \equiv V^{(2)} + 16V - 1. \tag{16}$$

$$\beta_{\pm}^{(5)} = \pm\frac{1}{2} Z\left[\left(24(\partial_\xi w)^2 \partial_\xi^2 w + \partial_\xi^4 w\right)\partial_\xi w \pm \left(24(\partial_\eta w)^2 \partial_\eta^2 w + \partial_\eta^4 w\right)\partial_\eta w\right]. \tag{17}$$

Notice that for the SG potential (3) the factor Z above vanishes identically; therefore, the anomalies vanish $\beta_{\pm}^{(3)} = 0$, and the relevant charges $q_{\pm}^{(3)}$ turn out to be the exactly conserved charges of the standard SG model at this order.

The properties of the quantities $q_{\pm}^{(2n+1)}$ and $\int dx \beta_{\pm}^{(2n+1)}$ in (11) will depend on the symmetry properties of the solitons, in particular on the space-reflection symmetry of $\beta_{\pm}^{(2n+1)}$, as we will see below. So, let us examine the space-reflection symmetry of them.

Let us write the anomalies in terms of the ∂_x and ∂_t derivatives. So, once the eq. of motion (1) is used to substitute $\partial_t^2 w \to \left[\partial_x^2 w - V'(w)\right]$, as well as, neglecting surface terms one has

$$\alpha_+^{(3)} \equiv -2\int dx f_+^{(3)}(x,t), \tag{18}$$

$$f_+^{(3)}(x,t) \equiv \left[V'' + 16V\right]\left\{\partial_x\left[(\partial_t w)^2\right] + \partial_x\left[(\partial_x w)^2\right]\right\}, \tag{19}$$

where we have defined the anomaly density $f_+^{(3)}$. Notice that for even parity potentials (8) and for definite parity (even or odd) fields w the density $f_+^{(3)}$ is an odd function, and thus the $x-$integrated anomaly $\alpha_+^{(3)}$ vanishes.

Following analogous procedure as above one has

$$\alpha_-^{(3)} = -4\int dx f_-^{(3)}(x,t), \tag{20}$$

$$f_{-}^{(3)}(x,t) \equiv \left[V'' + 16V\right]\left\{\partial_t w \partial_x^2 w + \partial_x w \partial_x \partial_t w\right\}, \tag{21}$$

where we have defined the anomaly density $f_{-}^{(3)}$. Notice that for even parity potentials (8) and for definite parity (even or odd) fields w the density $f_{-}^{(3)}$ is an even function, and thus the $x-$integrated anomaly $\alpha_{-}^{(3)}$ will not vanish solely by a space-reflection parity reason.

The anomalies $\alpha_{\pm}^{(3)}$ and $\int dt\, \alpha_{\pm}^{(3)}$ in (18) and (20), will be computed numerically for two-solitons and breather-like solutions below.

By direct construction it has been found new towers of anomalous charges in [8]. In the next subsections we will discuss those charges and anomalies in relation to the symmetry (4) and (5).

3.2 Second type of towers

The quasi-conservation laws [8].

$$\frac{d}{dt} Q_a^{(N)} = \mathrm{a}^{(N)}, \tag{22}$$

$$Q_a^{(N)} \equiv \int dx \left[\frac{1}{N}(\partial_\xi w)^N + V(\partial_\xi w)^{N-2}\right], \tag{23}$$

$$\mathrm{a}^{(N)} \equiv \int dx\, (N-2)(\partial_\xi w)^{N-3} \partial_\xi^2 w V, \quad N \geq 3, \tag{24}$$

define the asymptotically conserved charges $Q_a^{(N)}$ and the corresponding anomalies $\mathrm{a}^{(N)}$.

The dual quasi-conservation laws become

$$\frac{d}{dt} \tilde{Q}_a^{(N)} = \tilde{\mathrm{a}}^{(N)}, \tag{25}$$

$$\tilde{Q}_a^{(N)} \equiv \int dx \left[\frac{1}{N}\left(\partial_\eta w\right)^N + V\left(\partial_\eta w\right)^{N-2}\right], \tag{26}$$

$$\tilde{\mathrm{a}}^{(N)} \equiv \int dx\, (N-2)\left(\partial_\eta w\right)^{N-3} \partial_\eta^2 w V, \quad N \geq 3, \tag{27}$$

where we have introduced the dual asymptotically conserved charges $\tilde{Q}_a^{(N)}$ and the relevant anomalies $\tilde{\mathrm{a}}^{(N)}$.

The densities of the anomalies $\mathrm{a}^{(N)}$ and $\tilde{\mathrm{a}}^{(N)}$ in (24) and (27), respectively, possess odd parities under (4) and (5), so the quasi-conservation laws (22) and (25), respectively, allow the construction of asymptotically conserved charges.

3.3 Third type of towers

Let us define the quasi-conservation laws [8].

$$\frac{d}{dt} Q_a^{(N)} = \gamma^{(N)}, \tag{28}$$

$$Q_a^{(N)} \equiv \int dx \left[\frac{1}{2} V^{N-1} (\partial_\xi w)^2 + \frac{1}{N} V^N\right], \tag{29}$$

$$\gamma^{(N)} \equiv \int dx \, \frac{1}{2} (\partial_\xi w)^2 \partial_\eta V^{N-1}, \quad N \geq 2, \tag{30}$$

where we have introduced the asymptotically conserved charges $\hat{Q}_a^{(N)}$ and the corresponding anomalies $\gamma^{(N)}$.

The interchange $\eta \leftrightarrow \xi$ allows us to reproduce the dual quasi-conservation laws. So, one has

$$\frac{d}{dt} \tilde{Q}_a^{(N)} = \tilde{\gamma}^{(N)}, \tag{31}$$

$$\tilde{Q}_a^{(N)} \equiv \int dx \left[\frac{1}{2} V^{N-1} (\partial_\xi w)^2 + \frac{1}{N} V^N \right], \tag{32}$$

$$\tilde{\gamma}^{(N)} \equiv \int dx \, \frac{1}{2} \left(\partial_\eta w\right)^2 \partial_\xi V^{N-1}, \quad N \geq 2, \tag{33}$$

where we have defined the dual asymptotically conserved charges $\tilde{Q}_a^{(N)}$ and the anomalies $\tilde{\gamma}^{(N)}$.

Similarly, the densities of the anomalies $\gamma^{(N)}$ and $\tilde{\gamma}^{(N)}$ in (30) and (33), respectively, possess odd parities under (4) and (5), so the quasi-conservation laws (28) and (31), respectively, allow the construction of asymptotically conserved charges.

The relevant anomalies of the lowest order quasi-conservation laws of the above towers will be simulated below for 2-soliton interactions.

Remarkably, the above charges turn out to be anomalous even for the standard sine-Gordon model. In fact, the relevant 2-soliton solutions have been constructed analytically [4, 11] which possess a definite parity under (4)–(5), such that the odd anomaly densities vanish upon space–time integration. The usual explanation for the appearance of novel anomalous charges in the standard sine-Gordon model is the symmetry argument. The anomalous charges also appear in the standard KdV and its deformations [9].

These charges have been computed for soliton collisions in the treatment of soliton gases and formation of some structures in integrable systems, such as integrable turbulence and rogue waves. In the context of the usual KdV model it has been analyzed the behavior of the statistical moments defined by (see e.g. [16, 17]) $M_n(t) = \int_{-\infty}^{+\infty} v^n \, dx, n \geq 1$; where v is the KdV field. The $M_{1,2}$ cases are conserved charges. It is remarkable that the moments, $M_{3,4}$, respectively, in the interaction region of two-solitons, behave as the anomalous charges of the quasi-integrable KdV models [9]. In fact, in the quasi-integrable KdV models the moments $M_{2,3}$ are in fact anomalous charges [9]. So, since the two-soliton collision is an important ingredient in the formation of soliton turbulence and the dynamics of soliton gases, we can expect they will be important in the quasi-integrable counterparts. In the case of the SG soliton ensemble, to our knowledge, it is needed a further theoretical research.

4. Numerical simulations

Here we will check numerically the lowest order expressions of the various towers of quasi-conservation laws presented above. For this purpose we will numerically solve the Eq. (1) with the particular deformed potential (2). In the **Figures 1** and **2** we plot the kink-kink and kink-antikink collisions, respectively. Moreover, we show the first conserved charges, i.e. the energy and momentum for these field configurations.

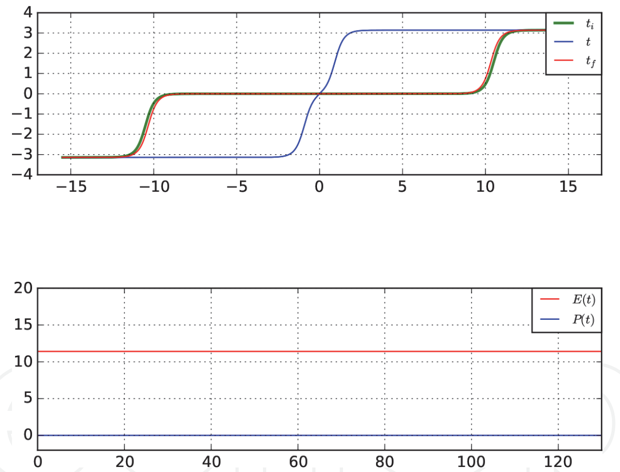

Figure 1.
Kink-kink with velocities $v_2 = -v_1 = 0.15$ and $q = 2.01$ in (2), for initial (green), collision (blue) and final (red) times. Bottom, the energy (E) and momentum (P) charges of the kink-kink.

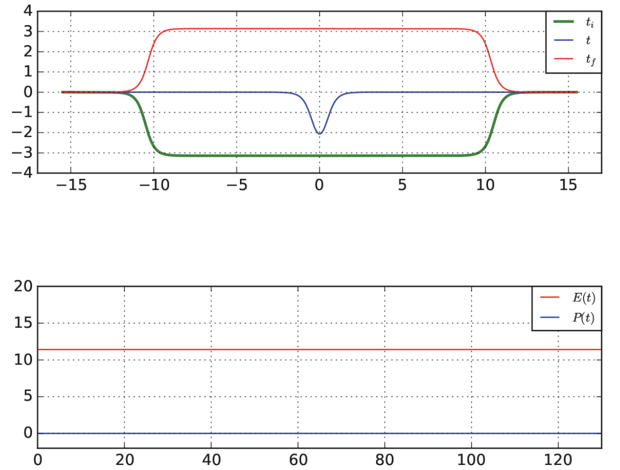

Figure 2.
Kink-antikink with velocities $v_2 = -v_1 = 0.15$ and $q = 2.01$ in (2), for initial (green), collision (blue) and final (red) times. Bottom, the energy (E) and momentum (P) charges of the kink-antikink.

4.1 First non-trivial anomalies of the SG-type quasi-conservation laws

We have checked our results by numerical simulation of the anomalies $\alpha_{\pm}^{(3)}$ (18)–(21) for kink-antikink, kink-kink and breather solutions of the model (2).

So, let us write (11) in the form

$$q_{a,\pm}^{(3)}(t) - q_{a,\pm}^{(3)}(t_0) = -\int_{t_0}^{t} dt \alpha_{\pm}^{(3)}(t), \tag{34}$$

where $\alpha_{\pm}^{(3)}(t)$ were defined in (18) and (20) and t_0 is the initial time.

The simulations of the kink-antikink, kink-kink and breather systems of the deformed SG model will consider, as the initial condition, two analytic solitary wave solutions presented in Eq. (1.2) of [4], located some distance apart and stitched together at the middle point.

4.1.1 Kink-antikink

In the **Figures 3** and **4** we show the results for kink-antikink system with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The plots of (19) and (21) as $f_{\pm}^{(3)}(x,t)\, vs\, x$ are

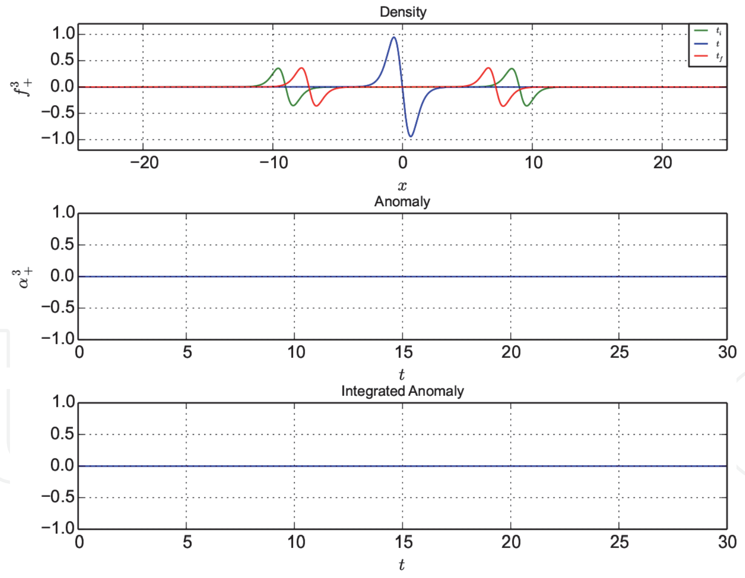

Figure 3.
$f_+^{(3)}$, $\alpha_+^{(3)}$ and $\int dt\alpha_+^{(3)}$ in (18) and (19) for kink-antikink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-antikink.

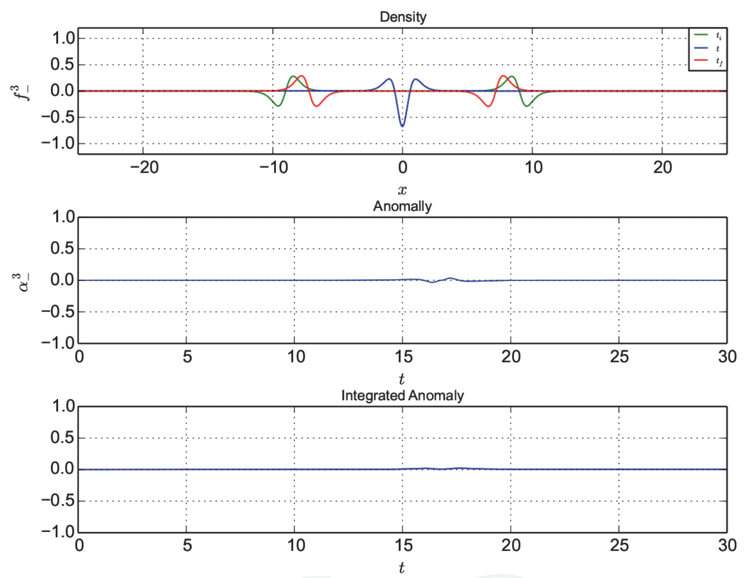

Figure 4.
$f_-^{(3)}$, $\alpha_-^{(3)}$ and $\int dt\alpha_-^{(3)}$ in (20)–(21) for kink-antikink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-antikink.

shown for three successive times (top figures). Their integration in space furnish vanishing $\alpha_+^{(3)}(t)$ and non-vanishing $\alpha_-^{(3)}(t)$ (middle figures). The bottom figures show $\int dt'\alpha_+^{(3)}(t')$, vanishing in **Figure 3** and $\int dt'\alpha_-^{(3)}(t')$ asymptotically vanishing in **Figure 4**, respectively. According to (34) our numerical simulations show the asymptotically conservation of the charge $q_{a,-}^{(3)}$ and the exact conservation of the charge $q_{a,+}^{(3)}$, within numerical accuracy.

4.1.2 kink-kink

In the **Figures 5** and **6** we show the results for kink-kink system with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The plots of (19) and (21) as $f_{\pm}^{(3)}(x,t) vs x$ are shown for three successive times (top figures). Their integration in space furnish vanishing $\alpha_+^{(3)}(t)$ and non-vanishing $\alpha_-^{(3)}(t)$ (middle figures). The bottom figures show

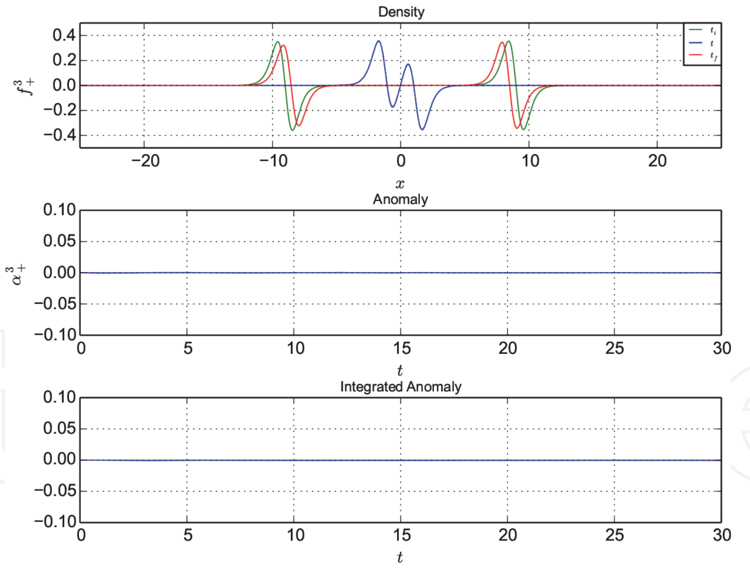

Figure 5.
$f_+^{(3)}$, $\alpha_+^{(3)}$ and $\int dt\alpha_+^{(3)}$ in (18) and (19) for kink-kink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-kink.

$\int dt'\alpha_+^{(3)}(t')$, vanishing in **Figure 5** and $\int dt'\alpha_-^{(3)}(t')$ asymptotically vanishing in **Figure 6**. According to (34) our numerical results show the asymptotically conservation of the charge $q_{a,-}^{(3)}$ and the exact conservation of the charge $q_{a,+}^{(3)}$, within numerical accuracy.

So, one can conclude that for kink-antikink (kink-kink) solution the definite parity related to the space-reflection symmetry is a necessary condition in order to achieve a conserved $q_{a,+}^{(3)}$ charge, within numerical accuracy.

The both kink-antikink and kink-kink solitons of the SG model with opposite and different velocities do not possess the required parity symmetry. However, it has been shown that in the center-of-mass reference frame (x', t') the parity symmetries are recovered, as discussed in [11]. So, the simulations performed in these reference frames, in the both kink-antikink and kink-kink cases, will provide vanishing $\alpha_+^{(3)}$ anomalies as shown above.

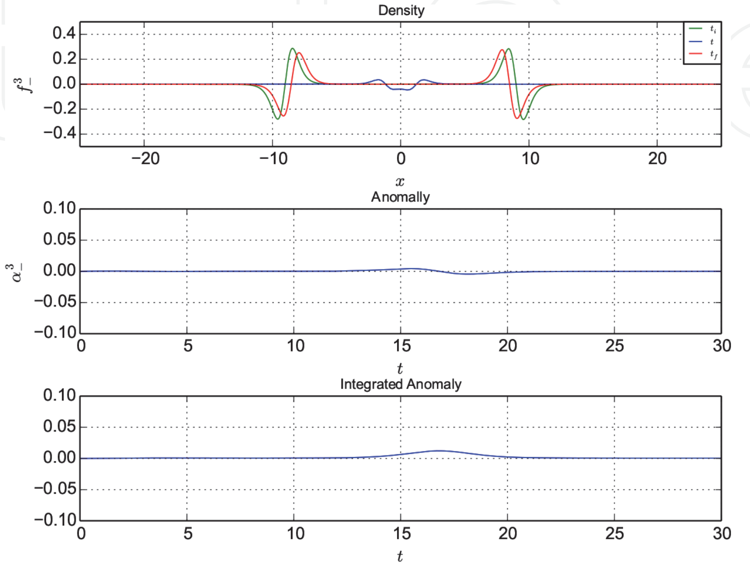

Figure 6.
$f_-^{(3)}$, $\alpha_-^{(3)}$ and $\int dt\alpha_-^{(3)}$ in (20) and (21) for kink-kink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-kink.

4.1.3 Breather: kink-antikink bound state

Figures 7 and **8** show the results for breather (kink-antikink bound state) with $\varepsilon = 0.06$. The densities $f_{\pm}^{(3)}(x,t)$ in (19) and (21), respectively, have been plotted as functions of x for three successive times (top figures). They show the vanishing $\alpha_{+}^{(3)}(t)$ and non-vanishing (periodic in time) $\alpha_{-}^{(3)}(t)$ (middle figures). The bottom figures of **Figures 7** and **8** show the vanishing $\int dt' \alpha_{+}^{(3)}(t')$ and periodic $\int dt' \alpha_{-}^{(3)}(t')$ expressions. According to (34) our numerical results show the oscillation of the charges $q_{a,-}^{(3)}$ around a fixed value and the exact conservation of the charge $q_{a,+}^{(3)}$, within numerical accuracy.

4.2 Lowest order anomalies of the second and third types of towers

We will compute the linear combinations of the lowest order anomalies of the second and third types of towers in (22)–(27) and (28)–(33), respectively,

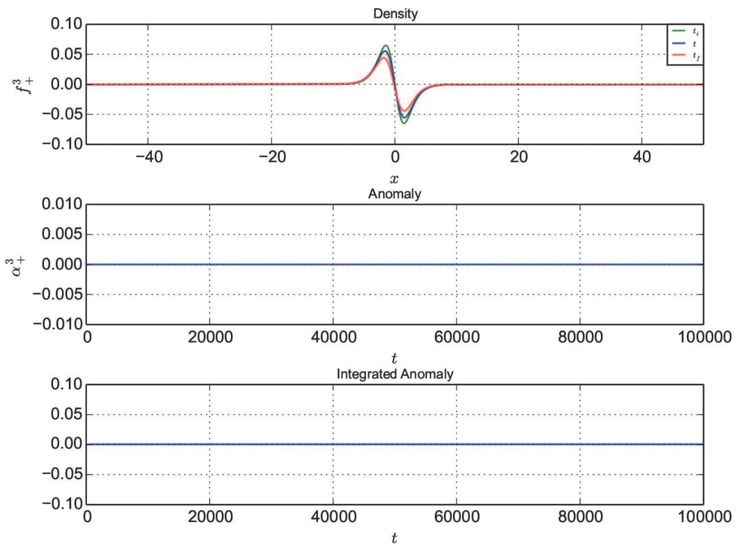

Figure 7.
$f_{+}^{(3)}$, $\alpha_{+}^{(3)}$ and $\int dt \alpha_{+}^{(3)}$ in (18) and (19) for breather with $\varepsilon = 0.06$. The density is shown for three times $t \in [t_f - T_0, t_f]$, $T_0 = 7.025$. The long-lived breather for $t_f \approx 10^5$.

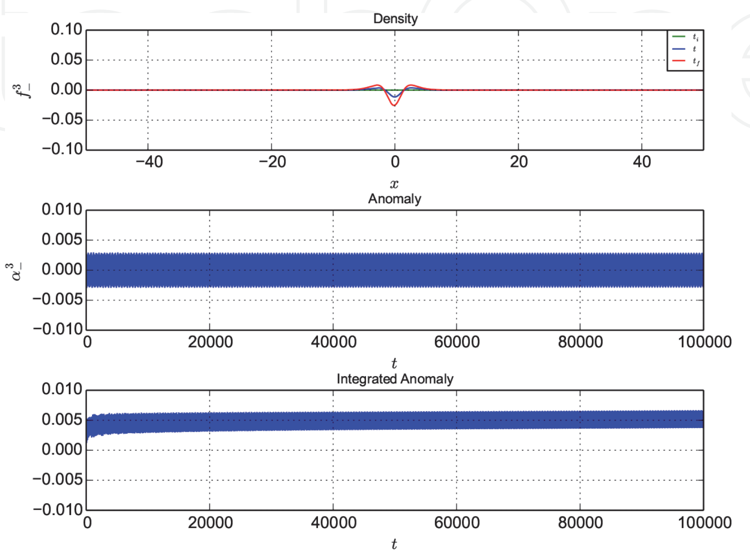

Figure 8.
$f_{-}^{(3)}$, $\alpha_{-}^{(3)}$ and $\int dt \alpha_{-}^{(3)}$ in (20) and (21) for breather with $\varepsilon = 0.06$. The density is shown for three times $t \in [t_f - T_0, t_f]$, $T_0 = 7.025$. The long-lived breather for $t_f \approx 10^5$.

$$\mathrm{a}_{\pm} \equiv \mathrm{a}^{(3)} \pm \tilde{\mathrm{a}}^{(3)}, \tag{35}$$

$$\gamma_{\pm} \equiv \gamma^{(2)} \pm \tilde{\gamma}^{(2)}. \tag{36}$$

4.2.1 Second and third types of towers and lowest order anomalies

The two anomalies in (35) can be written as

$$\mathrm{a}_{+} = \int dx\, 2\left[\partial_t^2 w + \partial_x^2 w\right] V, \tag{37}$$

$$\mathrm{a}_{-} = \int dx\, 4\left[\partial_t \partial_x w\right] V. \tag{38}$$

Similarly, the two anomalies in (36) can be written as

$$\gamma_{+} = \int dx \left[(\partial_t w)^2 - (\partial_x w)^2\right] \partial_t V, \tag{39}$$

$$\gamma_{-} = -\int dx \left[(\partial_t w)^2 - (\partial_x w)^2\right] \partial_x V. \tag{40}$$

Notice that under the space–time reflection transformation (4) and (5), the densities of the above anomalies $\mathrm{a}_{\pm}^{(3)}$ and $\gamma_{\pm}$, respectively, are odd; then they must vanish upon space–time integration. Therefore, one has asymptotically conserved charges associated to the relevant quasi-conservation laws.

Under the space-reflection symmetry (6) and (8), some of the densities of the above anomalies will present odd parities; therefore, they must vanish upon space integration. So, in such cases one can have exact conserved charges. These results will be verified for certain solutions as we will see below in the numerical simulations for the kink-kink and kink-antikink solutions.

Figures 9–**12** show the anomalies $\mathrm{a}_{\pm}$ and $\gamma_{\pm}$ and their corresponding densities. The anomalies a_{-} and γ_{-} vanish as shown in the **Figures 9** and **10**, respectively, for symmetric kink-antikink soliton (see **Figure 2**), within numerical accuracy, since their densities are odd under space reflection. Similarly, for anti-symmetric kink-kink soliton (see **Figure 1**) the anomalies a_{+} and γ_{-} vanish in the **Figures 11** and **12**, respectively, since their densities are odd under space reflection.

These results suggest that the quasi-integrable models set forward in the literature [4, 6, 7], and in particular the model (1), would possess more specific integrability structures, such as an infinite set of exactly conserved charges, and some type

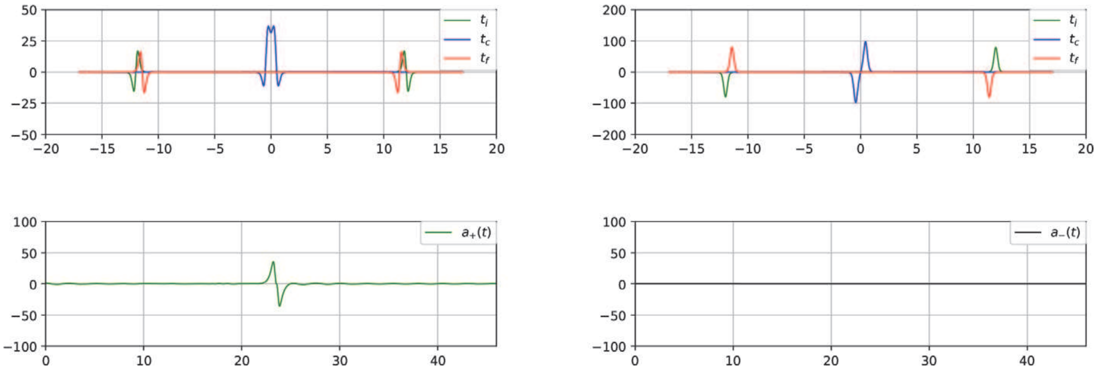

Figure 9.
*Top: The anomaly densities (37) and (38), respectively, plotted in x−coordinate for three times t_i(green), t_c (blue) and t_f (red). Bottom: The anomalies $a_{\pm}$ vs t, for kink-antikink collision shown in **Figure 2**.*

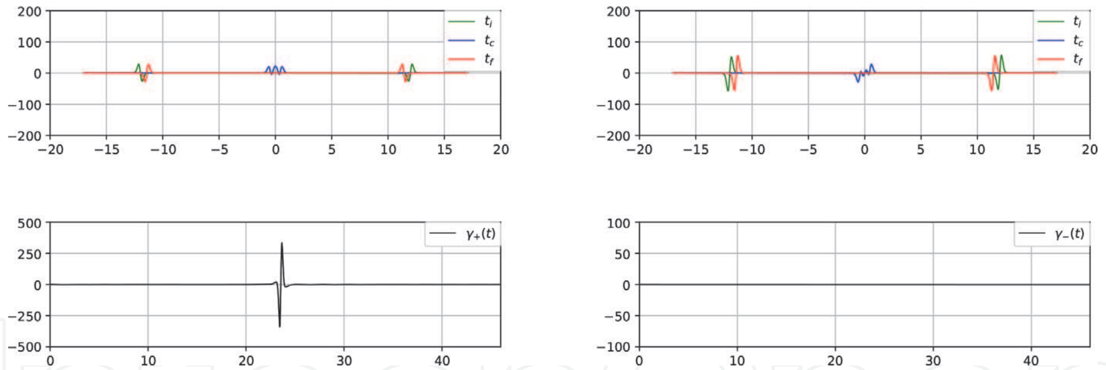

Figure 10.
*Top: Anomaly densities (39) and (40), respectively, plotted in x−coordinate for three times t_i(green), t_c(blue) and t_f (red). Bottom: Anomalies $\gamma_\pm$ vs t, for kink-antikink shown in **Figure 2**.*

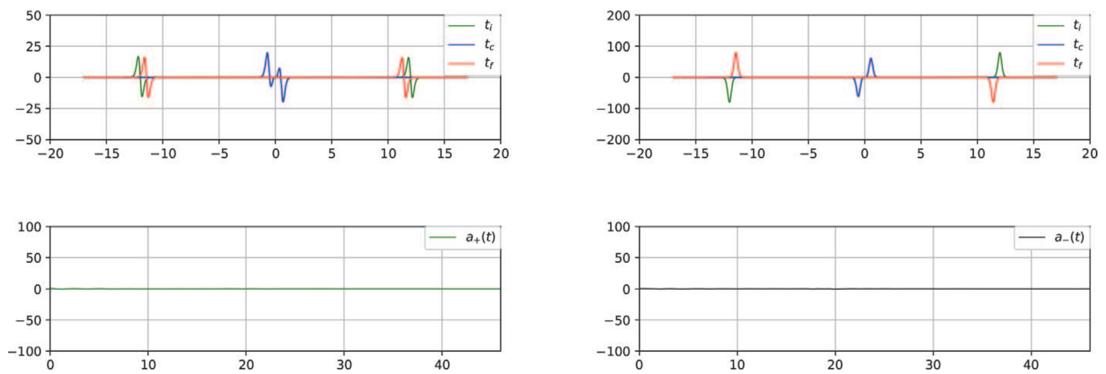

Figure 11.
*Top: Anomaly densities of (37) and (38), respectively, plotted in x−coordinate for three successive times t_i(green), t_c(blue) and t_f (red). Bottom figures show the relevant anomalies $a_\pm$ vs t, for kink-kink shown in **Figure 1**.*

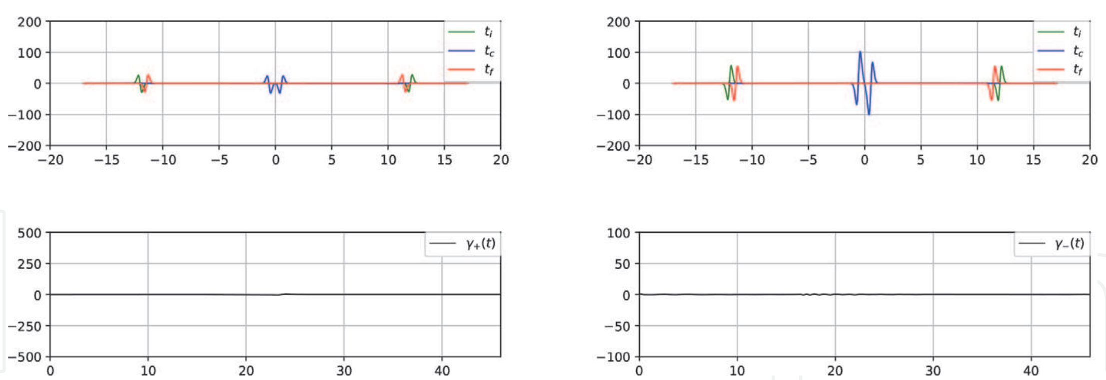

Figure 12.
*Top: Anomaly densities of (39) and (40), respectively, plotted in x−coordinate for three successive times t_i (green), t_c(blue) and t_f (red). Bottom: Anomalies $\gamma_\pm$ vs t, for kink-kink shown in **Figure 1**.*

of linear formulations for certain deformed potentials. So, in the next section we will tackle the problem of extending the Riccati-type pseudo-potential formalism to the deformed sine-Gordon model (1).

5. Riccati-type pseudo-potentials and non-local conservation laws

The Lax equations and Backlund transformations, as well as the conservation laws for the well-known non-linear evolution equations can be generated from the pseudo-potentials and the properties of the Riccati Equation [25–29].

So, in the next steps we consider a convenient deformation of the usual pseudo-potential approach to integrable field theories. Let us consider the system of Riccati-type equations

$$\partial_\xi u = -2\lambda^{-1} u + \partial_\xi w + \partial_\xi w\, u^2, \tag{41}$$

$$\partial_\eta u = -2\lambda(V-2)u - \frac{1}{2}\lambda V^{(1)} + \frac{1}{2}\lambda V^{(1)} u^2 + \psi, \tag{42}$$

and the next linear first order equation for ψ

$$\partial_\xi \psi + 2\lambda^{-1}\psi - 2u\partial_\xi w\,\psi = (2\lambda - 2u - \lambda\,\partial_\xi u)Z, \quad Z \equiv V^{(2)}(w) + 16V(w) - 1. \tag{43}$$

The compatibility condition $\partial_\eta(\partial_\xi u) - \partial_\xi(\partial_\eta u) = 0$ of the system (41) and (42), taking into account (43), provides the equation of motion of the DSG model (1). Moreover, the ordinary differential equation for ψ in the variable ξ can be integrated by quadratures [8]. Its expression will become highly non-local and, once inserted into (42), the system of Eqs. (41) and (42) will provide a non-local Riccati-type representation of the DSG model (1).

From the system (41) and (42) one can get a quasi-conservation law

$$\partial_\eta(u\,\partial_\xi w) + \partial_\xi\left(\lambda(V-2) - \frac{1}{2}\lambda u\, V^{(1)}\right) = -\lambda\partial_\xi w u Z + \partial_\xi w \psi. \tag{44}$$

This equation has been used to construct a tower of infinite number of quasi-conservation laws [8]. For the standard SG one has $Z = \psi = 0$; so the Eq. (44) can generate the well known conservation laws of the usual SG model.

5.1 Pseudo-potentials and a linear system associated to DSG

In this section we search for a linear system formulation of the DSG model. It is achieved by taking into account the Riccati Eq. (41) and the conservation law (44), as well as the Eq. (43). So, the following system of equations has been proposed as a linear formulation of the deformed SG model [8].

$$\mathcal{L}_1\Phi = 0, \qquad \mathcal{L}_2\Phi = 0, \tag{45}$$

$$\mathcal{L}_1 \equiv \partial_\xi - A_\xi, \;\; A_\xi \equiv \frac{\lambda}{2}(\partial_\xi w)^2 - 2\frac{(\partial_\xi w)^3}{\partial_\xi^2 w}, \tag{46}$$

$$\mathcal{L}_2 \equiv \partial_\eta - A_\eta, \;\; A_\eta \equiv -2\lambda - \lambda V + \zeta, \tag{47}$$

where the auxiliary non-local field ζ is defined as

$$\zeta = \int d\xi' \left[6V^{(1)} \frac{(\partial_{\xi'} w)^2}{\partial_{\xi'}^2 w} - 2V^{(2)} \frac{(\partial_{\xi'} w)^4}{\left(\partial_{\xi'}^2 w\right)^2} \right]. \tag{48}$$

In fact, taking into account the expression for the auxiliary field ζ, the compatibility condition of the linear problem (45) provides the equation

$$\Delta(\xi,\eta)\lambda - 6\frac{\partial_\xi w}{\partial_\xi^2 w}\Delta(\xi,\eta) + 2\frac{(\partial_\xi w)^2}{\left(\partial_\xi^2 w\right)^2}\partial_\xi \Delta(\xi,\eta) = 0, \tag{49}$$

with

$$\Delta(\xi,\eta) \equiv \partial_\xi \partial_\eta w + V^{(1)}(w). \tag{50}$$

In (49) the coefficient of the linear term in $\lambda\, \Delta(\xi,\eta)$ must vanish, providing the DSG equation of motion (1). The other terms in (49) must also vanish provided that $\Delta(\xi,\eta) = 0$ is imposed. So, $\mathcal{L}_1$ and $\mathcal{L}_2$ in (45) become a pair of linear operators associated to the DSG model (1).

5.2 Non-local conservation laws

For non-linear equations, not necessarily integrable, which can be derived from a compatibility condition of an associated linear system with spectral parameter, explicit expressions of local and non-local currents can be obtained (see e.g. [30, 31]). In the non-linear $\sigma-$model the non-local conserved charges imply the absence of particle production and the first non-trivial one alone fixes almost completely the on-shell dynamics of the model (see e.g. [3, 32]). These charges may be constructed through an iterative procedure [33]. Following this method one gets a set of infinite number of non-local conservation laws for the system (45). In fact, this system satisfies the properties: i) (A_ξ, A_η) is a "pure gauge"; i.e. $A_\mu = \partial_\mu \Phi \Phi^{-1}, \mu = \xi, \eta$; ii) $J_\mu = (A_\xi, A_\eta)$ is a conserved current satisfying

$$\partial_\eta A_\xi - \partial_\xi A_\eta = 0. \tag{51}$$

So, one can construct an infinite set of non-local conserved currents through an inductive procedure. Let us define the currents

$$J_\mu^{(n)} = \partial_\mu \chi^{(n)},\ \mu \equiv \xi, \eta;\ \ n = 0, 1, 2, \ldots \tag{52}$$

$$d\chi^{(1)} = A_\xi d\xi + A_\eta d\eta \equiv dI_0(\xi,\eta) + \lambda dI_1(\xi,\eta); \tag{53}$$

$$J_\mu^{(n+1)} = \partial_\mu \chi^{(n)} - A_\mu \chi^{(n)};\ \ \chi^{(0)} = 1, \tag{54}$$

where

$$dI_0(\xi,\eta) \equiv a_0(\xi,\eta)d\xi + b_0(\xi,\eta)d\eta,\ \ dI_1(\xi,\eta) \equiv a_1(\xi,\eta)d\xi + b_1(\xi,\eta)d\eta, \tag{55}$$

where

$$a_0 \equiv -2\frac{(\partial_\xi w)^3}{\partial_\xi^2 w};\ b_0 \equiv \zeta = \int d\xi' \left[6V^{(1)} \frac{(\partial_{\xi'} w)^2}{\partial_{\xi'}^2 w} - 2V^{(2)} \frac{(\partial_{\xi'} w)^4}{\left(\partial_{\xi'}^2 w\right)^2}\right]; \tag{56}$$

$$a_1 \equiv \frac{1}{2}(\partial_\xi w)^2; \quad b_1 \equiv -2 - V. \tag{57}$$

Then one can show by an inductive procedure that the (non-local) currents $J_\mu^{(n)}$ are conserved

$$\partial_\mu J^{(n)\mu} = 0,\ \ n = 1, 2, 3, \ldots, +\infty. \tag{58}$$

The first current conservation law $\partial_\mu J^{(1)\mu} = 0$ reduces to the Eq. (51), and then provides the first two conservation laws

$$\partial_\eta a_0 - \partial_\xi b_0 = 0, \quad \partial_\eta a_1 - \partial_\xi b_1 = 0. \tag{59}$$

The next conservation law $\partial_\mu J^{(2)\mu} = 0$, in powers of λ, furnishes

$$\partial_\eta(a_0 I_0) - \partial_\xi(b_0 I_0) = 0, \tag{60}$$

$$\partial_\eta(a_0 I_1 + a_1 I_0) - \partial_\xi(b_0 I_1 + b_1 I_0) = 0, \tag{61}$$

$$\partial_\eta(a_1 I_1) - \partial_\xi(b_1 I_1) = 0. \tag{62}$$

The construction of analogous linear systems have been performed for deformations of the KdV and NLS models [9, 10]. The construction of the classical Yangian as a Poisson-Hopf type algebra [34] for those non-local currents is worth to pursue in a future work.

6. Conclusions

Our work presents an in-depth demonstration of the quasi-integrability property of the modified sine-Gordon models and the presence of several towers of infinite number of asymptotically conserved charges for soliton configurations satisfying the space–time symmetry (4) and (5). In addition, it is observed that there exist a subset of towers of infinite number of exactly conserved charges, provided that some two-soliton configurations are eigenstates (even or odd) of the space-reflection symmetry (6)–(8).

Moreover, we have uncovered a linear system formulation (45) of the modified SG model, and an infinite set of exact non-local conservation laws (58) associated to that linear formulation.

The space–time and internal symmetries related to quasi-integrability deserve further investigations, due to their applications in several areas of non-linear science, but we hope that the results reported here have opened new lines of research in the context of the quasi-integrability phenomena.

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