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Chapter

A Posteriori Error Analysis in Finite Element Approximation for Fully Discrete Semilinear Parabolic Problems

Younis Abid Sabawi

Abstract

This Chapter aims to investigate the error estimation of numerical approximation to a class of semilinear parabolic problems. More specifically, the time discretization uses the backward Euler Galerkin method and the space discretization uses the finite element method for which the meshes are allowed to change in time. The key idea in our analysis is to adapt the elliptic reconstruction technique, introduced by Makridakis and Nochetto 2003, enabling us to use the a posteriori error estimators derived for elliptic models and to obtain optimal order in $L_\infty(H^1)$ for Lipschitz and non-Lipschitz nonlinearities. In this Chapter, some challenges will be addressed to deal with nonlinear term by employing a continuation argument.

Keywords: A posteriori error estimates, semilinear parabolic problems, finite element approximation, L_∞ (H^1) bounds in finite element approximation, fully discrete semilinear parabolic approximation

1. Introduction

The finite element method (FEM) consider is the most of flexibility common technique used for dealing with various kinds of application in many fields, for instance, in engineering, in chemistry and in biology. The derivation of a posteriori error estimates for linear and nonlinear parabolic problems are gaining increasing interest and there is a significant implementation of the method now are understandable and available in the literature [1–9]. However, There is less progress has been made comparatively in the proving of a posteriori error bounds for semilinear parabolic problems [10–13]. These estimations play a crucial rule in designing adaptive mesh refinement algorithms and consequently leading to a good accuracy while reducing the computational cost of the scheme.

The key technique used in the proofs is the elliptic reconstruction idea, introduced by Makridakis and Nochetto for spatially discrete conforming FEM [2] and extended to fully discrete conforming FEM by Lakkis and Makridakis [3] These ideas have been carried forward also to fully discrete schemes involving spatially non-conforming/dG methods in [14]. The choice of this technique for deriving a posteriori error for parabolic problem is motivated by the following factors.

First, elliptic reconstruction allows us to utilise the readily available elliptic a posteriori estimates [2] to bound the main part of the spatial error. Second, this technique combines the energy approach and appropriate pointwise representation of the error in order to arrive to optimal order a posteriori estimators in the $L_\infty(L^2)$-norm. As a result, this approach will lead to optimal order in both $L^2(H^1)$ and $L_\infty(L^2)$-type norms, while the results obtained by the standard energy methods are only optimal order in $L^2(H^1)$-type norms.

The aim of this Chapter is to derive a posteriori error bounds for the fully discrete in two cases Lipschitz and non Lipschitz. Continuation Argument will be used to deal with nonlinear forcing terms.

2. Preliminaries

Before we proceed with the error analysis, we require some auxiliary results that will be used in our analysis.

2.1 Functional spaces

Let $z(t, \ x)$ is a function of time t and space χ, we introduce the Bochner space $L_P(0, \ T, \cdot \ \mathrm{X})$ where (X is some real Banach space equipped with the norm $\| \cdot \|_X$) which is the collection of all measurable functions v: $(0, \ T) \rightarrow \mathrm{X}$, more precisely, for any number $r \geq 1$

$$L_P(0, \ T; \ X) = \left\{ z \ : \ (0, \ T) \rightarrow \mathrm{X} \ : \ \int_0^T \|z\|^2 dt \leq \infty \right\}, \tag{1}$$

such that

$$\begin{aligned} \|z\|_{L_P(0,T;X)} &:= \left(\int_0^T \|z\|^2 dt \right)^{1/2} < \infty \quad \text{for } 1 \leq p < \infty, \\ \|z\|_{L_P(0,T;X)} &:= \max_{t \in [0, \ T]} \|z(t)\|_X < \infty \quad \text{for} p = \infty. \end{aligned} \tag{2}$$

Lemma 1.1 (Continuous Gronwall inequality). Let $C_0, \ C_1 \in L^1(0, \ T)$ for all $T > 0$ and $z \in W^{1,1}$, then for almost every $t \in (0, \ T]$, reads

$$z'(t) \leq C_0(t) + C_1(t) z(t) \ , \tag{3}$$

then

$$z(t) \leq F(0, \ T) z(0) + \int_0^T F(0, \ T) z(s) ds, \tag{4}$$

where $F(0, \ T) = \exp\left(\int_0^T C_1(\xi(t) d\xi \right)$. Furthermore, if C_0 and C_1 are non-negatives, gives

$$z(T) \leq F(0, \ T) \left(z(0) + \int_0^T C_0(s) ds \right). \tag{5}$$

Proof: See [15].

Theorem 1.2 Given some $p \geq 2$, we have

$$\|v\|^p_{L^p(\Omega)} \leq C\|\nabla v\|^{\frac{pd-2d}{2}}\|_v\|^{\frac{2p+2d-pd}{2}}$$
$$\|v\|^p_{L^p(\Omega)}p \leq C\|\nabla v\|^{p-2}\|v\|^2, d = 2$$
$$\|v\|^p_{L^p(\Omega)}p \leq C\|\nabla v\|^{\frac{3p-6}{2}}\|v\|^{\frac{6-p}{2}}, \ \ d = 3, p \leq 6.$$

Proof: See [16].

3. Model problem

Consider the semilinear parabolic problem as

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u = f(u), && \text{in } \Omega \cup [0, \ \ T], \\ &u = 0, && \text{on } \partial\Omega, \\ &u(0, x) = u_0(x), && \text{on } \{0\} \times \Omega, \end{aligned} \tag{6}$$

where Ω is a plane convex domain subset of $\mathbb{R}^k$, $\Omega \subset \mathbb{R}^k$ with smooth boundary condition $\partial\Omega$, where $u_t = \partial u/\partial t$, $T > 0$ and $f \in C^1(\mathbb{R})$. Let $L_p(\omega)$, $1 \leq p \leq \infty$ and $H^r(\omega)$, $r \in \mathbb{R}$, denote the standard Lebesgue and Hilbertian Sobolev spaces on a domain $\omega \subset \Omega$. For brevity, the norm of $L_2(\omega) \equiv H^0(\omega)$, $\omega \subset \Omega$, will be denoted by $\|\cdot\|_\omega$, and is induced by the standard $L_2(\omega)$-inner product, denoted by $(\cdot, \cdot)_\omega$; when $\omega = \Omega$, we shall use the abbreviations $\|\cdot\| \equiv \|\cdot\|_\Omega$ and $(\cdot, \cdot) \equiv (\cdot, \cdot)_\Omega$.

Returning to the (6), multiplying by a test function $v \in H^1_0(\Omega)$ and then integrate by parts, we arrive to (7) in weak form, which reads: find $u \in L_2\left(0, \ \ T, H^1_0\right)(\Omega) \cap H^1_0(0, \ \ T, \ \ L_2(\Omega)$ for almost every $t \in (0, \ \ T]$, this becomes

$$\int_\Omega \frac{\partial z}{\partial t} v dx + D(t; z, v) = \int_\Omega f(z) v dx, \tag{7}$$

for all $v \in H^1_0(\Omega)$. Here,

$$D(t; z, \ \ v) = \int_\Omega \nabla z \cdot \nabla v dx. \tag{8}$$

By using Cauchy-Schwarz inequality, the convercitivity and continuity of the bilnear form D, viz.

$$\begin{aligned} &D(v, \ \ v) \geq C_{\text{coer}}\|\nabla v\|^2 \ \ \text{for all} \, v \in H^1_0(\Omega), \\ &|D(v, \ \ w)| \leq C_{\text{cont}}\|\nabla v\|\|\nabla w\| \ \ \text{for all } v, \ \ w \in H^1_0(\Omega), \end{aligned} \tag{9}$$

with $C_{\text{cont}}, C_{\text{coer}}$ positive constants independent of w, v.

4. Fully discrete backward Euler formulation

To introduce a backward Euler approximation of the time derivative paired with the standard conforming finite element method of the spatial operator. To this end, we will discretize the time interval $[0,\ T]$ into subintervals $(t_{n-1},\ t_n],\ n = 1,\ \dots,\ N$ with $t^0 = 0$ and $t_N = T$, and we denote by $\kappa_n = t_n - t_{n-1}$ the local time step. We associate to each time-step t_N a spatial mesh $\mathcal{T}^n$ and the respective finite element space $V^n;\ = V_h^p(\mathcal{T}^n)$. The fully discrete scheme is defined as follows. Set $Z(0)$ to be a projection of z_0 onto some space V^0 subordinate to a mesh $\mathcal{T}^0$ employed for the discretization of the initial condition. For $k = 1,\ \dots,\ n$, find $Z \in S^n$ such that the fully discrete, then reads as follows

$$\left(\frac{Z^n - Z^{n-1}}{K_n},\ \phi^n\right) + D(Z^n,\ \phi^n) = (\ f^n(Z^n),\ \phi^n),\ \forall \phi^n \in V^n \tag{10}$$

where $D^n(\cdot,\cdot) = D(t_n,\ \cdot,\ \cdot)$ denotes the cG bilinear form defined on the mesh $\mathcal{T}^n$. Since $Z^n \in V^n$, there exist $\alpha_i(t) \in \mathbb{R}, j = 0, 1, 2,\ \dots,\ N_h$, so that

$$Z^n(x,\ t) = \sum_{j=0}^{N_{loc}N_{el}} \alpha_j^n(t)\Phi_j(x),\ \ \Phi_j, \quad j = 0, 1, 2 \dots N_h \tag{11}$$

is the basis functions. After plugging (11) into (10), yields a nonlinear system of ordinary differential equations

$$\begin{aligned} &(M \ + \kappa_n A)\alpha_j^n(t) = M\alpha_j^{n-1}(t) + \kappa_n F \\ &\alpha(0) = \delta, \end{aligned} \tag{12}$$

where $M_{i,j} = (\Phi_j,\ \Phi_j)$ and $A_{i,j} = D(\Phi_j,\ \Phi_j)$ are called the mass and stiffness matrices with element $F_{\ j,k} = (\ f(\Phi_j),\ \Phi_k)$. We define the piecewise linear interpolant Z and time-dependent elliptic reconstruction $w(t)$ as by the linear interpolant with respect to t of the values Z^{n-1} and Z^n, viz.,

$$Z(t) \coloneqq \ell_{n-1}(t)Z^{n-1} + \ell_n(t)Z^n, \qquad w(t) \coloneqq \ell_{n-1}\mathrm{R}_{be}^{n-1}Z^{n-1} + \ell_n \mathrm{R}_{be}^n Z^n, \tag{13}$$

where $\{\ell_{n-1}, \ell_n\}$ denotes the linear Lagrange interpolation basis on the interval I_n are defined as

$$\ell_n \coloneqq \frac{t_n - t}{K_n},\quad \ell_{n-1} \coloneqq \frac{t - t_{n-1}}{K_n}. \tag{14}$$

We give here some essential definitions in the error analysis of the discrete parabolic equations.

i. L^2 projection operator Π_0^n; The operator defined $\Pi_0^n: L^2 \to V^n,\ \ 1 \le n \le N$ such that

$$(\Pi_0^n v,\ \ \phi^n) = (v,\ \ \phi^n) \quad \forall \phi^n \in V^n, \tag{15}$$

for all $v \in L^2(\Omega)$.

ii. Discrete elliptic operator: The elliptic operator defined $\mathrm{A}_h^n: H_0^1(\Omega) \to V^n$ such that for $v \in H_0^1(\Omega)$, reads

$$(\mathrm{A}_h^n v,\ \ \phi^n) = D(v,\ \ \phi^n)\ \ \forall \phi^n \in V^n. \tag{16}$$

Using the above projections, (10) can be expressed in distributional form as

$$\frac{Z^n - \Pi_0^n Z^{n-1}}{K_n} + \mathrm{A}_h^n Z^n = \Pi_0^n f^n(Z^n). \tag{17}$$

5. Elliptic reconstruction

The aim of this section will be introduced the elliptic reconstruction operator and then discuss the related aposteriori error analysis for the backward Euler approximation. To do this, we define the elliptic reconstruction $\mathrm{R}_{be}^n \in H_0^1(\Omega)$ of Z^n as the solution of elliptic problem

$$D\left(\mathrm{R}_{be}^n v, \ \phi\right) = (g^n, \ \phi), \tag{18}$$

for a given $v \in V^n$ and $g^n = \Pi_0^n f^n(Z^n) - \frac{Z^n - \Pi_0^n Z^{n-1}}{k_n}$. The crucial property, this operator R_{be}^n is orthogonal with respect to D such that

$$D\left(u - \mathrm{R}_{be}^n u, \ v\right) = 0 \quad u, \ v \in V^n. \tag{19}$$

The following lemma is the elliptic reconstruction error bound in the H^1 and L_2-norms To see the proof, we refer the reader to [3] for details.

Lemma 1.3 (Posteriori error estimates). For any $Z^n \in V^n$, the following elliptic a posteriori bounds hold:

$$\begin{aligned} &\| \langle R_{be}^n Z^n - Z^n \| \leq C\Phi_{n,L_2}^2 \\ &\| \nabla \left(\langle R_{be}^n Z^n - Z^n \right) \| \leq C\Phi_{n,\mathrm{H}^1}^2 \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Phi_{n,L_2}^2 &:= \| h_n^2 (g^n + \Delta^n Z^n) \| + \| h_n^{3/2} [\![Z^n]\!] \|_{\Sigma_n}, \\ \Phi_{n,\mathrm{H}^1}^2 &:= \| h_n (g^n + \Delta^n Z^n) \| + \| h_n^{1/2} [\![Z^n]\!] \|_{\Sigma_n}, \end{aligned} \tag{21}$$

and g^n defined in (18).

Lemma 1.4 (Main semilinear parabolic error equation). The following error bounds hold

$$\begin{aligned} \left(\frac{\partial \rho}{\partial t}, \ \psi \right) + D(\rho, \ \phi) &= (f(z) - f^n(Z^n), \ \phi) + \left(\frac{\partial \varepsilon}{\partial t}, \ \phi \right) + D(w - w^n, \ \phi) \\ &\quad + \left(\Pi_0^n f^n(Z^n) - f^n(Z^n) + \frac{\Pi_0^n Z^{n-1} - Z^{n-1}}{K_n}, \ \phi \right). \end{aligned} \tag{22}$$

Proof: To begin with, we first decompose the error as

$$e := \rho - \varepsilon, \quad \rho := z - w, \quad \varepsilon := w - Z. \tag{23}$$

By recalling (17), this becomes

$$\left(\frac{\partial Z}{\partial t}, \ \phi \right) + D(w^n, \ \phi) = \left(\frac{\Pi_0^n Z^{n-1} - Z^{n-1}}{k_n}, \ \phi \right) + \left(\Pi_0^n f^n(Z^n), \ \phi \right) \ \ \forall \phi \in H_0^1(\Omega), \tag{24}$$

where $\frac{\partial Z}{\partial t} = \frac{Z^{n-1} - Z^n}{\kappa_n}$. Subtracting (24) from (7), gives

$$\left(\frac{\partial}{\partial t}[Z - z],\ \phi\right) + D(w^n - z,\ \phi) = \left(\Pi_0^n f^n(Z^n) - f(z),\ \phi\right) + \left(\frac{\Pi_0^n Z^{n-1} - Z^{n-1}}{\kappa_n},\ \phi\right). \tag{25}$$

Using elliptic reconstruction to split the error, gives

$$\begin{aligned}\left(\frac{\partial}{\partial t}[-z - w + w + Z^n],\ \phi\right) + D(w^n - w + w - z,\ \phi) &= \left(\Pi_0^n f^n(Z^n) - f^n(Z^n),\ \phi\right)\\ +\left(f^n(Z^n) - f(z),\ \phi\right) + \left(\frac{\Pi_0^n Z^{n-1} - Z^{n-1}}{K_n},\ \phi\right).\end{aligned} \tag{26}$$

After using triangle inequality, the proof will be concluded.
The proof of the following Lemmas 1.5, 1.6, 1.7 in details, we refer to [3].
Lemma 1.5 (Temarol error estimate). Let $T_{n,1}, 1 \le n \le N$ be given by

$$T_{n,1} := \int_{t_{n-1}}^{t_n} \left| D\left(w - w^n,\ \frac{\partial \rho}{\partial t}\right)\right| dt, \tag{27}$$

then

$$T_{n,1} \le \left(\int_{t_{n-1}}^{t_n} \left\|\frac{\partial \rho}{\partial t}\right\|^2 dt\right)^{1/2} (\kappa_n)^{1/2} \Phi_{n,2}, \tag{28}$$

where

$$\Phi_{n,2} := \begin{cases} \frac{\sqrt{3}}{3}\partial\left(\left\|\prod_0^n f^n(Z^n) - \frac{Z^n - \prod_0^n Z^{n-1}}{k_n}\right\|\right) & \text{for}\, n \in [2:N], \\ \frac{\sqrt{3}}{3}\left(\left\|\prod_0^1 f^1(Z^1) - \frac{Z^1 - \prod_0^1 Z^0}{k1}\right\|\right) & \text{for}\, n = 1. \end{cases} \tag{29}$$

Lemma 1.6 (Space-mesh error estimate). Let $T_{n,2}, 1 \le n \le N$ is defined by

$$T_{n,2} := \int_{t_{n-1}}^{t_n} \left| \left(\frac{\partial \varepsilon}{\partial t},\ \frac{\partial \rho}{\partial t}\right) \right| dt, \tag{30}$$

we have

$$T_{n,2} \le \left(\int_{t_{n-1}}^{t_n} \left\|\frac{\partial \rho}{\partial t}\right\|^2 dt\right)^{1/2} (\kappa_n)^{1/2} \Upsilon_{n,2}, \tag{31}$$

where

$$\Upsilon_{n,2} := C\left(\frac{d}{dt}\|h_n^2(g^n + \Delta^n Z^n)\|\right) + C\|\tilde{h}_n^{3/2}[\![Z^n - Z^{n-1}]\!]\|_{\tilde{\Sigma}_n} + C\|\tilde{h}_n^{3/2}[\![Z^n - Z^{n-1}]\!]\|_{\tilde{\Sigma}_n\ \hat{\Sigma}_n}. \tag{32}$$

Lemma 1.7 (Mesh change estimates). Let $T_{n,3}, 1 \le n \le N$ is given by

$$T_{n,3} := \int_{t_{n-1}}^{t_n} \left| \left(\Pi_0^n f^n(Z^n) - f^n(Z^n) + \frac{\Pi_0^n Z^{n-1} - Z^{n-1}}{\kappa_n},\ \frac{\partial \rho}{\partial t}\right) \right| dt, \tag{33}$$

such that

$$T_{n,3} \le \kappa_n \max_{t\in[0,t_m]} \|\nabla\rho\| \left(\delta_{n,\infty} + \sum_{n=2}^{m} \kappa_n \delta_{n,1} + \delta_{\infty,1} \right) , \tag{34}$$

where

$$\begin{aligned} \delta_{n,1} &:= \|h_n^{\wedge}\partial\left(\Pi_0^n - I\right)\left(f^n(Z^n) - \kappa_n Z^{n-1}\right)\|, \\ \delta_{n,\infty} &:= \|h_n\left(\Pi_0^n - I\right)\left(f^n(Z^n) - \kappa_n Z^{n-1}\right)\|. \end{aligned} \tag{35}$$

6. A posteriori error bound for fully discrete semilinear parabolic problems

The aim of this section is to study a posteriori error bound in $L_\infty\left(H^1\right)$-norm for nonlinear forcing terms. Both globally and locally Lipschitz continuous nonlinearities are considered.

6.1 A posteriori error analysis for the globally Lipschitz continuity case

Let us suppose that f is defined on the whole of and satisfies globally Lipschitz continuous

$$|f(z_1) - f(z_2)| \le C_g |z_1 - z_2|, \tag{36}$$

where $|\cdot|$ denotes the standard Euclidean norm on $(R \ge 1)$.

Lemma 1.8 (Data approximation error estimate). Suppose that the nonlinear reaction f satisfying the globally Lipschitz continuous defined in (36), then, the following error bounds hold:

$$\begin{aligned} T_{n,4} = \int_{t_{n-1}}^{t_n} \left| \left(f(z) - f^n(Z^n), \frac{\partial\rho}{\partial t} \right) \right| dt &\le \frac{\sqrt{C_g}}{2\beta} \kappa_n \|\nabla\rho\|^2 + \frac{\beta\sqrt{C_g}}{2} \int_{t_{n-1}}^{t_n} \left\| \frac{\partial\rho}{\partial t} \right\|^2 dt \\ + \kappa_n \Psi_{n,1} \left(\int_{t_{n-1}}^{t_n} \left\| \frac{\partial\rho}{\partial t} \right\|^2 dt \right)^{1/2} &+ \kappa_n \Psi_{n,2} \left(\int_{t_{n-1}}^{t_n} \left\| \frac{\partial\rho}{\partial t} \right\|^2 dt \right)^{1/2}, \end{aligned} \tag{37}$$

where

$$\begin{cases} \Psi_{n,1} := \sqrt{C_g} \left\{ \|\varepsilon^{n-1}\|, \ \ \|\varepsilon^n\| \right\}, \\ \Psi_{n,2} := \dfrac{1}{\kappa_n} \displaystyle\int_{t_{n-1}}^{t_n} \|f(Z) - f^n(Z^n)\|. \end{cases} \tag{38}$$

Proof: Using triangle inequality, $T_{n,4}$ written as

$$\begin{aligned} T_{n,4} = \int_{t_{n-1}}^{t_n} \left| \left(f(z) - f^n(Z^n), \frac{\partial\rho}{\partial t} \right) \right| dt &\le \int_{t_{n-1}}^{t_n} \left| \left(f(z) - f(w), \frac{\partial\rho}{\partial t} \right) \right| dt \\ + \int_{t_{n-1}}^{t_n} \left| \left(f(w) - f(Z), \frac{\partial\rho}{\partial t} \right) \right| dt &+ \int_{t_{n-1}}^{t_n} \left| \left(f(Z) - f^n(Z^n), \frac{\partial\rho}{\partial t} \right) \right| dt \\ := L_{n,1} + L_{n,2} + L_{n,3}. \end{aligned} \tag{39}$$

Applying Cauchy–Schwarz inequality and (36) along with Young's inequality and Poincar'e-Friedrichs inequality, $L_{n,1}$ gives

$$L_{n,1} = \int_{t_{n-1}}^{t_n} \left| \left(f(z) - f(w), \frac{\partial \rho}{\partial t} \right) \right| dt \le \int_{t_{n-1}}^{t_n} \| f(z) - f(w) \| \left\| \frac{\partial \rho}{\partial t} \right\| dt$$
$$\le \frac{\sqrt{C_g}}{2\beta} \kappa_n \|\nabla \rho\|^2 + \frac{\beta \sqrt{C_g}}{2} \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt. \tag{40}$$

The second term $L_{n,2}$, reads

$$L_{n,2} = \int_{t_{n-1}}^{t_n} \left| \left(f(w) - f(Z), \frac{\partial \rho}{\partial t} \right) \right| dt \le \int_{t_{n-1}}^{t_n} \|w - Z\| \left\| \frac{\partial \rho}{\partial t} \right\| dt$$
$$\le \sqrt{C_g} \int_{t_{n-1}}^{t_n} \left(\left| \frac{t_n - t}{\kappa_n} \right| \|\varepsilon^{n-1}\| + \left| \frac{t - t_{n-1}}{\kappa_n} \right| \|\varepsilon^n\| \right) \left\| \frac{\partial \rho}{\partial t} \right\| dt \tag{41}$$
$$\le \frac{\sqrt{C_g}}{2} \kappa_n \left(\|\varepsilon^{n-1}\| + \|\varepsilon^n\| \right) \left(\int_{t_{n-1}}^{t_n} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt) \right)^{1/2}.$$

Finally, $L_{n,3}$ can be bounded by using Cauchy–Schwarz inequality, to obtain

$$L_{n,3} = \int_{t_{n-1}}^{t_n} \left| \left(f(Z) - f^n(Z^n), \frac{\partial \rho}{\partial t} \right) \right| dt \le \|f(Z) - f^n(Z^n)\| \left(\int_{t_{n-1}}^{t_n} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt) \right)^{1/2}. \tag{42}$$

Collecting all the results together, the proof will be finished.

Lemma 1.9 Let z be the exact solution of (7) and let Z^n be its finite element approximation obtained by the backward Euler approximation (10). Then, for $1 \le n \le N$, the following a posteriori error bounds hold:

$$\left(\max_{t \in [0, t_m]} \|\nabla \rho(t)\|^2 + \int_0^{t_m} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt \right)^{1/2} \le \left\{ 2\mathcal{E}_G(m) \|\nabla \rho\|^2 \right\}^{1/2} + 2\mathcal{E}_G(m) \left(\mathcal{F}_{1,m}^2 + \mathcal{F}_{2,m}^2 \right) \tag{43}$$

where

$$\mathcal{F}_{1,m} := 2 \max_{t \in [0, t_m]} \delta_{m,\infty} + 2 \sum_{n=2}^{m} \kappa_n \delta_{n,1},$$
$$\mathcal{F}_{2,m}^2 := \sum_{n=1}^{m} \kappa_n \left(\Phi_{n,2}^2 + \Upsilon_{n,2}^2 + \Psi_{n,1}^2 + \Psi_{n,2}^2 \right). \tag{44}$$

Proof: Now, setting $\phi = \frac{\partial \rho}{\partial t}$ in 22, gives

$$\frac{1}{2} \frac{d}{dt} \|\nabla \rho(t)\|^2 + \frac{C_{coer}}{2} \left\| \frac{\partial \rho}{\partial t} \right\|^2 \le \left| \left(\frac{\partial \varepsilon}{\partial t}, \frac{\partial \rho}{\partial t} \right) \right| + \left| \left(f(z) - f^n(Z^n), \frac{\partial \rho}{\partial t} \right) \right| + \left| D \left(w - w^n, \frac{\partial \rho}{\partial t} \right) \right|$$
$$+ \left| \left(\Pi_0^n f(Z^n) - f^n(Z^n) + \frac{P_0^n Z^{n-1} - Z^{n-1}}{k_n}, \frac{\partial \rho}{\partial t} \right) \right|. \tag{45}$$

Integrate the above from t_{n-1} to t_n then, we have

$$\frac{1}{2}\|\nabla\rho(t_n)\|^2 - \frac{1}{2}\|\nabla\rho(t_{n-1})\|^2 + \frac{C_{coer}}{2}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad (46)$$

where $T_{n,i}, i = 1, 2, 3, 4$ defined in Lemmas 1.5, 1.6, 1.7 and 1.8, respectively. Summing up over $n = 1{:}\, m$ so that

$$\|\nabla\rho(t_m)\|^2 + C_{coer}\int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le \|\nabla\rho(0)\|^2 + 2\sum_{n=1}^{m}(T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}). \quad (47)$$

By introducing

$$\|\nabla\left(\rho_m^*\right)\|; = \|\nabla\rho\left(t_m^*\right)\| = \max_{t\in[0,t_m]}\|\nabla\rho(t)\|, \quad (48)$$

therefore

$$\max_{t\in[0,t_m]}\|\nabla\rho(t)\| + C_{coer}\int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le 2\|\nabla\rho(0)\|^2 + 4\sum_{n=1}^{m}(T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}). \quad (49)$$

Now, using Lemmas 1.5, 1.6, 1.7 and 1.8, reads

$$\max_{t\in[0,t_m]}\|\nabla\rho(t)\|^2 \le 2\|\nabla\rho(0)\|^2 + \left(2\beta\sqrt{C_g} - C_{coer}\right)\int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt + 2\max_{t\in[0,t_m]}\|\nabla\rho(t)\|\mathcal{F}_{1,m}$$
$$+\frac{2\sqrt{C_g}}{\beta}\sum_{n=1}^{m}K_n\max_{t\in[0,t_m]}\|\nabla\rho(t)\|^2 + 4\left(\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt\right)^{1/2}(\kappa_n)^{1/2}(\Phi_{n,2} + \Upsilon_{n,2} + \Psi_{n,1} + \Psi_{n,2}). \quad (50)$$

Selecting now $\beta > 0$ be such that $\left(2\beta\sqrt{C_g} - C_{coer}\right) > 0$ and using Gronwall's inequality, imply

$$\max_{t\in[0,t_m]}\|\nabla\rho(t)\|^2 + \mathcal{E}_G(m)\int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le 2\mathcal{E}_G(m)\|\nabla\rho(0)\|^2 + 2\mathcal{E}_G(m)\max_{t\in[0,t_m]}\|\nabla\xi(t)\|\mathcal{F}_{1,m}$$
$$+4\mathcal{E}_G(m)\sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt\right)^{1/2}(\kappa_n)^{1/2}(\Phi_{n,2} + \Upsilon_{n,2} + \Psi_{n,1} + \Psi_{n,2}), \quad (51)$$

with $\mathcal{E}_G(m) := \left\{1, \sum_{n=1}^{m}\frac{2\sqrt{C_g}}{\beta}\kappa_n \exp\left(\frac{2\sqrt{C_g}}{\beta}\left(\Sigma_{n<j<m}k_j\right)\right)\right\}$. To finish the proof of lemma, we use a standard inequlty. For $(a_0, a_1, \ldots, a_n), (b_0, b_1, \ldots, b_n) \in \mathbb{R}^{m+1}$.

$$|a|^2 \le c^2 + ab, \quad (52)$$

then

$$|a| \le |c| + |b|, \quad (53)$$

and by taking

$$a_0 := \max_{t\in[0,t_m]} \|\nabla\rho(t)\|, \quad a_n := \left\{\mathcal{E}_G(m)\int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt\right\}^{1/2}, \quad c := \left\{2\mathcal{E}_G(m)\|\nabla\rho(0)\|^2\right\}^{1/2}$$
$$b_0 := \sqrt{2}\mathcal{E}_G(m)\mathcal{F}_{1,m}, \quad b_n := 4\mathcal{E}_G(m)\sum_{n=1}^{m}(\kappa_n)^{1/2}(\Phi_{n,2}+\Upsilon_{n,2}+\Psi_{n,1}+\Psi_{n,2}). \tag{54}$$

The proof already will be finished.

Theorem 1.10 Let z be the exact solution of (7) and let Z^n be its finite element approximation obtained by the backward Euler approximation (10). Then, for $1 \le n \le N$, the following a posteriori error bounds hold:

$$\max_{t\in[0,t_m]}\|\nabla(z(t)-Z(t))\|^2 \le 2\mathcal{E}_G(m)\left(\Phi^2_{n,\mathrm{H}^1}(0)+\|\nabla(z(0)-Z(0))\|^2\right)$$
$$+2\mathcal{E}_G(m)\left(\mathcal{F}^2_{1,m}+\mathcal{F}^2_{2,m}\right)) + 2\max_{t\in[0,t_m]}\Phi^2_{n,\mathrm{H}^1}, \tag{55}$$

where Φ^2_{n,H^1} defined in (20).

Proof: By decomposing $Z(t)-z(t)$ into ρ and ε, so that

$$\|\nabla(Z(t)-z(t))\|^2 \le 2\|\nabla\varepsilon\|^2 + 2\|\nabla\rho\|^2. \tag{56}$$

To be able to bound the first term on the right hand side of (56), using (13), this becomes

$$\begin{aligned}\|\nabla\varepsilon(t)\|^2 &= \|\nabla(w(t)-Z(t))\|^2 = \|\nabla\left(\ell_n \mathrm{R}^n_{be}Z^n + \ell_{n-1}\mathrm{R}^{n-1}_{be}Z^{n-1} - \ell_{n-1}(t)Z^{n-1} - \ell_n(t)Z^n\right)\|^2\\ &\le \ell_n\|\nabla\left(\mathrm{R}^n_{be}Z^n - Z^n\right)\|^2 + \ell_{n-1}\|\nabla\left(\mathrm{R}^{n-1}_{be}Z^n - Z^{n-1}\right)\|^2\\ &\le \max_{t\in[0,t_m]}\left\{\left\|\nabla\left(\mathrm{R}^{n-1}_{be}Z^{n-1}-Z^{n-1}\right)\right\|^2, \left\|\nabla\left(\mathrm{R}^n_{be}Z^n - Z^n\right)\right\|^2\right\}\\ &\le \max_{t\in[0,t_m]}\left\{\|\nabla\left(\mathrm{R}^n_{be}Z^n - Z^n\right)\|^2\right\}\\ &\le \max_{t\in[0,t_m]}\Phi^2_{n,\mathrm{H}^1}.\end{aligned} \tag{57}$$

and $\|\nabla\rho(0)\|^2 = \|\nabla(w(0)-z(0))\|^2 \le 2\|\nabla\varepsilon(0)\|^2 + 2\|\nabla(z(0)-Z(0))\|^2$. Finally, the second term on the right hand side of (56) will be estimated via Lemma 1.9.

6.2 A posteriori error analysis for the locally Lipschitz continuity case

Let $f: R \to R$ is locally Lipschitz continuous for a.e. $(x, t) \in \Omega \cup [0, T]$, in the sense that there exist real numbers $C_L > 0$ and $\gamma \ge 0$ such that

$$|f(u) - f(v)| = C_L(t)\,(1+|u|^\gamma + |v|^\gamma)|u-v|. \tag{58}$$

Lemma 1.11 (Estimation of the nonlinear term). If the nonlinear reaction f is satisfying the growth condition (58) with $0 \le r < 2$ for $d = 2$, and with $0 \le r \le 4/3$ for $d = 3$, we have the bound

$$\|f(z) - f^n(Z^n)\| \le \mathcal{N}_1(t)\left\{\mathcal{N}_2(Z)(\|\rho\|+\|\varepsilon\|) + \sqrt{3}\|\rho\|\|\nabla\rho\|^\gamma + \sqrt{5}\|\varepsilon\|\|\nabla\varepsilon\|^\gamma\right\}$$
$$+\Theta_{n,3}\left(\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt\right)^{1/2}, \tag{59}$$

where $\mathcal{N}_1(t) := \frac{1}{\sqrt{2}}C_L(t)\max\{1, 4^\gamma\}$, $N(Z) := \frac{1}{\sqrt{2}}\sqrt{1+4\gamma|Z|^{2\gamma}_\infty}$ and

$$\Theta_{n,3} := \frac{1}{\kappa_n}\int_{t_{n-1}}^{t_n} \|(f(Z) - f^n(Z^n)\|.$$

Proof: Applying triangle inequality, reads

$$\begin{aligned} T_{L,4} &= \int_{t_{n-1}}^{t_n} \left| \left(f(z) - f^n(Z^n), \frac{\partial \rho}{\partial t} \right) \right| dt \le \int_{t_{n-1}}^{t_n} \left| \left(f(z) - f(Z), \frac{\partial \rho}{\partial t} \right) \right| dt \\ &+ \int_{t_{n-1}}^{t_n} \left| \left(f(Z) - f^n(Z^n), \frac{\partial \rho}{\partial t} \right) \right| dt := \mathcal{J}_{n,1} + \mathcal{J}_{n,2}. \end{aligned} \tag{60}$$

$\mathcal{J}_{n,1}$ can be bounded as follows

$$\begin{aligned} \mathcal{J}_{n,1} &= \int_{t_{n-1}}^{t_n} \left(f(z) - f(Z), \frac{\partial \rho}{\partial t} \right) \le \int_{t_{n-1}}^{t_n} \|(f(z) - f(Z)\| \left\| \frac{\partial \rho}{\partial t} \right\| dt \\ &\le \frac{1}{2} \left\| (f(z) - f(Z) \right\|^2 + \frac{1}{2} \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \rho}{\partial t} \right\|^2 dt. \end{aligned} \tag{61}$$

Now, we have

$$\begin{aligned} \|f(z) - f(Z)\|^2 &= \int_{t_{n-1}}^{t_n} \|f(z) - f(Z)\|^2 dt \le \int_{t_{n-1}}^{t_n} \|f(z) - f(w)\|^2 dt + \int_{t_{n-1}}^{t_n} \|f(w) - f(Z)\|^2 dt \\ &:= Z_{1,n} + Z_{2,n}. \end{aligned} \tag{62}$$

To estimate $Z_{1,n}$ on the first term in the right hand side of (62), we use the Cauchy–Schwarz inequality and (58) to obtain

$$\begin{aligned} Z_{1,n} &= \int_{t_{n-1}}^{t_n} \|f(z) - f(w)\|^2 dt = C_L^2(t) \int_{t_{n-1}}^{t_n} \left(1 + |z|^{2\gamma} + |w|^{2\gamma}\right) |z - w|^2 \\ &\le \int_{t_{n-1}}^{t_n} \left(1 + |z|^{2\gamma}\right) |z - w|^2 dt + \int_{t_{n-1}}^{t_n} |w|^{2\gamma} |z - w|^2 dt. \end{aligned} \tag{63}$$

Applying the elementary inequality $|C_a + C_b|^{2\alpha} \le C\left(|C_a|^{2\alpha} + |C_b|^{2\alpha}\right)$ with $C_a = z - w$ and $C_b = w$, so that $|z|^{2\alpha} \le C|z - w|^{2\alpha} + C|w|^{2\alpha}$, this becomes

$$\begin{aligned} Z_{1,n} &\le C_L^2(t) C \int_{t_{n-1}}^{t_n} \left(1 + |z - w|^{2\gamma}\right) |z - w|^2 dt + C_L^2(t) C \int_{t_{n-1}}^{t_n} \left(2|w - Z|^{2\gamma} + 2|Z|^{2\gamma}\right) |z - w|^2 dt \\ &\le C_L^2(t) C \max\{1, 16^\gamma\} \left(\left(1 + 4^\gamma |Z|^{2r}\right) \|\rho\|^2 + \|\rho\|_{2+2\gamma}^{2+2\gamma} + 2\int_{t_{n-1}}^{t_n} \|\varepsilon\|^{2\gamma} \|\rho\|^2 \right). \end{aligned} \tag{64}$$

Similarly, $Z_{2,n}$ follows as

$$\begin{aligned} Z_{2,n} &= \int_{t_{n-1}}^{t_n} \|f(w) - f(Z)\|^2 dt = C_L^2(t) C \int_{t_{n-1}}^{t_n} \left(1 + |w|^{2\gamma} + |Z|^{2\gamma}\right) |w - Z|^2 \\ &\le C_L^2(t) C \max\{1, 16^\gamma\} \left(\left(1 + 4^r |Z|_\infty^{2\gamma}\right) \|\varepsilon\|^2 + \|\varepsilon\|_{2+2\gamma}^{2+2\gamma} \right). \end{aligned} \tag{65}$$

Collecting all these terms, we obtain

$$\|f(z)-f(Z)\|^2 \le C_L^2(t)C\max\{1,16^\gamma\}\left(1+4^\gamma|Z|_\infty^{2\gamma}\right)\left(\|\rho\|^2+\|\varepsilon\|^2\right)$$
$$+C_L^2(t)C\max\{1,16^\gamma\}\left(\|\rho\|_{2+2\gamma}^{2+2\gamma}+3\|\varepsilon\|_{2+2\gamma}^{2+2\gamma}+2\int_{t_{n-1}}^{t_n}\|\varepsilon\|^2\|\rho\|^{2\gamma}dt\right). \quad (66)$$

Using Holder's inequality and Young's inequality, we deduce that

$$\int_{t_{n-1}}^{t_n}\|\alpha\|^{2r}\|\beta\|^2dx \le \frac{\|\alpha\|_{2+2r}^{2+2r}}{r+1}+\frac{r\|\beta\|_{2+2r}^{2+2r}}{r+1}. \quad (67)$$

Therefore,

$$\int_{t_{n-1}}^{t_n}\|\varepsilon\|^{2r}\|\rho\|^2 \le \frac{\|\varepsilon\|_{2+2\gamma}^{2+2\gamma}}{\gamma+1}+\frac{\gamma\|\rho\|_{2+2\gamma}^{2+2\gamma}}{\gamma+1}$$
$$\le \|\varepsilon\|_{2+2\gamma}^{2+2\gamma}+\|\rho\|_{2+2\gamma}^{2+2\gamma}. \quad (68)$$

Substituting this into our grand inequality yields

$$\|f(z)-f(Z)\|^2 \le \mathcal{N}_1^2(t)\left(\mathcal{N}_2^2(Z)\left(\|\rho\|^2+\|\varepsilon\|^2\right)+3\|\rho\|_{2+2\gamma}^{2+2\gamma}+5\|\varepsilon\|_{2+2\gamma}^{2+2\gamma}\right)\ , \quad (69)$$

where $\mathcal{N}_1^2(t)=\frac{1}{2}C_L^2(t)C\max\{1,16^\gamma\}$ and $\mathcal{N}_2^2(Z)=\frac{1}{2}\left(1+4^r|Z|_\infty^{2r}\right)$. From Gagliardo-Nirenberg inequality in Theorem 1.2, implies that

$$\|\rho\|_{2+2\gamma} \le C\|\nabla\rho\|^{\frac{(2+2\gamma)d-2d}{2}}\|\rho\|^{\frac{4+4\gamma+2d-2d-2d\gamma}{2}}, \quad (70)$$

valid for all $\gamma \ge 0$ for $d=2$ and $0 \le \gamma \le 2$ for $d=3$. Combining this with the Poincar'e-Friedrichs inequality $\|\rho\| \le C\|\nabla\rho\|$, yields

$$\|\rho\|_{2+2\gamma} \le C\|\nabla\rho\|. \quad (71)$$

Finally,

$$\mathcal{J}_{n,2}=\int_{t_{n-1}}^{t_n}\left|\left(f(Z)-f^n(Z^n),\frac{\partial\rho}{\partial t}\right)\right|dt \le \left\|(f(Z)-f^n(Z^n)\|\left(\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\rho}{\partial t}\right\|^2dt\right)^{1/2}\right.. \quad (72)$$

Putting all of the results together the proof will be finished.

Theorem 1.12 Let z be the exact solution of (7) and let Z^n be its finite element approximation obtained by the backward Euler approximation (10). Then, for $1 \le n \le N$, the following a posteriori error bounds hold

$$\max_{t\in[0,t_m]}\|\nabla(z(t)-Z(t))\|^2 \le 4\mathcal{E}(t_n,\ Z)\left(\|\nabla(z(0)-Z(0))\|^2+\Phi_{n,\mathrm{H}^1}^2(0)\right)$$
$$+4\mathcal{E}(t_n,Z)\sum_{n=1}^m\mathcal{F}_{1,m}^2+4\mathcal{E}(t_n,\ Z)\sum_{n=1}^m\kappa_n^2\left\{\Phi_{n,2}^2+\Upsilon_{n,2}^2+\Psi_{n,1}^2+\Psi_{n,2}^2\right\} \quad (73)$$
$$+4\mathcal{N}_1^2(t)\mathcal{E}(t_n,\ Z)\sum_{n=1}^m\left(\mathcal{N}_2^2(Z)\Phi_{n,L_2}^2+\Phi_{n,L_2}^2\Phi_{n,\mathrm{H}^1}^{2\gamma}\right)+2\max_{t\in[0,t_m]}\Phi_{n,\mathrm{H}^1}^2,$$

where Φ_{n,L_2}^2 and Φ_{n,H^1}^2 are given in (20).

Proof: Now, setting $v = \frac{\partial \rho}{\partial t}$ in 22, and integrate from t_{n-1} to t_n along with summing up over $n = 1{:}\, m$ we have

$$\max_{t\in[0,t_m]} \|\nabla\rho(t)\|^2 + C_{coer}\int_0^{t_m} \|\frac{\partial\rho}{\partial t}\|^2 dt \le \|\nabla\rho(0)\|^2 + 2\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} \|f(z) - f^n(Z^n)\|^2 \\ +2\sum_{n=1}^{m}(T_{n,1} + T_{n,2} + T_{n,3}). \tag{74}$$

Using Lemma 1.11, along with lemmas 1.3, 1.5, 1.6 and 1.7, imply

$$\max_{t\in[0,t_m]} \|\nabla\rho(t)\|^2 + \int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le \|\nabla\rho(0)\|^2 + \sum_{n=1}^{m}\mathcal{F}^2_{1,m} \\ + \sum_{n=1}^{m}\kappa_n^2\left(\Phi^2_{n,2} + \Upsilon^2_{n,2} + \Psi^2_{n,1} + \Psi^2_{n,2}\right) + \mathcal{N}_1^2(t)\sum_{n=1}^{m}\left(\mathcal{N}_2^2(Z)\Phi^2_{n,L_2} + 5\Phi^2_{n,L_2}\Phi^{2\gamma}_{n,\mathrm{H}^1}\right) \\ + \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left(\mathcal{N}_1^2(t)\mathcal{N}_2^2(Z)\|\nabla\rho\|^2 + 3\mathcal{N}_1^2(t)\|\rho\|^2\|\nabla\rho\|^{2\gamma}\right). \tag{75}$$

Setting

$$\mathcal{F}(t_n,\ Z,\ \varepsilon)^2 := \|\nabla\rho(0)\|^2 + \sum_{n=1}^{m}\mathcal{F}^2_{1,m} + \sum_{n=1}^{m}\kappa_n^2\left\{\Phi^2_{n,2} + \Upsilon^2_{n,2} + \Psi^2_{n,1} + \Psi^2_{n,2}\right\} \\ +\mathcal{N}_1^2(t)\sum_{n=1}^{m}\left(\mathcal{N}_2^2(Z)\Phi^2_{n,L_2} + 5\Phi^2_{n,L_2}\Phi^{2\gamma}_{n,\mathrm{H}^1}\right). \tag{76}$$

Upon observing that

$$\int_{t_{n-1}}^{t_n}\|\nabla\rho\|^{2r}\|\rho\|^2 \le \max_{t\in[0,t_m]}\|\nabla\rho\|^{2\gamma}\int_{t_{n-1}}^{t_n}\|\rho\|^2)\mathrm{d}s \\ \le \left(\max_{t\in[0,t_m]}\|\nabla\rho\|^2 + \int_{t_{n-1}}^{t_n}\|\rho\|^2\right)dt)^{\gamma+1}. \tag{77}$$

Now combining two equations, we obtain

$$\max_{t\in[0,t_m]} \|\nabla\rho(t)\|^2 + \int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le \mathcal{F}(t_m,\ Z,\ \varepsilon)^2 + \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\mathcal{N}_1^2(t)\mathcal{N}_2^2(Z)\|\nabla\rho\|^2 \\ +3\mathcal{N}_1^2(t)\sum_{n=1}^{m}\left(\max_{t\in[0,t_m]}\|\nabla\rho(t)\|^2 + \int_{t_{n-1}}^{t_n}\|\rho\|^2 dt)\right)^{\gamma+1}. \tag{78}$$

To bound of the nonlinear term of above equation, we shall employ a continuation argument in the spirit of [17, 18]. To do that, we consider the set

$$\mathcal{M}_n = \left\{\lim_{t\in[0,t_m]}\|\nabla\rho(t)\|^2 + C_{coer}\int_0^{t_m}\left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \le 4\mathcal{F}(t_m,\ Z,\ \varepsilon)^2\mathcal{E}(t_m, Z)\right\}, \tag{79}$$

where $\mathcal{E}(t_m, \ Z) = exp\left(\int_0^{t_m} \mathcal{N}_1^2(t)\mathcal{N}_2^2(Z)dt\right)$. Since the left hand side of (78) depends continuously on t, and our aim is to show that $\mathcal{M}_n = [0, \ T]$. To do this, assuming $t_m^* = \max \mathcal{M}_n > 0$ and $t_m^* < T$, imply

$$\max_{t \in [0, t_m^*]} \|\nabla\rho(t)\|^2 + \int_0^{t_m^*} \left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \leq \mathcal{F}(t_n, \ Z, \ \varepsilon)^2 + \{4\mathcal{F}(t_m, \ Z, \ \varepsilon)\mathcal{E}(t_m, \ Z)\}^{\gamma+1} + \mathcal{N}_1^2(t)\mathcal{N}_2^2(Z)\int_0^{t_m^*} \|\nabla\rho\|^2 dt, \tag{80}$$

and Grönwall inequality, thus, implies

$$\max_{t \in [0, t_m^*]} \|\nabla\rho(t)\|^2 + \int_0^{t_m^*} \left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \leq \mathcal{E}\ (t_m, \ Z)\left\{\left(4\mathcal{N}_1^2(t)\mathcal{F}\left(t_m, Z, \ \varepsilon\right)^2\mathcal{E}(t_m, Z)\right)^{\gamma+1} + \mathcal{F}^2(t_m, \ Z, \ \varepsilon)^2\right\}. \tag{81}$$

Since $\mathcal{E}\left(t_m^*, \ Z\right) \leq \mathcal{E}(t_m, \ Z)$ and, suppose that the maximum size h_{max} of the mesh is small enough that, for $h < h_{\max}$, satisfy

$$\mathcal{F}(t_m, Z, \ \varepsilon) \leq \left(\frac{1}{\mathcal{N}_1^2(t)}\right)^{\gamma}\left(\frac{1}{4\mathcal{F}(t_m, Z, \varepsilon)^2\mathcal{E}(t_m, Z)}\right)^{\gamma+1}. \tag{82}$$

This leads to

$$\mathcal{N}_1^2(t)\ \left(4\mathcal{F}(t_m, \ Z, \ \varepsilon)^2\mathcal{E}(t_m, Z)\right)^{\gamma+1} \leq \mathcal{F}(t_m, \ Z, \ \varepsilon)^2. \tag{83}$$

Then, (81), becomes

$$\max_{t \in [0, t_m^*]} \|\nabla\rho(t)\|^2 + \int_0^{t_m^*} \left\|\frac{\partial\rho}{\partial t}\right\|^2 dt \leq 2\mathcal{E}(t_m, \ Z)\mathcal{F}(t_m, \ Z, \ \varepsilon)^2. \tag{84}$$

This leads to contradictions, because of t_m^* suppose to be $t_m^* = \max \mathcal{M}_n$. The triangle inequality along with Lemma 1.3, imply that

$$\begin{aligned} \max_{t \in [0, t_m]} \|\nabla e\|^2 &\leq 2 \max_{t \in [0, t_m]} \|\nabla\rho\|^2 + 2 \max_{t \in [0, t_m]} \|\nabla\varepsilon\|^2 \\ &\leq 4\mathcal{F}(t_m, \ Z, \ \varepsilon)^2\mathcal{E}(t_m, Z) + 2 \max_{t \in [0, t_m]} \Phi_{n,H^1}^2. \end{aligned} \tag{85}$$

By recalling (76), the proof already finished.

7. Adaptive algorithms

This section aims to explain an adaptive algorithm aiming to investigate the performance of the presented a posteriori bound from Theorems 1.10 and 1.12 for the backward-Euler cG method for the semilinear parabolic problem (6). To this end, the implementation of the adaptive algorithm will be based on the deal. II finite element library [19] to the present setting of semilinear problems. We shall write algorithm for Theorem 1.10. For the Theorem 1.12 will follow the same with some modifcations. To begin with, we have

$$\Psi^j_{ini} := \|\nabla(z(0) - Z(0))\| + \|\nabla\varepsilon(0)\|$$

$$\Psi^j_{time} := \sum_{j=1}^{m}\left(\kappa_j \frac{\sqrt{3}}{3}\partial\left\|\Pi^j_0 f^j(Z^j) - \frac{Z^j - \Pi^j_0 Z^{j-1}}{\kappa_j}\right\| + \int_{t_{j-1}}^{t_j} \|f(Z) - f^j(Z^j)\|\right) \quad (86)$$

$$\Psi^j_{space} := \|h_j(g^j + \Delta^j Z^j)\| + \|h_j^{1/2}[\![Z^j]\!]\|_{\Sigma j}.$$

The adaptive algorithm from [15], starts with an initial uniform mesh in space and with a given initial time step. Starting from a uniform square mesh of 16×16 elements, the algorithm adapts the mesh to improve approximation to the initial condition using the initial condition estimator Ψ_{ini} until some tolerance is satisfied. To adapt the timestep κ_j, the algorithm bisects a time interval not satisfying a user-defined temporal tolerance $\Psi^j_{time} \leq$ **ttol**, and leaves a time-interval unchanged if $\Upsilon^j_{time} \leq$ **ttol**.

Once the time-step is adapted, the algorithm performs spatial mesh refinement and coarsening, determined by the space indicator Ψ^j_{space} using the user-defined tolerances $\mathbf{stol}^+$ and $\mathbf{stol}^-$, corresponding to refinement and coarsening, respectively. More specifically, we select the elements with the largest local contributions which result to $\Psi^j_{space} > \mathbf{stol}^+$ for refinement. The spatial coarsening threshold is set to $\mathbf{stol}^- = 0.001 * \mathbf{stol}^+$; we select the elements with the smallest local contributions which result to $\Psi^j_{space} < \mathbf{stol}^-$ for coarsening. The algorithm iterates for each time-step. We refer to [15] for the algorithm's workflow and all implementation details. The following two algorithms give the backward Euler method to the ODE system (12) and space-time adaptivity for Theorem 1.10.

Algorithm 1. The backward Euler method for solving the semilinear parabolic equation

1: Create a mesh with n elements on the interval I_n.
2: We disctize I_n as $0 = t_1 < t_2 < t_{3,\dots,<t_n} = T$, where n is time step defined as $\kappa_n = t_n - t_{n-1}$.
3: Setting $\alpha^0 = \alpha(0)$.
4: **for** $k = 1, 2, \quad \dots, \quad n$ **do**
5: Calculate the mass and stiffness matrices M and A, and the load vector F with entries

$$M_{i,j} = \int_{I_n} \phi_j \phi_i dx, \quad A_{i,j} = \int_{I_n} \phi'_j \phi'_i dx, \quad F_{i,j} = \int_{I_n} f\left(\phi_j\right)\phi_i dx. \quad (87)$$

6: Solve

$$(M + \kappa_n A)\alpha^n_i(t) = M\alpha^{n-1}_i(t) + \kappa_n F. \quad (88)$$

7: **end for**

Algorithm 2. Space-time adaptivity.

Input $a,\ b, f, z^0,\ T,\ \Omega, n,\ \mathcal{T},\ ttol, stol^+, stol^-$
Pick $\kappa_1,\ \ldots,\ \kappa_n = \frac{T}{n}$.
Compute Z^0.
Compute Z^1 from Z^0.
while $\left(\Psi^1_{time}\right)^2 > ttol^+$ **or** $\max\left(\Psi^1_{space}\right)^2 > stol^+$ do bisction $\mathcal{T}^0$ by refining all elements such that $\left(\Psi^1_{space}\right)^2 > stol^+$ and coarsening all elements such that $\left(\Psi^1_{space}\right)^2 < stol^-$
if $\left(\Psi^1_{time}\right)^2 > ttol$, **then**.
$n - 1 \leftarrow n$.
$K_n = K_{n-1},\ \ldots,\ \kappa_2 = \kappa_1$.
$\kappa_2 = \frac{\kappa_1}{2}$.
$\kappa_1 \leftarrow \frac{\kappa_1}{2}$.
end if.
Compute Z^0.
Compute Z^1 from Z^0.
end while
put $j = 1,\ \mathcal{T}^1 = \mathcal{T}^0,\ time = \kappa_1$.
while $time < T$ do
Calculute Z^j from Z^{j-1}.
while $\left(\Psi^i_{time}\right)^2 > ttol$ **do**
if $\left(\Psi^1_{time}\right)^2 > ttol$ **then**
$n - 1 \leftarrow n$.
$\kappa_n = \kappa_{n-1}, \ldots, \kappa_{j+2} = \kappa_{j+1}$.
$\kappa_{j+1} = \frac{\kappa_j}{2}$.
$\kappa_j \leftarrow \frac{\kappa_j}{2}$.
end if
Compute Z^j from Z^{j-1}.
end while
Create $\mathcal{T}^j$ from $\mathcal{T}^{j-1}$ by refining all elements such that $\left(\Psi^i_{space}\right)^2 > stol^+$ and coarsening all elements such that $\left(\Psi^i_{space}\right)^2 < stol^-$.
Compute Z^j from Z^{j-1}.
$time \leftarrow time + \kappa_j$.
$j - 1 \leftarrow j$.
end while

8. Conclusion

The aim of this Chapter is to derive an optimal order a posteriori error estimates in term of the $L_\infty(H^1)$ for the fully semilinear parabolic problems in two cases when $f(u)$ Lipschitz and non Lipschitz are proved. The crucial tools in proving this error is the elliptic reconstruction techniques introduced by Makridakis and Nochetto 2003. This is consequently enabling us to use a posteriori error estimators derived for elliptic equation to obtain optimal order in terms of $L_\infty(H^1)$ norm for Lipschitz and non-Lipschitz nonlinearities. Some challenges have to be overcome due to non-linearity on the forcing term depending on Gronwall's Lemma and Sobolev embedding through continuation argument. Furthermore, this will give insight about designing adaptive algorithm, which allow use to control the cost of computations. In the future, this Chapter can be extended to the fully discrete case for semilinear parabolic interface problems in $L_\infty(L_2) + L_2(H^1)$ and $L_\infty(L_2)$ norms [18, 20–22].

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Author details

Younis Abid Sabawi[1,2]

1 Department of Mathematics, Faculty of Science and Health, Koya University Koya KOY45, Kurdistan Region – F.R. Iraq

2 Department of Mathematics Education, Faculty of Education, Tishk International University, Kurdistan – Iraq

*Address all correspondence to: younis.abid@koyauniversity.org; younis.sabawi@tiu.edu.iq

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