We are IntechOpen, the world's leading publisher of Open Access books
Built by scientists, for scientists

5,600
Open access books available

137,000
International authors and editors

170M
Downloads

154
Countries delivered to

Our authors are among the
TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Chapter

Solution of Nonlinear Partial Differential Equations by Mixture Adomian Decomposition Method and Sumudu Transform

Tarig M. Elzaki and Shams E. Ahmed

Abstract

This chapter is fundamentally centering on the application of the Adomian decomposition method and Sumudu transform for solving the nonlinear partial differential equations. It has instituted some theorems, definitions, and properties of Adomian decomposition and Sumudu transform. This chapter is an elegant combination of the Adomian decomposition method and Sumudu transform. Consequently, it provides the solution in the form of convergent series, then, it is applied to solve nonlinear partial differential equations.

Keywords: adomian decomposition method, sumudu transform, nonlinear partial differential equations

1. Introduction

Many of nonlinear phenomena are a necessary part in applied science and engineering fields [1]. The wide use of nonlinear partial differential equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically. In the past, active study attempts were given a large amount of attention to the study of getting exact or approximate solutions of this kind of equations.

Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations. For some examples of the traditional methods, such as, the separation of variables method, the method of characteristics, the σ-expansion method, the integral transforms and Hirota bilinear method [2–5]. Moreover, the recently developed methods like, Adomian decomposition method (ADM) [1, 6–9], He's semi – inverse method, the tanh method, the sinh – cosh method, the homotopy perturbation method (HPM) [3, 4, 10, 11], the differential transform method (DTM), the variational iteration method (VIM) [1, 5, 12], and the weighted finite difference.

In this chapter, our presentation will be based on applying the new method, namely the Adomian Decomposition Sumudu Transform Method (ADSTM) for solving the nonlinear partial differential equations. This method is an elegant combination of the Sumudu transform method and decomposition method.

IntechOpen

2. Sumudu Transform

A long time ago, differential equations wared a necessary part in all aspects of applied sciences and engineering fields. In this chapter we need to develop a new technique for help us to obtain the exact and approximate solutions of these differential equations.

Watugula [13] introduced a new integral transform and called it as Sumudu transform, which is defined as:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u} e^{\left(-\frac{t}{u}\right)} f(t)\, dt; \tag{1}$$

Watugula [13] applied this transforms to the solution of ordinary differential equations. Because of its useful properties, the Sumudu transforms helps in solving complex problems in applied sciences and engineering mathematics. Henceforward, is the definition of the Sumudu transforms and properties describing the simplicity of the transform.

Definition 1: The Sumudu transform of the function $f(t)$ is defined by:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u} e^{\left(-\frac{t}{u}\right)} f(t)\, dt \tag{2}$$

Or,

$$F(u) = S[f(t)] = \int_0^\infty f(ut)\, e^{-t}\, dt \tag{3}$$

For any function $f(t)$ and $-\tau_1 < u < \tau_2$.

3. The relation between Sumudu and Laplace transform

The Sumudu transform $F_s(u)$ of a function $f(t)$ defined for all real numbers $t \geq 0$. The Sumudu transform is essentially identical with the Laplace transform.

Given an initial $f(t)$ its Laplace transform $G(u)$ can be translated into the Sumudu transform $F_s(u)$ of f by means of the relation;

$$F(u) = \frac{G\left(\frac{1}{u}\right)}{u}, \text{ and it's inverse, } G(s) = \frac{F_s\left(\frac{1}{s}\right)}{s}$$

Theorem 1: Let $f(t)$ with Laplace transform $G(s)$, then, the Sumudu transform $F(u)$ of $f(t)$ is given by, $F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$.

Proof:

Form definition (1.1.1) we get:

$F(u) = \int_0^\infty e^{-t} f(ut)\, dt$, If we set $w = ut$ and $dt = \frac{dw}{u}$ then:

$$F(u) = \int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w) \frac{dw}{u} = \frac{1}{u} \int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)\, dw$$

By definition of the Laplace transform we get: $F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$.

Theorem 2: It deals with the effect of the differentiation of the function $f(t)$, k times on the Sumudu transform $F(u)$ if $S[f(t)] = F(u)$ then:

i. $S[f'(t)] = \frac{1}{u}[F(u) - f(0)]$

ii. $S[f''(t)] = \frac{1}{u^2}[F(u)] - \frac{1}{u^2}f(0) - \frac{1}{u}f'(u)$

iii. $S\left[f^{(n)}(t)\right] = \frac{1}{u^n}[F(u)] - \frac{1}{u^n}\sum_{k=0}^{n-1} u^k f^{(k)}(0) = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right]$

Where $f^{(0)}(0) = f(0)$, $f^{(k)}(0)$, $k = 1, 2, 3, \cdots, n-1$ are the nth-order derivatives of the function $f(t)$ evaluated at, $t = 0$.

Proof:

i. Using integration by parts,

ii.
$$S[f'(t)] = \left[\frac{1}{u}\exp\left(-\frac{t}{u}f(t)\right)\right]_0^\infty + \frac{1}{u}\int_0^\infty \frac{1}{u}\exp\left(-\frac{t}{u}\right)f(t)\,dt = -\frac{1}{u}f(0) + \frac{1}{u}F(u).$$

$$S[f'(t)] = \frac{1}{u}[F(u) - f(0)]$$

Using integration by parts;

$$S[f''(t)] = \left[\frac{1}{u}e^{\left(-\frac{t}{u}\right)}f'(t)\right]_0^\infty + \frac{1}{u}\int_0^\infty \frac{1}{u}e^{\left(-\frac{t}{u}\right)}f'(t)\,dt$$

From (i)

$$= -\frac{1}{u}f'(0) + \frac{1}{u}S[f'(t)]$$

$$S[f''(t)] = \frac{1}{u^2}[F(u)] - \frac{1}{u^2}f(0) - \frac{1}{u}f'(0).$$

iii. By definition the Laplace transform for $f^{(n)}(t)$ is given by

$$G_n(s) = s^n G(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0)$$

By using the relation between Sumudu and Laplace transform;

$$G_n\left(\frac{1}{u}\right) = \frac{G\left(\frac{1}{u}\right)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-(k+1)}}$$

Since $F_n(u) = \frac{G_n\left(\frac{1}{u}\right)}{u^n}$, we get:

$$u\,F_n(u) = \frac{u\,F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}\,u^{-1}}$$

$$F_n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}}$$

$$F_n(u) = u^{-n}F(u) - \sum_{k=0}^{n-1} u^{-n}u^k f^{(k)}(0)$$

$$S\left[f^{(n)}(t)\right] = F(u) = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right]$$

4. Adomian decomposition method

Many of nonlinear phenomena are a necessary part in applied science and engineering fields. Nonlinear equations are noticed in a different type of physical problems [1], such as fluid dynamics, plasma physics, solid mechanics, and quantum field theory.

The wide use of these equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically.

In the past, active study attempts were given a large amount of attention to the study of getting exact or approximate solutions of this kind of equations. It is significant to note that several powerful methods have been advanced for this purpose.

The Adomian decomposition method will be used in this chapter and in other chapters to deal with nonlinear equations. The Adomian decomposition method proves to be powerful, effective and successfully used to handle most types of linear or nonlinear ordinary or partial differential equations, and linear or nonlinear integral equations.

In the following, the Adomian scheme for calculating a wide variety of forms of nonlinearity.

5. Calculation of Adomian polynomials

It is well known that the Adomian decomposition method suggests the unknown linear function u may be represented by the decomposition series;

$$u = \sum_{n=0}^{\infty} u_n, \tag{4}$$

Where the components $u_n, n \geq 0$ can be elegantly computed in a recursive way. However, the nonlinear term $F(u)$, such as $u^2, u^3, u^4,\ \sin u, e^u, u u_x, {u_x}^2,$ etc., can be expressed by an infinite series of the so- called Adomian polynomials A_n given in the form;

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n). \tag{5}$$

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression;

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \ldots \tag{6}$$

Assuming that the nonlinear function is $F(u)$, therefore, by using (6), Adomian polynomials are given by;

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1 F'(u_0),\\
A_2 &= u_2 F'(u_0) + \frac{1}{2!}\,{u_1}^2 F''(u_0),\\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}\,{u_1}^3 F'''(u_0).
\end{aligned} \tag{7}$$

Other polynomials can be generated in a similar manner.
Substituting (7) into (5) gives;

$$F(u) = A_0 + A_1 + A_2 + A_3 + \ldots = F(u_0) + (u_1 + u_2 + u_3 + \ldots)F'(u_0)$$
$$+\frac{1}{2!}\left(u_1{}^2 + 2u_1u_2 + u_2{}^2 + \ldots\right)F''(u_0) + \frac{1}{3!}\left(u_1{}^3 + 3u_1{}^2u_2 + 3u_1{}^2u_3 + \ldots\right)F'''(u_0) + \ldots$$
$$= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u - u_0)^2F''(u_0) + \ldots.$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used.

In the following, we will calculate Adomian polynomials for several forms of nonlinearity.

5.1 Nonlinear polynomials

$$\text{If } F(u) = u^2$$

The polynomials can be found as follows:

$$A_0 = F(u_0) = u_0{}^2,\ \ A_1 = u_1F'(u_0) = 2u_0u_1,\ \ A_2 = u_2F'(u_0) + \frac{1}{2!}u_1{}^2F''(u_0) = 2u_0u_2 + u_1{}^2,$$
$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1{}^3F'''(u_0) = 2u_0u_3 + 2u_1u_2.$$

And so on. Proceeding as before, we find $u^3, u^4, u^5, \ldots$, etc.

5.2 Nonlinear derivatives

Case1. $F(u) = (u_x)^2$

$$A_0 = u_{0x}{}^2,\quad A_1 = 2u_{0x}u_{1_x},\quad A_2 = 2u_{0x}u_{2x} + u_{1x}{}^2,\quad A_3 = 2u_{0x}u_{3x} + 2u_{1x}u_{2x}.$$

And so on. In a similar, we get $u_x{}^3, u_x{}^4, u_x{}^5, \ldots$, etc.

Case 2. $F(u) = uu_x = \frac{1}{2}L_x(u^2)$

The A_n polynomials in this case given by;

$$A_0 = F(u_0) = u_0u_{0_x},\ \ A_1 = \frac{1}{2}L_x(2u_0u_1) = u_{0x}u_1 + u_0u_{1_x},$$
$$A_2 = \frac{1}{2}L_x\left(2u_0u_2 + u_1{}^2\right) = u_{0_x}u_2 + u_0u_{2_x}u_0 + u_1u_{1_x},$$
$$A_3 = \frac{1}{2}L_x(2u_0u_3 + 2u_1u_2) = u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0.$$

And so on.

5.3 Trigonometric nonlinearity

$$\text{If } F(u) = \sin u$$

The Adomian polynomials for this form nonlinearity are given by;

$$A_0 = \sin u_0, \quad A_1 = u_1 \cos u_0, \ A_2 = u_2 \cos u_0 - \frac{1}{2!} {u_1}^2 \sin u_0, \ A_3 = u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} {u_1}^3 \cos u_0.$$

And so on. In a similar way, we find $F(u) = \cos u$.

5.4 Hyperbolic nonlinearity

$$\text{If } F(u) = \sinh u$$

The A_n polynomials in this case are given by;

$$A_0 = \sinh u_0, \ A_1 = u_1 \cosh u_0, \ A_2 = u_2 \cosh u_0 + \frac{1}{2!} {u_1}^2 \sinh u_0, \ A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!} {u_1}^3 \cosh u_0.$$

And so on. In a parallel manner, Adomian polynomials can be calculated for $F(u) = \cosh u$.

5.5 Exponential nonlinearity

$$\text{If } F(u) = e^u$$

The Adomian polynomials in this form of nonlinearity are given by;

$$A_0 = e^{u_0}, \ A_1 = u_1 e^{u_0}, \ A_2 = \left(u_2 + \frac{1}{2!} {u_1}^2\right) e^{u_0}, \ A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} {u_1}^3\right) e^{u_0}.$$

And so on. Proceeding as a before, we find $F(u) = e^{-u}$.

5.6 Logarithmic nonlinearity

$$\text{If } F(u) = \ln u, u > 0$$

The A_n polynomials for logarithmic nonlinearity are given by;

$$A_0 = \ln u_0, \ A_1 = \frac{u_1}{u_0}, \ A_2 = \frac{u_2}{u_0} - \frac{1}{2} \frac{{u_1}^2}{{u_0}^2}, \ A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{{u_0}^2} + \frac{1}{3} \frac{{u_1}^3}{{u_0}^3}.$$

And so on. In a similar way, we find $F(u) = \ln(1+u), \ -1 < u \leq 1$.

6. A New algorithm for calculating Adomian polynomials (The alternative algorithm for calculating Adomian polynomials)

It is well known about the main disadvantage of the calculating Adomian polynomials A_n, that it is a difficult method to perform calculation so called nonlinear terms. There is an alternative algorithm to reduce the demerits of formula introduced before, which depends mainly on algebraic, trigonometric identities and on Taylor expansions.

In the alternative algorithm which is a simple and reliable may be employed to calculate Adomian Polynomials A_n.

The new algorithm will be clarified by discussing the following suitable forms of nonlinearity.

6.1 Nonlinear polynomials

$$\text{If } F(u) = u^2$$

We first set,

$$u = \sum_{n=0}^{\infty} u_n. \tag{8}$$

Substituting (8) into $F(u) = u^2$ gives;

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + \dots)^2. \tag{9}$$

Expanding the expression at the right- hand side gives;

$$F(u) = {u_0}^2 + 2u_0 u_1 + 2u_0 u_2 + {u_1}^2 + 2u_0 u_3 + 2u_1 u_2 + \dots \tag{10}$$

The expansion in (10) can be rearranged by grouping all terms with the sum of the subscripts of the components is the same. This means that we can rewrite (10) as;

$$F(u) = \underbrace{{u_0}^2}_{A_0} + \underbrace{2u_0 u_1}_{A_1} + \underbrace{2u_0 u_2 + {u_1}^2}_{A_2} + \underbrace{2u_0 u_3 + 2u_1 u_2}_{A_3} + \dots. \tag{11}$$

This gives Adomian polynomials for $F(u) = u^2$ by;

$$A_0 = {u_0}^2, \quad A_1 = 2u_0 u_1, \quad A_2 = 2u_0 u_2 + {u_1}^2, \quad A_3 = 2u_0 u_3 + 2u_1 u_2.$$

And so on. Proceeding as before, we get $u^3, u^4, u^5, \dots$, etc.

6.2 Nonlinear derivatives

Case 1. If $F(u) = {u_x}^2$.
We first set;

$$u_x = \sum_{n=0}^{\infty} u_{n\,x}. \tag{12}$$

Substituting (12) into $F(u) = {u_x}^2$ giving;

$$F(u) = (u_{0x} + u_{1x} + u_{2x} + u_{3x} + u_{4x} + \dots)^2. \tag{13}$$

Squaring the right – hand side gives;

$$F(u) = {u_{0x}}^2 + 2u_{0x} u_{1x} + 2u_{0x} u_{2x} + {u_{1x}}^2 + 2u_{0x} u_{3x} + 2u_{1x} u_{2x} + \dots \tag{14}$$

Grouping the terms as discussed above, we find;

$$F(u) = \underbrace{u_{0x}^{\ 2}}_{A_0} + \underbrace{2u_{0x}u_{1x}}_{A_1} + \underbrace{2u_{0x}u_{2x} + u_{1x}^{\ 2}}_{A_2} + \underbrace{2u_{0x}u_{3x} + 2u_{1x}u_{2x}}_{A_3} + \ldots \tag{15}$$

Adomian polynomials are given by;

$$A_0 = u_{0x}^{\ 2}, \quad A_1 = 2u_{0x}u_{1_x}, \ A_2 = 2u_{0x}u_{2x} + u_{1x}^{\ 2}, \quad A_3 = 2u_{0x}u_{3x} + 2u_{1x}u_{2x}.$$

Case 2. $F(u) = uu_x$

Note that this form of nonlinearity appears in advection problems and in nonlinear Burgers equations. We first set;

$$u = \sum_{n=0}^{\infty} u_n, u_x = \sum_{n=0}^{\infty} u_{nx}. \tag{16}$$

Substituting (16) into $F(u) = uu_x$ yields;

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + \ \ldots) \times (u_{0x} + u_{1x} + u_{2x} + u_{3x} + u_{4x} + \ \ldots); \tag{17}$$

Multiplying the two factors gives;

$$\begin{aligned} F(u) &= u_0u_{0x} + u_{0x}u_1 + u_0u_{1x} + u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0 + u_{0x}u_3 + u_{1x}u_2 + \\ &+ u_{2x}u_1 + u_{3x}u_0 + u_{0x}u_4 + u_0u_{4x} + u_{1x}u_3 + u_1u_{3x} + u_2u_{2x} + \ \ldots \end{aligned}. \tag{18}$$

Proceeding with grouping the terms are obtained;

$$\begin{aligned} F(u) = &\underbrace{u_0u_{0x}}_{A_0} + \underbrace{u_{0x}u_1 + u_0u_{1x}}_{A_1} + \underbrace{u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0}_{A_2} \\ &+ \underbrace{u_{0x}u_3 + u_{1x}u_2 + u_{2x}u_1 + u_{3x}u_0}_{A_3} \ldots \end{aligned} \tag{19}$$

Consequently, the Adomian polynomials are given by;

$$\begin{aligned} &A_0 = u_0u_{0_x}, \quad A_1 = u_{0x}u_1 + u_0u_{1_x}, \quad A_2 = u_{0_x}u_2 + u_0u_{2_x}u_0 + u_1u_{1_x}, \\ &A_3 = u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0. \end{aligned}$$

Proceeding as before, we find $F(u) = u^2u_x$.

6.3 Trigonometric nonlinearity

$$\text{If } F(u) = \sin u$$

First, we should be separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute;

$$u = \sum_{n=0}^{\infty} u_n; \tag{20}$$

Into $F(u) = \sin u$ to obtain;

$$F(u) = \sin\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \dots)\right]. \tag{21}$$

To separate A_0, recall the trigonometric identity;

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi. \tag{22}$$

This means that;

$$F(u) = \sin u_0 \cos(u_1 + u_2 + u_3 + u_4 + \dots) + \cos u_0 \sin(u_1 + u_2 + u_3 + u_4 + \dots). \tag{23}$$

Separating $F(u_0) = \sin u_0$ from other factors and using Taylor expansion for, $\cos(u_1 + u_2 + u_3 + u_4 + .\dots)$ and, $\sin(u_1 + u_2 + u_3 + u_4 + .\dots)$ gives;

$$\begin{aligned} F(u) = \sin u_0 &\left(1 - \frac{1}{2!}(u_1 + u_2 + \dots)^2 + \frac{1}{4!}(u_1 + u_2 + \dots)^4 - \dots\right) + \\ &+ \cos u_0\left((u_1 + u_2 + \dots) - \frac{1}{3!}(u_1 + u_2 + \dots)^3 + \dots\right) \end{aligned}, \tag{24}$$

So that;

$$\begin{aligned} F(u) = \sin u_0 &\left(1 - \frac{1}{2!}({u_1}^2 + 2u_1u_2 + \dots) + \dots\right) \\ &+ \cos u_0\left((u_1 + u_2 + \dots) - \frac{1}{3!}{u_1}^3 + \dots\right). \end{aligned} \tag{25}$$

The last expansion can be rearranged by grouping all terms with the same sum of subscripts. This leads to;

$$\begin{aligned} F(u) = \underbrace{\sin u_0}_{A_0} + \underbrace{u_1\cos u_0}_{A_1} + &\underbrace{u_2\cos u_0 - \frac{1}{2!}{u_1}^2\sin u_0 +}_{A_2} \\ &\underbrace{+ u_3\cos u_0 - u_1u_2\sin u_0 - \frac{1}{3!}{u_1}^3\cos u_0}_{A_3} + \dots \end{aligned} \tag{26}$$

This completes the calculation of the Adomian polynomials for nonlinear operator $F(u) = \sin u$, therefore we write;

$$\begin{aligned} &A_0 = \sin u_0,\ A_1 = u_1\cos u_0,\ A_2 = u_2\cos u_0 - \frac{1}{2!}{u_1}^2\sin u_0, \\ &A_3 = u_3\cos u_0 - u_1u_2\sin u_0 - \frac{1}{3!}{u_1}^3\cos u_0. \end{aligned}$$

And so on. In a similar way, we find $F(u) = \cos u$.

6.4 Hyperbolic nonlinearity

If $F(u) = \sinh u$ then, we first substitute

$$u = \sum_{n=0}^{\infty} u_n; \tag{27}$$

Into $F(u) = \sinh u$ to obtain;

$$F(u) = \sinh\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \dots)\right]. \tag{28}$$

To calculate A_0, recall the hyperbolic identity;

$$\sinh(\theta + \phi) = \sinh\theta \cosh\phi + \cosh\theta \sinh\phi. \tag{29}$$

Accordingly, Eq. (29) becomes;

$$\begin{aligned} F(u) = {} & \sinh u_0 \cosh(u_1 + u_2 + u_3 + u_4 + \dots) \\ & + \cosh u_0 \sinh(u_1 + u_2 + u_3 + u_4 + \dots). \end{aligned} \tag{30}$$

Separating $F(u_0) = \sinh u_0$ from other factors and using Taylor expansion for $\cosh(u_1 + u_2 + u_3 + u_4 + \dots)$ and $\sinh(u_1 + u_2 + u_3 + u_4 + \dots)$ gives;

$$\begin{aligned} F(u) = {} & \sinh u_0 \left(1 + \frac{1}{2!}(u_1 + u_2 + \dots)^2 + \frac{1}{4!}(u_1 + u_2 + \dots)^4 + \dots\right) \\ & + \cosh u_0 \left((u_1 + u_2 + \dots) + \frac{1}{3!}(u_1 + u_2 + \dots)^3 + \dots\right) \\ = {} & \sinh u_0 \left(1 + \frac{1}{2!}\left({u_1}^2 + 2u_1u_2 + \dots\right) + \dots\right) + \cosh u_0 \left((u_1 + u_2 + \dots) + \frac{1}{3!}{u_1}^3 + \dots\right) \end{aligned}$$

By grouping all terms with the same sum of subscripts we find

$$\begin{aligned} F(u) = {} & \underbrace{\sinh u_0}_{A_0} + \underbrace{u_1 \cosh u_0}_{A_1} + \underbrace{u_2 \cosh u_0 + \frac{1}{2!}{u_1}^2 \sinh u_0}_{A_2} \\ & + \underbrace{u_3 \cosh u_0 + u_1u_2 \sinh u_0 - \frac{1}{3!}{u_1}^3 \cosh u_0}_{A_3} + \dots \end{aligned}$$

Consequently, the Adomian polynomials for $F(u) = \sinh u$ are given by;

$$\begin{aligned} & A_0 = \sinh u_0, \; A_1 = u_1 \cosh u_0, \; A_2 = u_2 \cosh u_0 + \frac{1}{2!}{u_1}^2 \sinh u_0, \\ & A_3 = u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!}{u_1}^3 \cosh u_0. \end{aligned}$$

Similarly as before, we find $F(u) = \cosh u$.

6.5 Exponential nonlinearity

$$\text{If } F(u) = e^u.$$

Substituting

$$u = \sum_{n=0}^{\infty} u_n; \tag{31}$$

Into $F(u) = e^u$ gives;

$$F(u) = e^{(u_0+u_1+u_2+u_3+u_4+\dots)}. \tag{32}$$

Or equivalently;

$$F(u) = e^{u_0} \times e^{(u_1+u_2+u_3+u_4+\dots)}. \tag{33}$$

Keeping the term $F(u_0) = e^{u_0}$ and using Taylor expansion for the other factors we obtain;

$$F(u) = e^{u_0} \times \left(1 + (u_1 + u_2 + u_3 + \dots) + \frac{1}{2!}(u_1 + u_2 + u_3 + \dots)^2 + \dots\right). \tag{34}$$

By grouping all terms with an identical sum of subscripts we find

$$F(u) = \underbrace{e^{u_0}}_{A_0} + \underbrace{u_1 e^{u_0}}_{A_1} + \underbrace{\left(u_2 + \frac{1}{2!}u_1{}^2\right)e^{u_0}}_{A_2} + \underbrace{\left(u_3 + u_1 u_2 + \frac{1}{3!}u_1{}^3\right)e^{u_0}}_{A_3} + \dots. \tag{35}$$

It then follows that;

$$A_0 = e^{u_0}, \quad A_1 = u_1 e^{u_0}, \quad A_2 = \left(u_2 + \frac{1}{2!}u_1{}^2\right)e^{u_0}, \quad A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!}u_1{}^3\right)e^{u_0}.$$

And so on. Proceeding as a before, we find $F(u) = e^{-u}$.

6.6 Logarithmic nonlinearity

$$\text{If } F(u) = \ln u, u > 0$$

Substituting

$$u = \sum_{n=0}^{\infty} u_n; \tag{36}$$

Into $F(u) = \ln u$ gives;

$$F(u) = \ln(u_0 + u_1 + u_2 + u_3 + u_4 + \dots). \tag{37}$$

Eq. (34) can be written as;

$$F(u) = \ln\left(u_0\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right)\right). \tag{38}$$

Using the identity $\ln(\alpha\beta) = \ln\alpha + \ln\beta$, Eq. (38) becomes;

$$F(u) = \ln(u_0) + \ln\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right). \tag{39}$$

Separating $F(u_0) = \ln(u_0)$ and using Taylor expansion of the remaining term, we obtain;

$$F(u) = \ln(u_0) + \left\{ \begin{array}{l} \left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right) - \frac{1}{2}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right)^2 + \frac{1}{3}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right)^3 \\ -\frac{1}{4}\left(\frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right)^4 + \dots \end{array} \right\} \tag{40}$$

Proceeding as before, Eq. (40) can be rewritten as;

$$F(u) = \underbrace{\ln(u_0)}_{A_0} + \underbrace{\frac{u_1}{u_0}}_{A_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2}\frac{{u_1}^2}{{u_0}^2}}_{A_2} + \underbrace{\frac{u_3}{u_0} - \frac{u_1 u_2}{{u_0}^2} + \frac{1}{3}\frac{{u_1}^3}{{u_0}^3}}_{A_3} + \dots \tag{41}$$

Based on this, the Adomian polynomials are given by;

$$A_0 = \ln(u_0), \quad A_1 = \frac{u_1}{u_0}, \quad A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{{u_1}^2}{{u_0}^2}, \quad A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{{u_0}^2} + \frac{1}{3}\frac{{u_1}^3}{{u_0}^3}.$$

And so on. In a like manner, we obtain $F(u) = \ln(1+u), \ -1 < u \le 1$.

7. Adomian decomposition Sumudu transform method for solving nonlinear partial differential equations

In this section, we will concentrate our study on the nonlinear PDEs. There are many nonlinear partial differential equations which are quite useful and applicable in engineering and physics.

The nonlinear phenomena that appear in the many scientific fields' such as solid state physics, plasma physics, fluid mechanics and quantum field theory can be modeled by nonlinear differential equations. The significance of obtaining exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is yet an important problem that needs new methods to develop new techniques for obtaining analytical solutions. Several powerful mathematical methods are used for this purpose. We, propose a new method, namely Adomian Decomposition Sumudu Transform Method (ADSTM) for solving nonlinear equations. This method is a combination of Sumudu transform and decomposition method which was introduced by D. Kumar, J. Singh and S. Rathore.

(ADSTM) provides the solution for nonlinear equations in the form of convergent series. This forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

To illustrate the basic idea of this method, we consider a general non-homogeneous partial differential equation with the initial conditions of the form:

$$\begin{array}{c} DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \\ U(x,0) = h(x), U_t(x,0) = f(x). \end{array}; \tag{42}$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (42), we get:

$$S[DU(x,t)] + S[RU(x,t)] + S[N(x,t)] = S[g(x,t)]; \tag{43}$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have:

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + uf(x) - u^2 S[RU(x,t) + NU(x,t)]. \tag{44}$$

If we apply the inverse operator S^{-1} to both sides of Eq. (44), we obtain:

$$U(x,t) = G(x,t) - S^{-1}\left[u^2 S[RU(x,t) + NU(x,t)]\right]. \tag{45}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomain decomposition method:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \tag{46}$$

The nonlinear term can be decomposed as:

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U); \tag{47}$$

For some Adomain polynomials $A_n(U)$ that are given by:

$$A_n(U_0, U_1, U_2, \dots, U_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0}, n = 0, 1, 2, \dots$$

Substituting Eq. (46) and Eq. (47) in Eq. (45), we get:

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1}\left[u^2 S\left[R\sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n(U)\right]\right]. \tag{48}$$

Accordingly, the formal recursive relation is defined by:

$$\begin{aligned} U_0(x,t) &= G(x,t), \\ U_{k+1}(x,t) &= -S^{-1}\left[u^2 S[RU_k + A_k]\right].k \geq 0. \end{aligned} \tag{49}$$

The Adomian decomposition Sumudu transform method will be illustrated by discussing the following examples.

Example 1: Consider the following nonlinear partial differential equation:

$$U_t + UU_x = 0; \tag{50}$$

With the initial condition:

$$U(x,0) = x. \tag{51}$$

Taking the Sumudu transform of both sides of Eq. (50) and using the initial condition, we have:

$$S[U(x,t)] = x - uS[UU_x]. \tag{52}$$

Applying S^{-1} to both sides of Eq. (52) implies that:

$$U(x,t) = x - S^{-1}[u\,S[UU_x]]; \tag{53}$$

Following the technique, if we assume an infinite series of the form (53), we obtain:

$$\sum_{n=0}^{\infty} U_n(x,t) = x - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right]. \tag{54}$$

Where $A_n(U)$ are Adomian polynomials that represent the nonlinear terms. The first few components of the Adomian polynomials are given by;

$$\begin{aligned} &A_0(U) = U_0\,U_{0_x}, \\ &A_1(U) = U_0\,U_{1_x} + U_1\,U_{0_x}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{55}$$

This gives the recursive relation:

$$\begin{aligned} U_0(x,t) &= x, \\ U_{k+1}(x,t) &= -S^{-1}[u\,S[A_k]], k \geq 0. \end{aligned} \tag{56}$$

The first few components are given by:

$$\begin{aligned} U_0(x,t) &= x, \\ U_1(x,t) &= -S^{-1}[u\,S[A_0]] = -xt, \\ U_2(x,t) &= -S^{-1}[u\,S[A_1]] = xt^2, \\ U_3(x,t) &= -S^{-1}[u\,S[A_2]] = -xt^3. \end{aligned} \tag{57}$$

And so on. The solution in a series form is given by:

$$U(x,t) = x\left(1 - t + t^2 - t^3 + \ \ldots\right); \tag{58}$$

And in a closed form of:

$$U(x,t) = \frac{x}{1+t}. \tag{59}$$

Example 2: Consider the following nonlinear partial differential equation:

$$U_t + UU_x = x + xt^2; \tag{60}$$

With the initial condition:

$$U(x,0) = 0. \tag{61}$$

Proceeding as in **Example 1**, Eq. (60) becomes:

$$\sum_{n=0}^{\infty} U_n(x,t) = xt + \frac{xt^3}{3} - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right]. \tag{62}$$

The modified decomposition method admits the of a modified recursive relation given by:

$$U_0(x,t) = xt,$$
$$U_1(x,t) = \frac{xt^3}{3} - S^{-1}[uS[A_0]] \tag{63}$$
$$U_{k+1}(x,t) = -S^{-1}[uS[A_k]], k \geq 1.$$

Consequently, we obtain:

$$U_0(x,t) = xt,$$
$$U_1(x,t) = \frac{xt^3}{3} - S^{-1}\left[uS\left[xt^2\right]\right] = 0 \tag{64}$$
$$U_{k+1}(x,t) = 0, k \geq 1.$$

In few of Eq. (64), the exact solution is given by:

$$U(x,t) = xt. \tag{65}$$

Example 3: Consider the nonlinear partial differential equation:

$$U_{tt} + {U_x}^2 + U - U^2 = te^{-x}; \tag{66}$$

With the initial condition

$$U(x,0) = 0, U_t(x,0) = e^{-x}. \tag{67}$$

By taking Sumudu transform for (66) and using (67) we obtain:

$$S[U(x,t)] = u^3e^{-x} + ue^{-x} - u^2S\left[{U_x}^2 - U^2 + U\right]. \tag{68}$$

Applying S^{-1} to both sides of (68) we obtain;

$$U(x,t) = te^{-x} + \frac{1}{6}t^3e^{-x} - S^{-1}\left[u^2S\left[{U_x}^2 - U^2 + U\right]\right]. \tag{69}$$

Substituting;

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \tag{70}$$

And the nonlinear terms of;

$${U_x}^2 = \sum_{n=0}^{\infty} A_n, U^2 = \sum_{n=0}^{\infty} B_n. \tag{71}$$

Into (69) gives;

$$\sum_{n=0}^{\infty} U_n(x,t) = te^{-x} + \frac{1}{6}t^3e^{-x} - S^{-1}\left[u^2S\left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} U_n(x,t) - \sum_{n=0}^{\infty} B_n\right)\right] \tag{72}$$

This gives the modified recursive relation;

$$\begin{aligned} U_0(x,t) &= te^{-x}, \\ U_1(x,t) &= \frac{1}{6}t^3e^{-x} - \ L_t{}^{-1}(A_0 + U_0 - B_0) \\ U_{k+1}(x,t) &= -\ L_t{}^{-1}(A_k + U_k - B_k), k \geq 1. \end{aligned} \tag{73}$$

The first few of the components are given by;

$$\begin{aligned} U_0(x,t) &= te^{-x}, \\ U_1(x,t) &= \frac{1}{6}t^3e^{-x} - \ L_t{}^{-1}(A_0 + U_0 - B_0) = 0, \\ U_{k+1}(x,t) &= 0, k \geq 1. \end{aligned} \tag{74}$$

The solution in a closed form is given by;

$$U(x,t) = te^{-x}. \tag{75}$$

Example 4: Consider the following nonlinear partial differential equation,

$$U_{tt} + U^2 - U_x{}^2 = 0; \tag{76}$$

With the initial conditions

$$U(x,0) = 0\ , U_t(x,0) = e^x. \tag{77}$$

By taking Sumudu transform for (76) and using (77) we obtain:

$$S[U(x,t)] = ue^x + u^2 S\left[U_x{}^2 - U^2\right]. \tag{78}$$

By applying the inverse Sumudu transform of (78), we get:

$$U(x,t) = te^x + S^{-1}\left[u^2 S\left[U_x{}^2 - U^2\right]\right]; \tag{79}$$

This assumes a series solution of the function $U(x,t)$is given by:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \tag{80}$$

Using (80) into (79), we get:

$$\sum_{n=0}^{\infty} U_n(x,t) = \ te^x + \ S^{-1}\left[u^2 S\left[\sum_{n=0}^{\infty} A_n(U) - \sum_{n=0}^{\infty} B_n(U)\right]\right]. \tag{81}$$

Where $A_n(U)$ and $B_n(U)$ are Adomian polynomials that represents nonlinear terms.

The few components of the Adomian polynomials are given as follows:

$$\begin{aligned} A_0(U) &= U_{0_x}{}^2\ ,\ A_1(U) = 2U_{0_x}U_{1_x}, \\ B_0(U) &= U_0{}^2\ \ ,\ \ B_1(U) = 2U_0U_1\ , \end{aligned} \tag{82}$$

And so on. From the above equations we obtain:

$$U_0(x,t) = te^x, \\ U_{k+1}(x,t) = S^{-1}\left[u^2 S[A_k - B_k]\right], k \geq 0. \tag{83}$$

The first few terms of $U_n(x,t)$ follows immediately upon setting:

$$U_1(x,t) = S^{-1}\left[u^2 S[A_0 - B_0]\right] = S^{-1}\left[u^2 S\left[{U_{0_x}}^2 - {U^2}_0\right]\right] = 0 \\ U_{k+1}(x,t) = 0, k \geq 1. \tag{84}$$

Therefore the solution obtained by ADSTM is given as follows:

$$U(x,t) = te^x.$$

8. Nonlinear physical models

Now we will, concentrate our study on the linear and nonlinear particular applications that appear in applied science. The wide use of these physical significant problems is the most important reason why they have drawn mathematician's attention in recent years.

Nonlinear partial differential equations have witnessed remarkable improvement. Nonlinear problems appear in the many scientific fields' such as gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics, acoustics, and others. Several approaches have been used such as the Adomian decomposition method, the variational iteration method, and the characteristics method and perturbation techniques to examine these problems.

(ADSTM) gives the solution of nonlinear equations in the form of convergent series. The main advantage of this method is its potentiality of combining two powerful methods for deriving exact and approximate solution of nonlinear equations. This forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

The following section offers the effectiveness of the Adomian decomposition Sumudu transform method (ADSTM) in solving nonlinear physical models.

Example 5: Consider the following inhomogeneous advection problem:

$$U_t + UU_x = 2t + x + t^3 + xt^2; \tag{85}$$

With the initial condition:

$$U(x,0) = 0. \tag{86}$$

Following discussion presented above, we obtain the recursive relation;

$$U_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt}{3}, \\ U_{k+1}(x,t) = -S^{-1}[uS[A_k]], k \geq 0. \tag{87}$$

This gives;

$$U_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\ U_1(x,t) = -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8. \tag{88}$$

It is easily observed that two noise term appears in the components $U_0(x,t)$ and $U_1(x,t)$. By canceling these terms from $U_0(x,t)$, the remaining non-canceled term of $U_0(x,t)$ may provide the exact solution.

The exact solution is given by;

$$U(x,t) = t^2 + xt.$$

Example 6: Consider the following nonlinear Klein – Gordon equation:

$$U_{tt} - U_{xx} + U^2 = x^2 t^2; \tag{89}$$

Subject to the initial conditions:

$$U(x,0) = 0 \ , \ U_t(x,t) = x. \tag{90}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} U_0(x,t) &= \ xt + \frac{1}{12}\, x^2 t^4, \\ U_{k+1}(x,t) &= S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_k]\right], k \geq 0. \end{aligned} \tag{91}$$

So the Adomian polynomials A_n are given as follows;

$$\begin{aligned} A_0 &= \ {U_0}^2, \\ A_1 &= \ 2U_0 U_1, \\ A_2 &= 2U_0 U_2 + {U_1}^2. \end{aligned}$$

And so on. Using modified recursive relation Eq. (91) can be rewritten in the scheme;

$$\begin{aligned} U_0(x,t) &= \ xt, \\ U_1(x,t) &= \ \frac{1}{12}\, x^2 t^4 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right] \ , \\ U_{k+1}(x,t) &= S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_k]\right], k \geq 1. \end{aligned} \tag{92}$$

This lead to;

$$\begin{aligned} U_0(x,t) &= \ xt, \\ U_1(x,t) &= \ \frac{1}{12}\, x^2 t^4 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right] \ = 0, \\ U_{k+1}(x,t) &= \ 0, k \geq 1. \end{aligned} \tag{93}$$

Therefore, the exact solution is given by;

$$U(x,t) = xt.$$

Example 7: Consider the following Sine-Gordon equation with the given initial conditions:

$$U_{tt}(x,t) - \ U_{xx}(x,t) = \ \sin U; \tag{94}$$

Subject to the initial conditions;

$$U(x,0) = \frac{\pi}{2} \ , \ U_t(x,t) = 0. \tag{95}$$

Using the recursive scheme yields;

$$\begin{aligned} &U_0(x,t) = \frac{\pi}{2}, \\ &U_{k+1}(x,t) = S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_k]\right], k \geq 0. \end{aligned} \tag{96}$$

The first few the Adomian polynomials for $\sin U$ are given as;

$$\begin{aligned} A_0 &= \sin U_0, \\ A_1 &= U_1 \cos U_0, \\ A_2 &= U_2 \cos U_0 - \frac{1}{2!} U_1{}^2 \sin U_0, \\ A_3 &= U_3 \cos U_0 - U_1 U_2 \sin U_0 - \frac{1}{3!} U_1{}^3 \cos U_0. \end{aligned} \tag{97}$$

Combining (96) and (97) leads to;

$$\begin{aligned} &U_0(x,t) = \frac{\pi}{2}, \\ &U_1(x,t) = S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_0]\right] = \frac{t^2}{2!}, \\ &U_2(x,t) = S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_1]\right] = 0, \\ &U_3(x,t) = S^{-1}\left[u^2 S\left[(U_2)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_2]\right] = -\frac{1}{240} t^6. \end{aligned} \tag{98}$$

And so on. The series solution is;

$$U(x,t) = \frac{\pi}{2} + \frac{t^2}{2!} - \frac{1}{240} t^6 + \ldots$$

Example 8: Consider the following one – dimensional Burgers equation:

$$U_t = U_{xx} - UU_x; \tag{99}$$

Subject to the initial conditions:

$$U(x,0) = x. \tag{100}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} &U_0(x,t) = x, \\ &U_{k+1}(x,t) = S^{-1}\left[uS\left[(U_k)_{xx}\right]\right] - S^{-1}[uS[A_k]], k \geq 0. \end{aligned} \tag{101}$$

Using the Adomian polynomials we obtain;

$$\begin{aligned} &U_0(x,t) = x, \\ &U_1(x,t) = S^{-1}\left[a\,uS\left[(U_0)_{xx}\right]\right] - S^{-1}[uS[A_0]] = -xt, \\ &U_2(x,t) = S^{-1}\left[a\,uS\left[(U_1)_{xx}\right]\right] - S^{-1}[uS[A_1]] = xt^2, \\ &U_3(x,t) = S^{-1}\left[a\,uS\left[(U_2)_{xx}\right]\right] - S^{-1}[uS[A_2]] = -xt^3. \end{aligned} \tag{102}$$

Summing these iterates gives the series solution;

$$U(x,t) = x\left(1 - t + t^2 - t^3 + \ \ldots\right); \tag{103}$$

Consequently, the exact solution is given by;

$$U(x,t) = \frac{x}{1+t}.$$

Example 9: Consider the following nonlinear Schrodinger equation:

$$i\,U_t + U_{xx} - 2|U|^2U = 0; \tag{104}$$

Subject to the initial condition:

$$U(x,0) = e^{ix}. \tag{105}$$

Following the discussion presented above, we find;

$$\begin{aligned}
&U_0(x,t) = e^{ix},\\
&U_1(x,t) = S^{-1}\left[iuS\left[(U_0)_{xx}\right]\right] - S^{-1}[2iuS[A_0]] = -3ite^{ix},\\
&U_2(x,t) = S^{-1}\left[iuS\left[(U_1)_{xx}\right]\right] - S^{-1}[2iuS[A_1]] = \frac{1}{2!}\,(3it)^2e^{ix},\\
&U_3(x,t) = S^{-1}\left[iuS\left[(U_2)_{xx}\right]\right] - S^{-1}[2iuS[A_2]] = -\frac{1}{3!}\,(3it)^3e^{ix}.
\end{aligned} \tag{106}$$

In a few of (106), the series solution is given by;

$$U(x,t) = e^{ix}\left(1 - (3it) + \frac{1}{2!}\,(3it)^2 - \frac{1}{3!}\,(3it)^3 + \ \ \ldots\right); \tag{107}$$

The exact solution is;

$$U(x,t) = e^{i\,(x-3t)}$$

Example 9: Consider the following homogeneous nonlinear KdV equation:

$$U_t - 6UU_x + \ U_{xxx} = 0; \tag{108}$$

Subject to the initial condition;

$$U(x,0) = 6x. \tag{109}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned}
&U_0(x,t) = 6x,\\
&U_{k+1}(x,t) = -S^{-1}\left[uS\left[(U_k)_{xxx}\right]\right] + S^{-1}[6uS[A_k]], k \geq 0.
\end{aligned} \tag{110}$$

That gives the first few the components by;

$$\begin{aligned}
&U_0(x,t) = 6x,\\
&U_1(x,t) = -S^{-1}\left[uS\left[(U_0)_{xxx}\right]\right] + S^{-1}[6uS[A_0]] = 6^3xt,\\
&U_2(x,t) = -S^{-1}\left[b\,uS\left[(U_1)_{xxx}\right]\right] + S^{-1}[auS[A_1]] = 6^5xt^2,\\
&U_3(x,t) = -S^{-1}\left[buS\left[(U_2)_{xxx}\right]\right] + S^{-1}[6uS[A_2]] = 6^7xt^3.
\end{aligned} \tag{111}$$

In a few of (111), the series solution is given by;

$$U(x,t) = 6x\left(1 + (36t) + (36t)^2 + (36t)^3 + \ldots\right); \tag{112}$$

The exact solution is;

$$U(x,t) = \frac{6x}{1 - 36t}, |36t| < 1.$$

9. Conclusion

In this chapter, we have combined the Adomian decomposition method and Sumudu transform to solve some of the nonlinear partial differential equations. This method has advantages of converting to the exact or approximate solutions and can easily handle a wide class of nonlinear differential equations.

Author details

Tarig M. Elzaki[1]* and Shams E. Ahmed[2,3]

1 Mathematics Department, Faculty of Sciences and Arts, Alkamil, Jeddah, University of Jeddah, Kingdom of Saudi Arabia

2 Department of Mathematic, Faculty of Sciences and Arts-Tubarjal, Aljouf University, Saudi Arabia

3 Department of Mathematic, University of Gezira, Sudan

*Address all correspondence to: tarig.alzaki@gmail.com

IntechOpen

© 2021 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

References

[1] Wazwaz AM. Partial Differential Equations and Solitary Wave's Theory. Higher Education Press: Springer; 2009

[2] Maitama S. A new approach to linear and nonlinear Schrodinger equations using the natural decomposition method. International Mathematical Forum. 2014;**9**:835-847

[3] Yildirim A. on the solution of the nonlinear Korteweg-de Vries equation by the homotopy perturbation method. Communications in Numerical Methods in Engineering. 2009;**25**:1127-1136

[4] Tarig M. Elzaki and J. Biazar, Homotopy Perturbation Method and Elzaki Transform for Solving System of Nonlinear Partial Differential Equations, World Applied Sciences Journal 24 (7): 944–948, 2013. DOI: 10.5829/idosi.wasj.2013.24.07.1041

[5] Eman M. A. Hilal & Tarig M. Elzaki, Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method, Journal of Function Spaces,Volume 2014, pp. 1–5, http://dx.doi.org/10.1155/2014/790714.

[6] Wazwaz AM, El-sayed SM. a new modification of the Adomian decomposition method for linear and nonlinear operators. Applied Mathematics and Computation. 2001;**122**:393-405

[7] S. Ahmed and T. Elzaki; Solution of heat and wave-like equations by Adomian decomposition Sumudu transform method, British Journal of Mathematics and Computer Science, 8 (2015):101–111.

[8] Ahmed S, Elzaki T. The solution of nonlinear Volterra integro-differential equations of second kind by combine Sumudu transforms and Adomian decomposition method. International Journal of Advanced and Innovative Research. 2013;**2**:90-93

[9] Mohammed E. A. Rabie, & Tarig. M. Elzaki, A study of some systems of non linear partial differential equations by using Adomian and modified decomposition methods, academic Jornalsm, Afreican Journal of Mathematics and Computer Science Research, Vol. 7(6), pp, 61–67. (2014). DOI: 10.5897/AJMCSR2014.0541.

[10] Tarig M. Elzaki & Hwajoon Kim, The Solution of Burger's Equation by Elzaki Homotopy Perturbation Method, Applied Mathematical Sciences, Vol. 8, 2014, no. 59, 2931–2940, http://dx.doi.org/10.12988/ams.2014.44314.

[11] Musa R. Gad-Allah & Tarig. M. Elzaki, Application of the New Homotopy Perturbation Method (NHPM) for Solving Non-Linear Partial Differential Equations, Journal of Computational and Theoretical NanoscienceVol. 14, 1–9, 2017.

[12] Tarig M. Elzaki, chapter in the book (Differential Equations - Theory and Current Research) Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method, h ttp://dx.doi.org/10.5772/, intechopen, 73291,(2018)

[13] Weatugala GK. Sumudu transforms - a new integral transform to solve differential equations and control engineering problems. Mathematic Engineering. 1993;**61**:319-329