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A short solution of the kissing number problem in dimension three

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A SHORT SOLUTION OF THE KISSING NUMBER PROBLEM IN DIMENSION THREE

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Abstract. In this note, we give a short solution of the kissing number problem in dimension three.

1. Introduction

The problem of finding the maximum number of non-overlapping unit spheres tangent to a given unit sphere is known as *the kissing number problem*. Schütte and van der Waerden [13] settled the thirteen spheres problem (the kissing number problem for dimension three) that was the subject of the famous discussion between Isaac Newton and David Gregory in 1694. A sketch of an elegant proof was given by Leech [6]. The thirteen spheres problem continues to be of interest to mathematicians, and new proofs have been published in recent years [8, 2, 1, 9]. In other dimensions, the kissing number problem is solved only for $d = 8, 24$ [7, 11], and for $d = 4$ [10].

Theorem 1. [13] *The kissing number in dimension three is 12.*

For our proof, we use the linear programming approach. The method was discovered by Delsarte [3] for the Hamming space, then extended to the spherical case [4] and generalized by Kabatyansky and Levenshtein [5]. For the linear programming approach, we use the properties of Gegenbauer polynomials defined recursively as follows.

$$G_0^{(d)}(t) = 1, \quad G_1^{(d)}(t) = t, \quad G_k^{(d)}(t) = \frac{(d+2k-4)\,t\,G_{k-1}^{(d)}(t) - (k-1)\,G_{k-2}^{(d)}(t)}{d+k-3}.$$

In particular, the Delsarte method in the spherical case is based on the following proposition.

Proposition 1. [4, 5] *For any finite set $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{d-1}$ and any $k \geq 0$,*

$$\sum_{1 \leq i,j \leq N} G_k^{(d)}(\langle x_i, x_j \rangle) \geq 0.$$

2. A short proof of Theorem 1

Let $f(t) = 0.09465869 + 0.17273741\,G_1^{(3)}(t) + 0.33128438\,G_2^{(3)}(t) + 0.17275228\,G_3^{(3)}(t) + 0.18905584\,G_4^{(3)}(t) + 0.00334265\,G_5^{(3)}(t) + 0.03616728\,G_9^{(3)}(t)$ (see Figure 1 for the plot of $f(t)$).

Assume we have N non-overlapping unit spheres tangent to a given unit sphere $\mathbb{S}^2$. Then all pairwise angular distances between points of tangency $x_1, \ldots, x_N$ in $\mathbb{S}^2$ are at least $\pi/3$. If we show that for each i, $\sum_{j=1}^N f(\langle x_i, x_j \rangle) \leq 1.23$ then we can conclude the statement of the theorem. Indeed, on the one hand $\sum_{i,j=1}^N f(\langle x_i, x_j \rangle) \leq 1.23N$. On the other hand, Proposition 1 implies $\sum_{i,j=1}^N f(\langle x_i, x_j \rangle) \geq \sum_{i,j=1}^N 0.09465869 = 0.09465869N^2$. Therefore, $N \leq 1.23/0.09465869 \approx 12.99405263$.

Fix $x = x_i$. The polynomial f is negative on $[-1/\sqrt{2}, 1/2]$ so the positive contribution to the sum $\sum_{j=1}^N f(\langle x, x_j \rangle)$ can be made only by points x_j in the open spherical cap C with the center

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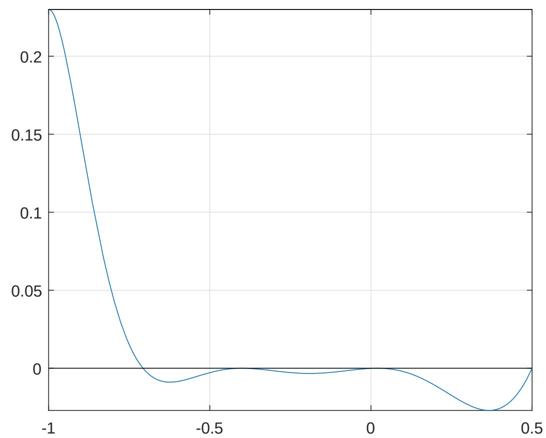

FIGURE 1. Plot of $f(t)$ for $t \in [-1, 1/2]$.

$-x$ and the angular radius $\pi/4$. No more than 3 points with pairwise angular distances at least $\pi/3$ can fit in C. Indeed, if there are at least 4 points y_1, y_2, y_3, y_4 in C then at least one angle $\angle(y_i, -x, y_j)$ is no greater than $\pi/2$. By the spherical law of cosines, the angular distance between y_i and y_j is less than $\pi/3$.

If there is exactly one point y in C, then

$$f(1) + f(\langle x, y\rangle) \leq f(1) + \max_{t \in [-1, -1/\sqrt{2}]} f(t) \leq 1.23.$$

For two points y, z in C, the angular distance between y and $-x$ is at least $\frac{\pi}{12}$ by the triangle inequality for $y, z, -x$. Hence if $\langle x, y\rangle = t$ then t cannot be less than $-\cos\frac{\pi}{12}$. By the triangle inequality, $\langle x, z\rangle \geq \alpha(t) = \frac{1}{2}t - \frac{\sqrt{3}}{2}\sqrt{1-t^2}$. Since f is decreasing on $I = [-\cos\frac{\pi}{12}, -1/\sqrt{2}]$,

$$f(1) + f(\langle x, y\rangle) + f(\langle x, z\rangle) \leq f(1) + \max_{t \in I}(f(t) + f(\alpha(t))) \leq 1.23.$$

For three points y, z, w in C, we use the monotonicity of f on I and move them as close as possible to $-x$. This way we get at least two of the three pairwise angular distances equal to $\pi/3$. Assume $\langle y, z\rangle = \langle z, w\rangle = 1/2$. Note that z, w, x cannot belong to the same large circle because otherwise y does not fit in C. This means we can always move w keeping $\langle w, z\rangle = 1/2$ and decreasing $\langle x, w\rangle$. The process stops in two possible cases: w reaches the boundary of C or $\langle y, w\rangle$ becomes $1/2$. In the former case we are left with the case of two points in C covered above. Now we can assume that $\langle y, z\rangle = \langle z, w\rangle = \langle y, w\rangle = 1/2$. Without loss of generality, $\langle x, y\rangle \leq \langle x, z\rangle \leq \langle x, w\rangle$. We keep the point y intact and rotate the regular triangle yzw so that $\langle x, z\rangle$ decreases. Since $\langle x, y\rangle \geq \langle x, z\rangle$, $\langle x, z\rangle \leq -\frac{\sqrt{2}}{4} - \frac{1}{2}$. Note that $\langle x, z\rangle + \langle x, w\rangle$ decreases in this case as well and, due to convexity and monotonicity of f on the interval $[-\frac{\sqrt{2}}{4} - \frac{1}{2}, -\frac{1}{\sqrt{2}}]$, $f(\langle x, z\rangle) + f(\langle x, w\rangle)$ increases. This process will stop either when w reaches the boundary of C or when $\langle x, z\rangle$ becomes equal to $\langle x, y\rangle$. In the former case, we are left with two points in C. In the latter case, if $\langle x, y\rangle = \langle x, z\rangle = t$ then $\langle x, w\rangle = \beta(t) = \frac{2}{3}t - \frac{2}{3}\sqrt{\frac{3}{2} - 2t^2}$. Given that $t \leq \langle x, w\rangle \leq -1/\sqrt{2}$, t must belong to $J = [-\frac{\sqrt{2}}{4} - \frac{1}{2}, -\sqrt{\frac{2}{3}}]$. Then

$$f(1) + f(\langle x, y\rangle) + f(\langle x, z\rangle) + f(\langle x, w\rangle) \leq f(1) + \max_{t \in J}(2f(t) + f(\beta(t))) \leq 1.23.$$

Remark 1. *This proof is similar to the proof in* [9] *and the solution of the kissing problem in dimension four* [10] *(see also* [12]*) but the function is chosen more carefully so the case analysis is much simpler.*

Remark 2. *The function $f(t)$ was found by using a fixed value of 1.23 and maximizing the constant term in the Gegenbauer expansion while imposing required conditions. All inequalities are easily verifiable. For convenience, their explicit forms are available in a separate file attached to the arXiv submission of the paper.*

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