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Generalized squared remainder minimization method for solving multi-term fractional differential equations

Mir Sajjad Hashemi[a] , **Mustafa Inc**[b,c,1] , **Somayeh Hajikhah**[a]

[a]Department of Mathematics, Basic Science Faculty, University of Bonab,
P.O. Box 55517-61167, Bonab, Iran
hashemi_math396@yahoo.com; so.hajikhah@gmail.com

[b]Department of Mathematics, Firat University, 23119 Elazig, Turkey
minc@firat.edu.tr

[c]Department of Medical Research, China Medical University Hospital,
China Medical University, Taichung, Taiwan

Abstract. In this paper, we introduce a generalization of the squared remainder minimization method for solving multi-term fractional differential equations. We restrict our attention to linear equations. Approximate solutions of these equations are considered in terms of linearly independent functions. We change our problem into a minimization problem. Finally, the Lagrange-multiplier method is used to minimize the resultant problem. The convergence of this approach is discussed and theoretically investigated. Some relevant examples are investigated to illustrate the accuracy of the method, and obtained results are compared with other methods to show the power of applied method.

Keywords: squared remainder minimization method, Lagrange-multiplier method, multi-term fractional differential equation.

1 Introduction

Fractional integration and differentiation are generalizations of integer-order calculus to noninteger ones. It is demonstrated in literature that fractional calculus can play a justifiable and beneficial role in the modeling of various phenomena e.g. science and engineering [2, 3, 6, 25, 29, 30, 35–37, 42]. The classical differential operators are defined as local operators, whereas the fractional differential operators are nonlocal. This significant property makes studying fractional differential equations an active area of research. There are a several number of definitions of fractional derivatives involving the kernel of the special functions, such as Mittag–Leffler function, Prabhakar function and so on. Some important ones are, Riemann–Liouville and Caputo fractional derivatives in [14, 41], Caputo–Fabrizio fractional derivative in [8, 13], and Atangana and Baleanu suggested

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[1]Corresponding author.

another version of fractional-order derivative, which uses the generalized Mittag–Leffler function with strong memory as nonlocal and nonsingular kernel in [5].

Multi-point fractional differential equations appear in different types of visco-elastic damping [40]. The multi-point boundary conditions in the fractional differential equations can be understood as the controllers at the end points dissipate or add energy according to censors located at intermediate points. Model equations proposed so far are almost always linear. Therefore, we concentrate on the following multi-point fractional differential equation:

$$
\begin{aligned}
{}^{C}\mathcal{D}_{x_0^+}^{\alpha} u(x) &= \sum_{j=1}^{k} \gamma_j \, {}^{C}\mathcal{D}_{x_0^+}^{\beta_j} u(x) + \gamma_{k+1} u(x) + f(x), \quad x \in [x_0, x_f], \\
u^{(i)}(x_0) &= a_i, \quad i = 0, 1, \ldots, \lceil \alpha \rceil - 1,
\end{aligned}
\tag{1}
$$

where $\gamma_j, j = 1, \ldots, k+1$ are real constant coefficients and $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$. Here, we consider this equation with most usual fractional derivative in Caputo sense. Equation (1) permits us to describe the model more accurately than the classical integer equation. The nonlocality of Caputo fractional derivative means that the next state of a system depends not only upon its current state but also upon all of its historical states.

Bhrawy et al. [11] applied the spectral algorithm based on generalized Laguerre tau (GLT) method with generalized Laguerre–Gauss (GQ) and generalized Laguerre–Gauss–Radau (GRQ) quadrature methods for Eq. (1). The shifted Chebyshev spectral tau (SCT) method based on the integrals of shifted Chebyshev polynomials is utilized to construct the approximate solutions of such problems [12]. Spectral tau method combined with the shifted Chebyshev polynomials are proposed to solve the Eq. (1) in [15]. Three alternative decomposition approaches are introduced for the approximate solution of Eq. (1) by Ford et al. [16]. More discussion about the approximate solution of Eq. (1) can be found in [27, 34, 38].

The paper is organized as follows: In Section 2, we briefly discuss some necessary definitions and mathematical preliminaries of fractional calculus, which will be needed in the forthcoming sections. The third section deals with a generalization of squared remainder minimization (GSRM) method for the multi-point fractional differential equations. Section 4 is devoted to the study of convergence analysis for the proposed method. Some illustrative examples are investigated in Section 5. Conclusion remarks provide our final section.

2 Preliminaries and notations

In this section, we review some preliminaries and properties of well-known fractional derivatives and integrals for the purpose of acquainting with sufficient fractional calculus theory. Among various definitions of fractional derivatives e.g. Caputo–Fabrizio [4,13,19], Atangana-Baleanu [7,17,18] and conformable fractional derivative [20,21,26], we introduce two most commonly used definitions, namely, the Riemann–Liouville and Caputo derivatives [9, 14, 33]. Various analytical and numerical methods are utilized to consider the fractional differential equations e.g. [1, 10, 22–24, 31, 32, 39].

Definition 1. Let $\alpha \in \mathbb{R}_+$. The operator $J_{x_0^+}^\alpha \in L_1[x_0, x_f]$ defined by

$$J_{x_0^+}^\alpha u(x) := \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} u(t)\,\mathrm{d}t, \quad x \in [x_0, x_f],$$

is called the Riemann–Liouville fractional integral operator.

Now, we discuss the interchange of the Riemann–Liouville fractional integration and limit operation, which is useful in the convergence analysis of further discussed method.

Lemma 1. *Let* $\alpha \in \mathbb{R}_+$. *Assume that* $\{u_n\}_1^\infty$ *is a uniformly convergent sequence of functions on* $C[x_0, x_f]$. *Then*

$$\left(J_{x_0^+}^\alpha \lim_{n\to\infty} u_n \right)(x) = \left(\lim_{n\to\infty} J_{x_0^+}^\alpha u_n \right)(x).$$

Definition 2. Let $\alpha \in \mathbb{R}_+$. The operator ${}^{RL}\mathcal{D}_{x_0^+}^\alpha$ defined by

$$
\begin{aligned}
{}^{RL}\mathcal{D}_{x_0^+}^\alpha u(x) &:= \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}x^{\lceil\alpha\rceil}} J_{x_0^+}^{\lceil\alpha\rceil-\alpha} u(x) \\
&= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}x^{\lceil\alpha\rceil}} \int_{x_0}^x (x-t)^{\lceil\alpha\rceil-\alpha-1} u(t)\,\mathrm{d}t, \quad x \in [x_0, x_f],
\end{aligned}
$$

is called the Riemann–Liouville fractional differential operator, where the ceiling function $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α.

Definition 3. Let $\alpha \in \mathbb{R}_+$. The operator ${}^{C}\mathcal{D}_{x_0^+}^\alpha$ defined by

$$
\begin{aligned}
{}^{C}\mathcal{D}_{x_0^+}^\alpha u(x) &:= J_{x_0^+}^{\lceil\alpha\rceil-\alpha} u^{(\lceil\alpha\rceil)}(x) \\
&= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_{x_0}^x (x-t)^{\lceil\alpha\rceil-\alpha-1} u^{(\lceil\alpha\rceil)}(t)\,\mathrm{d}t, \quad x \in [x_0, x_f],
\end{aligned}
$$

is called the Caputo fractional differential operator.

For the Caputo derivative, we have

$$
{}^{C}\mathcal{D}_{x_0^+}^\alpha (x-x_0)^\beta = \begin{cases} 0 & \text{if } \beta \in \{0, 1, \dots, \lceil\alpha\rceil - 1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-x_0)^{\beta-\alpha} & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geqslant \lceil\alpha\rceil \text{ or} \\ & \text{if } \beta \notin \mathbb{N} \text{ and } \beta > \lceil\alpha\rceil - 1. \end{cases}
$$

3 The GSRM method

For problem (1), we specify the linear operator

$$(\mathcal{O}u)(x) := {}^{C}\mathcal{D}_{x_0^+}^\alpha u(x) - \sum_{j=1}^k \gamma_j {}^{C}\mathcal{D}_{x_0^+}^{\beta_j} u(x) - \gamma_{k+1} u(x) - f(x). \tag{2}$$

Suppose $\psi_0(x), \psi_1(x), \ldots, \psi_n(x)$ are linearly independent functions on $[x_0, x_f]$ and $\Psi_n = \text{span}\{\psi_0(x), \psi_1(x), \ldots, \psi_n(x)\}$. Let $u_n(x) \in \Psi_n$ is the approximate solution of Eq. (1). Then there exist unknown constants $c_0, c_1, \ldots, c_n$ such that

$$u_n(x) = \sum_{l=0}^{n} c_l \psi_l(x). \tag{3}$$

Obviously, the approximate solution $u_n(x)$ needs to satisfy the following conditions:

$$\left\|(\mathcal{O}u_n)(x)\right\|_{L_2[x_0, x_f]} < \epsilon, \qquad u_n^{(i)}(x_0) = a_i, \quad i = 0, 1, \ldots, \lceil \alpha \rceil - 1. \tag{4}$$

Substituting (3) into Eq. (2) concludes

$$(\mathcal{O}u_n)(x) = \sum_{l=0}^{n} c_l \left({}^C\mathcal{D}_{x_0^+}^{\alpha} \psi_l(x) - \sum_{j=1}^{k} \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} \psi_l(x) - \gamma_{k+1} \psi_l(x) \right) - f(x)$$

$$= \sum_{l=0}^{n} c_l \omega_l(x) - f(x), \tag{5}$$

where $\omega_l(x) = {}^C\mathcal{D}_{x_0^+}^{\alpha} \psi_l(x) - \sum_{j=1}^{k} \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} \psi_l(x) - \gamma_{k+1} \psi_l(x)$.

For any $x \in [x_0, x_f]$, if $e_n(x) = u_n(x) - u(x)$, then the nth-order remaining terms can be given by

$$\mathcal{R}_n(x) = (\mathcal{O}u_n)(x) - (\mathcal{O}u)(x)$$

$$= {}^C\mathcal{D}_{x_0^+}^{\alpha} e_n(x) - \sum_{j=1}^{k} \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} e_n(x) - \gamma_{k+1} e_n(x).$$

Remark 1. If $\mathcal{R}_n(x) \to 0$, then $e_n(x) \to 0$ or, equivalently, $u_n(x) \to u(x)$.

Remark 2. If the linearly independent functions are supposed as

$$\psi_l(x) = x^l, \quad l = 0, 1, \ldots, n, \tag{6}$$

then Eq. (5) becomes

$$(\mathcal{O}u_n)(x) = \sum_{l=0}^{n} c_l \left({}^C\mathcal{D}_{x_0^+}^{\alpha} x^l - \sum_{j=1}^{k} \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} x^l - \gamma_{k+1} x^l \right) - f(x)$$

$$= \sum_{l=\lceil \alpha \rceil}^{n} c_l \frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)} x^{l-\alpha} - \sum_{j=1}^{k} \gamma_j \sum_{l=\lceil \beta_j \rceil}^{n} c_l \frac{\Gamma(l+1)}{\Gamma(l+1-\beta_j)} x^{l-\beta_j}$$

$$- \gamma_{k+1} \sum_{l=0}^{n} c_l x^l - f(x).$$

The important point to note here is the minimization problem in the GSRM method by the fact (4). That is, the problem is minimizing $(\mathcal{O}u_n)(x)$ in such a way that

$$u_n^{(i)}(x_0) = \sum_{l=0}^{n} c_l \psi_l^{(i)}(x_0) = a_i, \quad i = 0, 1, \ldots, \lceil \alpha \rceil - 1.$$

To do this, we introduce the real functions

$$\mathcal{J}(c_0, c_1, \ldots, c_n) = \left\| (\mathcal{O}u_n)(x) \right\|^2_{\mathcal{L}_2[x_0, x_f]},$$

and

$$\mathcal{I}_i(c_0, c_1, \ldots, c_n) = \sum_{l=0}^{n} c_l \psi_l^{(i)}(x_0) - a_i, \quad i = 0, 1, \ldots, \lceil \alpha \rceil - 1.$$

Therefore in order to find the unknown vector $C = (c_0, c_1, \ldots, c_n)$, we have to solve the minimization problem

$$\text{Min } \mathcal{J}(C) \quad \text{s.t. } \mathcal{I}_i(C) = 0, \ i = 0, 1, \ldots, \lceil \alpha \rceil - 1. \tag{7}$$

Remark 3. If we set polynomials (6) as basis functions, then constrains in minimization problem (7) become

$$\mathcal{I}_i(C) = \sum_{l=i}^{n} c_l l(l-1) \cdots (l-i+1) x_0^{l-i} - a_i, \quad i = 0, 1, \ldots, \lceil \alpha \rceil - 1.$$

We use the Lagrange-multiplier method [28] to minimize problem (7). By using this method, we solve the following system of algebraic equations[2]:

$$\nabla \mathcal{J}(C) + \Lambda^{\mathrm{T}} \nabla \mathcal{I}(C) = 0, \qquad \mathcal{I}(C) = 0, \tag{8}$$

where $\Lambda^{\mathrm{T}} = (\lambda_0, \lambda_1, \ldots, \lambda_{\lceil \alpha \rceil - 1})$ and $\mathcal{I}^{\mathrm{T}}(C) = (\mathcal{I}_0(C), \mathcal{I}_1(C), \ldots, \mathcal{I}_{\lceil \alpha \rceil - 1}(C))$. Equivalently, we can define the vector

$$\mathcal{K} = (\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_n) = \nabla \mathcal{J}(C) + \Lambda^{\mathrm{T}} \nabla \mathcal{I}(C),$$

where

$$\mathcal{K}_p = \frac{\partial \| (\mathcal{O}u_n)(x) \|^2_{\mathcal{L}_2[x_0, x_f]}}{\partial c_p} + \sum_{i=0}^{\lceil \alpha \rceil - 1} \lambda_i \frac{\partial \mathcal{I}_i(C)}{\partial c_p}$$

$$= 2 \int_{x_0}^{x_f} (\mathcal{O}u_n)(x) \frac{\partial (\mathcal{O}u_n)(x)}{\partial c_p} \, \mathrm{d}x + \sum_{i=0}^{\lceil \alpha \rceil - 1} \lambda_i \frac{\partial \mathcal{I}_i(C)}{\partial c_p}$$

$$= 2 \int_{x_0}^{x_f} \omega_p(x) \left(\sum_{l=0}^{n} c_l \omega_l(x) - f(x) \right) \mathrm{d}x + \sum_{i=0}^{\lceil \alpha \rceil - 1} \lambda_i \psi_p^{(i)}(x_0)$$

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[2]From the linearity of Eq. (1), this system of equations will be linear.

$$= 2 \sum_{l=0}^{n} c_l \int_{x_0}^{x_f} \omega_p(x)\omega_l(x)\,\mathrm{d}x - 2 \int_{x_0}^{x_f} \omega_p(x)f(x)\,\mathrm{d}x$$

$$+ \sum_{i=0}^{\lceil\alpha\rceil-1} \lambda_i \psi_p^{(i)}(x_0), \quad p = 0,1,\dots,n.$$

For simplicity of notations, if we use

$$\langle f,g\rangle = \int_{x_0}^{x_f} f(x)g(x)\,\mathrm{d}x \quad \text{and} \quad \theta_{pi} = \psi_p^{(i)}(x_0),$$

then

$$\mathcal{K}_p = 2 \sum_{l=0}^{n} c_l \langle \omega_p, \omega_l \rangle - 2\langle \omega_p, f \rangle + \sum_{i=0}^{\lceil\alpha\rceil-1} \lambda_i \theta_{pi}, \quad p = 0,1,\dots,n.$$

Therefore, the linear system of equations (8) becomes

$$2 \sum_{l=0}^{n} c_l \langle \omega_p, \omega_l \rangle + \sum_{i=0}^{\lceil\alpha\rceil-1} \lambda_i \theta_{pi} = 2\langle \omega_p, f \rangle,$$

$$\sum_{l=0}^{n} c_l \theta_{li} = a_i, \quad i = 0,1,\dots,\lceil\alpha\rceil-1,$$

or, in the abstract form,

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_2^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} C \\ \Lambda \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix}, \tag{9}$$

where

$$\mathcal{A}_1 = \begin{pmatrix} 2\langle\omega_0,\omega_0\rangle & 2\langle\omega_0,\omega_1\rangle & \cdots & 2\langle\omega_0,\omega_n\rangle \\ 2\langle\omega_1,\omega_0\rangle & 2\langle\omega_1,\omega_1\rangle & \cdots & 2\langle\omega_1,\omega_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 2\langle\omega_n,\omega_0\rangle & 2\langle\omega_n,\omega_1\rangle & \cdots & 2\langle\omega_n,\omega_n\rangle \end{pmatrix} \in M_{(n+1)\times(n+1)},$$

$$\mathcal{A}_2 = \begin{pmatrix} \theta_{00} & \theta_{01} & \cdots & \theta_{0(\lceil\alpha\rceil-1)} \\ \theta_{10} & \theta_{11} & \cdots & \theta_{1(\lceil\alpha\rceil-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{n0} & \theta_{n1} & \cdots & \theta_{n(\lceil\alpha\rceil-1)} \end{pmatrix} \in M_{(n+1)\times(\lceil\alpha\rceil)},$$

$$\mathcal{F}_1^{\mathrm{T}} = \begin{bmatrix} 2\langle\omega_0,f\rangle, 2\langle\omega_1,f\rangle, \dots, 2\langle\omega_n,f\rangle \end{bmatrix} \in M_{1\times(n+1)},$$

$$\mathcal{F}_2^{\mathrm{T}} = \begin{bmatrix} a_0, a_1, \dots, a_{\lceil\alpha\rceil-1} \end{bmatrix} \in M_{1\times(\lceil\alpha\rceil)}.$$

Therefore, our main goal is finding the unknown vector C from Eq. (9).

4 Convergence analysis

This section is a discussion about the convergence of the GSRM method for the multi-point fractional differential equations of the form (1).

Theorem 1. *Suppose that $u(x)$ is the exact solution of fractional differential equation* (1) *defined on $[x_0, x_f]$, and $u_n(x)$ is the corresponding approximate solution of problem given by the GSRM method. If there exists a polynomial $p_n(x) \in \mathcal{P}_n[x_0, x_f]$ such that for any $x \in [x_0, x_f]$, $p_n(x) \to u(x)$, then*

$$\left\| (\mathcal{O} u_n)(x) \right\|_{L_2[x_0, x_f]} \to 0.$$

Proof. One can easily conclude that for any $n \in \mathbb{N}$, we have

$$0 \leqslant \left\| (\mathcal{O} u_n)(x) \right\|_{L_2[x_0, x_f]} \leqslant \left\| (\mathcal{O} p_n)(x) \right\|_{L_2[x_0, x_f]}.$$

Therefore

$$0 \leqslant \lim_{n \to \infty} \left\| (\mathcal{O} u_n)(x) \right\|_{L_2[x_0, x_f]} \leqslant \lim_{n \to \infty} \left\| (\mathcal{O} p_n)(x) \right\|_{L_2[x_0, x_f]}. \tag{10}$$

Moreover, from the continuity of norms and $p_n(x) \to u(x)$ we get

$$\lim_{n \to \infty} \left\| (\mathcal{O} p_n)(x) \right\|_{L_2[x_0, x_f]}$$
$$= \left\| \lim_{n \to \infty} (\mathcal{O} p_n)(x) \right\|_{L_2[x_0, x_f]}$$
$$= \left\| \lim_{n \to \infty} \left({}^C\mathcal{D}_{x_0^+}^\alpha p_n(x) - \sum_{j=1}^k \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} p_n(x) - \gamma_{k+1} p_n(x) - f(x) \right) \right\|_{L_2[x_0, x_f]}$$
$$= \left\| {}^C\mathcal{D}_{x_0^+}^\alpha \lim_{n \to \infty} p_n(x) - \sum_{j=1}^k \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} \lim_{n \to \infty} p_n(x) \right.$$
$$\left. - \gamma_{k+1} \lim_{n \to \infty} p_n(x) - f(x) \right\|_{L_2[x_0, x_f]}$$
$$= \left\| {}^C\mathcal{D}_{x_0^+}^\alpha u(x) - \sum_{j=1}^k \gamma_j {}^C\mathcal{D}_{x_0^+}^{\beta_j} u(x) - \gamma_{k+1} u(x) - f(x) \right\|_{L_2[x_0, x_f]}$$
$$= \left\| (\mathcal{O} u)(x) \right\|_{L_2[x_0, x_f]}. \tag{11}$$

From Eqs. (1), (2), obviously, we have $(\mathcal{O} u)(x) = 0$ and therefore $\|(\mathcal{O} u)(x)\|_{L_2[x_0, x_f]} = 0$. Hence, from (10) and (11) we obtain

$$0 \leqslant \lim_{n \to \infty} \left\| (\mathcal{O} u_n)(x) \right\|_{L_2[x_0, x_f]} \leqslant 0,$$

which completes the proof. □

5 Numerical results

In this section, we present the results of the GSRM method on five test problems. We perform our computations using Maple 18 software with 30 digits.

Example 1. Consider the equation

$$
{}^{C}\mathcal{D}_{0^+}^{\alpha} u(x) + u(x)
$$
$$
= x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}x^{4-\alpha}, \quad 0 < \alpha < 1, \ x \in [0, 10],
$$

with initial condition $u(0) = 0$. Exact solution of this equation is given by $u(x) = x^4 - x^3/2$. Figure 1 shows the absolute errors for various α by the GSRM method
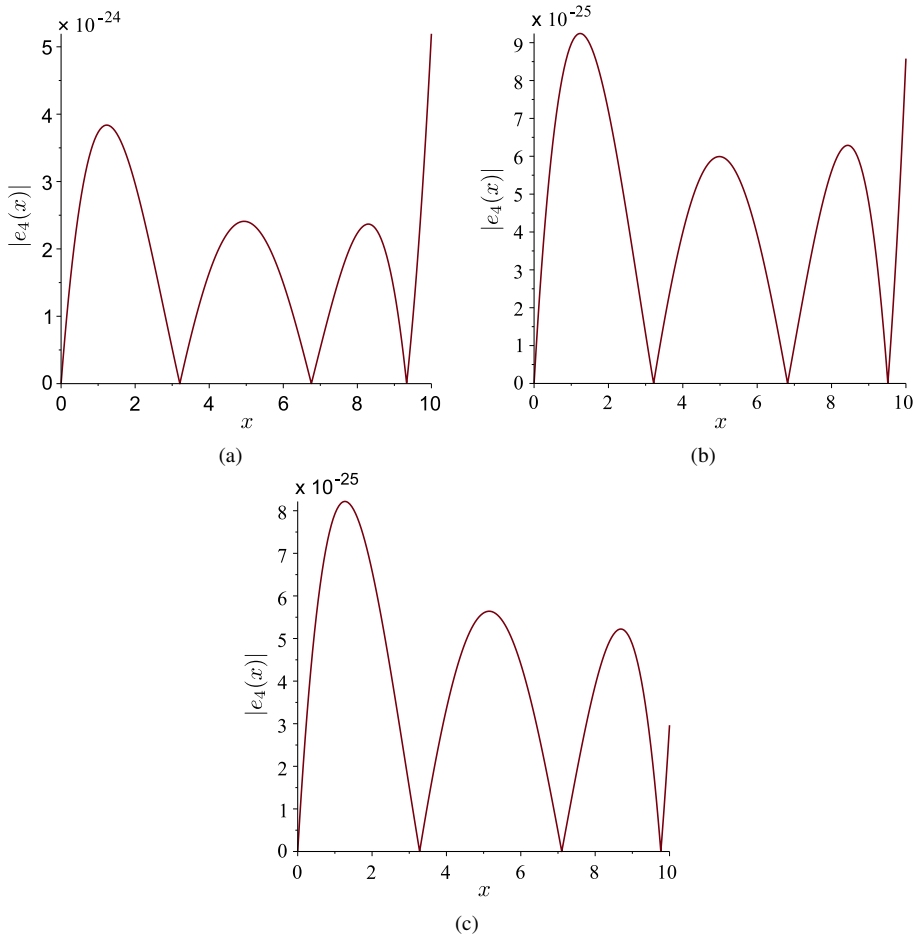
Figure 1. Absolute errors of example 1 for (a) $\alpha = 0.01$, (b) $\alpha = 0.5$ and (c) $\alpha = 0.99$.

**Table 1.** Comparison of maximum absolute values for Example 1 and various differential orders α.

α	GLT(GQ)	GLT(GRQ)	SCT	GSRM
0.01	$5.58\mathrm{E}-07$	$1.58\mathrm{E}-06$	$1.30\mathrm{E}-13$	$5.19\mathrm{E}-24$
0.50	$1.59\mathrm{E}-05$	$3.56\mathrm{E}-05$	$2.20\mathrm{E}-13$	$9.24\mathrm{E}-25$
0.99	$1.42\mathrm{E}-06$	$1.77\mathrm{E}-06$	$2.10\mathrm{E}-16$	$8.21\mathrm{E}-25$

with polynomial bases as (6) and $n = 4$. In Table 1, the absolute errors obtained by present method are compared with GLT method [11] with $N = 50$ and SCT method [12] with $N = 64$. From this table it may be concluded that the GSRM method is more accurate than the mentioned approaches. The CPU time used in this example is 0.842, and condition number of coefficient matrix in (9) w.r.t. $\alpha = 0.01$ is 2.61.

Example 2. Let us consider the following fractional equation:

$$^{C}\mathcal{D}_{0^{+}}^{\alpha} u(x) + u(x) = \Gamma(\alpha + 2)x + x^{\alpha+1}, \quad x \in [0,1],\ 0 < \alpha \leqslant 2,$$

with initial conditions $u(0) = 0$ when $0 < \alpha \leqslant 1$ and $u(0) = u'(0) = 0$ when $1 < \alpha \leqslant 2$. The reported exact solution of this fractional equation is $u(x) = x^{\alpha+1}$. In this example, we suppose $\Psi_3 = \mathrm{span}\{x^{l/2+1},\ l = 0,\dots 3\}$. Maximum absolute error in the best case $\mathrm{E}-03$ is reported w.r.t. $\alpha = 1.5$ by Bhrawy et al. [11], whereas $5.93\mathrm{E}-18$ and $1.31\mathrm{E}-21$ are extracted by the present method for $\alpha = 0.5$ and $\alpha = 1.5$, respectively. High accuracy of the GSRM method for both choices of α can be found from Fig. 2. In Fig. 3, the approximate solutions obtained by the proposed method are plotted with respect to different values of fractional-order α. The CPU time used in this example w.r.t. $\alpha = 0.5$ and $\alpha = 1.5$ are $0.375s$ and $0.421s$, respectively.
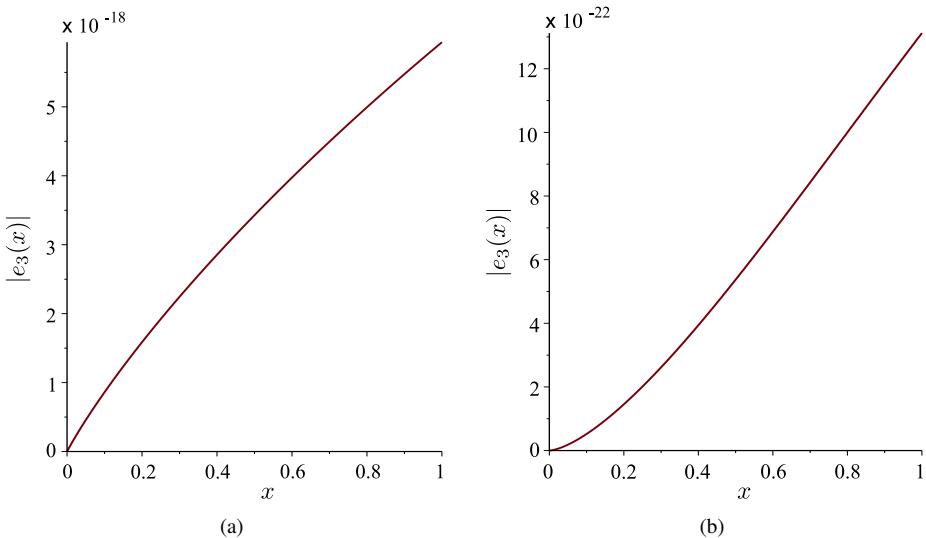
Figure 2. Absolute errors of example 2 for (a) $\alpha = 0.5$ and (b) $\alpha = 1.5$.
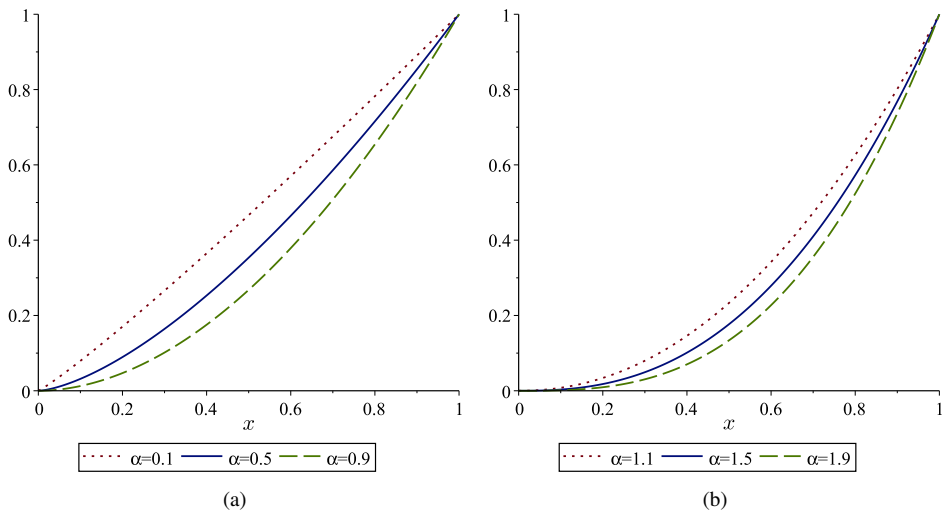
Figure 3. Approximate solutions of example 2 for various values of α.

Example 3. Consider the Bagley–Torvik equation

$$u''(x) + {}^{C}\mathcal{D}_{0^+}^{0.5}u(x) + u(x) = x^2 + 2 + \frac{2}{0.75 \cdot \Gamma(0.5)}x^{1.5}, \quad x \in [0, 10],$$

with initial conditions $u(0) = 0$ and $u'(0) = 0$. Exact solution of this equation is given by $u(x) = x^2$. For this problem, we choose the polynomial linear independent functions with $n = 2$, and we obtain

$$\mathcal{A}_1 = \begin{pmatrix} 20 & 147.5766431 & 1087.279811 \\ 147.5766431 & 1364.910338 & 11184.58608 \\ 1087.279811 & 11184.58608 & 97877.15350 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{F}_1 = \begin{pmatrix} 1087.279811 \\ 11184.58608 \\ 97877.15350 \end{pmatrix}, \qquad \mathcal{F}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By solving the resultant system, we get $c_0 = c_1 = 0$ and $c_2 = 1$. Therefore we obtain the exact solution for this example by using the GSRM method. The best maximum absolute errors by GLT(GQ) and GLT(GRQ) reported in [11] w.r.t. $N = 90$, are $7.06\mathrm{E} - 06$ and $1.75\mathrm{E} - 05$, respectively. The Haar wavelet collocation method obtained the error $4.23\mathrm{E} - 22$ for $h = 1/512$ in [38]. By using an operational matrix method [34] it can be found the maximum absolute error $1.89\mathrm{E} - 12$. The best reached error value among the utilized methods in [16] is $1.54\mathrm{E} - 05$, which is reported for method 2 w.r.t. $h = 1/512$. So, the present method is most reliable and powerful approach for this example.

Example 4. Let us consider the equation

$$u''(x) + {}^{C}\mathcal{D}_{0+}^{0.75}u(x) + u(x) = x^3 + 6x + \frac{6}{0.703125 \cdot \Gamma(0.25)}x^{2.25}, \quad x \in [0,1],$$

with initial conditions $u(0) = u'(0) = 0$. Exact solution of this equation is given by $u(x) = x^3$. Numerical results will not be presented since the exact solution is achieved by choosing $n = 3$.

Regarding Example 4 and in [16], the best result is attained with 512 steps, and the maximum absolute errors are $6.93\mathrm{E}-05$, $1.18\mathrm{E}-04$ and $3.10\mathrm{E}-06$ by using method 1, method 2 and method 3, respectively. Obtained results by GLT(GQ) and GLT(GRQ) methods [11] with $N = 64$ are $1.43\mathrm{E}-05$ and $1.80\mathrm{E}-05$, respectively. Moreover, in [38], the absolute error $1.86\mathrm{E}-09$ is reported by HWCM method, and $3.39\mathrm{E}-13$ is reported by SCT method in [15].

Example 5. As the last test problem, we consider

$$ {}^{C}\mathcal{D}_{0+}^{2.2}u(x) + {}^{C}\mathcal{D}_{0+}^{1.25}u(x) + {}^{C}\mathcal{D}_{0+}^{0.75}u(x) + u(x) = f(x), \quad x \in [0,1],$$

with initial conditions $u(0) = u'(0) = u''(0) = 0$ and $f(x) = x^3/3 + 2x^{0.8}/\Gamma(1.8) + 2x^{1.75}/\Gamma(2.75) + 2x^{2.25}/\Gamma(3.25)$. Exact solution of this equation is reported as $u(x) = x^3/3$. By choosing $n = 4$ we obtain the absolute error plot of this equation in Fig. 4.

In [27], the maximum absolute error by the Haar wavelet operational matrix method is $1.12\mathrm{E}-02$, and reported error in [38] is $2.91\mathrm{E}-03$, w.r.t. $N = 64$. Whereas by the GSRM method, we obtain $4.99\mathrm{E}-28$. The CPU time used in this example is $0.811s$, and condition number of coefficient matrix in (9) is 849.54.
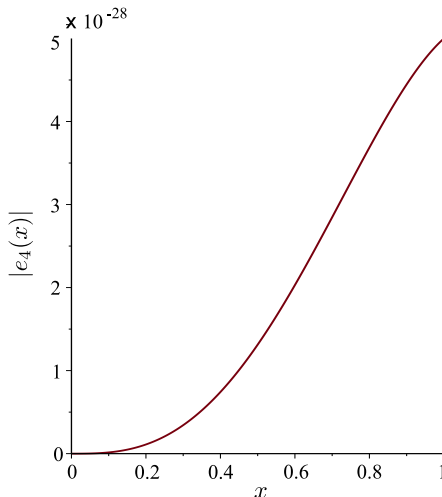
Figure 4. Absolute error of Example 5.

6   Conclusion

In the present paper, the squared remainder minimization method is developed to the multi-term fractional differential equations. A minimization problem is manifested and it considered by the Lagrange-multiplier method. Convergence of the GSRM method is theoretically proved. Five test problems are investigated. For some of the given examples, exact solutions by the present method are extracted. Accuracy and reliability of the GSRM method is revealed by the reported figures and tables.

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