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Connectedness of a suborbital graph for congruence subgroups

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Abstract
In this paper, we give necessary and sufficient conditions for the graph $H_{u,n}$ to be connected and a forest.

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1 Introduction

Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ be the extended rationals and $\Gamma = PSL(2,\mathbb{Z})$ be the modular group acting on $\hat{\mathbb{Q}}$ as with the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z = \frac{x}{y} \to \frac{az+b}{cz+d} = \frac{ax+by}{cx+dy},$$

where a, b, c, and d are rational integers and $ad - bc = 1$.

Jones, Singerman, and Wicks [1] used the notion of the imprimitive action [2–4] for a Γ-invariant equivalence relation induced on $\hat{\mathbb{Q}}$ by the congruence subgroup $\Gamma_0(n) = \{g \in \Gamma : c \equiv 0 \pmod n\}$ to obtain some suborbital graphs and examined their connectedness and forest properties. They left the forest problem as a conjecture, which was settled down by the second author in [5].

In this paper we introduce a different Γ-invariant equivalence relation by using the congruence subgroup $\Gamma_1(n)$ instead of $\Gamma_0(n)$ and obtain some results for the newly constructed subgraphs $H_{u,n}$. In Section 4 we will prove our main theorems on $H_{u,n}$ which give conditions for $H_{u,n}$ to be connected or to be a forest, and we work out some relations between the lengths of circuits in $H_{u,n}$ and the elliptic elements of the group $\Gamma_1(n)$. As Γ only has finite order elements of orders 2 and 3, the same is true for $\Gamma_1(n)$.

Here, it is worth noting that these concepts are very much related to the binary quadratic forms and modular forms in [6] and [7, 8] respectively.

2 Preliminaries

Let $\Gamma_1(n) = \{g \in \Gamma : a \equiv d \equiv 1 \pmod n, c \equiv 0 \pmod n\}$, which is one of the congruence subgroups of Γ. Then $\Gamma_\infty < \Gamma_1(n) \leq \Gamma$ for each n, where Γ_∞ is the stabilizer of ∞ generated by the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and second inclusion is strict if $n > 1$.

Since, by [1], Γ acts transitively on $\hat{\mathbb{Q}}$, any reduced fraction $\frac{r}{s}$ in $\hat{\mathbb{Q}}$ equals $g(\infty)$ for some $g \in \Gamma$. Hence, we get the following imprimitive Γ-invariant equivalence relation on $\hat{\mathbb{Q}}$ by

$\Gamma_1(n)$:

$$\frac{r}{s} \sim \frac{x}{y} \quad \text{if and only if } g^{-1}h \in \Gamma_1(n),$$

where $g = \left(\begin{smallmatrix} r & * \\ s & * \end{smallmatrix}\right)$ and h is similar.

Here, as in [1], the imprimitivity means that the above relation is different from the identity relation ($a \sim b$ if and only if $a = b$) and the universal relation ($a \sim b$ for all $a, b \in \hat{\mathbb{Q}}$).

From the above, we can easily verify that

$$\frac{r}{s} \sim \frac{x}{y} \quad \text{if and only if } x \equiv r \ (\operatorname{mod} n), y \equiv s \ (\operatorname{mod} n).$$

The equivalence classes are called blocks and the block containing $\frac{x}{y}$ is denoted by $[\frac{x}{y}]$.

Here we must point out that the above equivalence relation is different from the one in [1]. This is because we take the group $\Gamma_1(n)$ instead of $\Gamma_0(n)$. The main reason of changing the equivalence relation lies in the fact that in the case of $\Gamma_1(n)$, as we will see below, the elliptic elements do not necessarily correspond to circuits of the same order. It was the case in [5].

3 Subgraphs $H_{u,n}$

The modular group Γ acts on $\hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$ through $g : (\alpha, \beta) \to (g(\alpha), g(\beta))$. The orbits are called suborbitals. From the suborbital $O(\alpha, \beta)$ containing (α, β) we can form the suborbital graph $G(\alpha, \beta)$ whose vertices are the elements of $\hat{\mathbb{Q}}$ and edges are the pairs $(\gamma, \delta) \in O(\alpha, \beta)$, which we will denote by $\gamma \to \delta$ and represent them as hyperbolic geodesics in $\mathcal{H}$.

Since Γ acts transitively on $\hat{\mathbb{Q}}$, every suborbital $O(\alpha, \beta)$ contains a pair $(\infty, \frac{u}{n})$ for $\frac{u}{n} \in \hat{\mathbb{Q}}$, $n \geq 0$, $(u, n) = 1$. In this case, we denote the suborbital graph by $G_{u,n}$ for short.

As Γ permutes the blocks transitively, all subgraphs corresponding to blocks are isomorphic. Therefore we will only consider the subgraph $H_{u,n}$ of $G_{u,n}$ whose vertices form the block $[\infty] = [\frac{1}{0}]$, which is the set $\{\frac{x}{y} \in \hat{\mathbb{Q}} \mid x \equiv 1 \ (\operatorname{mod} n) \text{ and } y \equiv 0 \ (\operatorname{mod} n)\}$. The following two results were proved in [1].

Theorem 1 *There is an edge $\frac{r}{s} \to \frac{x}{y}$ in $G_{u,n}$ if and only if either*
1. $x \equiv ur \ (\operatorname{mod} n)$, $y \equiv us \ (\operatorname{mod} n)$ *and* $ry - sx = n$ *or*
2. $x \equiv -ur \ (\operatorname{mod} n)$, $y \equiv -us \ (\operatorname{mod} n)$ *and* $ry - sx = -n$.

Lemma 1 $G_{u,n} = G_{v,m}$ *if and only if* $n = m$ *and* $u \equiv v \ (\operatorname{mod} n)$.

The suborbital graph $F := G_{1,1}$ is the familiar Farey graph with $\frac{a}{b} \to \frac{c}{d}$ if and only if $ad - bc = \pm 1$.

As it is illustrated in Figure 1, the pattern is periodic of period 1. That is, if $x \to y$ is an edge, then $x + 1 \to y + 1$ is an edge as well.

Lemma 2 *No edges of F cross in $\mathcal{H}$.*

Theorem 1 clearly gives the following.

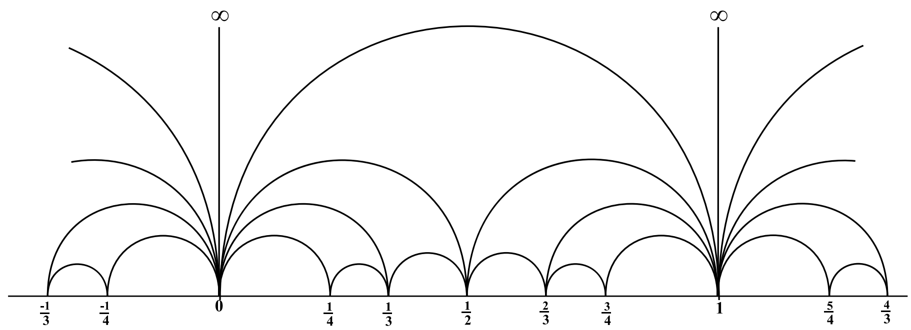

Figure 1 Farey graph.

Theorem 2 *Let $\frac{r}{s}$ and $\frac{x}{y}$ be in $[\infty]$. Then there is an edge $\frac{r}{s} \to \frac{x}{y}$ in $H_{u,n}$ if and only if*

1. $x \equiv ur \pmod n$, $ry - sx = n$, *or*
2. $x \equiv -ur \pmod n$, $ry - sx = -n$.

Theorem 3 *Let $\frac{r}{s}$ and $\frac{x}{y}$ be in $[\infty]$. Then there is an edge $\frac{r}{s} \to \frac{x}{y}$ in $H_{u,n}$ if and only if*

1. $u = 0$ *and* $ry - sx = 1$ *or*
2. $u = 1$ *and* $ry - sx = n$ *or*
3. $u = n - 1$ *and* $ry - sx = -n$.

Proof Let $\frac{r}{s} \to \frac{x}{y}$ be an edge in $H_{u,n}$. Since $\frac{r}{s}$ and $\frac{x}{y}$ are in $[\infty]$, $x, r \equiv 1 \pmod n$. Therefore, according to Theorem 2, we have $1 \equiv u \pmod n$, $ry - sx = n$ or $1 \equiv -u \pmod n$, $ry - sx = -n$. The first implies that $u = 0$ and $u = 1$, which proves (1). The second assures that $u = 0$, $n = 1$ or $u = 1$, $n = 2$ or $u = n - 1$, which gives (3).

For the converse, it is enough to verify (3) only. For this, let $u = n - 1$ and $ry - sx = -n$. Then $x \equiv -r(n-1) \equiv 0 \pmod n$. This, by Theorem 2, completes the proof. □

Theorem 4 $\Gamma_1(n)$ *permutes the vertices and the edges of $H_{u,n}$ transitively.*

Proof

1. Let v and w be vertices in $H_{u,n}$. Then $w = g(v)$ for some $g \in \Gamma$. Since $v \sim \infty$, $g(v) \sim g(\infty)$, that is, $w \sim g(\infty)$. Therefore, $g(\infty)$ lies in the block $[\infty]$ and so g is in $\Gamma_1(n)$.
2. The proof for edges is similar. □

Definition 1 Let $H_{u,n}$ and $H_{v,m}$ be two suborbital graphs. If the map ϕ is an injective function from the vertex set of $H_{u,n}$ to that of $H_{v,m}$ and sends the edges of $H_{u,n}$ to the edges of $H_{v,m}$, then ϕ is called a suborbital graph homomorphism (*homomorphism* for short) and it will be denoted by $\phi : H_{u,n} \to H_{v,m}$.

Theorem 5

1. *If $m \mid n$, then $\phi(v) = \frac{nv}{m}$ is a homomorphism from $H_{u,n}$ to $H_{u,m}$.*
2. *Let $m \mid n$ and $m \neq n$; then the homomorphism in* (1) *is not an isomorphism.*
3. *Let $\phi : H_{1,n} \to H_{n-1,n}$, given by $\phi(a) = a$ for all vertices and $\phi(a \to b) = b \to a$, be an isomorphism.*

Proof

1. Let $\frac{r}{sn} \to \frac{x}{yn}$ be in $H_{u,n}$. To see $\frac{r}{sm} \to \frac{x}{ym}$ is in $H_{u,m}$ is an easy consequence of Theorem 2.
2. Conversely, suppose that $h : H_{u,n} \to H_{u,m}$, $h(v) = \frac{nv}{m}$ is an isomorphism. Then there exists a vertex v in $H_{u,n}$ such that $h(v) = \frac{m+1}{m}$. Therefore, $v = \frac{m+1}{n}$. But, since $m \mid n$ and $m \neq n$, $m + 1 \not\equiv 1 \pmod{n}$. That is, $\frac{m+1}{n}$ is not a vertex in $H_{u,n}$. This gives the proof.
3. Since the subgraphs $H_{1,n}$ and $H_{n-1,n}$ have same set of vertices, ϕ is well defined. Now suppose $\frac{r}{sn} \to \frac{x}{yn}$ is an edge in $H_{1,n}$. Then, by Theorem 3(2), $ry - sx = n$. So, $sx - ry = -n$. That is, using Theorem 3(3), $\frac{x}{yn} \to \frac{r}{sn}$ is an edge in $H_{n-1,n}$. □

Corollary 1 *If $m \mid n$, then $H_{u,n} \to H_{u,m}$, $v \to \frac{nv}{m}$ is an isomorphism if and only if $m = n$.*

Corollary 2 *$\phi : H_{u,n} \to F$, given by $v \to nv$, is a homomorphism.*

Proof Since $H_{u,1} = F$, Theorem 5(1) gives the result. □

Corollary 3 *No edges of $H_{u,n}$ cross in $\mathcal{H}$.*

Proof By Corollary 2 there is an isomorphism from $H_{u,n}$ to a subgraph of F. Also, by Lemma 2, no edges of F cross in $\mathcal{H}$. Therefore the result follows. □

4 Main calculations

In this final section, we state all conditions for $H_{u,n}$ to be connected and a forest.

Definition 2 For $m \in \mathbb{N}$, $m \geq 2$, let $v_1, v_2, \ldots, v_m$ be vertices of $H_{u,n}$. The configuration $v_1 \to v_2 \to \cdots \to v_m \to v_1$ (some arrows, not all, may be reversed) is called a circuit of length m.

If $m = 3$, the circuit is said to be a triangle. If $m = 2$, we call the self paired edge a 2-gon.
A graph is called a forest if it contains no circuits other than 2-gons.

As in examples $\infty \to \frac{1}{2} \to \infty$ is a 2-gon in $H_{1,2}$ and $\infty \to 1 \to 2 \to \infty$ is a triangle in $H_{1,1}$ and furthermore we will see below that $\infty \to v_1 \to v_2 \to \cdots$ never becomes a circuit in $H_{1,n}$ for $n \geq 2$.

We now prove the connectedness of $H_{1,n}$ separately as follows.

Theorem 6 *$H_{1,2}$ is connected.*

Proof Since the situation, only for this case, coincides with the situation in [1], it is not necessary to give a proof. □

To understand subsequent proofs better, we start by giving the following example.

Example 1 The subgraph $H_{1,3}$ is not connected.

Solution 1 Since $\infty \to \frac{1}{3}$ is an edge in $H_{1,3}$ and $H_{1,3}$ is periodic with period 1, we just need consider the strip $\frac{1}{3} \leq \operatorname{Re} z \leq \frac{4}{3}$. It is clear that ∞ is adjacent to $\frac{1}{3}$ and $\frac{4}{3}$ in $H_{1,3}$, but to no intermediate vertices. We will show that no vertices of $H_{1,3}$ in the interval $[\frac{2}{3}, 1]$ are

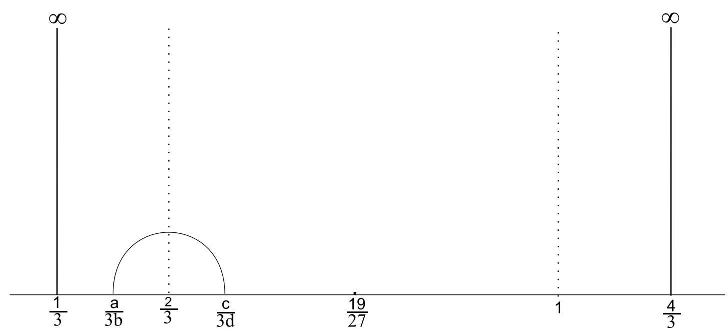

Figure 2 $H_{1,3}$.

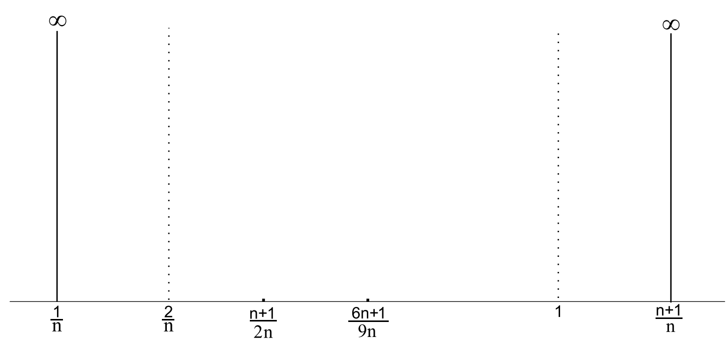

Figure 3 $H_{1,n}$.

adjacent to vertices of $H_{1,3}$ outside this interval. Of course, there is some vertex of $H_{1,3}$, such as $\frac{19}{27}$, in $[\frac{2}{3},1]$.

As in Figure 2, suppose that the edge $\frac{a}{3b} \to \frac{c}{3d}$ in $H_{1,3}$ crosses $\mathrm{Re}\, z = \frac{2}{3}$. Then Corollary 2 implies that $\frac{a}{b} \to \frac{c}{d}$ is an edge in F and furthermore $\frac{a}{b} < 2 < \frac{c}{d}$. This proves that the edges $\frac{a}{b} \to \frac{c}{d}$ and $\infty \to 2$ cross in F, a contradiction. A similar argument shows that no edges of $H_{1,3}$ cross $\mathrm{Re}\, z = 1$. These conclude that $H_{1,3}$ is not connected.

Note 1 The graphs $H_{1,3}$ and $H_{2,3}$ have at least two connected components.

Proof Example 1 and Theorem 5(3) give the result. □

We now give the following.

Theorem 7 *$H_{1,n}$ is not connected if $n \geq 3$.*

Proof Since $H_{1,n}$ is periodic with period 1, we can, again, work in the strip $\frac{1}{n} \leq \mathrm{Re}\, z \leq \frac{n+1}{n}$. Note that ∞ is adjacent to $\frac{1}{n}$ and $\frac{n+1}{n}$ in $H_{1,n}$, but to no intermediate vertices. We will show that no vertices in $H_{1,n}$, between $\frac{2}{n}$ and 1, are adjacent to vertices outside this interval. We note that there are vertices of $H_{1,n}$ in $(\frac{2}{n},1)$ for $n \in \mathbb{N}$. Indeed as in Figure 3, if n is odd, take the vertex $\frac{6n+1}{9n}$ in $(\frac{2}{n},1)$ and if n is even, take the vertex $\frac{n+1}{2n}$ in $(\frac{2}{n},1)$.

Suppose that an edge crosses $\mathrm{Re}\, z = \frac{2}{n}$, whence that it joins $v = \frac{a}{nb}$ to $w = \frac{c}{nd}$. By Corollary 2, nv and nw must be adjacent in F. As in Example 1, this is a contradiction. A similar argument shows that no edge crosses $\operatorname{Re} z = 1$, and since vertices between $\frac{2}{n}$ and 1 are not adjacent to ∞, it follows that $H_{1,n}$ is not connected.

Consequently, since there is no circuit like $\infty \rightarrow \frac{1}{n} \leftarrow v_1 \leftarrow \cdots \leftarrow \frac{n+1}{n} \leftarrow \infty$ in $H_{1,n}$, $H_{1,n}$ is not connected for $n \geq 3$. □

Theorem 8 *$H_{u,n}$ is connected if and only if $n \leq 2$.*

Proof If $n = 1, 2$, it follows from [1]; otherwise, it follows from Theorem 7. □

Theorem 9 *$H_{u,n}$ contains a triangle if and only if $n = 1$.*

Proof Let D be a triangle in $H_{u,n}$. From Theorem 3, $u = 1$ or $u = n - 1$. Using Theorem 5(3), we may only work in $H_{1,n}$. By Theorem 4, we may suppose that D has the form $\infty \rightarrow v_1 \rightarrow v_2 \rightarrow \infty$ or $\infty \rightarrow v_1 \leftarrow v_2 \rightarrow \infty$. Let us do calculations only for the first triangle. We easily see that $v_1 = \frac{x}{n}$ and $v_2 = \frac{y}{n}$ for some $x, y \in \mathbb{Z}$. If $\frac{x}{n} < \frac{y}{n}$, then $x - y = -1$. Since $\frac{x}{n}$ and $\frac{y}{n} \in [\infty]$, $x - y \equiv 0 \pmod{n}$. So, $n = 1$. If $\frac{x}{n} > \frac{y}{n}$, then $x - y = 1$. Therefore, again, $n = 1$.

Conversely, if $n = 1$, then $u = 0$ or 1. But since $H_{0,1} = H_{1,1}$, we have the triangle $\frac{1}{0} \rightarrow \frac{1}{1} \rightarrow \frac{2}{1} \rightarrow \frac{1}{0}$. □

Theorem 10 *$H_{u,n}$ contains a 2-gon if and only if $n = 1$ or 2.*

Proof Suppose $\frac{x}{kn} \rightarrow \frac{y}{ln} \rightarrow \frac{x}{kn}$ is a 2-gon in $H_{u,n}$. Then, by Theorem 3, it is easily seen that $n = 1$ or 2.

Conversely, if $n = 1$ or 2, it is clear that $\frac{1}{0} \rightarrow \frac{1}{n} \rightarrow \frac{1}{0}$ is a 2-gon. □

We now give one of our main theorems.

Theorem 11 *If $n \geq 2$, then $H_{1,n}$ and $H_{n-1,n}$ are forests.*

Proof Let $n = 2$. Assume that $H_{1,2}$ is not a forest. Therefore we suppose that there exists a circuit D, other than 2-gon, in $H_{1,2}$. By Theorem 4 and Theorem 3, we may assume that D has the form $\infty \rightarrow v_1 \rightarrow \cdots \rightarrow v_k \rightarrow \infty$, where the vertices $v_1, v_2, \ldots, v_k$ are all different. Here, since the pattern for the subgraph $H_{u,n}$ is periodic with period 1, we may choose the vertices of D, apart from ∞, in the interval $[\frac{1}{2}, \frac{3}{2}]$. By Theorem 3, $v_1 = \frac{1}{2}$ or $\frac{3}{2}$. If $v_1 = \frac{1}{2}$, then $v_k = \frac{2a+1}{2} \in [\frac{1}{2}, \frac{3}{2}]$ and $v_1 \neq v_k$ give that $v_k = \frac{3}{2}$. Since 1 is not a vertex in $H_{1,2}$, Corollary 2 implies that such a circuit D does not occur. Similarly, we can show that there is not a circuit D in the case where $v_1 = \frac{3}{2}$. That is, $H_{1,2}$ is a forest.

Now let $n \geq 3$. If $H_{1,n}$ is not a forest, then, as we will see now by Theorem 3, D must be of the form $\infty \rightarrow v_1 \leftarrow \cdots \leftarrow v_k \leftarrow \infty$ or $\infty \rightarrow v_1 \rightarrow \cdots \rightarrow v_k \leftarrow \infty$. As above, we choose the vertices in the finite interval $[\frac{1}{n}, \frac{n+1}{n}]$. By Theorem 3, $v_1 = \frac{1}{n}$ or $\frac{n+1}{n}$. If $v_1 = \frac{1}{n}$, then, as above, v_k must be $\frac{n+1}{n}$. In this case D has the form $\infty \rightarrow \frac{1}{n} \leftarrow \cdots \leftarrow \frac{n+1}{n} \leftarrow \infty$. As 1 is not a vertex in $H_{1,n}$, Corollary 2 implies that such a circuit D does not occur. Similarly, we can show that if $v_1 = \frac{n+1}{n}$, then there does not exist a circuit D like $\infty \rightarrow \frac{n+1}{n} \rightarrow \cdots \rightarrow \frac{1}{n} \leftarrow \infty$. Therefore $H_{1,n}$ is a forest. Using Theorem 5(3), we see that the subgraph $H_{n-1,n}$ is a forest as well. Therefore the proof is completed. □

Theorem 12 *If $H_{u,n}$ contains a triangle, then $\Gamma_1(n)$ contains an elliptic element of order 3.*

Proof If $H_{u,n}$ contains a triangle, then by Theorem 9, $n = 1$. So, $\Gamma_1(1) = \Gamma$ and $\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \in \Gamma_1(1)$ is an elliptic element of order 3. □

Remark 1 In general, the converse of Theorem 12 is not true. For example, the element $\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \in \Gamma_1(3)$ is an elliptic element of order 3, but $H_{1,3}$ does not contain a triangle. And also, by Theorem 5(3), $H_{2,3}$ does not contain a triangle either.

Remark 2 In [5], it is shown that the elliptic elements in $\Gamma_0(n)$ correspond to circuits in the subgraph $F_{u,n}$ of the same order and vice versa. Here, in the case of $\Gamma_1(n)$, owing to Theorem 12 triangles in the subgraph $H_{u,n}$ correspond to elliptic elements in $\Gamma_1(n)$ of order 3. But the converse is not true as shown in Remark 1.

Theorem 13 *$H_{u,n}$ contains a* 2*-gon if and only if* $\Gamma_1(n)$ *contains an elliptic element of order* 2.

Proof If $H_{u,n}$ contains a 2-gon, then by Theorem 10, $n = 1$ or 2. So, $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ is an elliptic element of order 2 in both $\Gamma_1(1)$ and $\Gamma_1(2)$.

Conversely, assume that $\Gamma_1(n)$ contains an elliptic element of order 2. Then there is an element of $\Gamma_1(n)$ of the form $\begin{pmatrix} 1+an & b \\ cn & 1+dn \end{pmatrix}$ such that $2 + (a + d)n = 0$. From this we get $n = 1$ or 2. Hence, the proof now follows from Theorem 10. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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