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Going in Circles: An Exploration of Functions into the Circle Group

Honors Thesis

Submitted

In Partial Fulfillment

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By

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Going in Circles: An Exploration of Functions into the Circle Group

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Abstract: We investigate the properties of functions into the circle group. The circle group is given by the factor group $\mathbb{R}/\mathbb{Z}$. Functions into the circle group have real-valued domain and $\mathbb{R}/\mathbb{Z}$ co-domain. Every real-valued function has an analogous function into the circle group. By wrapping the graph of a real-valued function around a horizontal cylinder with a circumference of one, we visualize the analogous function into the circle group. How does wrapping a real-valued function around such a cylinder affect the function outputs, limits, continuity, and rate of change? Function outputs are naturally transformed to reside on a circle with a circumference of one. Consequently, while every real-valued function has an analogous function into the circle group, this transformation is not one-to-one, i.e. two non-equal real-valued functions may have equal analogous functions into the circle group. Function limits, continuity, and rate of change are preserved with respect to this transformation. More interestingly, we find that for some real-valued functions non-existent limits become existent for the analogous functions into the circle group. Similarly, some discontinuous real-valued functions have continuous analogous functions into the circle group, and some non-differentiable real-valued functions have differentiable analogous functions into the circle group.

Keywords: circle group, R/Z, wrapping functions, wrapped functions, factor group

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1. What is the Circle Group?

The simplest way to conceptualize the circle group is to imagine an infinitely looping circle with a circumference of one. Much like the real number line, the position along the circle is marked numerically. Figure 1 compares the circle group to the real number line. Note how the real number line has a unique identifier for each position while the circle group has multiple identifiers, as it loops onto itself. For convenience, we will primarily use the numbers in the interval [0,1) to mark position on the circle. More formally, in this paper, the circle group is represented by the factor group $\mathbb{R}/\mathbb{Z}$ under co-set addition. Before proceeding, we will briefly review factor groups.

Any collection of objects can be considered a mathematical *set*. Each object in a set is an *element*. On its own, a set lacks structure. We are allowed to reorder the elements, but we have no method for combining elements. An *operation* is a rule for

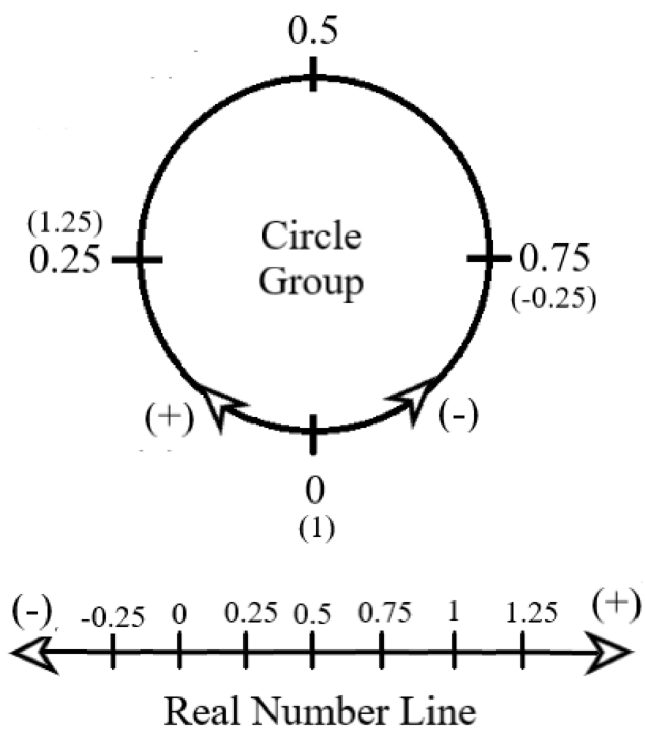


Figure 1. A comparison of the circle group and the real number line.

combining elements in a set. Consider the set $A = \{1, 2, 3\}$. The set is named A, and 1, 2, and 3 are the elements of A. We could combine the elements of A by adding them in the traditional fashion, e.g. $1 + 1 = 2$ and $2 + 3 = 5$. Notice that when we combine $2 \in A$ (read "2 in A") and $3 \in A$ using this addition, we obtain $5 \notin A$ (read "5 not in A"). Since 5 is not in A, we say A is not closed under this addition operation, i.e. we can combine two elements in A and obtain an element outside our set A. It is difficult to work with a set and operation if the set is not closed under the operation. When we equip a set with a "nice" operation that abides by specific constraints, we call the paired set and operation a *group*. Since groups play by rules, they are far more structured than sets on their own.

Definition. Let G be a set with an operation. We use $+$ to denote the operation here although the operation is not necessarily conventional addition. We say G is a *group* under the operation if the following four properties are satisfied.

(1) *Binary Operation.*

The group is closed under the operation, i.e. $a + b \in G$ for all a, $b \in G$.

(2) *Associativity.*

The operation is associative, i.e. $(a + b) + c = a + (b + c)$ for all a, b, $c \in G$.

(3) *Identity.*

There is an identity element $e \in G$ such that $a + e = e + a = a$ for all $a \in G$.

(4) *Inverses.*

For each element $a \in G$, there is an element $b \in G$ (called the *inverse* of a) such that $a + b = b + a = e$.

Furthermore, a group is an *Abelian group* if the operation is commutative; that is to say for all a, $b \in G$, we have $a + b = b + a$. Sometimes we may refer to a group G without acknowledging the operation explicitly. In other situations, we may refer to a group G under a specified operation $+$ by the notation $(G, +)$ [4].

Example 1.1. The set of real numbers, $\mathbb{R}$, is a group under addition. We confirm the fulfillment of each of the properties given above.

Binary Operation. Let a, $b \in \mathbb{R}$. Clearly, $a + b \in \mathbb{R}$ since a and b are real numbers.

Associativity. Let a, b, $c \in \mathbb{R}$. Again, the associative property of real numbers under addition is a familiar concept. We know that $(a + b) + c = a + (b + c)$ since a, b, and c are real numbers.

Identity. Let $a \in \mathbb{R}$. We identify the additive identity $0 \in \mathbb{R}$, as we know $a + 0 = 0 + a = a$.

Inverses. Let $a \in \mathbb{R}$. We identify the additive inverse of a as $-a$ since $a + (-a) = -a + a = 0$.

Thus, we have confirmed, albeit somewhat simplistically, that $\mathbb{R}$ is a group under addition. Further, for all a, $b \in \mathbb{R}$, we know $a + b = b + a$, so we can assert $\mathbb{R}$ is an Abelian group under addition.

Definition. Let $H \subset G$ where G is a group. If H is also a group under the same operation as G, we call H a *subgroup* of G [4].

While every subgroup is a subset of a group, not every subset of a group fulfills the requirements of a group, i.e. not every subset is a subgroup. The following example illustrates one way a subset of a group can fail to be a subgroup.

Example 1.2. Recall from Example 1.1 that $\mathbb{R}$ is a group under addition. Consider $\mathbb{R} - \{0\}$, the set of all real numbers solely excluding 0. While $\mathbb{R} - \{0\}$ is certainly a subset of $\mathbb{R}$, we see that $\mathbb{R} - \{0\}$ is not a subgroup of $\mathbb{R}$ under addition because the additive identity $0 \notin \mathbb{R} - \{0\}$.

When trying to confirm whether a subset of a group is a subgroup, it is redundant to check for each of the four group requirements separately. The following theorem offers a simple test to check a subset for subgroup standing. Note that in keeping with our previous use of $(+)$ to denote a general operation, we use $a - b$ to mean $a + (-b)$ where $-b$ is the inverse of b.

Theorem 1.3 (One-Step Subgroup Test)**.** *Let H be a non-empty subset of a group G. If $a - b \in H$ whenever a, $b \in H$, then H is a subgroup of G* [4]*.*

Example 1.4. Is $\mathbb{Z}$, the set of integers, a subgroup of $\mathbb{R}$ under addition? Certainly, $\mathbb{Z} \subset \mathbb{R}$. To simplify the investigation, we use the subgroup test offered in Theorem 1.3. Let a, $b \in \mathbb{Z}$. Then, $a - b = a + (-b) \in \mathbb{Z}$ since the negative of the integer b is an integer, and the sum of integers is also an integer. Thus, $\mathbb{Z}$ is a subgroup of $\mathbb{R}$ under addition. Note that $\mathbb{Z}$ inherits the additive identity 0 and inverse elements in $\mathbb{Z}$ are identified in the same fashion as they were for $\mathbb{R}$. Recall, that $\mathbb{R}$ is an Abelian group under addition, i.e. $c + d = d + c$ for all c, $d \in \mathbb{R}$. Since $\mathbb{Z} \subset \mathbb{R}$, the commutative property is inherited, i.e. $\mathbb{Z}$ is an Abelian subgroup of $\mathbb{R}$ under addition.

Definition. A subgroup H of a group G is called a *normal subgroup* of G if $a + H = H + a$ for all a in G. Here $a + H = \{a + h : h \in H\}$ is called the *left coset of H in G containing a* while $H + a = \{h + a : h \in H\}$ is the *right coset of H in G containing a* [4].

Theorem 1.5. *If H is a subgroup of an Abelian group G, then H is a normal subgroup of G* [4]*.*

Proof of Theorem 1.5. Let G be an Abelian group and H a subgroup of G. Let $a \in$. Let $h_1 \in H$. Then, certainly a, $h_1 \in G$ and G is Abelian, so $a + h_1 = h_1 + a$. Thus,

$$a + H = \{a + h : h \in H\} = \{h + a : h \in H\} = H + a.$$

By definition, H is a normal subgroup of G. □

Example 1.6. By Example 1.1 and Example 1.4, we know $\mathbb{Z}$ is a sugroup of the Abelian group $\mathbb{R}$ under addition. Now, by Theorem 1.5, we know that $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$ under addition.

Definition. Let G be a group and let H be a normal subgroup of G. Then, the set $G/H = \{a + H : a \in G\}$ is a *factor group* with operation

$$(a + H) + (b + H) = (a + b) + H. [4]$$

Each element of a factor group, G/H is a co-set, a set of elements that have been deemed equivalent to one another based off their relationship to the normal subgroup, H, that is "factored" out of the original group, G. If G is a completed puzzle, the elements of G are the individual pixels, and the co-sets are the puzzle pieces. See Figure 2 for a visualization of the factor group concept.

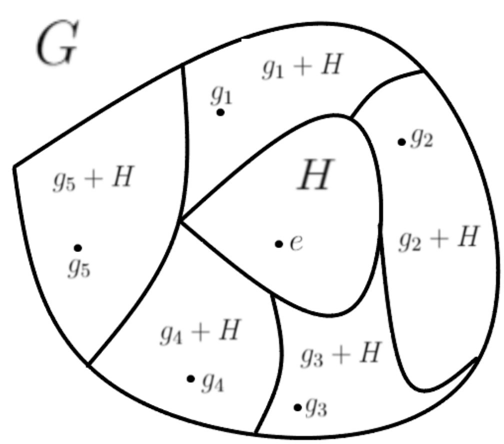


FIGURE 2. A model showing how the factor group G/H partitions G. Note e is the identity, H is a normal subgroup for G, and $e, g_1, g_2, g_3, g_4, g_5 \in G$ are the co-set representatives for $H,\ g_1 + H,\ g_2 + H,\ g_3 + H,\ g_4 + H,\ g_5 + H \in G/H$.

Consider the following:

$$0.1 + 0.3 = 0.4$$

$$1.1 + 1.3 = 0.1 + 0.3 + 1 + 1 = 0.4 + 2 = 2.4$$

$$99.1 + 99.3 = 0.1 + 0.3 + 99 + 99 = 0.4 + 198 = 198.4$$

Since $\mathbb{R}$ is a group under addition, we are allowed to represent addition of real numbers in this fashion. We can chop off the integer portion of each number and add the integer portions and decimal portions separately. Notice in the example calculations above, that the sum of the integer portions has no bearing on the sum of the decimal portions. Since $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$ it is self-contained (the sum of the integer portions always yields another integer) and combines with the decimal portions in the final step in the same fashion each time. We can reduce all the equations above, and an infinite number of equivalent equations with different integer values, to
(0.1+ integer) + (0.3 + integer) = (0.1 + 0.3)+integer= 0.4+ integer, i.e.

$$(0.1 + \mathbb{Z}) + (0.3 + \mathbb{Z}) = (0.1 + 0.3) + \mathbb{Z} = 0.4 + \mathbb{Z}.$$

This property is at the heart of the circle group, $\mathbb{R}/\mathbb{Z}$. In this section, we have been building to the conclusion that $\mathbb{R}/\mathbb{Z}$ is an Abelian factor group under co-set addition defined by $(a+\mathbb{Z})+(b+\mathbb{Z}) = a+b+\mathbb{Z}$ for all a, $b \in \mathbb{R}$. This is a direct consequence of $\mathbb{Z}$ being a normal subgroup for $\mathbb{R}$ under addition as delineated in Example 1.6. The elements of the real number line $\mathbb{R}$ are grouped into co-sets based off their position relative to the integers. All the integers are deemed equivalent, and the set $\mathbb{Z}$ is the identity element for $\mathbb{R}/\mathbb{Z}$. There are an infinite number of numbers between any adjacent integers, so there are an infinite number of distinct co-sets in $\mathbb{R}/\mathbb{Z}$. A basic schematic of $\mathbb{R}/\mathbb{Z}$ is included in Figure 3.

$\mathbb{R}/\mathbb{Z}$ will be worked with extensively throughout this paper, so note the following properties of this factor group.

Properties.

(1) $\mathbb{R}/\mathbb{Z} = \{a + \mathbb{Z} : a \in \mathbb{R}\}$

(2) For $a \in \mathbb{R}$, $a + \mathbb{Z} = \{a + k : k \in \mathbb{Z}\}$

(3) $(a + \mathbb{Z}) + (b + \mathbb{Z}) = (a + b) + \mathbb{Z}$

(4) For a, $b \in \mathbb{R}$, $a + \mathbb{Z} = b + \mathbb{Z} \iff a - b \in \mathbb{Z}$

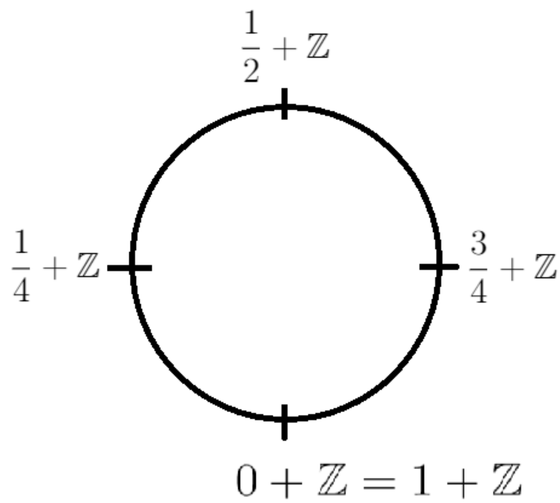


FIGURE 3. A basic schematic depicting the circular structure of $\mathbb{R}/\mathbb{Z}$. Note that the circumference of the circle is 1.

Example 1.7. $\mathbb{R}/\mathbb{Z}$ has identity element $0 + \mathbb{Z}$.

$$0 + \mathbb{Z} = \{0 + k : k \in \mathbb{Z}\} = \{k : k \in \mathbb{Z}\} = \mathbb{Z}.$$

Since the set of integers $\mathbb{Z}$ are closed under addition,

$$... = -2 + \mathbb{Z} = -1 + \mathbb{Z} = 0 + \mathbb{Z} = 1 + \mathbb{Z} = 2 + \mathbb{Z} = ...$$

Realize that for any integer n, $n - 0 = n \in \mathbb{Z}$, so by (4) above $\mathbb{Z} = 0 + \mathbb{Z} = n + \mathbb{Z}$. Thus, there are a countably infinite number of ways to represent the identity element.

In the previous example, we see that we can represent the identity element for $\mathbb{R}/\mathbb{Z}$ with an infinite number of co-set representatives (namely the integers). This property is not unique to the identity. In fact, every element of $\mathbb{R}/\mathbb{Z}$ has a countably infinite number of co-set representatives. Let $a \in \mathbb{R}$ and consider $a + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. Let $n \in \mathbb{Z}$. Then, $(a + n) - a = n \in \mathbb{Z}$, so by (4) above, we have $a + \mathbb{Z} = (a + n) + \mathbb{Z}$. Further note that $a + n \in a + \mathbb{Z}$ which is the basis for property (4). Because n is an arbitrary integer and the integers are countably infinite, there are a countably infinite number of co-set representatives that can be used to represent the element $a + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. To eliminate some redundancy and fit with our conceptualization of the circle group as a looping circle with circumference 1, in this paper $\bar{a}$ represents the element in the co-set $a+\mathbb{Z}$ such that $\bar{a} \in [0, 1)$. Note that the choice of the interval $[0, 1)$ is somewhat arbitrary in position but not diameter. Before formally defining $\bar{a}$, we define the floor function used in this paper [10].

Definition. Let $a \in \mathbb{R}$. Then, $\lfloor a \rfloor$ is the greatest integer less than or equal to a.

The difference between a number and its floor is sometimes called the fractional part [5]. In this paper we refer to this difference as the circle representative because it indiscriminately gives a value in the interval [0,1).

Definition. Let $a \in \mathbb{R}$. The circle representative for $a + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ is $\bar{a} = a - \lfloor a \rfloor$.

Theorem 1.8. *Let* a, b, $c \in \mathbb{R}$.

Then,

(1) $\bar{a} \in [0, 1)$

(2) $a + \mathbb{Z} = \bar{a} + \mathbb{Z}$

(3) $a + \mathbb{Z} = b + \mathbb{Z}$ *if and only if* $\bar{a} = \bar{b}$,

Proof of Theorem 1.8. Let a, b, $c \in \mathbb{R}$.

(1) Recall that $\lfloor a \rfloor$ is the greatest integer less than or equal to a.
Thus, $\lfloor a \rfloor \le a < \lfloor a \rfloor + 1$. Then subtracting $\lfloor a \rfloor$ from all portions of the inequality, we have $0 \le a - \lfloor a \rfloor < 1$. Hence $\bar{a} = a - \lfloor a \rfloor \in [0, 1)$.

(2) This result follows directly from the definition of the circle representative.

$$a - \bar{a} = a - (a - \lfloor a \rfloor) = \lfloor a \rfloor \in \mathbb{Z}.$$

Since the difference between a and $\bar{a}$ is an integer, we know $a + \mathbb{Z} = \bar{a} + \mathbb{Z}$.

(3) Suppose $a + \mathbb{Z} = b + \mathbb{Z}$.
By (2) above, $a + \mathbb{Z} = \bar{a} + \mathbb{Z}$ and $b + \mathbb{Z} = \bar{b} + \mathbb{Z}$, so transitively, we have

$\bar{a} + \mathbb{Z} = \bar{b} + \mathbb{Z}$. This means $\bar{a} - \bar{b} \in \mathbb{Z}$. By (1) above, we also have that $\bar{a}$, $\bar{b} \in [0, 1)$. Thus, $|\bar{a} - \bar{b}| < 1$. The only integer with absolute value strictly less than 1 is 0. Hence, $\bar{a} - \bar{b} = 0$ which implies $\bar{a} = \bar{b}$.

Now, suppose $\bar{a} = \bar{b}$.

Then, $a - \lfloor a \rfloor = b - \lfloor b \rfloor$.

Subtracting b and adding $\lfloor a \rfloor$, we have $a - b = \lfloor a \rfloor - \lfloor b \rfloor$.

Since $\lfloor a \rfloor$, $\lfloor b \rfloor \in \mathbb{Z}$, $a - b = \lfloor a \rfloor - \lfloor b \rfloor \in \mathbb{Z}$.

Hence, $a + \mathbb{Z} = b + \mathbb{Z}$.

□

2. Alternate Structure and Lie Group Representations

The circle group is often alternatively depicted by the complex unit circle under complex multiplication. The complex unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ where $|z| = \sqrt{a^2 + b^2}$ for $z = a + bi$. See Figure 4 for clarification. The complex unit circle under multiplication and the factor group $\mathbb{R}/\mathbb{Z}$ under co-set addition are group isomorphic. Isomorphic is a fancy way of saying the same shape and same properties. We have different ways of notating the elements of the complex unit circle and the elements of $\mathbb{R}/\mathbb{Z}$, but their elements combine analogously, and the geometric underpinnings of the groups are identical. For all practical purposes, working with one of these groups is the same as working with the other. We need a few more tools from group theory to prove this isomorphism.

Definition. Let $\sigma : G \to K$ be a map from a group G to a group K.

If $\sigma(a + b) = \sigma(a) + \sigma(b)$ for all a, $b \in G$, then σ is a *group homomorphism.* In other words, a *homomorphism* preserves a group's operation. Additionally, the *kernel* of a homomorphism $\sigma : G \to K$, denoted $\ker \sigma$, is the subset of G that σ maps to the identity of K. In set notation,

$$\ker \sigma = \{x \in G : \sigma(x) = e \in K\}$$

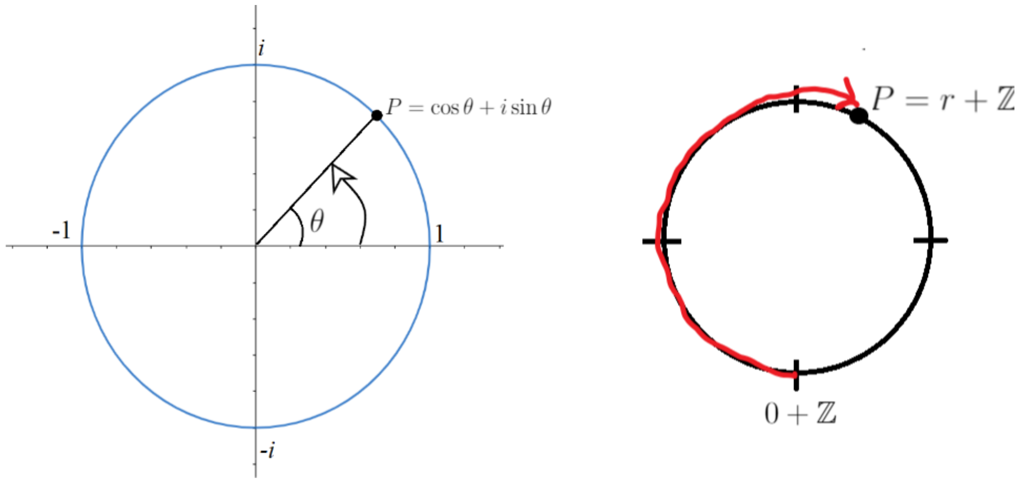


FIGURE 4. A comparison of point P identification for S_1 (left) and $\mathbb{R}/\mathbb{Z}$ (right). In S_1, P is identified by an angle θ measured counterclockwise from the positive real axis. In $\mathbb{R}/\mathbb{Z}$, the point P is identified by arc length r measured clockwise from the south pole $0 + \mathbb{Z}$. For S_1, the specific point P can be identified by any of the angles: θ, $\theta + 2\pi$, $\theta + 4\pi$,..., $\theta + 2\pi k$. Analogously, in $\mathbb{R}/\mathbb{Z}$, the specific point P can be identified by any of the arc lengths: r, $r + 1$, $r + 2$,..., $r + k$ where k is an integer in both cases.

where e denotes the identity element for K [4].

Definition. Building on the idea of a group homomorphism, a *group isomorphism* is a group homomorphism that is one-to-one and onto [4].

Theorem 2.1 (First Isomorphism Theorem)**.** *Suppose $\sigma : G \to H$ is a group homomorphism. Then, the mapping from $G/\ker\sigma$ to $\sigma(G)$ given by $g + \ker\sigma \to \sigma(g)$, is an isomorphism. This means that $G/\ker\sigma$ is isomorphic to $\sigma(G)$* [4].

The complex unit circle under multiplication and the factor group $\mathbb{R}/\mathbb{Z}$ under co-set addition are group isomorphic.

Proof. In set notation, the complex unit circle can be written $\{z \in \mathbb{C} : |z| = 1\}$. Note that $\{z \in \mathbb{C} : |z| = 1\} = \{\cos x + i\sin(x) : x \in \mathbb{R}\} = \{e^{ix} : x \in \mathbb{R}\}$ [9]. Then consider the function $\sigma : \mathbb{R} \to \{e^{ix} : x \in \mathbb{R}\}$ defined by $\sigma(x) = e^{2\pi ix}$.

First, let $x, y \in \mathbb{R}$. Then,

$$\begin{aligned}\sigma(x+y) &= e^{2\pi i(x+y)} \\ &= e^{2\pi ix+2\pi iy)} \\ &= e^{2\pi ix}e^{2\pi iy} \\ &= \sigma(x)\sigma(y).\end{aligned}$$

Thus, σ is a group homomorphism.

Now, let $y_1 \in \{e^{ix} : x \in \mathbb{R}\}$. Then, $y_1 = e^{ir}$ for some $r \in \mathbb{R}$.

Consider $x_1 = \frac{r}{2\pi}$. Since $r, 2\pi \in \mathbb{R}$, $x_1 \in \mathbb{R}$.

Further note that, $\sigma(x_1) = e^{2\pi ix_1} = e^{2\pi ir/2\pi} = e^{ir} = y_1$. Thus, σ is onto.

Finally, consider the kernel of σ. Note that

$$\begin{aligned}\ker(\sigma) &= \{x \in \mathbb{R} : \sigma(x) = 1\} \\ &= \{x \in \mathbb{R} : e^{2\pi ix} = 1\} \\ &= \{x \in \mathbb{R} : \cos(2\pi x) + i\sin(2\pi x) = 1\} \\ &= \{x \in \mathbb{R} : \cos(2\pi x) = 1, \sin(2\pi x) = 0\} \\ &= \{x \in \mathbb{R} : 2\pi x = 2\pi k, k \in \mathbb{Z}\} \\ &= \mathbb{Z}.\end{aligned}$$

By the First Isomorphism Theorem, $\mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} : |z| = 1\}$. $\square$

The circle group can be outfitted with additional structure and considered a Lie group. A Lie group is a group that is also a differentiable manifold. We have already shown that the circle group is a group (it is in the name). Showing the circle group is a differentiable manifold is outside the scope of this paper. However, the applicability of the circle group as a Lie group is well within our scope. Lie groups can be considered continuous groups that encode continuous symmetries; consequently, Lie groups have applications in physical sciences including chemistry, biology, and physics. The differential manifold portion of the definition ensures the Lie group has local similarity to Euclidean space even if there are global dissimilarities. Lie groups are further characterized by their actions on vector spaces, these actions are called representations [2]. Representations bridge the gap between the abstract form of a Lie group and its physical application, as representations are typically encoded in matrices as a set of linear transformations [14]. The real representations of the circle group are rotations of $\mathbb{R}^2$ about the origin. The complex unit circle form easily translates to rotations. Recall that the complex unit circle $S^1 = \{cos\theta + i\sin\theta : \theta \in \mathbb{R}\}$ where geometrically θ is the central angle measured from the positive real axis. For a point $(x, y) \in \mathbb{R}^2$, we can rotate this point counterclockwise about the origin by angle θ and attain the rotated coordinates (x', y') by the following matrix multiplication operation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

This is a basic transformation found in introductory linear algebra texts [14]. To properly encode all the possible rotations given by the circle group as well as their combinations, the real representations of S^1 are given by

$$\rho_n(e^{i\theta}) = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \quad n \in \mathbb{N}.$$

3. Metric for the Circle Group

We will return to our work on $\mathbb{R}/\mathbb{Z}$. Instead of approaching $\mathbb{R}/\mathbb{Z}$ as a Lie group, we will restrict ourselves to $\mathbb{R}/\mathbb{Z}$ as a metric space and group. A metric space is a set with a function to measure distance between set members. The function is called a metric and must abide by three rules: (1) non-negativity, (2) symmetry, and (3) the triangle inequality [1]. Non-negativity means that the distance between non-equal set members is positive and the distance between an element and itself is 0, e.g. the distance between the park and the mall is 2 miles, but the distance between the park and the park is 0 miles. Symmetry means that the distance between two elements is equal regardless of which element we start with, e.g. the distance between the park and the mall is the same as the distance between the mall and the park. The triangle inequality ensures the distance is measured efficiently, e.g. the direct route between the park and the mall is shorter than if we take a detour and go from the park to the movie theater and then from the movie theater to the mall (assuming the movie theater is not on the way). These rules are formalized in the folowing definition, but they are fairly intuitive if we realize that the metric is a measure of distance.

Definition. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ that fulfills the following conditions:

(1) *Non-negativity*: For all a, $b \in X$, $d(a, b) \geq 0$ with $d(a, b) = 0$ only when $a = b$.

(2) *Symmetry*: For all a, $b \in X$, $d(a, b) = d(b, a)$.

(3) *The Triangle Inequality*: For all a, b, $c \in X$, $d(a, b) \leq d(a, c) + d(c, b)$.

Because of the broad definition of a metric, there are multiple potential metrics that could be defined for the circle group [3]. However, a meaningful metric on the circle group would be preferred. Since we are working with a "circle," a metric based on arc length is intuitive. Because $\mathbb{R}/\mathbb{Z}$ is a factor group, it is possible to have elements $a + \mathbb{Z} = b + \mathbb{Z}$ with $a \neq b$. Thus, it is important to ensure our metric is well-defined. For any a, b, $c \in \mathbb{R}/\mathbb{Z}$ if $a + \mathbb{Z} = b + \mathbb{Z}$, then we should have $d(a + \mathbb{Z}, c + \mathbb{Z}) = d(b + \mathbb{Z}, c + \mathbb{Z})$. An effective metric sends each member of $\mathbb{R}/\mathbb{Z}$ to its circle representative. Elements are then positioned on a circle of circumference 1 by their circle representatives in $[0, 1)$. The metric computes the arc-length between the elements based off a circle of circumference 1. Consider the proposed metric for $\mathbb{R}/\mathbb{Z}$ given by $d : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ below.

$$d(a + \mathbb{Z}, b + \mathbb{Z}) = \begin{cases} |\bar{a} - \bar{b}| & \text{if } |\bar{a} - \bar{b}| \leq 0.5 \\ 1 - |\bar{a} - \bar{b}| & \text{if } |\bar{a} - \bar{b}| > 0.5. \end{cases}$$

The following lemma aids us in proving this proposed metric is indeed a metric.

Lemma 3.1.

Let a, b, $c \in \mathbb{R}$. *Then,* $|\bar{a} - \bar{b}| + |\bar{a} - \bar{c}| + |\bar{b} - \bar{c}| < 2$.

Proof of Lemma 3.1. Suppose $\bar{a} \leq \bar{b} \leq \bar{c}$. Since $\bar{a}$, $\bar{c} \in [0, 1)$, we have

$$|\bar{a} - \bar{c}| < 1.$$

Then, because $\bar{a} \leq \bar{b} \leq \bar{c}$, we know $|\bar{a} - \bar{c}| = |\bar{a} - \bar{b}| + |\bar{b} - \bar{c}|$. Substituting into the previous inequality for $|a - c|$, we have

$$|\bar{a} - \bar{b}| + |\bar{b} - \bar{c}| < 1.$$

Combining the two inequalities, we have $|\bar{a} - \bar{c}| + |\bar{a} - \bar{b}| + |\bar{b} - \bar{c}| < 2$. Note that the original order of $\bar{a}$, $\bar{b}$, $\bar{c}$ has no bearing on this final inequality. □

Now we will prove that this proposed metric d, given in the piecewise function above, fulfills the three requirements for a defined metric.

Proof of Metric. Let $A, B \in \mathbb{R}/\mathbb{Z}$. Then, $A = a + \mathbb{Z}$ and $B = b + \mathbb{Z}$ for some $a, b \in \mathbb{R}$. Then, $\bar{a}$, $\bar{b} \in [0, 1)$.

Non-negativity:

Suppose $A \neq B$, i.e. $a + \mathbb{Z} \neq b + \mathbb{Z}$. Then by the properties of the circle representative, $\bar{a} \neq \bar{b}$. Since $\bar{a} \neq \bar{b}$, $|\bar{a} - \bar{b}| > 0$. Because $\bar{a}, \bar{b} \in [0, 1)$, $1 > |\bar{a} - \bar{b}|$. Subtracting $|\bar{a} - \bar{b}|$ from both sides of the inequality, we attain $1 - |\bar{a} - \bar{b}| > 0$.

Therefore, regardless of whether $d(A, B) = |\bar{a} - \bar{b}|$ or $d(A, B) = 1 - |\bar{a} - \bar{b}|$, we know $d(A, B) > 0$. Furthermore, $|\bar{a} - \bar{a}| = 0 < 0.5$, so $d(A, A) = |\bar{a} - \bar{a}| = 0$.

Symmetry:

Note that $|\bar{a} - \bar{b}| = |\bar{b} - \bar{a}|$, so $1 - |\bar{a} - \bar{b}| = 1 - |\bar{b} - \bar{a}|$.

Suppose $d(A, B) = |\bar{a} - \bar{b}| \leq 0.5$.

Then, $d(A, B) = |\bar{a} - \bar{b}| = |\bar{b} - \bar{a}| = d(B, C)$.

Now suppose $d(A, B) = 1 - |\bar{a} - \bar{b}| < 0.5$.

Then, $d(A,B) = 1 - |\bar{a} - \bar{b}| = 1 - |\bar{b} - \bar{a}| = d(b + \mathbb{Z}, a + \mathbb{Z})$.

Either way, $d(A,B) = d(B,C)$.

The Triangle Inequality:

Let $C \in \mathbb{R}/\mathbb{Z}$. Then, $C = c + \mathbb{Z}$ for some $c \in \mathbb{R}$. Then, $\bar{c} \in [0,1)$. We can split the proof of the triangle equality into three sub-cases depending on which half of the piece-wise metric defines the distances between A, B, and C.

First, suppose $d(A,C) = |\bar{a} - \bar{c}|$ and $d(C,B) = |\bar{c} - \bar{b}|$.

$$\begin{aligned} \text{Then, } d(A,C) + d(C,B) &= |\bar{a} - \bar{c}| + |\bar{c} - \bar{b}| \\ &\geq |\bar{a} - \bar{c} + \bar{c} - \bar{b}| \\ &= |\bar{a} - \bar{b}|. \end{aligned}$$

If $d(A,B) = |\bar{a} - \bar{b}|$, we have $d(A,C) + d(C,B) \geq d(A,B)$.

On the other hand, if $d(A,B) = 1 - |\bar{a} - \bar{b}|$, then $|\bar{a} - \bar{b}| > 1 - |\bar{a} - \bar{b}|$.

$$\begin{aligned} \text{Then we still have } d(A,C) + d(C,B) &\geq |\bar{a} - \bar{b}| \\ &> 1 - |\bar{a} - \bar{b}| \\ &= d(A,B). \end{aligned}$$

Either way, $d(A,C) + d(C,B) \geq d(A,B)$.

Now, suppose instead that $d(A,C) = 1 - |\bar{a} - \bar{c}|$ and $d(C,B) = |\bar{c} - \bar{b}|$.

$$\begin{aligned} \text{Then, } d(A,C) + d(C,B) &= 1 - |\bar{a} - \bar{c}| + |\bar{c} - \bar{b}| \\ &= 1 - |\bar{a} - \bar{b} - \bar{c} + \bar{b}| + |\bar{c} - \bar{b}| \\ &\geq 1 - (|\bar{a} - \bar{b}| + |\bar{c} - \bar{b}|) + |\bar{c} - \bar{b}| \\ &= 1 - |\bar{a} - \bar{b}| - |\bar{c} - \bar{b}| + |\bar{c} - \bar{b}| \\ &= 1 - |\bar{a} - \bar{b}|. \end{aligned}$$

If $d(A,B) = 1 - |\bar{a} - \bar{b}|$, $d(A,C) + d(C,B) \geq 1 - |\bar{a} - \bar{b}| = d(A,B)$.

If $d(A,B) = |\bar{a} - \bar{b}|$, $d(A,C) + d(C,B) \geq 1 - |\bar{a} - \bar{b}| \geq |\bar{a} - \bar{b}| = d(A,B)$.

Finally, suppose $d(A,C) = 1 - |\bar{a} - \bar{c}|$ and $d(C,B) = 1 - |\bar{c} - \bar{b}|$.

Because $\bar{a}, \bar{b}, \bar{c} \in [0,1)$, we now know $2 > |\bar{a} - \bar{b}| + |\bar{a} - \bar{c}| + |\bar{b} - \bar{c}|$.

$$\begin{aligned} \text{Now, } d(A,C) + d(C,B) &= 1 - |\bar{a} - \bar{c}| + 1 - |\bar{c} - \bar{b}| \\ &= 2 - |\bar{a} - \bar{c}| - |\bar{c} - \bar{b}| \end{aligned}$$

$$\geq |\bar{a}-\bar{b}| + |\bar{a}-\bar{c}| + |\bar{c}-\bar{b}| - |\bar{a}-\bar{c}| - |\bar{c}-\bar{b}|$$
$$= |\bar{a}-\bar{b}|.$$

If $d(A,B) = |\bar{a}-\bar{b}|$, we have $d(A,C) + d(C,B) \geq |\bar{a}-\bar{b}| = d(A,B)$.

If $d(A,B) = 1 - |\bar{a}-\bar{b}|$, we have $d(A,C) + d(C,B) \geq |\bar{a}-\bar{b}| > 1 - |\bar{a}-\bar{b}|$
$$= d(A,B).$$

□

In the following sections, we will use this arc-length metric extensively, as we compare the properties of real-valued functions and functions in the circle group. By outfitting $\mathbb{R}/\mathbb{Z}$ with a metric we are able to approach questions of analysis including limits, continuity, and differentiability. The metric tool enables us to concretely define key abstract ideas such as infinitesimally close, as illustrated in the following example [8].

Example 3.2. If we want to consider a subset of $\mathbb{R}/\mathbb{Z}$ that is arbitrarily close to a specific point, say $a+\mathbb{Z}$ for some $a \in \mathbb{R}$, we can construct an ϵ-ball centered at $a+\mathbb{Z}$ using the arc-length metric, d, and any real-valued $\epsilon > 0$. The ϵ-ball centered at $a+\mathbb{Z}$ is notated $B_d(a+\mathbb{Z}, \epsilon)$ and is defined in set notation here:

$$B_d(a+\mathbb{Z}, \epsilon) = \{b+\mathbb{Z} \in \mathbb{R}/\mathbb{Z} : d(a+\mathbb{Z}, b+\mathbb{Z}) < \epsilon\}.$$

See Figure 5 for a visualization of the ϵ-ball centered at $a+\mathbb{Z}$. By making ϵ arbitrarily small, i.e. arbitrarily close to 0 yet positive, we are allowed to identify sets of points in $\mathbb{R}/\mathbb{Z}$ that are infinitesimally close to $a+\mathbb{Z}$.

4. Functions in the Circle Group

Before approaching functions in the circle group, we will recall the definition of a real-valued function. In general, a function $f : X \to Y$ is a map between an input set X and an output set Y such that for each input value $x \in X$ we have exactly one output value $y \in Y$, notated $f(x) = y$ [13]. We call the input set X the domain of f and the output set Y the co-domain of f. If we specify $X \subset \mathbb{R}$ and $Y = \mathbb{R}$,

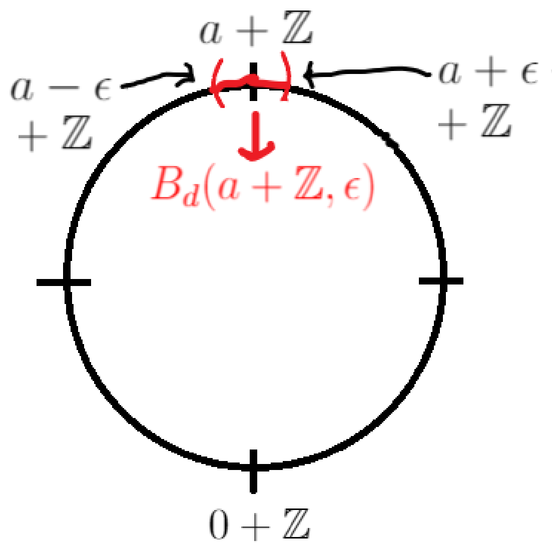


FIGURE 5. A visualization of the ϵ-ball centered at $a + \mathbb{Z}$. This ball can be made tighter by reducing the value of $\epsilon > 0$ to attain a sub-set of $\mathbb{R}/\mathbb{Z}$ arbitrarily close to $a + \mathbb{Z}$.

we obtain a real-valued function. We graph real-valued functions using a rectangular coordinate system. A two-dimensional plane is constructed around two perpendicular real-number lines that we call axes, a horizontal x-axis for input values and a vertical y-axis for output values. A real-valued function is graphed by plotting ordered pairs (x, y) for each input $x \in \mathbb{R}$ and corresponding output $y = f(x) \in \mathbb{R}$.

Example 4.1. A simple linear function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is graphed in the rectangular coordinate system in Figure 6. Some of the points on the graph are enlarged and labeled to emphasize the plotting system. In future figures, the grid my be omitted from graphs for clarity.

We progress into an exploration of functions in the circle group with real-valued domain, i.e functions of the form $f : A \to \mathbb{R}/\mathbb{Z}$ where $A \subset \mathbb{R}$. Geometrically, these functions are attained by wrapping real-valued functions around the circle. Note that the rectangular coordinate system is not sufficient for graphing functions in the circle group, as the outputs lie on a circle rather than a real number line. Our

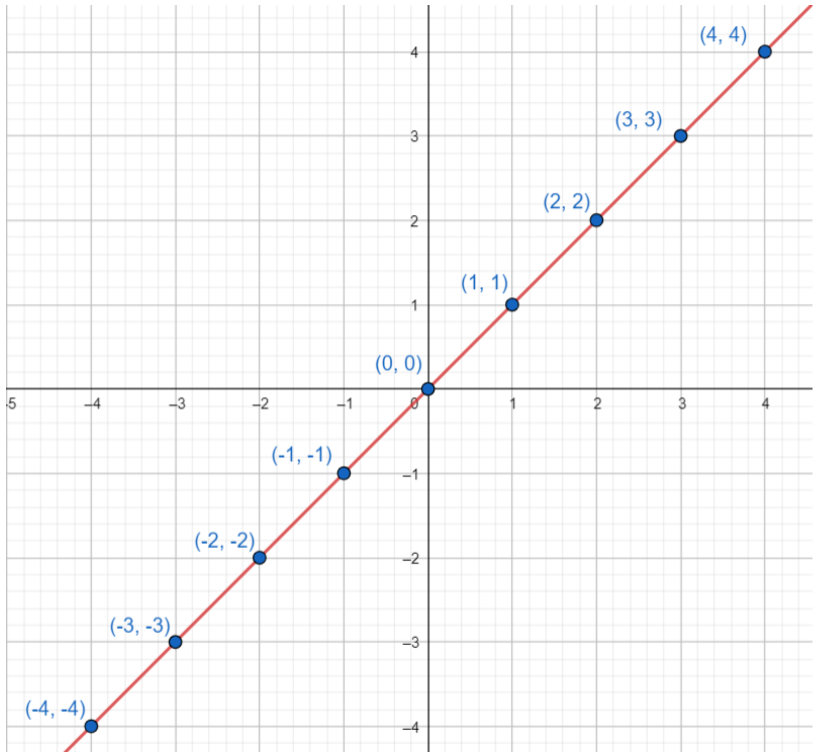


FIGURE 6. A graph of the real-valued function $f(x) = x$ (the red line) with several points enlarged and labeled (in blue).

graphing methods must be adapted to accommodate a one-dimensional input and two-dimensional output, i.e. our final graph will use three-dimensions for visualization. For every function $F : A \to \mathbb{R}/\mathbb{Z}$ where $A \subset \mathbb{R}$, there exists a real-valued function $f : A \to \mathbb{R}$ such that $F(x) = f(x) + \mathbb{Z}$ for all $x \in A$. This claim can be validated by constructing f element-wise as follows. Let $a \in A$. Then $F(a) \in \mathbb{R}/\mathbb{Z}$, so $F(a) = b+\mathbb{Z}$ for some $b \in \mathbb{R}$. Define $f(a) = b$ ensuring $F(a) = b+\mathbb{Z} = f(a)+\mathbb{Z}$. Note that there are countably infinite choices for $f(a)$ since $b+\mathbb{Z} = (b+k)+\mathbb{Z}$ for any $k \in \mathbb{Z}$. Thus, while we can ensure that there exists a real-valued function f such that $F(x) = f(x)+\mathbb{Z}$ for all $x \in A$, this real-valued function is not unique. However, when given a real-valued function, the analogous function in the circle group, as defined below, is unique.

Definition. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a real-valued function. Then, *the analogous function for* f *in the* $\mathbb{R}/\mathbb{Z}$ is denoted $F : A \to \mathbb{R}/\mathbb{Z}$ and defined by

$$F(x) = f(x) + \mathbb{Z}.$$

Corollary 4.2 (Corresponding to Theorem 1.8)**.** *Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two real-valued functions and define $\bar{f}(x) = f(x) - \lfloor f(x) \rfloor$ and $\bar{g}(x) = g(x) - \lfloor g(x) \rfloor$. Then,*

(1) *The analogous function for f in $\mathbb{R}/\mathbb{Z}$ is equal to the analogous function for $\bar{f}$ in $\mathbb{R}/\mathbb{Z}$, i.e. for all $x \in A$, we have $f(x) + \mathbb{Z} = \bar{f}(x) + \mathbb{Z}$.*

(2) *The analogous functions for f and g in $\mathbb{R}/\mathbb{Z}$ are equal if and only if $\bar{f}$ and $\bar{g}$ are equal.*

Proof of Corollary 4.2. Let $f,\ g : A \to \mathbb{R}$ be functions where $A \subset \mathbb{R}$. Define $\bar{f} : A \to \mathbb{R}$ by $\bar{f}(x) = f(x) - \lfloor f(x) \rfloor$, and likewise, define $\bar{g} : A \to \mathbb{R}$ by $\bar{g}(x) = g(x) - \lfloor g(x) \rfloor$. Let $a \in A$.

(1) Since $f : A \to \mathbb{R}$ and $a \in A$, we know $f(a) \in \mathbb{R}$, so by Theorem 1.8,

$$f(a) + \mathbb{Z} = f(a) - \lfloor f(a) \rfloor + \mathbb{Z}.$$

By definition, of $\bar{f}$, we have $f(a) + \mathbb{Z} = \bar{f}(a) + \mathbb{Z}$. Since a was selected arbitrarily in A, we know know that for all $x \in A$, we have $f(x)+\mathbb{Z} = \bar{f}(x)+\mathbb{Z}$. In other words, the analogous functions for f and $\bar{f}$ in $\mathbb{R}/\mathbb{Z}$ are equal.

(2) Suppose the analogous functions for f and g in $\mathbb{R}/\mathbb{Z}$ are equal. Then, for $f(a),\ g(a) \in \mathbb{R}$, we have $f(a) + \mathbb{Z} = g(a) + \mathbb{Z}$, so by Theorem 1.8, we have $\overline{f(a)} = \overline{g(a)}$, i.e. $f(a) - \lfloor f(a) \rfloor = g(a) - \lfloor g(a) \rfloor$. So, by the definition of $\bar{f}$ and $\bar{g}$, we have $\bar{f}(a) = \bar{g}(a)$. As above, this implies $\bar{f} = \bar{g}$. Now suppose instead that $\bar{f} = \bar{g}$, so clearly the analogous functions for $\bar{f}$ and $\bar{g}$ in $\mathbb{R}/\mathbb{Z}$ are equal. By (1), we know that the analogous functions for f and $\bar{f}$ in $\mathbb{R}/\mathbb{Z}$ are equal and similarly the analogous functions for g and $\bar{g}$ in $\mathbb{R}/\mathbb{Z}$ are equal. Then, transitively, we have that the analogous functions for f and g in $\mathbb{R}/\mathbb{Z}$ are equal.

□

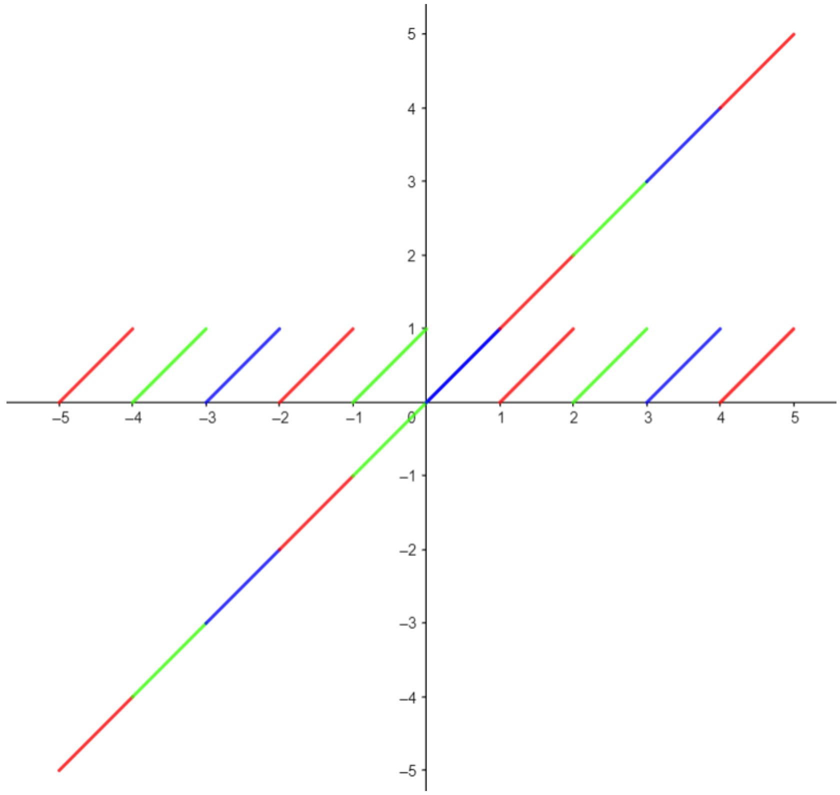


FIGURE 7. A graph showing the correspondence between the real-valued functions $f(x) = x$ and $\bar{f}(x) = x - \lfloor x \rfloor$ [6].

Definition. For a real-valued function $f : A \to \mathbb{R}$, the corresponding real-valued function $\bar{f} : A \to [0, 1)$ defined by $\bar{f}(x) = f(x) - \lfloor f(x) \rfloor$ gives the values of the circle representatives for $F(x) = f(x) + \mathbb{Z}$, so we say $\bar{f}$, defined in this fashion, is *the circle representative function for* f.

Example 4.3. Consider the real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Then, the analogous function for f in the circle group is $F(x) = f(x) + \mathbb{Z} = x + \mathbb{Z} = (x - \lfloor x \rfloor) + \mathbb{Z}$. The graphs of $f(x) = x$ and $\bar{f}(x) = x - \lfloor x \rfloor$ are combined below in Figure 7. Corresponding segments of $f(x) = x$ and $\bar{f}(x) = x - \lfloor x \rfloor$ are color-coded to emphasize the transformation of the linear function, f, with range $(-\infty, \infty)$ to it's circle representative function, $\bar{f}$, with range $[0, 1)$. Note that for $0 \leq x < 1$, we have $f(x) = x = x - \lfloor x \rfloor = \bar{f}(x)$. Omitting the redundancy of $f(x) = x$, we visualize the circle representative function $\bar{f}(x) = x - \lfloor x \rfloor$ on its own in Figure 8.

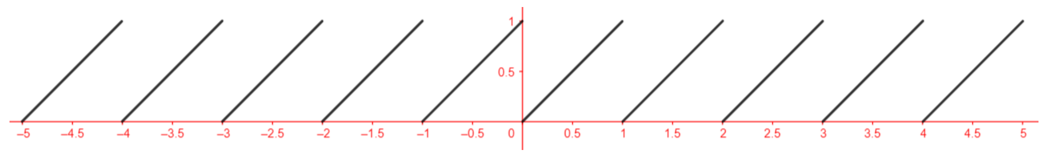


FIGURE 8. A graph of $\bar{f}(x) = x - \lfloor x \rfloor$ with range $[0, 1)$ [6].

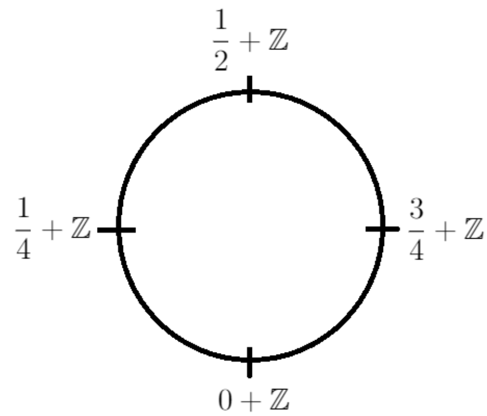
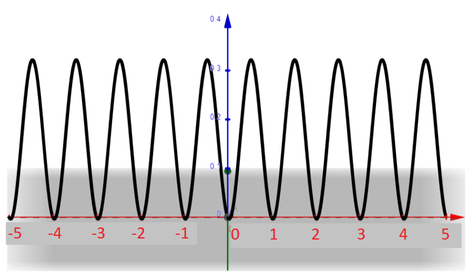


FIGURE 9. A visual representation of $\mathbb{R}/\mathbb{Z}$ as a metric space equipped with an arc length metric .

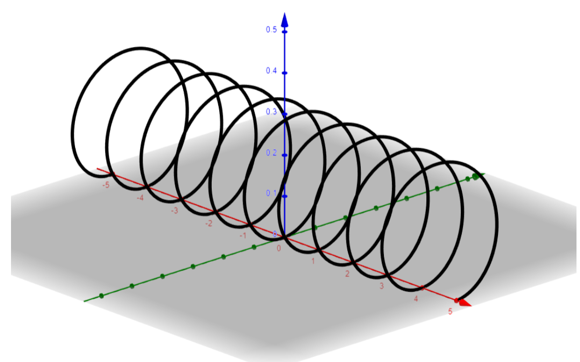
In general, while $\bar{f}$ is useful for clearly identifying the circle representatives of F, this graphical representation does not fully capture the looping nature of $\mathbb{R}/\mathbb{Z}$. That is to say, the graph of the real-valued function $\bar{f}$ fails to show how $0 + \mathbb{Z} = 1 + \mathbb{Z}$. Instead, the graph of $\bar{f}$ features repeated jumps between 0 and 1. Ideally, a model of a function in $\mathbb{R}/\mathbb{Z}$ would show the function outputs along the perimeter of an infinitely looping circle with circumference 1 instead of along a real number line. If we view $\mathbb{R}/\mathbb{Z}$ as the metric space previously described, the circle below would serve as an accurate representation of $\mathbb{R}/\mathbb{Z}$. Then, functions from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$ would trace along the surface of a cylinder rather than a plane. To acquire a visual model for a function in $\mathbb{R}/\mathbb{Z}$, we need to extend our endeavors to a third dimension. Refer to the graph of $\bar{f}(x) = x - \lfloor x \rfloor$ in Example 4.3 above. Imagine the two-dimensional representation of $\bar{f}$, wrapped around a horizontal cylinder with circumference 1 such that the lines $y = 1$ and $y = 0$ are overlapping. The resulting three-dimensional figure would depict the graph of $F(x) = x + \mathbb{Z}$ where the output of F for each input x is marked by the position along the circumference of the cylinder. To obtain a visual model of this abstract idea, we need a parametrization corresponding to our function. Generally speaking, for a real-valued function $f : A \to \mathbb{R}$, use the parametrization below to graph the analogous function of f in $\mathbb{R}/\mathbb{Z}$ defined by $F(x) = f(x) + \mathbb{Z}$.

$$x = x,\ y = -\frac{1}{2\pi}\cos\left[2\pi\left(f(x)\right) - \frac{\pi}{2}\right],\ z = \frac{1}{2\pi}\sin\left[2\pi\left(f(x)\right) - \frac{\pi}{2}\right] + \frac{1}{2\pi},\ x \in A$$

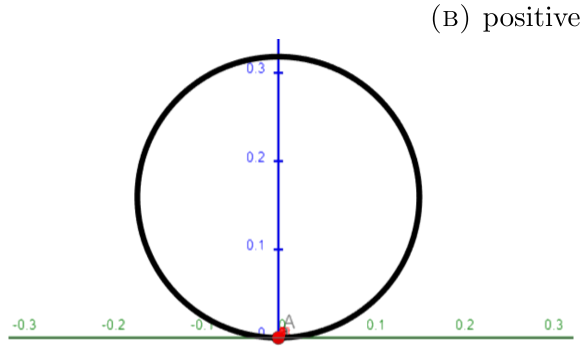
Example 4.4. Return to the real valued function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Using the parametrization delineated above, restricted by $-5 \leq x \leq 5$, we obtain the following three-dimensional graph of $F(x) = x + \mathbb{Z}$ shown from multiple angles in Figures 10a, 10b, and 10c.



(A) xz-plane



(B) positive xyz-octant



(C) yz-plane

FIGURE 10. Graph of $F(x) = x + \mathbb{Z}$ [6].

Figure 10a features the xz-plane. Notice that the vertical axis no longer corresponds to the circle representatives of F; however, the zeroes on the x-axis still correspond to the zeroes of F. In other words, we can see $F(0) = 0 + \mathbb{Z}$, $F(1) = 1 + \mathbb{Z} = 0 + \mathbb{Z}$, $F(2) = 2 + \mathbb{Z} = 0 + \mathbb{Z}$, etc.

Figure 10b, with the positive xyz-octant in the foreground, depicts the shape of $F(x) = x + \mathbb{Z}$ most accurately, as we can see how increasing and decreasing values of x are wrapped around the circle representing $\mathbb{R}/\mathbb{Z}$. Note the even spacing of the helix's rungs. Since $f(x) = x$ increases at a constant rate, the analogous function in $\mathbb{R}/\mathbb{Z}$ wraps around the circle at a constant rate. This feature is also apparent in the graph of $\bar{f} = x - \lfloor x \rfloor$ in Example 4.3.

Figure 10c shows the yz-plane and the circle embedded in it that represents $\mathbb{R}/\mathbb{Z}$. The parametrization is such that this circle corresponds directly with the the representation of $\mathbb{R}/\mathbb{Z}$ given in Figure 9. The circle has radius $\frac{1}{2\pi}$ and, consequently, a circumerfence of 1. The lowest point on the circle corresponds to $0+\mathbb{Z}$ with the circle representative values increasing in a clockwise fashion, so $\frac{1}{4}+\mathbb{Z}$ is at the leftmost point, $\frac{1}{2} + Z$ is at the topmost point, and $\frac{3}{4} + \mathbb{Z}$ is at the rightmost point. Note that $1 + \mathbb{Z}$ has looped back around to $0 + \mathbb{Z}$ as intended, and arc length serves as a reasonable measure of distance between points on this circle, so our previous conceptualization of $\mathbb{R}/\mathbb{Z}$ as a metric space remains intact.

Example 4.5. Consider the four following real-valued functions $f,\ g,\ h,\ k : \mathbb{R} \to \mathbb{R}$ defined below:

$$f(x) = x^2,$$

$$g(x) = \begin{cases} x^2 & x \neq 0 \\ x^2 + 5 & x = 0 \end{cases},$$

$$h(x) = x^2 + 5,$$

$$k(x) = \begin{cases} x^2 & x < 0 \\ x^2 + 5 & x \geq 0 \end{cases}$$

The graphs of these real-valued functions are found in Figure 17. Notice that $f \neq g \neq h \neq k$.

However,

$\bar{f}(x) = x^2 - \lfloor x^2 \rfloor$; $\quad \bar{h}(x) = x^2 + 5 - \lfloor x^2 + 5 \rfloor = x^2 + 5 - \lfloor x^2 \rfloor - 5 = x^2 - \lfloor x^2 \rfloor$;

$$\bar{g}(x) = \begin{cases} x^2 - \lfloor x^2 \rfloor & x \neq 0 \\ x^2 + 5 - \lfloor x^2 + 5 \rfloor & x = 0 \end{cases} = \begin{cases} x^2 - \lfloor x^2 \rfloor & x \neq 0 \\ x^2 + 5 - \lfloor x^2 \rfloor - 5 & x = 0 \end{cases} = x^2 - \lfloor x^2 \rfloor;$$

$$\bar{k}(x) = \begin{cases} x^2 - \lfloor x^2 \rfloor & x < 0 \\ x^2 + 5 - \lfloor x^2 + 5 \rfloor & x \geq 0 \end{cases} = \begin{cases} x^2 - \lfloor x^2 \rfloor & x < 0 \\ x^2 + 5 - \lfloor x^2 \rfloor - 5 & x \geq 0 \end{cases} = x^2 - \lfloor x^2 \rfloor.$$

Since the circle representative functions $\bar{f}$, $\bar{g}$, $\bar{h}$, and $\bar{k}$ are equal (see Figure 12), we know from Corollary 4.2, that the analogous functions in $\mathbb{R}/\mathbb{Z}$ are equal. Once again,

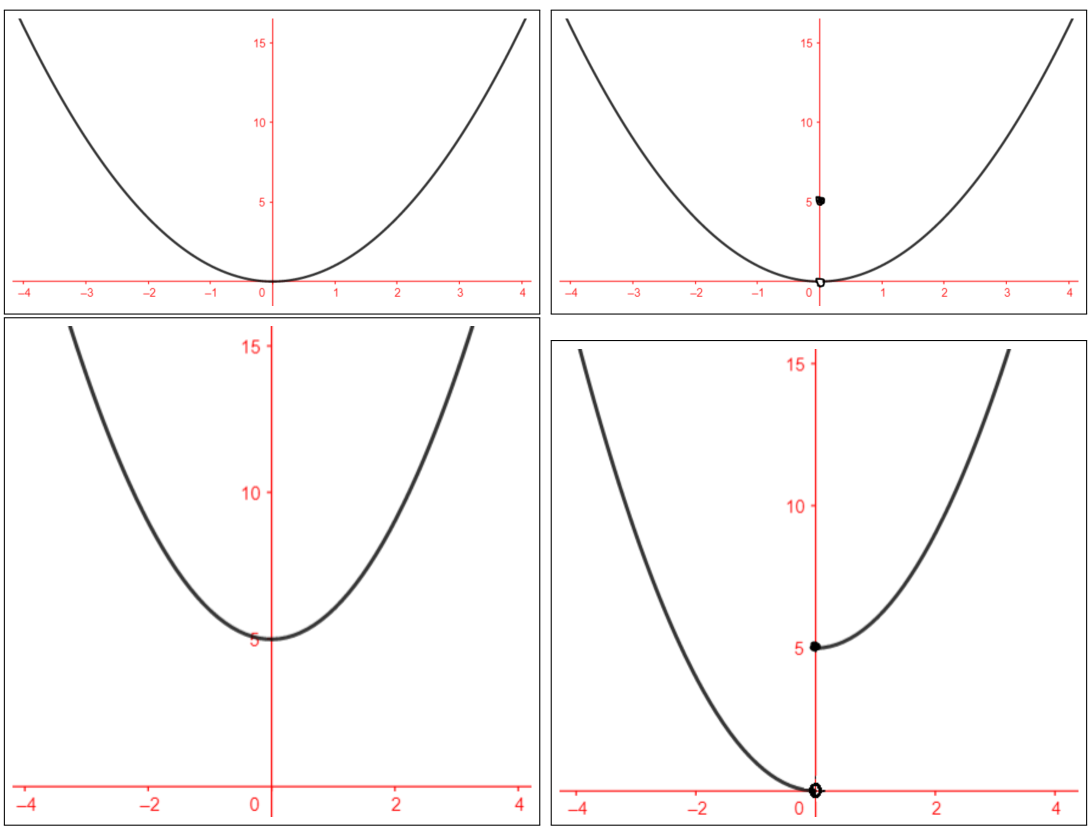


FIGURE 11. Graphs of the related real-valued functions f, g, (top) and h, k (bottom) defined in Example 4.5 [6].

we can use parametrization to visualize the graph of $F = G = H = K : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ defined by $F(x) = x^2 + \mathbb{Z} = x^2 - \lfloor x^2 \rfloor + \mathbb{Z}$ (see Figures 13a and 13b). Notice that f, g, h, and k are only four of the real-valued functions with the common analogous function $F(x) = x^2 + \mathbb{Z}$. Each output of F has a countably infinite number of potential coset representatives, so there are a countably infinite number of real-valued functions that only differ at a single output value that share the common analogous function F. However, when we also consider that our domain, $\mathbb{R}$ in this case scenario, is uncountable, we are theoretically able to produce an uncountably infinite number of real-valued functions with the common analogous function F.

Example 4.6. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$. Compared to our previous examples, the cubic function increases rapidly. The analogous function in the circle group $F(x) = x^3 + \mathbb{Z}$ consequently has increasingly frequent zeroes. Notice that on the interval $(0, 1]$, F has 1 zero at $F(1) = 1 + \mathbb{Z} = 0 + \mathbb{Z}$. On the interval $(1, 2]$, F has 7 zeroes. On the interval, $(2, 3]$, F has 19 zeroes. Since f is a strictly increasing

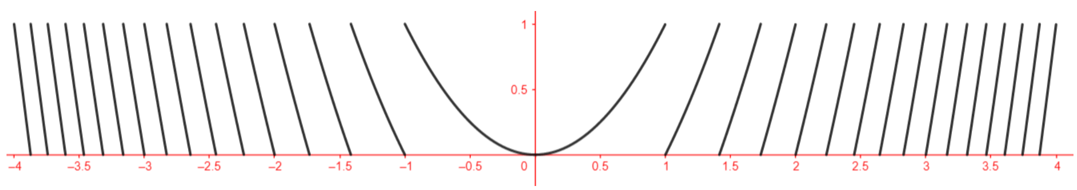
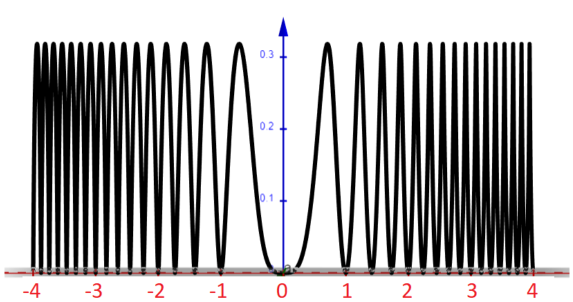
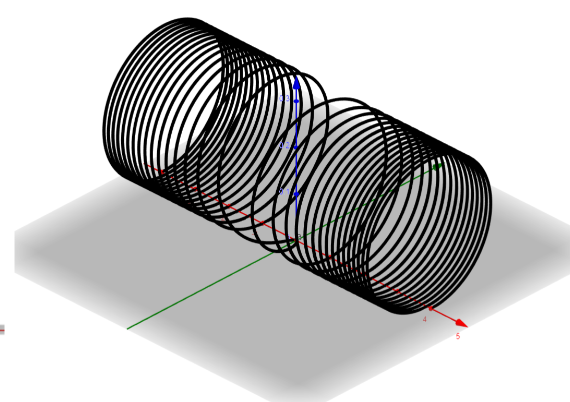


FIGURE 12. A graph of $\bar{f}(x) = \bar{g}(x) = \bar{h}(x) = \bar{k}(x) = x^2 - \lfloor x^2 \rfloor$ [6].



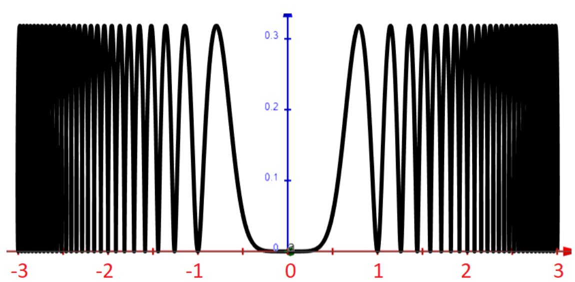
(A) xz-plane.



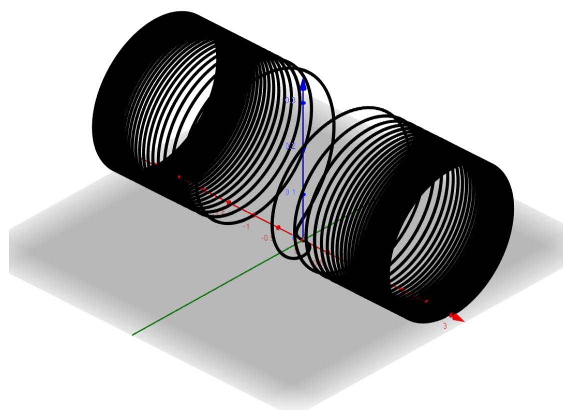
(B) positive xyz-octant

FIGURE 13. Graph of $F(x) = x^2 + \mathbb{Z}$ [6].

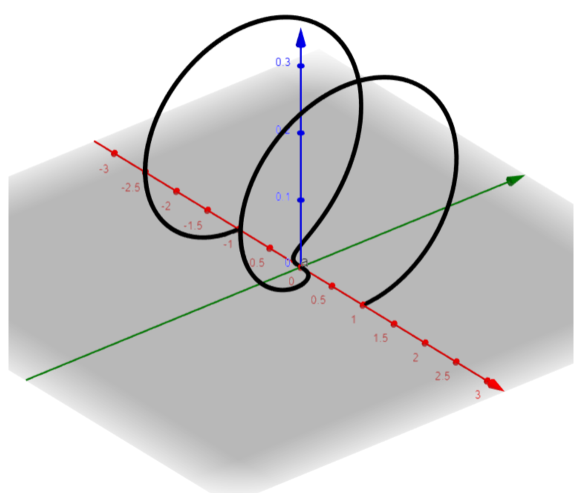
function on $\mathbb{R}$, the number of zeroes, n, on any interval $(a, b]$ can be computed by $n = \lfloor f(b) \rfloor - \lfloor f(a) \rfloor$. Notice that visualizing $F(x) = x^3 + \mathbb{Z}$ on large intervals is increasingly cumbersome, as most of these zeroes corresponds to a full traversal of the circle. On the interval $-3 \leq x \leq 3$ (Figures 14a and 14b), the circle is traversed 54 times. In this situation, restricting our parametrized x-interval can create a clearer graph of a portion of F. Figure 14c shows $F(x) = x^3 + \mathbb{Z}$ on the interval $[-1, 1]$. With respect to the positive x-axis, notice that both of the circles in Figure 14c are traversed in a clockwise manner.



(A) xz-plane

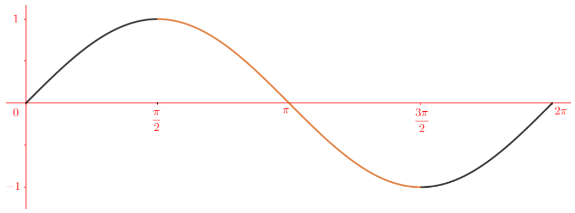


(B) positive xyz-octant

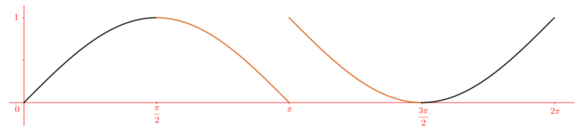


(C) positive xyz-octant with $-1 \leq x \leq 1$

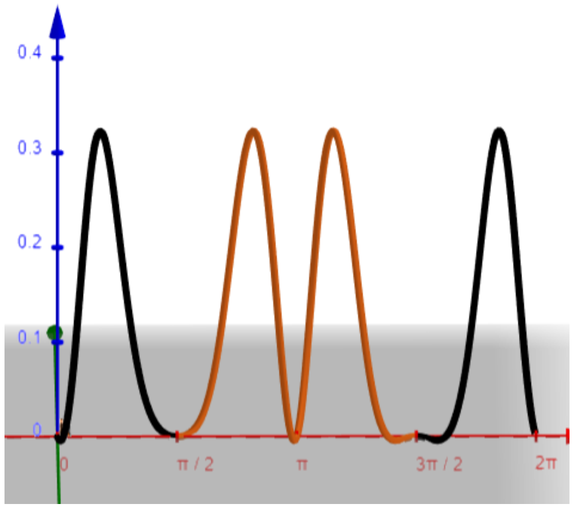
FIGURE 14. A graph of $F(x) = x^3 + \mathbb{Z}$ [6].



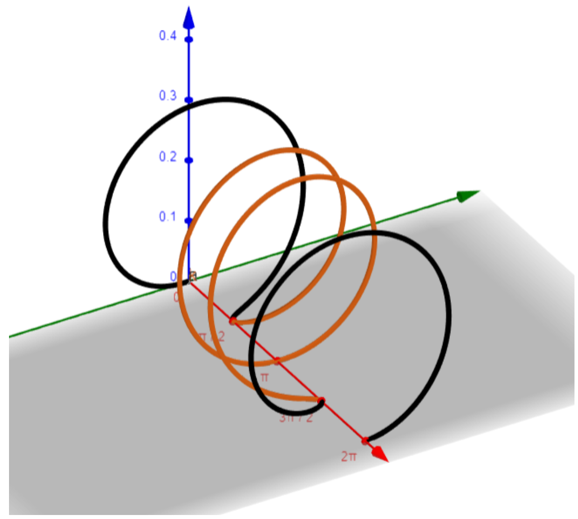
(A) $f(x) = \sin x$



(B) $\bar{f}(x) = \sin x - \lfloor \sin x \rfloor$



(C) $F(x) = \sin x + \mathbb{Z}$, xz-plane



(D) $F(x) = \sin x + \mathbb{Z}$, positive xyz-octant

FIGURE 15. Graphs of $f(x) = \sin x$, $\bar{f}$, and F on $[0, 2\pi]$ [6].

Example 4.7. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$. On $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$ f is increasing, and on $(\frac{\pi}{2}, \frac{3\pi}{2})$ f is decreasing (Figure 15a). Note that the circle representative function $\bar{f}$ (Figure 15b) is similarly increasing on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$. However, $\bar{f}$ is decreasing on $(\frac{\pi}{2}, \pi)$ and $(\pi, \frac{3\pi}{2})$. It does not make sense to say $\bar{f}$ is decreasing at π. The real-valued function's integer outputs are breaking points for the circle representative function's preservation of increasing and decreasing. For example, when a real-valued function decreases from 0 to -0.25, the corresponding circle representatives go from 0 to $-0.25 - \lfloor -0.25 \rfloor = 0.25 - (-1) = 0.75$. However, the analogous function in $\mathbb{R}/\mathbb{Z}$ does preserve increasing and decreasing intervals. When a real-valued function is increasing, it's analogous function in $\mathbb{R}/\mathbb{Z}$ is tracing around the circle in Figure 9 in a clockwise manner. When a real-valued function is decreasing, it's analogous function in $\mathbb{R}/\mathbb{Z}$ is tracing around the circle in Figure 9 in a counter-clockwise manner. The graphs of $F(x) = \sin x + \mathbb{Z}$ in Figures 15c and 15d illustrate this concept. Notice that the first circle on $(0, \frac{\pi}{2})$ is traced out clockwise, the next two circles on $(\frac{\pi}{2}, \frac{3\pi}{2})$ are counterclockwise, and the final circle on $(\frac{3\pi}{2}, 2\pi)$ is clockwise, with respect to the positive x-axis.

5. Limits of Functions in the Circle Group

A limit point for a subset of $\mathbb{R}$ is a real number for which there is no measurable gap between that real number and some other member of the subset, i.e. any open interval containing a limit point will also contain some other member of the subset. Note that a limit point for a set may or may not belong to the set. The endpoints of an interval are readily identified as limit points for that interval regardless of whether or not the endpoints are included in the interval. It is important to recognize that the interior points of an interval are also limit points because any open interval containing one interior point will also contain a neighboring interior point. The following definition formalizes the definition of a limit point for a real-valued subset.

Definition. Let $A \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is a *limit point* for A if for all $\epsilon > 0$ there exists some $a \in A$ with $a \neq x$ such that $|x - a| < \epsilon$ [12].

Example 5.1. Consider the followings subsets of $\mathbb{R}$: $A = [0, 1]$, $B = (0, 1)$, $C = (0, 1]$, and $D = [0, 1)$. The endpoints 0 and 1 are limit points for all 4 sets regardless of whether or not these endpoints are included. Additionally, all of the points in the interval $(0, 1)$ are limit points for all 4 sets.

The limit of a real-valued function f at a limit point in the domain is defined as the value $f(x)$ "approaches" as x "approaches" the limit point. It is important to note in the case that the limit point is included in the domain, that the value of f at the limit point has no bearing on the the limit of f; we only care about the value f approaches. Again, the following definition clarifies what is meant by "approaches."

Definition. Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a real-valued function. Suppose a is a limit point for A. The *limit of f at a* is l provided that for all $\epsilon > 0$, there exists a

$\delta > 0$ for which $|f(x) - l| < \epsilon$ for all $x \in A$ satisfying $0 < |x - a| < \delta$. The limit of f at a is denoted $\lim_{x\to a} f(x) = l$ [12].

Example 5.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 5 & x = 0 \end{cases}$$

Figure 16 displays the graph of f. Notice the hole in the graph at $x = 0$. Despite the fact that $f(0) = 5$, we have $\lim_{x\to 0} f(x) = 0$ because $f(x)$ approaches 0 as x approaches 0 from the left and right.

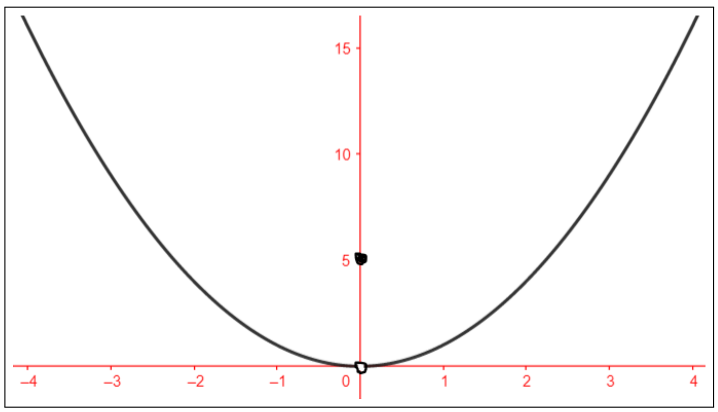


FIGURE 16. A graph of the real-valued function f defined in Example 5.2 [6].

Note that for a limit of a function to exist, the function has to approach the same limit value from the left- and right-hand sides of the limit point. We can also specify a left-hand limit $\lim_{x\to a^-} f(x) = l_1$ and a right-hand limit $\lim_{x\to a^+} f(x) = l_2$ which only specify that $f(x)$ approaches l_1 as x approaches a from the left-hand side and that $f(x)$ approaches l_2 as x approaches a from the right-hand side, respectively. In the case that $l_1 = l_2$, we have $\lim_{x\to a} f(x) = l_1 = l_2$. However, in the case that $l_1 \neq l_2$, we have that $\lim_{x\to a} f(x)$ does not exist. The following example illustrates how non-matching left- and right-hand limits can present.

Example 5.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & x < 0 \\ x^2 + 5 & x \geq 0 \end{cases}$$

For this function, we have

$$\lim_{x \to 0^-} f(x) = 0 \neq 5 = \lim_{x \to 0^+} f(x).$$

Since the left- and right-hand limits are non-equal at $x = 0$, we have that $\lim_{x \to 0} f(x)$ does not exist. In practical terms, we can not say what value $f(x)$ approaches as x approaches 0 because it approaches two distinct values depending on which side of 0 we approach from.

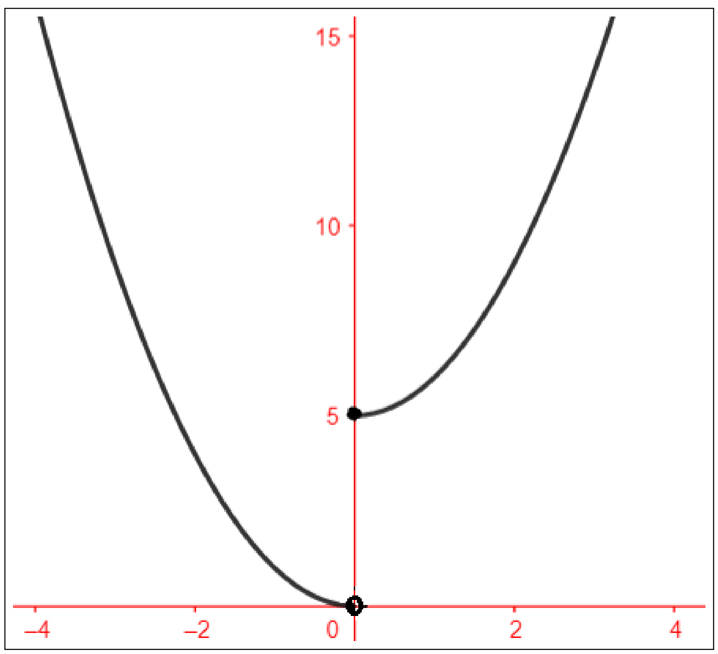


FIGURE 17. A graph of the real-valued function f defined in Example 5.3 illustrating non-matching left- and right-hand limits at $x = 0$ [6].

Definition. Let $A \subset \mathbb{R}$ and let $F : A \to \mathbb{R}/\mathbb{Z}$ be a function. Suppose a is a limit point for A. The *limit of F at a* is $L \in \mathbb{R}/\mathbb{Z}$ provided that given any $\epsilon > 0$, there exists a $\delta > 0$ for which $d(F(x), L) < \epsilon$ for all $x \in A$ satisfying $0 < |x - a| < \delta$. The limit of F at a is denoted $\lim_{x \to a} F(x) = L$.

Theorem 5.4. *Let $A \subset \mathbb{R}$. Let f be a real-valued function on A and let a be a limit point for A. Define $F(x) = f(x) + \mathbb{Z}$. If $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} F(x) = l + \mathbb{Z}$.*

Proof of Theorem 5.4. Suppose $\lim_{x\to a} f(x) = l$.

Let $\epsilon > 0$. Choose $\epsilon' = \min(\epsilon, 0.5)$.

Since $\lim_{x\to a} f(x) = l$, there exists some $\delta > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - a| < \delta$, $|f(x) - l| < \epsilon'$.

Let $x \in \mathbb{R}$ and suppose $|x - a| < \delta$.

Then, $|f(x) - l| < \epsilon' \leq 0.5$.

Then, $|\lfloor f(x)\rfloor - \lfloor l\rfloor| = 0$ or $|\lfloor f(x)\rfloor - \lfloor l\rfloor| = 1$.

First, suppose $|\lfloor f(x)\rfloor - \lfloor l\rfloor| = 0$.

Then,
$$\begin{aligned}|(f(x) - \lfloor f(x)\rfloor) - (l - \lfloor l\rfloor)| &= |f(x) - l - \lfloor f(x)\rfloor + \lfloor l\rfloor| \\ &\leq |f(x) - l| + |\lfloor f(x)\rfloor - \lfloor l\rfloor| \\ &= |f(x) - l| + 0 \\ &= |f(x) - l| \\ &\leq 0.5.\end{aligned}$$

Thus,
$$\begin{aligned}d(\bar{f}(x), l + \mathbb{Z}) &= d(f(x) + \mathbb{Z}, l + \mathbb{Z}) \\ &= |(f(x) - \lfloor f(x)\rfloor) - (l - \lfloor l\rfloor)| \\ &\leq |f(x) - l| \\ &< \epsilon' \leq \epsilon.\end{aligned}$$

Now, suppose instead that $|\lfloor f(x)\rfloor - \lfloor l\rfloor| = 1$. Assume, $l < f(x)$ without loss of generality. Then, we also have $\lfloor l\rfloor < \lfloor f(x)\rfloor$. Hence, $\lfloor f(x)\rfloor - \lfloor l\rfloor = 1$ and $0 < f(x) - l < 0.5$.

Therefore,
$$\begin{aligned}1 - |(f(x) - \lfloor f(x)\rfloor) - (l - \lfloor l\rfloor)| &= 1 - |f(x) - l - \lfloor f(x)\rfloor + \lfloor l\rfloor| \\ &= 1 - |f(x) - l - 1| \\ &= 1 - |1 - (f(x) - l)| \\ &= 1 - (1 - (f(x) - l)) \\ &= 1 - 1 + (f(x) - l) \\ &= f(x) - l\end{aligned}$$

$$= |f(x) - l|$$
$$< 0.5.$$

Thus, $d(\bar{f}(x), l + \mathbb{Z}) = d(f(x) + \mathbb{Z}, l + \mathbb{Z})$

$$= 1 - |(f(x) - \lfloor f(x) \rfloor) - (l - \lfloor l \rfloor)|$$
$$= |f(x) - l| < \epsilon' \leq \epsilon.$$ □

Notice that the converse of Theorem 5.4 is not necessarily true. It is possible to have a real-valued function $f : A \to \mathbb{R}$ where the $\lim_{x\to a} f(x)$ does not exist for which the analogous function in the circle, $F(x) = f(x) + \mathbb{Z}$, does have a defined limit as x approaches $a \in A$.

Example 5.5. Consider the following real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Note that $\lim_{x\to 0^-} f(x) = 0$ while $\lim_{x\to 0^+} = 1$. However, $\lim_{x\to 0} f(x)$ does not exist since the left and right limits are not equal. Turning to the analogous function in $\mathbb{R}/\mathbb{Z}$, we see that $f(x) + \mathbb{Z} = 0 + \mathbb{Z}$. Thus,

$$\lim_{x\to 0^-} (f(x) + \mathbb{Z}) = \lim_{x\to 0^+} (f(x) + \mathbb{Z}) = \lim_{x\to 0} (f(x) + \mathbb{Z}) = 0 + \mathbb{Z}.$$

Definition. Let $E \subset \mathbb{R}$ and let $F : E \to \mathbb{R}/\mathbb{Z}$ be a function. Suppose p is a limit point of $E \cap (p, \infty)$. The function F *has a right limit at* p if there is a number $L \in \mathbb{R}/\mathbb{Z}$ such that given any $\epsilon > 0$, there exists a $\delta > 0$ for which $d(F(x), L) < \epsilon$ for all $x \in E$ satisfying $p < x < p + \delta$. The right limit of F at p, if it exists, is denoted $F(p^+)$, and we write

$$F(p^+) = \lim_{x\to p^+} F(x).$$

Similarly, if p is a limit point of $E \cap (-\infty, p)$, the *left limit of* F *at* p, if it exists, is denoted $F(p^-)$, and we write

$$F(p^-) = \lim_{x\to p^-} F(x).$$

Theorem 5.6. *Let $A \subset \mathbb{R}$. Let f be a real-valued function on A and let a be a limit point for $A \cap (a, \infty)$. Define $F(x) = f(x) + \mathbb{Z}$. If $\lim_{x \to a^+} f(x) = L$, then $\lim_{x \to a^+} F(x) = L + \mathbb{Z}$.*
Now instead let a be a limit point for $A \cap (-\infty, a)$.
If $\lim_{x \to a^-} f(x) = L$, then $\lim_{x \to a^-} F(x) = L + \mathbb{Z}$.

6. Continuity of Functions in the Circle Group

In general terms, a real-valued function is *continuous* if its graph has no breaks, holes, or gaps [7]. A real-valued continuous function can be drawn from left to right or right to left without lifting pen from paper. Most physical phenomena are modeled by continuous functions. A function is continuous if the function values equal the function limits. The definition of continuity for a real-valued function is further formalized below.

Definition. Let $A \subset \mathbb{R}$. Then a function $f : A \to \mathbb{R}$ is continuous if for all $a \in A$ and all $\epsilon > 0$ there exists some $\delta > 0$ such that for all x in the domain satisfying $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$ [12].

Notice that this definition is a slight alteration of the definition of a limit of a function where the limit l has been replaced with the function value $f(a)$. In other words, for a continuous function, the limit of the function at a is equal to the value of f at a:

$$\lim_{x \to a} f(x) = f(a) \text{ [11].}$$

Examples 5.2 and 5.3 in the previous section present functions that are continuous for $x \in \mathbb{R} - \{0\}$ and discontinuous at $x = 0$. The following example gives an everywhere continuous function, i.e. continuous at every $x \in A$.

Example 6.1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous for all $x \in R$. Compare the graph of $f(x) = x^2$ given in Figure 18 to the graphs of the functions given in Examples 5.2 and 5.3 which are both discontinuous at $x = 0$.

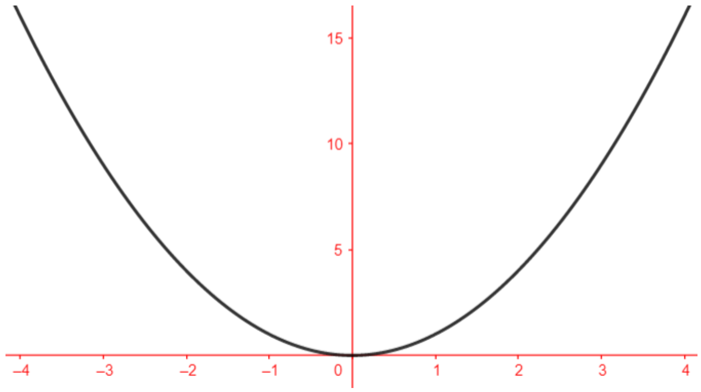


FIGURE 18. A graph of the continuous real-valued function $f(x) = x^2$ [6].

By replacing the usual metric on $\mathbb{R}$, $|x - y|$, with the arc-length metric d that we designed for $\mathbb{R}/\mathbb{Z}$, we can define continuity for a function from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$.

Definition. Let $A \subset \mathbb{R}$. Then a function $F : A \to \mathbb{R}/\mathbb{Z}$ is continuous if for all $a \in A$ and all $\epsilon > 0$ there exists some $\delta > 0$ such that for all x in the domain satisfying $|x - a| < \delta$, we have $d(F(x), F(a)) < \epsilon$.

In geometric terms, if we take the graph of a continuous real-valued function and wrap it around a horizontal cylinder to attain the analogous function in $\mathbb{R}/\mathbb{Z}$, we would expect that graph to also have no holes, gaps, or breaks. Wrapping the graph around a cylinder does not rip or tear the original curve. The following theorem states this in formal terms, and the proof follows from the definitions of continuity for real-valued functions and functions into $\mathbb{R}/\mathbb{Z}$.

Theorem 6.2. *Let $A \subset \mathbb{R}$. If $f : A \to \mathbb{R}$ is continuous in $\mathbb{R}$, then $F : A \to \mathbb{R}/\mathbb{Z}$ defined by $F(x) = f(x) + \mathbb{Z}$ is continuous in $\mathbb{R}/\mathbb{Z}$.*

Proof of Theorem 6.2. Suppose $f : A \to \mathbb{R}$ is continuous in $\mathbb{R}$.

Let $a \in A$. Let $\epsilon > 0$. Choose $\epsilon' = \min\{\epsilon, 0.5\}$.

Since f is continuous in $\mathbb{R}$, there exists some $\delta > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - a| \leq \delta$, $|f(x) - f(a)| < \epsilon'$.

Let $x \in \mathbb{R}$ and suppose $|x - a| \leq \delta$.

Then, $|f(x) - f(a)| < \epsilon' \leq 0.5$.

Then, $|\lfloor f(x) \rfloor - \lfloor f(a) \rfloor| = 0$ or $|\lfloor f(x) \rfloor - \lfloor f(a) \rfloor| = 1$.

First, suppose $|\lfloor f(x) \rfloor - \lfloor f(a) \rfloor| = 0$.

Then,
$$\begin{aligned}|(f(x) - \lfloor f(x) \rfloor) - (f(a) - \lfloor f(a) \rfloor)| &= |f(x) - f(a) - \lfloor f(x) \rfloor + \lfloor f(a) \rfloor| \\ &\leq |f(x) - f(a)| + |\lfloor f(x) \rfloor - \lfloor f(a) \rfloor| \\ &= |f(x) - f(a)| + 0 \\ &= |f(x) - f(a)| \\ &< 0.5.\end{aligned}$$

Thus,
$$\begin{aligned}d(F(x), F(a)) &= d(f(x) + \mathbb{Z}, f(a) + \mathbb{Z}) \\ &= |(f(x) - \lfloor f(x) \rfloor) - (f(a) - \lfloor f(a) \rfloor)| \\ &\leq |f(x) - f(a)| \\ &< \epsilon' \leq \epsilon.\end{aligned}$$

Now suppose instead that $|\lfloor f(x) \rfloor - \lfloor f(a) \rfloor| = 1$. Assume, $f(a) < f(x)$ without loss of generality. Then, we also have $\lfloor f(a) \rfloor < \lfloor f(x) \rfloor$. Hence, $\lfloor f(x) \rfloor - \lfloor f(a) \rfloor = 1$ and $0 < f(x) - f(a) < 0.5$.

Therefore,
$$\begin{aligned}1 - |(f(x) - \lfloor f(x) \rfloor) - (f(a) - \lfloor f(a) \rfloor)| &= 1 - |f(x) - f(a) - \lfloor f(x) \rfloor + \lfloor f(a) \rfloor| \\ &= 1 - |f(x) - f(a) - 1| \\ &= 1 - |1 - (f(x) - f(a))| \\ &= 1 - (1 - (f(x) - f(a))) \\ &= 1 - 1 + (f(x) - f(a)) \\ &= f(x) - f(a) \\ &= |f(x) - f(a)| \\ &< 0.5.\end{aligned}$$

Thus,
$$\begin{aligned}d(F(x), F(a)) &= d(f(x) + \mathbb{Z}, f(a) + \mathbb{Z}) \\ &= 1 - |(f(x) - \lfloor f(x) \rfloor) - (f(a) - \lfloor f(a) \rfloor)| \\ &= |f(x) - f(a)| \\ &< \epsilon' \leq \epsilon.\end{aligned}$$

$\square$

The converse to Theorem 6.2 is not necessarily true. As delineated in the example below, it is possible to have a real-valued function that is not continuous in $\mathbb{R}$ with an analogous function that is continuous in $\mathbb{R}/\mathbb{Z}$.

Example 6.3. Take $f : \mathbb{R} \to \mathbb{R}$ to be $f(x) = \lfloor x \rfloor$. Then, we have the function $F : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ to be $F(x) = \lfloor x \rfloor + \mathbb{Z}$. We know the floor function, f, is not continuous in $\mathbb{R}$. We will prove that the analogous function F is continuous in $\mathbb{R}/\mathbb{Z}$ using two different techniques.

Proof. Let $a \in \mathbb{R}$. Let $\epsilon > 0$. Choose $\delta = 1 > 0$.
Let $x \in \mathbb{R}$ with $|x - a| < \delta$.
Recall that for any integer $k \in \mathbb{Z}$, we know $k + \mathbb{Z} = 0 + \mathbb{Z}$ since $k - 0 = k \in \mathbb{Z}$.
Then,
$$\begin{aligned} d(F(x), F(a)) &= d(\lfloor x \rfloor + \mathbb{Z}, \lfloor a \rfloor + \mathbb{Z}) \\ &= d(0 + \mathbb{Z}, 0 + \mathbb{Z}) \\ &= 0 < \epsilon. \end{aligned}$$
□

Proof. Alternatively, consider the constant function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 0$. Since the constant function g is continuous in $\mathbb{R}$, the analogous function for g in the circle group, namely $G(x) = 0 + \mathbb{Z}$, is continuous in $\mathbb{R}/\mathbb{Z}$ by Theorem 6.2. Now,

$$F(x) = f(x) + \mathbb{Z} = (f(x) - \lfloor f(x) \rfloor) + \mathbb{Z} = (\lfloor x \rfloor - \lfloor \lfloor x \rfloor \rfloor) + \mathbb{Z} = 0 + Z = G(x).$$

Since $F = G$ and G is continuous in $\mathbb{R}/\mathbb{Z}$, F is continuous in $\mathbb{R}/\mathbb{Z}$. □

Example 6.4. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Note that while f is nowhere continuous in $\mathbb{R}$ due to the density of both the rational and irrational numbers, f is integer-valued, so $f(x) + \mathbb{Z} = 0 + \mathbb{Z}$. Thus, for all $a \in \mathbb{R}$,

$$\lim_{x \to a}(f(x) + \mathbb{Z}) = \lim_{x \to a^+}(f(x) + \mathbb{Z}) = \lim_{x \to a^-}(f(x) + \mathbb{Z}) = 0 + \mathbb{Z}.$$

However, for all $a \in \mathbb{R}$, the $\lim_{x \to a} f(x)$, $\lim_{x \to a^+} f(x)$, and $\lim_{x \to a^-} f(x)$ do not exist.

7. Derivative for Functions in the Circle Group

Once again, before defining a derivative for functions in the circle group, we will review the definition of the derivative for a real-valued function. The derivative is the instantaneous rate of change of a function. Geometrically, the derivative evaluated at an input gives the slope of the tangent line at that point. In plain language, for a real-valued function, the derivative tells you how steep the curve is at any point.

Definition. Let $f : I \to \mathbb{R}$ be a real-valued function on an open interval $I \subset \mathbb{R}$. Then, the *derivative* of f at $a \in I$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists [12].

Definition. Let $F : I \to \mathbb{R}/\mathbb{Z}$ be continuous at $a \in A$ where $I \subset \mathbb{R}$ is an open interval. Then there exists some function $f : I \to \mathbb{R}$ such that $F(x) = f(x) + \mathbb{Z}$.
If $F(a) = 0 + \mathbb{Z}$, then the derivative of F at a if it exists is defined by

$$F'(a) = \lim_{x \to a} \frac{\overline{f(x) + 0.5} - \overline{f(a) + 0.5}}{x - a}.$$

If $F(a) \neq 0 + \mathbb{Z}$, then the derivative of F at a if it exists is defined by

$$F'(a) = \lim_{x \to a} \frac{\overline{f(x)} - \overline{f(a)}}{x - a}.$$

It is necessary that we confirm this definition is well-defined. That is, for $a \in I \subset \mathbb{R}$, $f, g : I \to \mathbb{R}$ if $g(x) + \mathbb{Z} = f(x) + \mathbb{Z}$, we should have that $F'(a) = G'(a)$ provided that F and G are differentiable at a. Since we know $\overline{f(x)} = \overline{g(x)}$ if and only if $f(x) + \mathbb{Z} = g(x) + \mathbb{Z}$, we need only prove the following claim. If $f(x) + \mathbb{Z} = g(x) + \mathbb{Z}$, then $\overline{f(x) + 0.5} = \overline{g(x) + 0.5}$.

Proof. Since $g(x) + \mathbb{Z} = f(x) + \mathbb{Z}$, $(g(x) + \mathbb{Z}) + (0.5 + \mathbb{Z}) = (f(x) + \mathbb{Z}) + (0.5 + \mathbb{Z})$, so $g(x) + 0.5 + \mathbb{Z} = f(x) + 0.5 + \mathbb{Z}$. Recall $a + \mathbb{Z} = b + \mathbb{Z} \iff \bar{a} = \bar{b}$. Thus, for all x, we have $\overline{g(x) + 0.5} = \overline{f(x) + 0.5}$. □

Theorem 7.1. *If $f : I \to \mathbb{R}$ is differentiable at a in the interval $I \subset \mathbb{R}$, then $F : I \to \mathbb{R}/\mathbb{Z}$ defined by $F(x) = f(x) + \mathbb{Z}$ is differentiable at a with $F'(a) = f'(a)$.*

Lemma 7.2. *Let $f : A \to \mathbb{R}$ be real-valued function where $A \subset \mathbb{R}$. If f is continuous at $a \in A$ and $f(a) \notin \mathbb{Z}$, then there exists some $\delta > 0$ such that for all $x \in A$ whenever $|x - a| < \delta$, we have $\lfloor f(x) \rfloor = \lfloor f(a) \rfloor$.*

Proof of Lemma 7.2. Suppose f is continuous at a and $f(a) \notin \mathbb{Z}$. Since $f(a) \notin \mathbb{Z}$, by properties of the greatest integer function, we know $1 + \lfloor f(a) \rfloor > \lfloor f(a) \rfloor$. Subtracting $\lfloor f(a) \rfloor$ from both sides of $f(a) > \lfloor f(a) \rfloor$, we obtain $f(a) - \lfloor f(a) \rfloor > 0$. Similarly, by subtracting $f(a)$ from both sides of $1 + \lfloor f(a) \rfloor > f(a)$, we obtain $1 + \lfloor f(a) \rfloor - f(a) > 0$. Thus, both $\overline{f(a)} = f(a) - \lfloor f(a) \rfloor$ and $1 - \overline{f(a)} = 1 + \lfloor f(a) \rfloor - f(a)$ are strictly positive. Choose $\epsilon = \min\{\overline{f(a)}, 1 - \overline{f(a)}\}$. Then, certainly $\epsilon > 0$.

Since f is continuous at a, there exists some $\delta > 0$ such that for all $x \in A$ whenever $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$.

Let $x \in A$ with $|x - a| < \delta$.

Then we have $|f(x) - f(a)| < \epsilon$; eliminating the absolute value function and adding $f(a)$ to all parts of this inequality, we obtain

$$f(a) - \epsilon < f(x) < f(a) + \epsilon.$$

By the selection of ϵ, we have simultaneously that $\epsilon \leq \overline{f(a)}$ and $\epsilon \leq 1 - \overline{f(a)}$. Combining this information with the inequality above, we obtain

$$\begin{aligned} f(a) - (f(a) - \lfloor f(a) \rfloor) &= f(a) - \overline{f(a)} \\ &\leq f(a) - \epsilon \\ &< f(x) \\ &< f(a) + \epsilon \\ &\leq f(a) + 1 - \overline{f(a)} \\ &= f(a) + 1 + \lfloor f(a) \rfloor - f(a). \end{aligned}$$

Simplifying, we have that

$$\lfloor f(a) \rfloor < f(x) < 1 + \lfloor f(a) \rfloor$$

which implies $\lfloor f(x) \rfloor = \lfloor f(a) \rfloor$ by properties of the floor function. □

Proof of Theorem 7.1. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function that is differentiable at $a \in A$. Since f is differentiable at a, we know f is continuous at a. Thus, by Theorem 6.2, $F(x) = f(x) + \mathbb{Z}$ is continuous.

First suppose $f(a) \notin \mathbb{Z}$. Then $F(a) \neq 0 + \mathbb{Z}$, so if F is differentiable at a,

$$F'(a) = \lim_{x \to a} \frac{\overline{f(x)} - \overline{f(a)}}{x - a}.$$

Let $\epsilon > 0$.

Since f is differentiable at a, there exists some δ' such that for all $x \in I$ whenever $0 < |x - a| < \delta'$, we have $\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon$. Similarly, since f is continuous at a and $f(a) \notin \mathbb{Z}$, by Lemma 7.2, there exists some $\delta'' > 0$ such that for all $x \in I$ whenever $|x - a| < \delta''$ we have $\lfloor f(x) \rfloor = \lfloor f(a) \rfloor$.

Choose $\delta = \min\{\delta', \delta''\}$ and let $x \in I$ with $0 < |x - a| < \delta$. By the selection of δ, certainly $\delta > 0$, and we simultaneously have that $\lfloor f(x) \rfloor = \lfloor f(a) \rfloor$ and

$\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon$. Then,

$$\begin{aligned}\left|\frac{\overline{f(x)} - \overline{f(a)}}{x-a} - f'(a)\right| &= \left|\frac{f(x) - \lfloor f(x) \rfloor - f(a) + \lfloor f(a) \rfloor}{x-a} - f'(a)\right| \\ &= \left|\frac{f(x) - f(a)}{x-a} - f'(a)\right| \\ &< \epsilon.\end{aligned}$$

Thus, $\lim_{x\to a} \frac{\overline{f(x)}-\overline{f(a)}}{x-a} = f'(a)$, i.e. $F'(a) = f'(a)$ provided that $f(a) \notin \mathbb{Z}$.

Now suppose instead that $f(a) \in \mathbb{Z}$. Then $F(a) = 0 + \mathbb{Z}$, so if F is differentiable at a,

$$F'(a) = \lim_{x\to a} \frac{\overline{f(x)+0.5} - \overline{f(a)+0.5}}{x-a}.$$

Define a function $g : I \to \mathbb{R}$ by $g(x) = f(x)+0.5$. Since $f(a) \in \mathbb{Z}$, $g(a) = f(a)+0.5 \notin \mathbb{Z}$. Additionally, since f is differentiable at a, we know g is differentiable at a. Then by above,

$$g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{\overline{g(x)} - \overline{g(a)}}{x-a} = \lim_{x\to a} \frac{\overline{f(x)+0.5} - \overline{f(a)+0.5}}{x-a} = F'(a).$$

Because $g(x) = f(x) + 0.5$, we have that $g'(a) = f'(a)$, and by above, $g'(a) = F'(a)$, so transitively, $F'(a) = f'(a)$. Therefore, regardless of whether $f(a) \in \mathbb{Z}$ or $f(a) \notin \mathbb{Z}$, we have that $F'(a) = f'(a)$. □

Example 7.3. Consider the following real-valued function.

$$f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

Note that $f'(0)$ does not exist because f is not continuous at 0.

However, $F(x) = f(x) + \mathbb{Z} = \overline{f(x)} + \mathbb{Z} = 0 + Z$, and the derivative of the constant real-valued function $g(x) = 0$ is 0 for all $x \in \mathbb{R}$, so by Theorem 7.1, $F'(0) = 0$.

Theorem 7.4. *Suppose $f : I \to \mathbb{R}$ is continuous at $a \in I$ where $I \subset \mathbb{R}$ is an interval. Define $F : I \to \mathbb{R}/\mathbb{Z}$ by $F(x) = f(x) + \mathbb{Z}$. If F is differentiable at a, then f is differentiable at a with $f'(a) = F'(a)$.*

Proof. Let $I \subset \mathbb{R}$ be an interval. Suppose the function $f : I \to \mathbb{R}$ is continuous at $a \in I$ and $F : I \to \mathbb{R}/\mathbb{Z}$ is differentiable at $a \in I$.

First suppose $F(a) \neq 0 + \mathbb{Z}$. Then $f(a) \notin \mathbb{Z}$

Let $\epsilon > 0$.

Since f is continuous at a and $f(a) \notin \mathbb{Z}$, by the lemma, there exists some $\delta' > 0$ such that for all $x \in I$ whenever $|x - a| < \delta'$, we have $\lfloor f(x) \rfloor = \lfloor f(a) \rfloor$.

Similarly, since F is differentiable at a, there exists some $\delta'' > 0$ such that for all $x \in I$ whenever $0 < |x - a| < \delta''$, we have $\left| \frac{\overline{f(x)} - \overline{f(a)}}{x-a} - F'(a) \right| < \epsilon$.

Choose $\delta = \min\{\delta', \delta''\}$.

Let $x \in A$ with $0 < |x - a| < \delta$.

Since $0 < |x - a| < \delta \leq \delta''$, we know $\left| \frac{\overline{f(x)} - \overline{f(a)}}{x-a} - F'(a) \right| < \epsilon$.

Combined with the knowledge that $\lfloor f(x) \rfloor = \lfloor f(a) \rfloor$, we know

$$\begin{aligned} \left| \frac{f(x) - f(a)}{x - a} - F'(a) \right| &= \left| \frac{(f(x) - \lfloor f(x) \rfloor) - (f(a) - \lfloor f(a) \rfloor)}{x - a} - F'(a) \right| \\ &= \left| \frac{\overline{f(x)} - \overline{f(a)}}{x - a} - F'(a) \right| \\ &< \epsilon \end{aligned}$$

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Thus, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = F'(a)$, i.e. $f'(a) = F'(a)$ provided that $F(a) \neq \mathbb{Z}$.

Now suppose instead that $F(a) = 0 + \mathbb{Z}$; then $f(a) \in \mathbb{Z}$.

Let $\epsilon > 0$.

Define a function $g : I \to \mathbb{R}$ by $g(x) = f(x) + 0.5$.

Since f is continuous at a, we know that g is also continuous at a.

Additionally, note that $f(a) \in \mathbb{Z}$, so $g(a) = f(a) + 0.5 \notin \mathbb{Z}$.

Since g is continuous at a and $g(a) \notin \mathbb{Z}$, by the lemma, there exists some $\delta' > 0$ such that for all $x \in A$ whenever $|x - a| < \delta'$, we have $\lfloor g(x) \rfloor = \lfloor g(a) \rfloor$, i.e. $\lfloor f(x) + 0.5 \rfloor = \lfloor f(a) + 0.5 \rfloor$.

Similarly, since F is differentiable at a with $F(a) = 0 + \mathbb{Z}$, there exists some $\delta'' > 0$ such that for all $x \in A$ whenever $0 < |x - a| < \delta''$, we have $\left| \frac{\overline{f(x)+0.5} - \overline{f(a)+0.5}}{x-a} - F'(a) \right| < \epsilon$.

Choose $\delta = \min\{\delta', \delta''\}$. Let $x \in I$ with $0 < |x - a| < \delta$.

Since $0 < |x - a| < \delta \leq \delta''$, we have $\left| \frac{\overline{f(x)+0.5} - \overline{f(a)+0.5}}{x-a} - F'(a) \right| < \epsilon$.

Combined with the knowledge that $\lfloor f(x) + 0.5 \rfloor = \lfloor f(a) + 0.5 \rfloor$, we know

$$\begin{aligned}
\left| \frac{f(x) - f(a)}{x - a} - F'(a) \right| &= \left| \frac{f(x) + 0.5 - (f(a) + 0.5)}{x - a} - F'(a) \right| \\
&= \left| \frac{(f(x) + 0.5 - \lfloor f(x) + 0.5 \rfloor) - (f(a) + 0.5 - \lfloor f(a) + 0.5 \rfloor)}{x - a} - F'(a) \right| \\
&= \left| \frac{\overline{f(x) + 0.5} - \overline{f(a) + 0.5}}{x - a} - F'(a) \right| \\
&< \epsilon.
\end{aligned}$$

Therefore, $f'(a) = F'(a)$. □

We previously saw that functions in the circle group can "repair" some discontinuities of analogous real-valued functions. Theorems 7.1 and 7.4 imply that differentiability is not repairable past discontinuity, i.e. if a real-valued function f is not differentiable and its analogous function in $\mathbb{R}/\mathbb{Z}$ is differentiable, then f was originally lacking continuity.

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