THE ENERGY METHOD AND NON-LINEAR STABILITY

A Project Report Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF SCIENCE

in

Mathematics and Computing

by

Gavhale Siddharth Balu
(Roll No. MA14MSCST11004)

Supervisor - **Dr. P A L Narayana**

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
HYDERABAD - 502 285, INDIA

Apr 2016

Declaration

I declare that this written submission represents the ideas in my own words, and wherever the ideas or the words of others have been included, I have adequately cited and referred the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea, data, fact and source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Gavhale Siddharth

MA14MSCST11004

Approval Sheet

This Thesis **THE ENERGY METHOD AND NON-LINEAR STABILITY** by **Mr. Gavhale Siddharth** is approved for the degree of Master of Science from Indian Institute of Technology Hyderabad.

Dr. P. A. L. Narayana
Supervisor
Department of Mathematics
Indian Institute of Technology Hyderabad

To kajal

Abstract

This thesis is primarily a presentation of energy stability results obtained in some standard partial differential equations by means of an integral inequality technique. We are interested in the problem of stability or instability of different partial differential equations. Suppose for a given equation we have a solution. It is the stability of that solution, we wish to investigate. The idea is that, for solution to be stable that must be stable against any disturbance to which that may be subjected. Damping the disturbance rapidly is our ultimate goal, for this, energy method is very useful. To show that the solution is unstable, it is sufficient to find at least one disturbance that grows in amplitude or remains bounded away from the solution. Linear instability analysis and nonlinear stability analysis are the main parts. Nonlinear stability we mean that following two condition are satisfied. Firstly, we can find an arbitrarily small bound on the size of any perturbation whose initial magnitude is small enough. Secondly, any perturbation whose initial magnitude is less than some critical value converges to 0 with time.

In Chapters 2 and 3, the linear diffusion equation and additional linear source term in diffusion equation have been considered respectively. From the diffusion equation, we have understood that over a infinite region, zero solution is always unconditionally stable for periodic perturbation disturbance. In Chapter 3, we show that the zero solution to diffusion equation with additional source term is always unstable for finite spatial region i.e. over $(0, 1)$. We find the necessary condition for stability by using energy method, for that, an eigenvalue problem has been derived.

In Chapter 4, we explored the effect of a nonlinear term on the stability of solution to convection-diffusion equation $(u_t + uu_x = u_{xx} + \beta u^2)$, with some boundary-initial conditions. The effect of quadratic nonlinear term i.e. βu^2 is to destabilize and for the convective term i.e. uu_x in certain cases acts to stabilize. To check this, we used energy method with some standard inequalities. we showed that if $||u_0|| < 2\beta^{-1}$, then solution is stable.

In Chapter 5, we explained basic terminology related to porous media. A nonlinear stability analysis is performed by using energy method for the thermal convection problem in a fluid saturated porous medium when the medium is rotating. We show that the nonlinear stability holds unconditionally. In nonlinear analysis, we find additional information about the boundary conditions. Then using energy method for the nondimensional perturbation governing equation of given problem, we get an eigenvalue problem. After solving this eigenvalue problem we end up with sharp nonlinear stability threshold.

Contents

1

Introduction

This thesis is primarily a presentation of energy stability results obtained in some standard partial differential equations by means of an integral inequality technique. We are interested in the problem of stability or instability of different partial differential equations. Suppose for a given equation we have a solution. It is the stability of that solution, we wish to investigate. The idea is that, for solution to be stable that must be stable against any disturbance to which that may be subjected. Solution is stable means all the perturbations decay to zero as time passes. Damping the disturbance rapidly is our ultimate goal, for this, energy method is very useful as it probably guaranties exponential decay. To show instability for the solution, it is sufficient to find at least one disturbance that grows in amplitude or remains bounded away from the solution.

1.1 Linear instability analysis

In this section when we talk about stability, we will always mean a linear stability. Our first job is to linearize our perturbation equations, this allows us to decompose our perturbation into *normal modes*, as described below. Consider a particular perturbation. We say that our steady state is stable with respect to this perturbation if the amplitude of the perturbation diminishes with time and the steady state is recovered. However, the perturbation may grow instead and our steady state may never be recovered. In this case we say that our steady state is unstable with respect to this perturbation. We say that our steady state is stable only if it is stable with respect to all possible perturbations.

Of course, it would be quite a bad job to have to check all possible perturbations to find out if our steady solution is stable. Instead we decompose our perturbation into normal modes. For example, if u_i is a velocity perturbation we write

$$u_i(x,t) = \sum_{k=1}^{\infty} u_i^k(x) \exp(\sigma_k t)$$

where $\sigma_k \in \mathbb{C}$ is called the *growth rate*, or *modal frequency*. u_i^k is called the *mode shape*. The steady solution will be unstable if it is unstable with respect to just one of these modes. So we look for the most unstable mode.

Consider a particular mode with growth rate σ. If $\Re(\sigma) > 0$ (where we use $\Re(\sigma)$ to denote the real part of σ) then the perturbation will grow with time and if $\Re(\sigma) < 0$ it will fade away with time until the steady state is reached again. We will be interested in finding the condition under which $\Re(\sigma) = 0$. This is known as *marginal stability*. If $\sigma \in \mathbb{C} - \mathbb{R}$ then we say mode is *oscillatory*. If $\sigma \in \mathbb{R}$ then our mode is *stationary* and we say that the *principle of exchange of stabilities* holds. If this is the case then we can substitute $\sigma = 0$ into our equations to find sufficient conditions for marginal linear instability (since $\Re(\sigma) = 0 \iff \sigma = 0$ in this case).

1.2 Nonlinear stability analysis

Here, obviously, we do not linearize our perturbation equation as we did above. Let $v(X,t)$ be a vector of our perturbation variables with $v = 0$ corresponding to the steady state solution. We want to find a function $E(v)$ which is continuous with continuous first order partial derivatives. Also we ask that it is positive definite, i.e.

$$E(v = 0) = 0 \quad \text{and} \quad E(v) > 0 \ \text{ otherwise.}$$

If $\frac{dE}{dt} \leq -kE, k$ a positive constant, then E converges rapidly to 0 as t increases and the steady state solution is stable. This procedure is known as the energy method and the difficulty is to construct a suitable function E. In this way we determine sufficient conditions for nonlinear stability. By nonlinear stability we mean that following two condition are satisfied. Firstly, we can find an arbitrarily small bound on the size of any perturbation whose initial magnitude is small enough. Secondly, any perturbation whose initial magnitude is less than some critical value converges to 0 with time. If we have convergence no matter how large the initial magnitude, we say that we have unconditional nonlinear stability. We can find a linear instability boundary above which the steady state is (linearly) unstable and we can find a nonlinear stability boundary below which we have nonlinear stability. The idea is to choose E carefully so that these two boundaries are as close together as possible (so that we only have a small region where we know nothing about stability).

2

Stability of solution to the Diffusion equation

Let u be the solution of one dimensional diffusion equation,

$$u_t = u_{xx}, \tag{2.1}$$

where for now $-\infty < x < \infty, \quad t > 0$ and we are interested in the behaviour of u with initial data as given below,

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty. \tag{2.2}$$

2.1 Infinite region case

The zero solution $(u \equiv 0)$ is a solution to (2.1). To check whether zero solution is stable or unstable, using linear theory, we proceed as follows. Since (2.1) is linear, consider a perturbation to zero solution to (2.1) of the form

$$u(x, t) = e^{\sigma t} e^{ikx}, \tag{2.3}$$

where k is any real number. Since u is periodic in x, $u(x, t)$ may be written as Fourier series,

$$u(x, t) = \sum_{n=0}^{\infty} A_n e^{\sigma_n t} e^{ik_n x}. \tag{2.4}$$

(2.3) is referred to as Fourier mode. To cause instability only one destabilizing disturbance is sufficient. We would pick up the most destabilizing term in (2.4). Hence we consider (2.3) only. We have,

$$u(x, t) \;\; = \;\; e^{\sigma t} e^{ikx}.$$

thus, we get

$$\begin{aligned} u_t &= \sigma e^{\sigma t} e^{ikx} = \sigma u(x,t), \\ u_{xx} &= i^2 k^2 e^{\sigma t} e^{ikx} = -k^2 u(x,t), \end{aligned}$$

substituting these in (2.1), we get

$$\sigma = -k^2.$$

Since $k \in \mathbb{R}$, $\sigma < 0$ which means as time goes to infinity perturbation approach to zero. So there is no unstable mode and all solution of the form (2.3) decay. Therefore the zero solution to (2.1) is always stable (unconditionally stable) for periodic perturbation disturbance.

2.2 Spatial region (0,1)

If we consider any subclass of (2.3) then, that will be the solution to (2.1). Suppose (2.1) holds in the region $(0,1)$ with boundary conditions as given below,

$$u(0,t) = u(1,t) = 0 \tag{2.5}$$

The only difference between infinite region case to (2.1) and above region is that k can no longer take values in $\mathbb{R}$. All function of x must vanish at $x = 0, 1$. As all solution of the form (2.3) are stable, so all solutions of this subclass are stable. Now using the energy method, we are showing that $u \equiv 0$ is stable solution to (2.1),(2.5).

Let u be the solution to (2.1), (2.5) which satisfy arbitrary initial condition $u_0(x)$. Define energy function $E(t)$ by

$$E(t) = \frac{1}{2}||u(t)||^2. \tag{2.6}$$

where $||\ .\ ||$ denotes the norm on $L^2(0,1)$, i.e., $||f||^2 = \int_0^1 f^2 dx$. Differentiate $E(t)$ with respect to t and using (2.1), we get

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\frac{d||u(t)||^2}{dt} \\ &= \frac{1}{2}\int_0^1 \frac{du^2}{dt}\, dx \\ &= \frac{1}{2}\int_0^1 2uu_t\, dx \\ &= \int_0^1 uu_{xx}\, dx \end{aligned}$$

now use of integration by parts yields,

$$
\begin{aligned}
\int_0^1 u u_{xx}\,dx &= u\int_0^1 u_{xx}\,dx - \int_0^1 \frac{du}{dx}\left(\int_0^1 u_{xx}\,dx\right)\,dx \\
&= u u_x|_0^1 - \int_0^1 u_x^2\,dx \\
\frac{dE}{dt} &= 0 - ||u_x||^2
\end{aligned}
$$

hence the energy equation becomes,

$$
\begin{aligned}
\frac{dE}{dt} + ||u_x||^2 &= 0 \\
\frac{d\left(\frac{1}{2}||u||^2\right)}{dt} + ||u_x||^2 &= 0
\end{aligned}
$$

using the Poincare inequality i.e. $(\pi^2||u||^2 \leqq ||u_x||^2)$ (Appendix (5.1)) we get,

$$
\begin{aligned}
\frac{d||u||^2}{dt} + 2\pi^2||u||^2 &\leqq 0 \\
e^{2\pi^2 t}\frac{d||u||^2}{dt} + ||u||^2 e^{2\pi^2 t}2\pi^2 &\leqq 0 \\
\frac{d\left(e^{2\pi^2 t}||u||^2\right)}{dt} &\leqq 0
\end{aligned}
$$

which leads to,

$$
||u||^2 \leqq e^{-2\pi^2 t}||u_0||^2 \tag{2.7}
$$

Hence, $||u(t)||^2 \longrightarrow 0$ at least exponentially and the zero solution to (2.1),(2.5) is stable.

3

Stability of solution to diffusion equation with a linear source term

Consider the diffusion equation with a linear source term,

$$u_t = u_{xx} + au, \tag{3.1}$$

where a is a positive constant, with initial data given by

$$u(x, 0) = u_0(x). \tag{3.2}$$

We are interested in the stability of zero solution to (3.1). When $a = 0$, it has been shown in the last section that it is stable always. For this equation having linear source term, we have to change according to the region unlike last section.

3.1 Spatial Region $x \in \mathbb{R}$

Here we use normal mode analysis on the perturbation as follows,

$$\begin{aligned} u(x,t) &= e^{\sigma t + ikx} \\ u_t &= \sigma u(x,t) \\ u_{xx} &= -k^2 u(x,t) \end{aligned}$$

using this in (3.1), we get

$$\sigma = -k^2 + a. \tag{3.3}$$

Since $x \in \mathbb{R}$, we have to look among all periodic disturbances, i.e. $k \in \mathbb{R}$ From (3.3) it is clear that, whenever $\sigma > 0$ we have instability. In other words, $k^2 < a$ implies instability. Since $a > 0$, we can find some k^2 such that $k^2 < a$ hold. So in this spatial region, $x \in \mathbb{R}$ the zero solution i.e. $u = 0$ to (3.1) is always unstable for any a $(a > 0)$.

3.2 Finite Spatial Region

Consider $x \in (0,1)$, u satisfies (3.1), moreover $u(0,t) = u(1,t) = 0, \forall t > 0$. Here the situation is same as (3.3) except vanishing of solution i.e $u = 0$ at $x = 0, 1$. The solution may be thought of as being periodic over $\mathbb{R}$ but must satisfy given condition. For example,

$$u(x,t) = e^{\sigma t} \sin kx,$$

now, $u = 0$ at $x = 0, 1$ which implies $k = n\pi, \quad n = \pm 1, \pm 2, ...$ We can look it as half range Fourier series since cosine term do not satisfy boundary conditions. Therefore for (3.3)

$$k^2 = n^2\pi^2, \quad n = \pm 1, \pm 2, ...$$

depending on a our stability criteria changes as follows. We have,

$$\sigma = -k^2 + a.$$

Thus, the solutoin will be unstable if $\sigma > 0 \Rightarrow k^2 < a$ with $k^2_{min} = \pi^2$. Hence if $a > \pi^2$ then solution is unstable. The mode $e^{\sigma t} \sin kx$ will grow in this case. For stability σ should be less than zero. That is possible only if $k^2 > 0$ i.e. $a < \pi^2$. In this case all modes decay. Hence solution is stable. If $a = \pi^2$ the region is called stability instability boundary, often called neutral stability boundary.

3.3 Stability of solution using energy method

We have boundary-initial value problem,

$$u_t = u_{xx} + au, \quad x \in (0,1), \quad t > 0, \tag{3.4}$$

$$u(0,t) = u(1,t) = 0, \quad \forall t \geqq 0,$$

with the initial condition is, $u(x,0) = u_0(x)$.
To study stability of the zero solution, develop energy method as follows. Multiply both side of (3.4) with u and integrating over $(0,1)$, we get

$$\int_0^1 u\frac{\partial u}{\partial t}dx = \int_0^1 u\frac{\partial^2 u}{\partial x^2}dx + a\int_0^1 u^2 dx$$

we know,

$$E(t) = \frac{1}{2}||u(t)||^2 \quad \left(= \frac{1}{2}\int_0^1 u^2(x,t)dx\right).$$

Differentiating with respect to t and using (3.4), we get

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{2}\int_0^1 \frac{du^2(x,t)}{dt}\,dx\\ &= \int_0^1 uu_t\,dx\\ &= \int_0^1 u(u_{xx}+au)\,dx\\ &= \int_0^1 uu_{xx}\,dx + a\int_0^1 u^2\,dx\end{aligned}$$

now using integration by parts, on the right side of the above

$$\begin{aligned}\frac{dE}{dt} &= -||u_x||^2 + a||u||^2\\ &= -a||u_x||^2\left(\frac{1}{a}-\frac{||u||^2}{||u_x||^2}\right), ||u_x||^2\neq 0\end{aligned}$$

$$\leq -a||u_x||^2\left(\frac{1}{a}-\overset{max}{\mathcal{H}}\frac{||u||^2}{||u_x||^2}\right), |||u_x||^2\neq 0 \tag{3.5}$$

where $\mathcal{H}$ is the space of admissible functions over which we look for a maximum, i.e.

$$\mathcal{H}=\{u\in C^2(0,1)|u=0\ when\ x=0,1\}$$

Let us define R_E by, $\frac{1}{R_E}=\overset{max}{\mathcal{H}}\frac{||u||^2}{||u_x||^2}$. So we rewrite (3.5) as,

$$\frac{dE}{dt}\leq -a||u_x||^2\left(\frac{1}{a}-\frac{1}{R_E}\right),$$

suppose $a<R_E\Longrightarrow\left(\frac{1}{a}-\frac{1}{R_E}\right)>0$. $\left(\frac{1}{a}-\frac{1}{R_E}\right)=c(>0)$ say, and putting this in energy equation (3.5), leads to

$$\frac{dE}{dt}\leq -ac||u_x||^2$$

by Poincare inequality,

$$\begin{aligned}\frac{dE}{dt} &\leq -ac\pi^2||u||^2\\ \frac{dE}{dt}+\pi^2acE &\leq 0,\\ e^{2\pi^2act}\frac{dE}{dt}+Ee^{2\pi^2act}\pi^2ac &\leq 0\\ \frac{d\left(e^{2\pi^2act}E\right)}{dt} &\leq 0\end{aligned}$$

integrate above, we get

$$E(t) \leq E(0)e^{-2\pi^2 act} \tag{3.6}$$

if $a < R_E$ then,

$$E(t) = \frac{1}{2}||u(t)||^2 \longrightarrow 0 \quad as\; t \longrightarrow \infty.$$

This tells us, $E(t) \longrightarrow 0$ as $t \longrightarrow \infty$ atleast exponentially.

3.3.1 Eigenvalue problem for R_E

Now, next question is, what is R_E? In the last section, we have defined

$$\frac{1}{R_E} = \max \frac{||u||^2}{||u_x||^2}$$

To obtain R_E, we are establishing Euler equation for eigenvalue problem. Let $I_1 = ||u||^2, I_2 = ||u_x||^2$. The Euler-Lagrange equations are found from,

$$\begin{aligned} \frac{d}{d\epsilon}\left.\frac{I_1(u+\epsilon\eta)}{I_2(u_x+\epsilon\eta_x)}\right|_{\epsilon=0} &= \delta\left(\frac{I_1}{I_2}\right) \\ &= \frac{(I_2\delta I_1) - (I_1\delta I_2)}{(I_2)^2} \\ &= \frac{1}{I_2}\left(\delta I_1 - \delta I_2 \left.\frac{I_1}{I_2}\right|_{max}\right) \\ &= \frac{1}{I_2}\left(\delta I_1 - \delta I_2 \frac{1}{R_E}\right). \end{aligned}$$

Here, $\frac{I_1}{I_2}$ will be a stationary value, since δ refer to 'derivative' evaluated at $\epsilon = 0$. Therefore,

$$\delta I_1 - \delta I_2 \frac{1}{R_E} = 0. \tag{3.7}$$

we know,

$$\delta I_1 = \frac{d}{d\epsilon}\int_0^1 (u+\epsilon\eta)^2 dx\Big|_{\epsilon=0} = \frac{d}{d\epsilon}||I_1||^2$$

where η is an arbitrary $C^2(0,1)$ function with $\eta(0) = \eta(1) = 0$, moreover

$$\delta I_2 = \frac{d}{d\epsilon}\int_0^1 (u_x+\epsilon\eta_x)^2 dx\Big|_{\epsilon=0} = \frac{d}{d\epsilon}||I_2||^2$$

substituting this in (3.7), we get

$$\begin{aligned} \delta I_1 - \delta I_2 \frac{1}{R_E} &= \frac{d}{d\epsilon}\int_0^1 (u+\epsilon\eta)^2 - \left(\frac{1}{R_E}(u_x+\epsilon\eta_x)^2\right) dx \\ &= \frac{d}{d\epsilon}\int_0^1 (u^2+\epsilon^2\eta^2+2u\epsilon\eta) - \left(\frac{1}{R_E}({u_x}^2+\epsilon^2{\eta_x}^2+2u_x\epsilon_x\eta_x)\right) dx \\ &= \int_0^1 (2\epsilon\eta^2+2u\eta) - \left(\frac{1}{R_E}(2\epsilon{\eta_x}^2+2u_x\eta_x)\right) dx \end{aligned}$$

evaluating at $\epsilon = 0$ gives,

$$\delta I_1 - \delta I_2 \frac{1}{R_E} = \int_0^1 (2u\eta) - \underline{\left(\frac{1}{R_E}(2u_x\eta_x)\right)dx}$$

Let integrate underline part (say S), by using integration by parts and boundary conditions,

$$\begin{aligned} S &= \frac{1}{R_E}\left(u_x \int_0^1 \eta_x dx - \int_0^1 \left(\frac{du_x}{dx}\int \eta_x dx\right) dx\right) \\ S &= \frac{1}{R_E}\left(\int_0^1 \eta u_{xx} dx\right) \end{aligned}$$

substituting the value of S and simplifying, we get

$$\int_0^1 \eta \left(u + \frac{1}{R_E}u_{xx}\right) dx = 0$$

η is arbitrary, from the fundamental theorem of calculus of variations, we get

$$u_{xx} + uR_E = 0, \quad u(0) = u(1) = 0. \tag{3.8}$$

This is the Euler equation which enable us to solve eigenvalue problem for R_E. (3.8) is a simple second order differential equation, which has a general solution,

$$u = Asin(x\sqrt{R_E}) + Bcos(x\sqrt{R_E})$$

to find A and B, use boundary conditions i.e. $u(0) = u(1) = 0$

$$u(0) = Asin(0) + Bcos(0) \quad \Longrightarrow B = 0$$

so,

$$u = sin(x\sqrt{R_E})$$

we have taken $A = 1$ since we are primarily interested in R_E not in u, so second condition show that,

$$u(1) = 0 \quad \Longrightarrow \quad \sin\sqrt{R_E} = 0 \quad \Longrightarrow \quad \sqrt{R_E} = n\pi, \qquad n = \pm 1, \pm 2, ...$$

for stability we required $a < R_{Emin}$ therefore,

$$\sqrt{R_E} = \pi \quad \Longrightarrow \quad R_E = \pi^2 \quad \Longrightarrow \quad a < \pi^2$$

hence $\;a < \pi^2\;$ provide the stability of zero solution to $(3.1), (3.2)$.

4

Effect of nonlinear term on stability of solution

To study the effect of a nonlinear term on the stability of solution to partial differential equation, we start with Diffusion equation but here with additional term i.e. quadratic nonlinear term on right hand side and convective nonlinear term on other side. With same boundary-initial conditions given in (2.1). Therefore the system that we are going to study is,

$$u_t + uu_x = u_{xx} + \beta u^2 \qquad x \in (0,1), \;\; t > 0 \tag{4.1}$$
$$u(0,t) = u(1,t) = 0, \quad \forall t \geq 0$$
$$u(x,0) = u_0(x)$$

where β is a positive constant.

4.1 Nonlinear conditional stability

Linear stability analysis

If we start with this, for $u = 0$, then since any perturbation is assumed such as $|u|, |u_x| << 1$ moreover u^2, uu_{xx} may be neglected. Then we have only linear stability analysis of diffusion equation. For this zero solution is always stable as we have seen in second section.

Nonlinear stability analysis

Here nonlinear terms can not be neglected. The effect of quadratic nonlinear term i.e. βu^2 is to destabilize and for the convective term i.e. uu_x in certain cases acts to stabilize. To see the effect of u^2 multiply (4.1) by u and integrate over $(0,1)$

$$\underbrace{\int_0^1 uu_t dx}_{A} + \underbrace{\int_0^1 u^2 u_x dx}_{B} = \underbrace{\int_0^1 uu_{xx} dx}_{C} + \underbrace{\int_0^1 \beta u^3 dx}_{D} \tag{4.2}$$

The above integrals can be evaluated easily except D. Where

$$\begin{aligned} A &= \frac{1}{2}\int_0^1 2uu_t dx = \frac{1}{2}\int_0^1 \frac{du^2}{dt}dx = \frac{1}{2}\frac{d}{dt}||u||^2 \\ B &= \int_0^1 u^2 u_x dx = 0 \\ C &= \int_0^1 u u_{xx} dx = -||u_x||^2 \end{aligned}$$

For D, we do not have idea about sign of $u(x,t)$ i.e. either it is positive or negative. So D may cause an instability. If $||u|| << 1$ the solution is always stable. Hence,

$$D = \beta \int_0^1 u^2 u \, dx \le \beta \left(\int_0^1 u^4 dx\right)^{\frac{1}{2}} \left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \tag{4.3}$$

by Cauchy-schwarz inequality. Now use of Sobolev embedding inequality (Appendix (5.2)) gives us,

$$\int_0^1 u^4 dx \le \frac{1}{4}\left(\int_0^1 u_x^2 dx\right)^2$$

using this further simplifying (4.3), we get

$$\begin{aligned} D &\le \frac{\beta}{2}\left(\int_0^1 u_x^2 dx\right)\left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \\ D &\le \frac{\beta}{2}||u_x||^2||u|| \end{aligned}$$

now, (4.2) gives

$$\frac{1}{2}\frac{d}{dt}||u||^2 \le -||u_x||^2 + \frac{\beta}{2}||u_x||^2||u||$$

$$\frac{1}{2}\frac{d}{dt}||u||^2 \le -||u_x||^2\left(1 - \frac{\beta}{2}||u||\right) \tag{4.4}$$

assume that,

$$||u_0||^2 \le 2\beta^{-1} \quad \Rightarrow \quad \int_0^1 u_0^2 dx \le 4\beta^{-2}$$

Then we can say either,
(i) $||u(t)||^2 \le 2\beta^{-1}, \quad \forall t > 0$ (i.e. perturbation decay as time passes) or
(ii) there exist an $\eta < \infty$ such that,

$$||u(\eta)|| = 2\beta^{-1}, \quad \text{with}$$

$$||u(\eta)|| < 2\beta^{-1}, \text{on}[0,\eta).$$

i.e. perturbation remain constant throughout time (instability).
Suppose (ii) holds. Then on $[0,\eta)$, $(1 - \frac{\beta}{2}||u|| > 0)$ put this in (4.4) which gives us,

$$\frac{1}{2}\frac{d}{dt}||u||^2 \le -\underbrace{||u_x||^2}_{+ve}\underbrace{\left(1 - \frac{\beta}{2}||u||\right)}_{+ve}$$

$$\frac{1}{2}\frac{d}{dt}||u||^2 < 0 \qquad \text{for } 0 \leq t < \eta \tag{4.5}$$

hence,

$$||u(t)||^2 \leq ||u(0)||^2 = ||u_0||^2 < 4\beta^{-2}, \quad t \in [0, \eta)$$

since $||u(t)||$ is assumed continuous in t, this means $||u(\eta)|| \neq 2\beta^{-1}$, a contradiction. Hence (ii) is false and (i) holds. We are assuming $u \in C^2$ in x, $u \in C^1$ in t. Therefore,

$$||u_0|| < 2\beta^{-1} \quad \Longrightarrow \quad ||u(t)|| < 2\beta^{-1}, \quad \forall t \geq 0.$$

Further, (4.5) holds $\forall t \geq 0$ and hence

$$\begin{aligned} ||u(t)||^2 &\leq ||u_0||^2 \qquad \forall t \geq 0. \\ -\frac{\beta}{2}||u_0||^2 &\leq -\frac{\beta}{2}||u(t)||^2 \\ 0 < \left(1 - \frac{\beta}{2}||u_0||^2\right) &\leq \left(1 - \frac{\beta}{2}||u(t)||^2\right) \end{aligned}$$

put this into (4.3) then we have,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u||^2 &\leq -||u_x||^2\left(1 - \frac{\beta}{2}||u(t)||\right) \\ \frac{1}{2}\frac{d}{dt}||u||^2 &\leq -||u_x||^2\left(1 - \frac{\beta}{2}||u_0||\right) \end{aligned}$$

once again use Poincare inequality (i.e. $||u_x||^2 \geq \pi^2||u||^2$)

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2 \leq -\pi^2\left(1 - \frac{\beta}{2}||u_0||\right)||u(t)||^2$$

set $A = \pi^2\left(1 - \frac{\beta}{2}||u_0||\right)$

$$\begin{aligned} A||u(t)||^2 + \frac{1}{2}\frac{d}{dt}||u(t)||^2 &\leq 0 \\ e^{2At}2A||u(t)||^2 + e^{2At}\frac{d}{dt}||u(t)||^2 &\leq 0 \\ \frac{d}{dt}\left(e^{2At}||u(t)||^2\right) &\leq 0 \\ ||u(t)||^2 &\leq e^{-2At}||u_0||^2 \end{aligned}$$

Here, we showed that if $||u_0|| < 2\beta^{-1}$, then $||u(t)|| \longrightarrow 0$ at least exponentially fast. We have a condition for initial data, hence it known as nonlinear conditional stability.

5

Nonlinear stability in Rotating Porous Convection (RPC)

The study of natural convection in a rotating porous media is motivated by it's applications in engineering. Among the applications, food process industry, chemical industry, solidification and centrifugal casting of metal and rotating machinery, are few to quote.

The equations governing the flow and heat transfer in a porous medium can be obtain via an averaging procedure of the Navier-Stokes and energy equation over a representative elementary volume (REV). A set of new parameters is introduce such as porosity (ratio of pore volume to the volume of porous matrix), permeability (a measure of the ability of a porous material to allow fluids to pass through it). Standard notation are used throughout, together with the Einstein summation convection for repeated indices. Standard vector or tensor notation is also used wherever necessary. For example,

$$u_x \equiv \frac{\partial u}{\partial x} \equiv u_{,x}, \qquad u_{i,t} \equiv \frac{\partial u_i}{\partial t}, \qquad u_{i,i} \equiv \frac{\partial u_i}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i},$$

$$u_j u_{i,j} \equiv u_j \frac{\partial u_i}{\partial x_j} \equiv \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}, \qquad i = 1, 2 \ or\ 3.$$

Note that, $u_j u_{i,j} \equiv (\mathbf{u} \cdot \nabla)\mathbf{u}$ and $u_{i,i} \equiv$ div $\mathbf{u}$.

ϵ_{ijk} a set of 27 numbers $\epsilon_{111}, \epsilon_{112,...}$

$$\begin{aligned} \epsilon_{ijk} &= 0 \quad \text{if any 2 indices are same} \\ \epsilon_{ijk} &= +1 \quad \text{if } (ijk) \text{ are (123) in cyclic order(3 cases)} \\ \epsilon_{ijk} &= -1 \quad \text{if } (ijk) \text{ are (213) in cyclic order(3 cases)} \end{aligned}$$

5.1 Description of problem

Let the porous medium occupy the horizontal layer, where gravity acting in the negative z direction. The layer is rotating about z-axis, then the nonlinear equation for convection in a saturated porous medium, derived from Vadasz(1998a) as,

$$\frac{1}{Va}\frac{\partial u_i}{\partial t} = -\frac{\partial \pi}{\partial x_i} + R\theta k_i - T(\mathbf{k}\times\mathbf{u})_i - u_i, \tag{5.1}$$

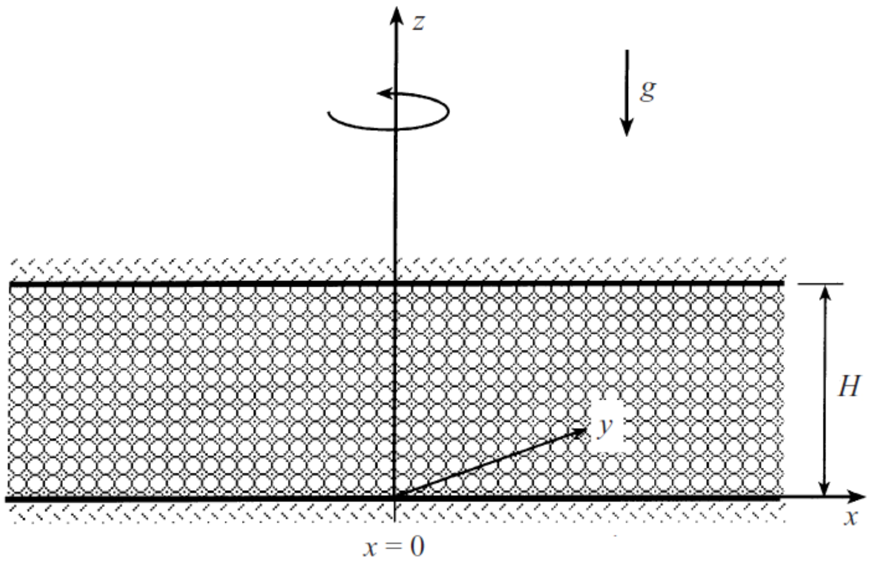
$$\frac{\partial u_i}{\partial x_i} = 0, \tag{5.2}$$

$$\frac{\partial \theta}{\partial t} + u_i\frac{\partial \theta}{\partial x_i} = Rw + \Delta\theta. \tag{5.3}$$

Domain of above governing system is $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$, where $\mathbf{k} = (0, 0, 1)$; u_i, π and θ are representing deviation to the velocity, pressure and temperature fields, here $\mathbf{u} = (u, v, w)$, Δ is a Laplace operator. R, T and Va are non-dimensional numbers and R^2 is Rayleigh number, T^2 is Taylor number (measuring rate of rotation of layer) and $Va = \phi Pr/Da$ is the Vadasz number. Here ϕ is the porosity, Pr is the prandtl number and Da is Darcy the number. The Darcy number represent ration of permeability to depth of layer i.e $Da = k/H^2$. Permeability(k) is a measure of the ability of a porous media to transmit fluids. Vadasz(1998a) pointed out that there is no loss in ignoring the acceleration term in (5.1) , i.e $Va \longrightarrow \infty$. Boundary conditions are,

$$w(x, t) = 0, \qquad \theta(x, t) = 0, \qquad z = 0, 1 \tag{5.4}$$

and assume u_i, θ, π satisfy a plane tiling periodic boundary condition in x and y.



A rotating fluid saturated porous layer

5.2 Stability analysis of RPC

Consider that the three-dimensional disturbance occupies a cell V in the porous layer. Let $||.||$ and $(.,.)$ represent the norm and inner product on $L^2(V)$. According to Vadasz(1998a) $Va \longrightarrow \infty$, rewriting the governing equation for this problem,

$$\begin{aligned} u_i + T(\mathbf{k}\times\mathbf{u})_i &= -\frac{\partial \pi}{\partial x_i} + R\theta k_i, &(5.5)\\ \frac{\partial u_i}{\partial x_i} &= 0, &(5.6)\\ \frac{\partial \theta}{\partial t} + u_i\frac{\partial \theta}{\partial x_i} &= Rw + \Delta\theta. &(5.7) \end{aligned}$$

Now we take curl of equation (5.5) and curl curl of equation (5.5), to get

$$\begin{aligned} \omega_i &= T\frac{\partial u_i}{\partial z} + R(\theta_{,y}\delta_{i1} - \theta_{,x}\delta_{i2}), &(5.8)\\ \Delta u_i + T\frac{\partial \omega_i}{\partial z} &= R[k_i\Delta^*\theta - \theta_{,xz}\delta_{i1} - \theta_{,yz}\delta_{i2}], &(5.9) \end{aligned}$$

respectively, where ω_i is the vorticity and $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Vorticity is curl of the velocity field and is hence a measure of local rotation of the fluid.
Now use the energy method to study nonlinear stability. Multiply equation (5.5) by u_i and integrating over V, we get

$$||u||^2 = R(\theta, w). \qquad (5.10)$$

It is clear that the effect of rotation is lost and so any nonlinear analysis which incorporates the stability effect of rotation will need more than the usual kinetic energy approach.

5.3 Linearized analysis

If $u_i = e^{\sigma t}u_i(x)$, with a similar time representation for velocity and temperature field, our task is to show that $\sigma \in \mathbb{R}$. By observation of equation (5.5) to (5.9) we can say that the linearized perturbation satisfy the equations,

$$\begin{aligned} \sigma\theta &= Rw + \Delta\theta, &(5.11)\\ \omega_3 &= Tw_{,z}, &(5.12)\\ \Delta w + T\frac{\partial \omega}{\partial z} &= R\Delta^*\theta, &(5.13) \end{aligned}$$

subjected to, $\theta = w = 0$ on $z = 0, 1$.

5.3.1 Proof of $\sigma \in \mathbb{R}$

For time being assume that the σ, u_i, θ are complex. Now take Δ^* of equation (5.11), multiplying by the conjugate θ^* of θ and integrating over V to get,

$$\sigma||\nabla^*\theta||^2 = R(\nabla^* w, \nabla^*\theta^*) - ||\nabla^*\nabla\theta||^2, \tag{5.14}$$

where $||\ .\ ||$ is norm* on the complex Hilbert space $L^2(V)$ (*note: consider this norm for this proof only) and $\nabla^* \equiv (\partial/\partial x, \partial/\partial y, 0)$. Similarly for equation (5.13), multiply it by conjugate w^* of w, integrate over V and using (5.12), we obtain

$$0 = R(\nabla^*\theta, \nabla^* w^*) - ||\nabla w||^2 - T^2||w_{,z}||^2.$$

Let $||\nabla w||^2 + T^2||w_{,z}||^2 = C$(say).

$$0 = R(\nabla^*\theta, \nabla^* w^*) - C \tag{5.15}$$

Add (5.14) and (5.15) to get

$$\sigma||\nabla^*\theta||^2 = R[(\nabla^*\theta, \nabla^* w^*) + (\nabla^* w, \nabla^*\theta^*)] - ||\nabla^*\nabla\theta||^2 - C. \tag{5.16}$$

Since σ is complex ($\sigma = \sigma_r + i\sigma_i$), the imaginary part of immediate above equation is

$$\sigma_i||\nabla^*\theta||^2 = 0. \tag{5.17}$$

Hence $\sigma \in \mathbb{R}$.

5.4 Nonlinear stability analysis

We required further information about the boundary condition. To this end, note that

$$\omega_i = \epsilon_{ijk} u_{k,j} \equiv (w_{,y} - v_{,z}, u_{,z} - w_{,x}, v_{,x} - u_{,y}). \tag{5.18}$$

From (5.8) we have,

$$\omega_1 = Tu_{,z} + R\theta_{,y}, \qquad \omega_2 = Tv_{,z} + R\theta_{,x}. \tag{5.19}$$

To find ω_1 & ω_2 on the boundary, we are using $\theta \equiv 0, w \equiv 0$ on $(z = 0, 1)$, we find

$$\omega_1 = -v_{,z} \quad \text{and} \quad \omega_1 = Tu_{,z} \quad \text{on} z = 0, 1, \tag{5.20}$$

$$\omega_2 = u_{,z} \quad \text{and} \quad \omega_2 = Tv_{,z} \quad \text{on} z = 0, 1. \tag{5.21}$$

These equations clearly show that

$$u_{,z} = v_{,z} = 0 \quad \text{on} \quad z = 0, 1, \tag{5.22}$$

and finally we get

$$\omega_1 = \omega_2 = 0 \quad \text{on} \quad z = 0, 1. \tag{5.23}$$

Now take a look for ω_3, from (5.8) we have

$$\omega_3 = T\frac{\partial w}{\partial z}, \tag{5.24}$$

and so

$$T\frac{\partial^2 w}{\partial z^2} = v_{,xz} - u_{,yz}, \tag{5.25}$$

and hence from (5.22), we have

$$w_{,zz} = 0 \quad \text{on} \quad z = 0, 1. \tag{5.26}$$

On boundary$(z = 0, 1)$ $w \equiv \theta \equiv 0$, from (5.7) we can directly get

$$w_{,zz} = 0 \quad \text{on} \quad z = 0, 1. \tag{5.27}$$

Let us set $w^{(m)} = \partial^m w / \partial z^m$, from (5.7) we derive

$$\theta_{,t}^{(2n)} + \sum_{r=0}^{2n} {}^{2n}C_r[u^{(r)}\theta_{,x}^{(2n-r)} + v^{(r)}\theta_{,y}^{(2n-r)} + w^{(r)}\theta^{(2n+1-r)}] = Rw^{(2n)} + \Delta^*\theta^{(2n)} + \theta^{(2n+2)} \tag{5.28}$$

Again using the boundary conditions $\theta \equiv 0$ and $w \equiv 0$ together with the (5.26), (5.27) we get

$$\theta^{(4)} = 0 \quad \text{on} \quad z = 0, 1. \tag{5.29}$$

Now differentiate (5.19) with respect to z an even number of times; ω_1 and ω_2 with respect to z an odd number of times, repeating the process leads to (5.20) and (5.21), to get

$$\omega_{1,zz} = -v_{,zzz} \quad \text{and} \quad \omega_{1,zz} = Tu_{,zzz} \quad \text{on} \quad z = 0, 1,$$
$$\omega_{2,zz} = u_{,zzz} \quad \text{and} \quad \omega_{2,zz} = Tv_{,zzz} \quad \text{on} \quad z = 0, 1.$$

This gives

$$u^{(3)} \equiv v^{(3)} \equiv 0 \quad \text{on} \quad z = 0, 1;$$

then, from (5.24) and (5.25), to

$$w^{(4)} = 0 \quad \text{on} \quad z = 0, 1. \tag{5.30}$$

This process is repeated to derive the boundary conditions

$$w^{(2n)} = 0, \quad \theta^{(2n)} = 0, \quad z = 0, 1, \quad n = 0, 1, ... \tag{5.31}$$

5.4.1 Energy method and eigen value problem of RPC

Let us rewrite energy equation and take, curl and curlcurl of (5.5)

$$\begin{aligned}\frac{\partial\theta}{\partial t}+u_i\frac{\partial\theta}{\partial x_i} &= Rw+\Delta\theta &(5.32)\\ \omega_i &= T\frac{\partial u_i}{\partial z}+R(\theta_{,y}\delta_{i1}-\theta_{,x}\delta_{i2}), &(5.33)\\ \Delta u_i+T\frac{\partial\omega_i}{\partial z} &= R[k_i\Delta^*\theta-\theta_{,xz}\delta_{i1}-\theta_{,yz}\delta_{i2}], &(5.34)\end{aligned}$$

Now multiplying (5.32) by θ and integrating over V to get

$$\frac{1}{2}\frac{d}{dt}||\theta||^2=R(w,\theta)-||\nabla\theta||^2. \tag{5.35}$$

Similarly for (5.34), multiply it with w for $i=3$ and integrating over V, then simplifying that using (5.33) with $i=3$ reduces to

$$0=R(\nabla^*\theta,\nabla^*w)-||\nabla w||-T^2||w_{,z}||^2. \tag{5.36}$$

Add (5.35) to $\xi\times$ (5.36) for a coupling parameter $\xi(>0)$ to obtain

$$\frac{1}{2}\frac{d}{dt}||\theta||^2=RI-D, \tag{5.37}$$

where I and D are given by

$$I=(w,\theta)+\xi(\nabla^*\theta,\nabla^*w),\qquad D=\xi(||\nabla w||+T^2||w_{,z}||^2)+||\nabla\theta||^2$$

Now define R_E as per following manner

$$\frac{1}{R_E}=\overset{max}{\mathcal{H}}\left(\frac{I}{D}\right) \tag{5.38}$$

where $\mathcal{H}$ is the space of all admissible functions over which we look for a maximum. Now further simplify (5.37) to obtain

$$\frac{1}{2}\frac{d}{dt}||\theta||^2\le -D\left(\frac{R_E-R}{R_E}\right). \tag{5.39}$$

Consider $R<R_E$, then using Poincare inequality ($D\ge\pi^2||\theta||^2$) which implies that RHS of (5.39) is negative and after solving it we get $||\theta(t)||\to 0$ at least exponentially. Similarly using this result and (5.10) we obtain $||u||\le R||\theta||$, gives $||u(t)||\to 0$ exponentially.

5.5 Eigenvalue problem for R_E

What is R_E? In the last section, we have defined $\frac{1}{R_E}=\overset{max}{\mathcal{H}}\left(\frac{I}{D}\right)$. Solution of this variation problem gives the nonlinear stability threshold. Using Euler-Lagrange equation we can derive

$$\begin{aligned}\pi_{,i} &= R_Ek_i(\theta-\xi\Delta^*\theta)+2k_i\xi(\Delta w+T^2w_{,zz}), &(5.40)\\ 0 &= R_E(w-\xi\Delta^*w)+2\Delta\theta, &(5.41)\end{aligned}$$

where $\pi(x)$ is a Lagrange multiplier. Now we try to simplify the expression for R_E. For this take curl curl of (5.40) and assume a plane tiling form

$$w = W(z)g(x,y), \;\; \theta = \Omega(z)g(x,y) \quad \text{where} \quad \Delta^* g + a^2 g = 0,$$

a being a wavenumber, the wavenumber has derived by reasoning from observed facts to be non-zero(Chandrasekhar 1981). From (5.40) and (5.41) we have following equations which are satisfied by W and Ω

$$2\xi[(1+T^2)D^2 - a^2]W + R_E(1+\xi a^2)\Omega = 0 \tag{5.42}$$

$$2(D^2 - a^2)\Omega + R_E(1+\xi a^2)W = 0. \tag{5.43}$$

Using the boundary condition (5.31), we can take $W = \sin n\pi z$ with similar representation of Ω. Now use this in (5.42) and (5.43) for R_E, to get

$$R_E^2 = \frac{4\xi[n^2\pi^2(1+T^2) + a^2][n^2\pi^2 + a^2]}{(1+\xi a^2)^2}$$

If we consider above equation as a function of n; then we required the minimum, hence take $n = 1$ in above equation, which give

$$R_E^2 = 4\xi\left[\frac{(\pi^2+a^2)^2}{(1+\xi a^2)^2} + \frac{T^2\pi^2(\pi^2+a^2)}{(1+\xi a^2)^2}\right]. \tag{5.44}$$

Set $\xi = 1/a^2$ to obtain

$$R_E^2 = \frac{(\pi^2+a^2)^2}{a^2} + \frac{T^2\pi^2(\pi^2+a^2)}{a^2}. \tag{5.45}$$

(5.45) show that the nonlinear stability threshold. We could directly select $a^2 = \sqrt{T^2+1} = \frac{1}{\xi}$ in (5.44).

6

Appendix

The inequalities presented here, are not in the most general forms, but given as per our necessity. That have been found to be very useful in energy stability theory.

6.1 Poincare inequality

The Poincare inequality allows one to obtain bounds on a function using bounds on its derivatives. Such bounds are of great importance in the modern, direct methods of the calculus of variations.
Let V be a cell in three dimension. Suppose for simplicity V is the cell

$$0 \leq x < 2a_1, \quad 0 \leq y < 2a_2, \quad 0 < z < 1,$$

and suppose u is a function periodic in x, y of period $2a_1, 2a_2$ respectively, and $u = 0$ on $z = 0, 1$. Then the Poincare inequality may be written,

$$< u^2 > \leq \frac{1}{\pi^2} < u_{i,j} u_{i,j} >, \tag{6.1}$$

where $< . >$ denotes integration over V. In general, the constant, $\frac{1}{\pi^2}$ in (5.1), depends on the geometry and size of the domain V.

6.2 Sobolev inequality

The one of frequent use in energy stability theory is the following. Let Ω be a bounded domain in $\mathbb{R}^3$ with the boundary $\delta\Omega$. Then for function u with $u = 0$ on $\delta\Omega$,

$$\left(\int_\Omega u^6 dV \right)^{\frac{1}{3}} \leq C \int_\Omega |\nabla u|^2 dV,$$

where the constant C is independent of th domain, in fact $C = \frac{2^{\frac{2}{3}}}{3^{\frac{1}{2}} \pi^{\frac{2}{3}}}$.

7

References

- B. Straughan, (2001) *Sharp nonlinear stability threshold in rotating porous convection*, Royal Society JSTOR. **vol. 457**, No.2005, pp.87-93.

- Vadasz, P. (1998) *Coriolis effect on gravity-driven convection in a rotating porous layer heated from below*, Journal of fluid mechanics. **376**, pp.351-375.

- ○ Brian Straughan 1992. The Energy Method, Stability, and Nonlinear Convection. Springer-Verlag, Heidelberg.

- ○ P. G. Drazin and W. H. Reid 1981. Hydrodynamic Stability, Cambridge university press.

- ○ S. Chandrasekhar 1961, Hydrodynamic and Hydromagnetic Stability, Dover.