Lattice Operations on Fuzzy Implications and the preservation of the Exchange Principle

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Abstract

In this work, we solve an open problem related to the preservation of the exchange principle (EP) of fuzzy implications under lattice operations (Problem 3.1, Fuzzy Sets and Systems 261(2015), 112–123.). We show that generalizations of the commutativity of antecedents (CA) to a pair of fuzzy implications (I, J), viz., the generalized exchange principle and the mutual exchangeability are sufficient conditions for the solution of the problem. Further, we determine conditions under which these become necessary too. Finally, we investigate the pairs of fuzzy implications from different families such that (EP) is preserved by the join and meet operations.

Keywords: the commutativity of antecedents, fuzzy implication, the exchange principle, the generalized exchange principle, the mutual exchangeability, lattice operations.
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1. Introduction

Fuzzy implications are one of the important logical connectives in fuzzy logic. These operators generalize the classical implication from $\{0, 1\}$-setting to many valued setting. Fuzzy implications on the unit interval $[0, 1]$ are defined as follows:

Definition 1.1 ([1], Definition 1.1.1). A function $I\colon [0,1]^2 \longrightarrow [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:

$$\text{if } x_1 \le x_2, \text{ then } I(x_1, y) \ge I(x_2, y)\ , \tag{I1}$$

$$\text{if } y_1 \le y_2, \text{ then } I(x, y_1) \le I(x, y_2)\ , \tag{I2}$$

$$I(0,0) = 1\ , I(1,1) = 1\ , I(1,0) = 0\ . \tag{I3}$$

Let $\mathbb{I}$ denote the set of fuzzy implications defined on $[0, 1]$.

1.1. Fuzzy Implications and the exchange principle(EP)

Let $\longrightarrow$ denote the classical implication. Then, from classical logic, it follows that

$$p \longrightarrow (q \longrightarrow r) \equiv q \longrightarrow (p \longrightarrow r), \tag{CA}$$

which is known as the commutativity of antecedents(CA).

Note that, a straightforward generalization of (CA) to the many-valued setting need not hold true always and thus leads to the notion of *the exchange principle*(EP) of a fuzzy implication, which is defined as follows:

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Name	Formula	(EP)
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$	✓
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$	✓
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$	✓
Kleene-Dienes	$I_{\mathbf{KD}}(x,y) = \max(1-x, y)$	✓
Goguen	$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$	✓
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$	×
Fodor	$I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x, y), & \text{if } x > y \end{cases}$	✓
Least FI	$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1 \\ 0, & \text{if } x > 0 \text{ and } \ y < 1 \end{cases}$	×
Greatest FI	$I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } \ y > 0 \\ 0, & \text{if } x = 1 \text{ and } \ y = 0 \end{cases}$	✓

Table 1: Examples of fuzzy implications (cf. Table 1.3 in [1])

Definition 1.2 (cf. [1], Definition 1.3.1). A fuzzy implication I is said to satisfy *the exchange principle* (EP), if for all $x, y, z \in [0,1]$,

$$I(x, I(y,z)) = I(y, I(x,z)) \ . \tag{EP}$$

Let $\mathbb{I}_{\mathbf{EP}}$ denote the set of fuzzy implications satisfying (EP).

Table 1 (see also, Table 1.3 in [1]) lists some examples of basic fuzzy implications along with whether they satisfy the exchange principle (EP) or not.

1.2. Motivation for this work

In this work, we investigate the problem (Problem 3.1 in [3], which is stated below as Problem 1.4) of preservation of (EP) by the lattice operations of fuzzy implications, which are defined as in the following result:

Theorem 1.3 ([1], Theorem 6.1.1)**.** *The family* $(\mathbb{I}, \leq)$ *is a complete, completely distributive lattice with the lattice operations*

$$(I \vee J)(x,y) := \max(I(x,y), J(x,y)) \ , \qquad x, y \in [0,1] \ ,$$
$$(I \wedge J)(x,y) := \min(I(x,y), J(x,y)) \ , \qquad x, y \in [0,1] \ ,$$

where $I, J \in \mathbb{I}$.

The fact that the lattice operations of fuzzy implications do not preserve (EP) (see, Remark 6.1.5 in [1]) has led to the following open problem.

Problem 1.4 ([3], Problem 3.1). Characterize the subfamily of all fuzzy implications ((S,N)- implications, R- implications, etc.) for whose elements the lattice operations preserve (EP).

While the scope of the originally proposed problem was limited to some families of fuzzy implications, noting that the families of (S, N)-implications, R-implications obtained from left-continuous t-norms, the Yager's families of f- and g-implications do satisfy (EP), in this work, we attempt to find the solutions of Problem 1.4 in a more general setting. Hence we consider the following modified problem:

Problem 1.5 (cf. [3], Problem 3.1). From the class of all fuzzy implications satisfying (EP), characterize the pairs of fuzzy implications which satisfy (EP) under the lattice operations of meet and join, i.e., find all pairs $I, J \in \mathbb{I}_{\mathbf{EP}}$, such that both $I \wedge J, I \vee J \in \mathbb{I}_{\mathbf{EP}}$.

This forms the main motivation of this work.

1.3. Outline of the paper

The organization of the paper is as follows: In Section 2, we show that the solutions of Problem 1.5 do exist, in general, by providing some suitable examples of $I, J \in \mathbb{I}_{\mathbf{EP}}$ such that $I \vee J$ and $I \wedge J$ satisfy (EP) and investigate some basic conditions for a pair (I, J) to satisfy the same. We recall also some important generalizations of (CA), viz., the generalized exchange principle(GEP) and the mutual exchangeability(ME). Following this, in Section 3, we show that either of (GEP) or (ME) is a sufficient condition for the lattice operations to preserve (EP). Later on, in Section 4, we show that the properties (GEP) and (ME) are also necessary under some conditions, namely, the Lattice Exchangeability Inequlaities (LEE). Finally, we investigate pairs from some specific but major families of fuzzy implications that satisfy (ME) and (GEP) in Sections 5 and 6, respectively.

2. Preliminaries

In this section, we first present some examples of $I, J \in \mathbb{I}_{\mathbf{EP}}$ which show that (EP) of $I \vee J$ does not always imply that of $I \wedge J$ and vice versa. Next, we show that there exist nontrivial solutions of Problem 1.5 by presenting some suitable examples. Finally, we recall two important generalizations of (CA), namely, (GEP) and (ME), that were already proposed in different contexts and discuss their independence.

To start with, in the following, we present a family of fuzzy implications whose elements do satisfy (EP).

For an $\alpha \in [0, 1]$, let us define I_α as follows:

$$I_\alpha(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y = 0, \\ \alpha, & \text{otherwise.} \end{cases} \tag{1}$$

Note that $I_\alpha \in \mathbb{I}$ for every $\alpha \in [0, 1]$. Further, observe that when $\alpha = 0$, $I_\alpha = I_{\mathbf{0}}$, is the least fuzzy implication and when $\alpha = 1$, $I_\alpha = I_{\mathbf{1}}$, is the greatest fuzzy implication (For formulae, see, Table 1). It is easy to see that $I_\alpha \in \mathbb{I}_{\mathbf{EP}}$, for all $\alpha \in [0, 1]$.

In the following, we present an example which shows that (EP) of $I \vee J$ and that of $I \wedge J$ are, indeed, independent. i.e., given $I, J \in \mathbb{I}_{\mathbf{EP}}$, the fact that $I \vee J$ satisfies (EP) need not imply that $I \wedge J$ also satisfies (EP) and vice versa.

Example 2.1. (i) Let $I = I_\alpha \in \mathbb{I}_{\mathbf{EP}}$ be defined as in (1) for some $\alpha \in [0, 1]$ and $J = I_{\mathbf{RC}} \in \mathbb{I}_{\mathbf{EP}}$. Then $I_\alpha \vee J$ and $I_\alpha \wedge J$, respectively, are as given below:

$$(I_\alpha \vee J)(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y = 0, \\ \max(\alpha, 1 - x + xy) & \text{otherwise,} \end{cases}$$

$$(I_\alpha \wedge J)(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y = 0, \\ \min(\alpha, 1 - x + xy), & \text{otherwise.} \end{cases}$$

$$(I_\alpha \wedge J)\Big(x,(I_\alpha \wedge J)(y,z)\Big) = \begin{cases} 1, & \text{if } x=0 \text{ or } y=0 \text{ or } z=1,\\ 0, & \text{if } x=1 \text{ and } y=1 \text{ and } z=0,\\ \min(\alpha, 1-xy+xyz), & \text{otherwise,} \end{cases}$$
$$= (I_\alpha \wedge J)\Big(y,(I_\alpha \wedge J)(x,z)\Big),$$

which implies that $I_\alpha \wedge J$ satisfies (EP) for all $\alpha \in [0,1]$. Now, we have that

$$(I_\alpha \vee J)\Big(x,(I_\alpha \vee J)(y,z)\Big) = \begin{cases} 1, & \text{if } x=0 \text{ or } y=0 \text{ or } z=1,\\ 0, & \text{if } x=1 \text{ and } y=1 \text{ and } z=0,\\ \max(1-x+\alpha x, 1-xy+xyz), & \text{otherwise,} \end{cases}$$
$$(I_\alpha \vee J)\Big(y,(I_\alpha \vee J)(x,z)\Big) = \begin{cases} 1, & \text{if } x=0 \text{ or } y=0 \text{ or } z=1,\\ 0, & \text{if } x=1 \text{ and } y=1 \text{ and } z=0,\\ \max(1-y+\alpha y, 1-xy+xyz), & \text{otherwise.} \end{cases}$$

However, it can be shown that $I_\alpha \vee J$ does not satisfy (EP) for any $\alpha \in (0,1)$. In fact, let $x=1, y=1-\alpha$ and $z=0$. Then we have

$$(I_\alpha \vee J)\Big(x,(I_\alpha \vee J)(y,z)\Big) = \alpha\ ,$$

while $(I_\alpha \vee J)\Big(y,(I_\alpha \vee J)(x,z)\Big) = \alpha + \alpha(1-\alpha) \neq \alpha$ for any $\alpha \in (0,1)$.

Of course, in the case $\alpha = 0$ we have $I_\alpha \vee J = I_0 \vee J = J = I_{\mathbf{RC}}$, which satisfies (EP). Similarly, if $\alpha = 1$ we have $I_\alpha \vee J = I_{\mathbf{1}} \vee J = I_{\mathbf{1}}$, which satisfies (EP).

(ii) Let $I, J \in \mathbb{I}$ be defined as follows (See, [2]):

$$I(x,y) = \begin{cases} 0, & \text{if } x=1 \text{ and } y<1,\\ 1, & \text{otherwise,} \end{cases} \quad \text{and } J(x,y) = \begin{cases} 0, & \text{if } x>0 \text{ and } y=0,\\ 1, & \text{otherwise.} \end{cases}$$

Then from Example 1.5.10 in [1], it follows that $I, J \in \mathbb{I}_{\mathbf{EP}}$. Also note that the point-wise maximum of I, J, i.e., $I \vee J$ is equal to $I_{\mathbf{1}}$, the greatest fuzzy implication (see Table 1 for formula), which satisfies (EP). However, the point-wise minimum of I, J, i.e., $I \wedge J$, which is given as follows does not satisfy (EP).

$$(I \wedge J)(x,y) = \begin{cases} 0, & \text{if } (x>0 \text{ and } y=0) \text{ or } (x=1 \text{ and } y<1),\\ 1, & \text{otherwise.} \end{cases}$$

To see that $I \wedge J$ does not satisfy (EP), let $x=1, y=0.2$ and $z=0.3$. Then we get,

$$(I \wedge J)(x,(I \wedge J)(y,z)) = 1 \neq 0 = (I \wedge J)(y,(I \wedge J)(x,z)).$$

2.1. Existence of the solutions of Problem 1.5

In this section, we first show that there exist solutions of Problem 1.5. Consequently, we obtain the equivalent formulations of solutions of Problem 1.5.

Example 2.2. (i) Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ and $I \leq J$ under the usual point-wise ordering of functions. Clearly, $I \vee J = J$ and $I \wedge J = I$, which satisfy (EP). Thus when I, J are comparable, $I \vee J$ and $I \wedge J$ always satisfy (EP).

(ii) However, there do exist $I, J \in \mathbb{I}_{\mathbf{EP}}$ that are not comparable but whose join and meet satisfy (EP). For instance, let $I, J \in \mathbb{I}$ be defined as follows:

$$I(x,y) = \begin{cases} 1, & \text{if } x=0,\\ \sin(\frac{\pi y}{2}), & \text{if } x>0, \end{cases} \quad \text{and } J(x,y) = \begin{cases} 1, & \text{if } x=0,\\ y^2, & \text{if } x>0. \end{cases}$$

Clearly, I, J are not comparable, but as can be easily seen $I, J, I \vee J$ and $I \wedge J \in \mathbb{I}_{\mathbf{EP}}$.

In fact, one can generalize Example 2.2(ii) to obtain further solutions of Problem 1.5 as in the following.

Let Φ denote the set of increasing functions φ defined on $[0,1]$ such that $\varphi(0)=0$ and $\varphi(1)=1$. Let $\mathbb{S}$ be the set of all fuzzy implications of the following form:

$$I(x,y)=\begin{cases}1, & \text{if } x=0,\\ \varphi(y), & \text{if } x>0,\end{cases}$$

where $\varphi\in\Phi$. Then it is easy to check that any $I\in\mathbb{S}$ satisfies (EP) and hence $\mathbb{S}\subsetneq\mathbb{I}_{\mathbf{EP}}$. Further, for any $I,J\in\mathbb{S}$, the implications $I\vee J, I\wedge J\in\mathbb{I}_{\mathbf{EP}}$. In the case, if φ,ψ are incomparable then I,J are also incomparable.

Remark 2.3. From Example 2.2(i), it follows that the lattice operations always preserve (EP) if the pair (I,J) under consideration are comparable fuzzy implications. Clearly, these are largely trivial solutions and hence in the following we investigate the solutions of Problem 1.5, in general.

Now, in the following, we present some important results that will be useful in the investigations of pairs of fuzzy implications for which the lattice operations do preserve (EP).

Proposition 2.4 ([1], Propositions 7.2.15 and 7.2.26)**.** *For a function $I\colon[0,1]^2\longrightarrow[0,1]$ the following statements are equivalent:*

(i) *I is increasing in the second variable, i.e., I satisfies* (I2).
(ii) *I satisfies $I(x,\min(y,z))=\min(I(x,y),I(x,z))$ for all $x,y,z\in[0,1]$.*
(iii) *I satisfies $I(x,\max(y,z))=\max(I(x,y),I(x,z))$ for all $x,y,z\in[0,1]$.*

In the following, we present some equivalent formulations of the fact that $I\vee J, I\wedge J$ satisfy (EP), both when $I,J\in\mathbb{I}$ and when $I,J\in\mathbb{I}_{\mathbf{EP}}$, which will be useful in the sequel. Towards this, we have the following result, which follows from Proposition 2.4.

Lemma 2.5. *Let $I,J\in\mathbb{I}$. Then the following statements are equivalent:*

(i) *$I\wedge J\in\mathbb{I}_{\mathbf{EP}}$.*
(ii) *I,J satisfy the following equation for all $x,y,z\in[0,1]$:*

$$\begin{aligned}&\min\Big\{I(x,I(y,z)),I(x,J(y,z)),J(x,I(y,z)),J(x,J(y,z))\Big\}\\ &\qquad=\min\Big\{I(y,I(x,z)),I(y,J(x,z)),J(y,I(x,z)),J(y,J(x,z))\Big\}\ .\end{aligned}\tag{2}$$

Proof. Let $I,J\in\mathbb{I}$ and $K_1=I\wedge J$. Let also $x,y,z\in[0,1]$. Then we get

$$\begin{aligned}K_1(x,K_1(y,z))&=K_1(x,\min(I(y,z),J(y,z))), \qquad \lceil\ \text{By using definition of } I\wedge J=K_1\\ &=\min(I(x,\min(I(y,z),J(y,z))),J(x,\min(I(y,z),J(y,z)))),\\ &\qquad\qquad \lceil\ \text{By using definition of } I\wedge J=K_1\\ &=\min(\min(I(x,I(y,z)),I(x,J(y,z))),\min(J(x,I(y,z)),J(x,J(y,z))))\\ &\qquad\qquad \lceil\ \text{By using Proposition 2.4}\\ &=\min(I(x,I(y,z)),I(x,J(y,z)),J(x,I(y,z)),J(x,J(y,z))).\end{aligned}$$

Similarly, one can get, for all $x,y,z\in[0,1]$, that

$$K_1(y,K_1(x,z))=\min(I(y,I(x,z)),I(y,J(x,z)),J(y,I(x,z)),J(y,J(x,z))).$$

Now, it follows directly that (i) $\Longleftrightarrow$ (ii). $\square$

Further, if $I, J \in \mathbb{I}_{\mathbf{EP}}$, then we see that (2) becomes (3) and we have the following corollary:

Corollary 2.6. *Let $I, J \in \mathbb{I}_{\mathbf{EP}}$. Then the following statements are equivalent:*

(i) $I \wedge J \in \mathbb{I}_{\mathbf{EP}}$.

(ii) *I, J satisfy the following equation:*

$$\min\Big\{I(x, I(y,z)), I(x, J(y,z)), J(x, I(y,z)), J(x, J(y,z))\Big\}$$
$$= \min\Big\{I(x, I(y,z)), I(y, J(x,z)), J(y, I(x,z)), J(x, J(y,z))\Big\}\ . \tag{3}$$

Along the lines of Lemma 2.5 and Corollary 2.6, we get the following result:

Lemma 2.7. *Let $I, J \in \mathbb{I}$. Then the following statements are equivalent:*

(i) $I \vee J \in \mathbb{I}_{\mathbf{EP}}$.

(ii) *I, J satisfy the following equation for all $x, y, z \in [0,1]$:*

$$\max\Big\{I(x, I(y,z)), I(x, J(y,z)), J(x, I(y,z)), J(x, J(y,z))\Big\}$$
$$= \max\Big\{I(y, I(x,z)), I(y, J(x,z)), J(y, I(x,z)), J(y, J(x,z))\Big\}\ . \tag{4}$$

Further, if $I, J \in \mathbb{I}_{\mathbf{EP}}$, then we see that (4) *is equivalent to*

$$\max\Big\{I(x, I(y,z)), I(x, J(y,z)), J(x, I(y,z)), J(x, J(y,z))\Big\}$$
$$= \max\Big\{I(x, I(y,z)), I(y, J(x,z)), J(y, I(x,z)), J(x, J(y,z))\Big\}\ . \tag{5}$$

Note that the results contained in Lemmas 2.5 - 2.7 are only some equivalent formulations of the fact that $I \vee J, I \wedge J \in \mathbb{I}_{\mathbf{EP}}$, in general. It should be emphasized that these results do not contain any characterizations of fuzzy implications I, J such that $I \vee J, I \wedge J \in \mathbb{I}_{\mathbf{EP}}$.

2.2. Some generalizations of (CA)

In the following, we recall two important generalizations of (CA) proposed in different contexts, which play an important role in the sequel.

2.2.1. Generalized Exchange Principle (GEP)

Definition 2.8. Let $\mathcal{A}$ be a subset of $[0,1]^3$. A pair (I, J) of fuzzy implications is said to satisfy the *generalized exchange principle* (GEP) on $\mathcal{A}$, if for all $(x, y, z) \in \mathcal{A}$,

$$\left.\begin{array}{l} I(x, J(y,z)) = I(y, J(x,z))\ , \\ J(x, I(y,z)) = J(y, I(x,z))\ . \end{array}\right\} \tag{GEP}$$

In the case if $\mathcal{A} = [0,1]^3$, for simplicity, we often state that the pair (I, J) satisfies (GEP).

Remark 2.9. Note that, in the original definition of (GEP) in [4], the pair (I, J) satisfies (GEP) if only the first of the above two conditions, viz., $I(x, J(y,z)) = I(y, J(x,z))$, is true and with $\mathcal{A} = [0,1]^3$. In that sense, given $I, J \in \mathbb{I}$, our definition requires both the pairs (I, J) and (J, I) to satisfy (GEP). However, to avoid cumbersome repetitions, we continue to consider the definition given in Definition 2.8 in this work.

Example 2.10. In the following, we present some examples related to (GEP).

(i) Let $I, J \in \mathbb{I}$ be defined as follows:

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq 0.4, \\ 1 - x + xy, & \text{if } x > 0.4, \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x \leq 0.4, \\ \max(1-x, y), & \text{if } x > 0.4. \end{cases}$$

Then the pair (I, J) satisfies (GEP) on $\mathcal{A} = [0, 0.4] \times [0, 0.4] \times [0, 1]$.

(ii) Let $I, J \in \mathbb{I}$ be defined as follows:

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y^3, & \text{if } x > 0, \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y^4, & \text{if } x > 0. \end{cases}$$

Then the pair (I, J) satisfies (GEP).

(iii) Let $I = I_{\mathbf{RS}}$ and $J \in \mathbb{I}$ be defined as follows:

$$J(x,y) = \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then it follows that, for all $x, y, z \in [0, 1]$,

$$I(x, J(y,z)) = I(y, J(x,z)) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \text{ or } z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

However, when $x = 0.3, y = 0.2$ and $z = 0.25$, it follows that

$$J(x, I(y,z)) = 1 \neq 0 = J(y, I(x,z)).$$

Thus, the two equations in (GEP) are mutually independent.

(iv) Let $I = I_{\mathbf{KD}}, J = I_{\mathbf{RC}}$. Then for $x = 0.2, y = 0.4 = z$, we get

$$I(x, J(y,z)) = 0.8 \neq 0.88 = I(y, J(x,z))$$

$$\text{and for } x = 1, y = 0.3, z = 0.4, \; J(x, I(y,z)) = 0.7 \neq 0.82 = J(y, I(x,z)),$$

which clearly shows that I, J satisfy none of the equations of (GEP).

2.2.2. Mutual Exchangeability (ME) *of a pair of Fuzzy Implications*

Another important generalization of (CA) is the mutual exchangeability (ME) of a pair (I, J) of fuzzy implications, which has been proposed in the context of preservation of (EP) under the $\circledast$-composition of fuzzy implications. For more about this, please see [5, 8].

For our context, we redefine (ME) of a pair (I, J) of fuzzy implications as follows.

Definition 2.11. Let $\mathcal{B}$ be a subset of $[0,1]^3$. A pair (I, J) of fuzzy implications is said to be *mutually exchangeable* on $\mathcal{B}$, if for all $(x, y, z) \in \mathcal{B}$,

$$I(x, J(y,z)) = J(y, I(x,z)). \tag{ME}$$

In the case if $\mathcal{B} = [0,1]^3$, we just say that the pair (I, J) satisfies (ME), which is same as the one presented in Definition 3.9 in [8].

In the following, we present an example related to (ME).

Example 2.12. (i) The pair (I, J) of fuzzy implications defined in Example 2.10(i) satisfies (ME) on $\mathcal{B} = [0, 0.4] \times [0, 0.4] \times [0, 1]$.

(ii) Let $0 \leq \epsilon, \delta < 1$. Now, consider the following fuzzy implications:

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq \epsilon \ , \\ y^2, & \text{if } x > \epsilon \ , \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x \leq \delta \ , \\ y^3, & \text{if } x > \delta \ . \end{cases}$$

Now, it is easy to verify that

$$I(x, J(y,z)) = \begin{cases} 1, & \text{if } x \leq \epsilon \text{ or } y \leq \delta, \\ z^6, & \text{if } x > \epsilon \text{ and } \ y > \delta, \end{cases} = J(y, I(x,z)),$$

which implies that the pair (I, J) satisfies (ME), i.e., $\mathcal{B} = [0,1]^3$.

(iii) Let $I = I_{\mathbf{KD}}, J = I_{\mathbf{RC}}$ and $x = 0.2, y = 0.4 = z$. Then it follows that,

$$I(x, J(y,z)) = 0.8 \neq 0.92 = J(y, I(x,z)),$$

which clearly shows that the pair (I, J) does not satisfy (ME).

So far, in this section, we have discussed two important generalizations of (CA), viz., (ME) and (GEP). In the following, we present some pairs (I, J) of fuzzy implications which establish the independence of the concepts (ME) and (GEP).

Remark 2.13. (i) Let $I, J \in \mathbb{I}$ be defined as in Example 2.12(i), with $\epsilon = \delta$. Then, it is easy to verify that the pair (I, J) satisfies both (GEP) and (ME) on $\mathcal{A} = \mathcal{B} = [0,1]^3$.

(ii) Now, we present a pair (I, J) of fuzzy implications satisfying (GEP) and not satisfying (ME). Let $\varepsilon \in [0,1)$. Define $I, J \in \mathbb{I}$ as follows:

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq \varepsilon \ , \\ y^3, & \text{if } x > \varepsilon \ , \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x \leq \varepsilon \ , \\ \sin(\frac{\pi y}{2}), & \text{if } x > \varepsilon \ . \end{cases}$$

Now, from definition of I and J, it follows that

$$I(x, J(y,z)) = \begin{cases} 1, & \text{if } x \leq \epsilon \text{ or } y \leq \epsilon, \\ \sin^3(\frac{\pi z}{2}), & \text{if } x > \epsilon \text{ and } \ y > \epsilon, \end{cases} = I(y, J(x,z)),$$

$$\text{and } J(x, I(y,z)) = \begin{cases} 1, & \text{if } x \leq \epsilon \text{ or } y \leq \epsilon, \\ \sin(\frac{\pi z^3}{2}), & \text{if } x > \epsilon \text{ and } \ y > \epsilon, \end{cases} = J(y, I(x,z)),$$

for all $x, y, z \in [0,1]$. Thus the pair (I, J) satisfies (GEP). However, it can be noticed that

$$I(x, J(y,z)) = \begin{cases} 1, & \text{if } x \leq \epsilon \text{ or } y \leq \epsilon, \\ \sin^3(\frac{\pi z}{2}), & \text{if } x > \epsilon \text{ and } \ y > \epsilon, \end{cases}$$

$$\text{while, } J(y, I(x,z)) = \begin{cases} 1, & \text{if } x \leq \epsilon \text{ or } y \leq \epsilon, \\ \sin(\frac{\pi z^3}{2}), & \text{if } x > \epsilon \text{ and } \ y > \epsilon, \end{cases}$$

which implies that the pair (I, J) does not satisfy (ME).

(iii) Let $I, J \in \mathbb{I}$ be defined as in Example 2.12(ii). Then, it follows that the pair (I, J) satisfies (ME). However, in the case $\epsilon \neq \delta$, we get that

$$I(x, J(y,z)) = \begin{cases} 1, & \text{if } x \leq \epsilon \text{ or } y \leq \delta, \\ z^6, & \text{if } x > \epsilon \text{ and } \ y > \delta, \end{cases} \neq \begin{cases} 1, & \text{if } x \leq \delta \text{ or } y \leq \epsilon, \\ z^6, & \text{if } x > \delta \text{ and } \ y > \epsilon, \end{cases} = I(y, J(x,z))$$

implies that the pair (I, J) does not satisfy (GEP).

(iv) From above points, it follows that (ME) defined in Definition 2.11 is different from (GEP) defined in Definition 2.8. However, when $I = J \in \mathbb{I}$ both (ME) and the (GEP) reduce to the usual (EP) of I.

3. Sufficient conditions on $I, J \in \mathbb{I}_{\mathbf{EP}}$ such that $I \vee J, I \wedge J$ satisfy (EP)

Let $\mathcal{A}, \mathcal{B} \subseteq [0,1]^3$ be such that $\mathcal{A} \cup \mathcal{B} = [0,1]^3$. Then, in this section, for any $I, J \in \mathbb{I}_{\mathbf{EP}}$, we show that (GEP) of (I, J) on $\mathcal{A}$ and (ME) of (I, J) on $\mathcal{B}$, or vice-versa, is a sufficient condition for a pair (I, J) to be a solution of Problem 1.5. Towards this, we have the following result.

Theorem 3.1. *Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ and the pair (I, J) satisfy* (GEP) *on $\mathcal{A}$ and* (ME) *on $\mathcal{B}$ where $\mathcal{A}, \mathcal{B}$ are such that $\mathcal{A} \cup \mathcal{B} = [0,1]^3$. Then both $I \wedge J$ and $I \vee J \in \mathbb{I}_{\mathbf{EP}}$.*

Proof. Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ and the pair (I, J) satisfy (GEP) on $\mathcal{A}$ and (ME) on $\mathcal{B}$ where $\mathcal{A}, \mathcal{B}$ are such that $\mathcal{A} \cup \mathcal{B} = [0,1]^3$. In the following, we show only that $I \wedge J \in \mathbb{I}_{\mathbf{EP}}$. Let $K_1 = I \wedge J$ and $x, y, z \in [0,1]$.

- Let $(x, y, z) \in \mathcal{A}$. Then (I, J) satisfies (GEP). Now, from (EP) of I, J and (GEP) of (I, J) and Corollary 2.6, it follows that

$$\begin{aligned} K_1(x, K_1(y,z)) &= \min(I(x, I(y,z)), I(x, J(y,z)), J(x, I(y,z)), J(x, J(y,z))) \\ &\qquad \lceil \text{From Corollary 2.6 and Proposition 2.4} \\ &= \min(I(y, I(x,z)), I(y, J(x,z)), J(y, I(x,z)), J(y, J(x,z))) \\ &\qquad \lceil \text{ By using (EP) of } I, J \text{ and (GEP) of } (I, J) \\ &= K_1(y, K_1(x,z)), \qquad \lceil \text{From Corollary 2.6 and Proposition 2.4} \end{aligned}$$

  or equivalently, $K_1 = I \wedge J$ satisfies (EP) on $\mathcal{A}$.

- Let $(x, y, z) \in \mathcal{B}$. Then (I, J) satisfies (ME). Then, once again, by using (EP) of I, J and (ME) of (I, J) and Corollary 2.6, we get

$$\begin{aligned} K_1(x, K_1(y,z)) &= \min(I(x, I(y,z)), I(x, J(y,z)), J(x, I(y,z)), J(x, J(y,z))) \\ &\qquad \lceil \text{From Corollary 2.6 and Proposition 2.4} \\ &= \min(I(y, I(x,z)), J(y, I(x,z)), I(y, J(x,z)), J(y, J(x,z))) \\ &\qquad \lceil \text{ By using (EP) of } I, J \text{ and (ME) of } (I, J) \\ &= K_1(y, K_1(x,z)), \qquad \lceil \text{From Corollary 2.6 and Proposition 2.4} \end{aligned}$$

  or equivalently, $K_1 = I \wedge J$ satisfies (EP) on $\mathcal{B}$.

□

Remark 3.2. Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ and the pair (I, J) satisfy (GEP) on $\mathcal{A}$ and (ME) on $\mathcal{B}$ where $\mathcal{A}, \mathcal{B}$ are such that $\mathcal{A} \cup \mathcal{B} = [0,1]^3$. Then we have the following conditions:

(i) If $\mathcal{A} = \emptyset$ then the pair (I, J) satisfies (ME). This implies that (ME) is a sufficient condition for lattice operations to preserve (EP).
(ii) If $\mathcal{B} = \emptyset$ then the pair (I, J) satisfies (GEP). This implies that (GEP) is a sufficient condition for lattice operations to preserve (EP).

Thus we have the following corollary emphasizing the importance of (GEP) and (ME).

Corollary 3.3. *Let $I, J \in \mathbb{I}_{\mathbf{EP}}$. If the pair (I, J) satisfies either* (GEP) *or* (ME)*, then both $I \wedge J$ and $I \vee J \in \mathbb{I}_{\mathbf{EP}}$.*

Remark 3.4. The converse of Corollary 3.3 need not be true always. i.e., if for given $I, J \in \mathbb{I}$, if their lattice operations $I \vee J$ and $I \wedge J$ satisfy (EP) then the pair (I, J) need not satisfy (ME) or (GEP) or both always. To see this, let $I = I_{\mathbf{KD}}$ and $J = I_{\mathbf{RC}}$, which satisfy (EP) (see, Table 1.4 in [1]). Since, $I_{\mathbf{KD}} < I_{\mathbf{RC}}$, with usual point-wise ordering (see, Example 1.1.6 in [1]), it follows that $I \vee J = I_{\mathbf{RC}}$ and $I \wedge J = I_{\mathbf{KD}}$ also satisfy (EP). Moreover, from Examples 2.10(iv) and 2.12(iii), it follows that the pair (I, J) does not satisfy either (GEP) or (ME), respectively.

Clearly, this also means that the conditions in Theorem 3.1 are, again, only sufficient but not necessary.

4. Necessary conditions on $\boldsymbol{I}, \boldsymbol{J} \in \mathbb{I}_{\mathbf{EP}}$ such that $\boldsymbol{I} \vee \boldsymbol{J}, \boldsymbol{I} \wedge \boldsymbol{J}$ satisfy (EP)

Let $I, J \in \mathbb{I}_{\mathbf{EP}}$. In Theorem 3.1, we have shown that (GEP) and (ME) of (I, J) on $\mathcal{A}, \mathcal{B}$ respectively, is a sufficient condition for $I \vee J$ and $I \wedge J$ to satisfy (EP). In this section, we show that these properties also become necessary under some conditions. Towards this end, we define the following:

Definition 4.1. Let $I, J \in \mathbb{I}$. Then we say that the pair (I, J) satisfies Lattice Exchangeable Equations (LEE) if it satisfies the following equations, for all $x, y, z \in [0,1]$:

$$\max(I(x, J(y,z)), J(x, I(y,z))) = \max(I(y, J(x,z)), J(y, I(x,z))) \ , \tag{LEE-1}$$

$$\min(I(x, J(y,z)), J(x, I(y,z))) = \min(I(y, J(x,z)), J(y, I(x,z))) \ . \tag{LEE-2}$$

In the following, we present an example regarding (LEE).

Example 4.2. Let $I_{\alpha_1}, I_{\alpha_2} \in \mathbb{I}_{\mathbf{EP}}$ be defined as in (1) and $I, J \in \mathbb{I}_{\mathbf{EP}}$ be defined as follows:

$$I(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy \text{ and } J(x,y) = 1 - x + xy^2 \ .$$

It can be verified that the pair of fuzzy implications $(I_{\alpha_1}, I_{\alpha_2})$ does satisfy the (LEE) equations, while the pair (I, J) does not, for example, at $x = 0.4, y = 0.7$ and $z = 0.6$.

In the following, we take up the task of investigating the solutions of (LEE-1) and (LEE-2). Before doing so, let us define the following set.

$$\mathcal{D} = \{(x,y,z) \in [0,1]^3 | I(x, J(y,z)) \geq J(x, I(y,z))\}.$$

Let us denote by

- $\mathcal{D}_1 = \{(x,y,z) \in \mathcal{D} | (y,x,z) \in \mathcal{D}\}$
- $\mathcal{D}_2 = \{(x,y,z) \in \mathcal{D} | (y,x,z) \in \mathcal{D}^c\}$
- $\mathcal{D}_3 = \{(x,y,z) \in \mathcal{D}^c | (y,x,z) \in \mathcal{D}\}$
- $\mathcal{D}_4 = \{(x,y,z) \in \mathcal{D}^c | (y,x,z) \in \mathcal{D}^c\}$

Note that $\mathcal{D}_i$'s form a partition of $[0,1]^3$.

Theorem 4.3. *Let $I, J \in \mathbb{I}$. If the pair (I, J) satisfies* (LEE-1) *and* (LEE-2)*, then (I, J) satisfies* (GEP) *on $\mathcal{D}_1 \cup \mathcal{D}_4$ and* (ME) *on $\mathcal{D}_2 \cup \mathcal{D}_3$.*

Proof. Let $I, J \in \mathbb{I}$ and the pair (I, J) satisfy (LEE-1) and (LEE-2).

(i) Let $(x,y,z) \in \mathcal{D}_1 \cup \mathcal{D}_4$.

- Let $(x,y,z) \in \mathcal{D}_1$. Then from the definition of $\mathcal{D}_1$, it follows that both $(x,y,z), (y,x,z) \in \mathcal{D}$. Again, from the definition of $\mathcal{D}$, we get the inequalities $I(x, J(y,z)) \geq J(x, I(y,z))$ and $I(y, J(x,z)) \geq J(y, I(x,z))$. Now, from these inequalities, (LEE-1) and (LEE-2) become $I(x, J(y,z)) = I(y, J(x,z))$, $J(x, I(y,z)) = J(y, I(x,z))$, respectively, which together imply the satisfiability of (GEP) by the pair (I, J) on $\mathcal{D}_1$.
- Let $(x,y,z) \in \mathcal{D}_4$. Then from the definition of $\mathcal{D}_4$, it follows that both $(x,y,z), (y,x,z) \in \mathcal{D}^c$. Again, from the definition of $\mathcal{D}$, we get the inequalities $I(x, J(y,z)) \leq J(x, I(y,z))$ and $I(y, J(x,z)) \leq J(y, I(x,z))$. Now, from these inequalities, (LEE-1) and (LEE-2) become $J(x, I(y,z)) = J(y, I(x,z))$, $I(x, J(y,z)) = I(y, J(x,z))$, respectively, which together imply the satisfiability of (GEP) by the pair (I, J) on $\mathcal{D}_4$.

(ii) Let $(x,y,z) \in \mathcal{D}_2 \cup \mathcal{D}_3$.

- Let $(x, y, z) \in \mathcal{D}_2$. Then from the definition of $\mathcal{D}_2$, it follows that $(x, y, z) \in \mathcal{D}$ and $(y, x, z) \in \mathcal{D}^c$. Again, from the definition of $\mathcal{D}$, we get the inequalities $I(x, J(y, z)) \geq J(x, I(y, z))$ and $I(y, J(x, z)) \leq J(y, I(x, z))$. Now, from these inequalities, (LEE-1) becomes $I(x, J(y, z)) = J(y, I(x, z))$, which implies the satisfiability of (ME) by the pair (I, J) on $\mathcal{D}_2$.
- Let $(x, y, z) \in \mathcal{D}_3$. Then from the definition of $\mathcal{D}_3$, it follows that $(x, y, z) \in \mathcal{D}^c$ and $(y, x, z) \in \mathcal{D}$. Again, from the definition of $\mathcal{D}$, we get the inequalities $I(x, J(y, z)) \leq J(x, I(y, z))$ and $I(y, J(x, z)) \geq J(y, I(x, z))$. Now, from these inequalities, (LEE-2) becomes $I(x, J(y, z)) = J(y, I(x, z))$, which implies the satisfiability of (ME) by the pair (I, J) on $\mathcal{D}_3$.

This completes the proof. □

Corollary 4.4. *Let the pair (I, J) satisfy* (LEE-1) *and* (LEE-2)*. Then the following conditions hold true:*
(i) *If either $\mathcal{D}_1 = [0, 1]^3$ or $\mathcal{D}_4 = [0, 1]^3$ then the pair (I, J) satisfies* (GEP).
(ii) *If either $\mathcal{D}_2 = [0, 1]^3$ or $\mathcal{D}_3 = [0, 1]^3$ then the pair (I, J) satisfies* (ME).

Proof. Proof follows directly from Theorem 4.3 and Definitions 2.8, 2.11. □

Recall from Theorem 3.1, that for a given $I, J \in \mathbb{I}_{\mathbf{EP}}$, the satisfiability of (GEP) on $\mathcal{A}$ and (ME) on $\mathcal{B}$, where $\mathcal{A} \cup \mathcal{B} = [0, 1]^3$, is a sufficient condition for $I \vee J, I \wedge J \in \mathbb{I}_{\mathbf{EP}}$. Further, in Theorem 4.3, we have shown that the above sufficient conditions also become necessary under some conditions, namely, (LEE-1) and (LEE-2). Thus, the characterizations of solutions of Problem 1.5, under the assumptions of (LEE-1) and (LEE-2), can be summarized as given below, whose proof directly follows from Theorems 3.1 and 4.3.

Theorem 4.5. *Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ and the pair (I, J) satisfy* (LEE-1) *and* (LEE-2)*. Then the following statements are equivalent:*
(i) $I \vee J, I \wedge J \in \mathbb{I}_{\mathbf{EP}}$.
(ii) *The pair (I, J) satisfies* (GEP) *on $\mathcal{A}$ and* (ME) *on $\mathcal{B}$, where $\mathcal{A} = \mathcal{D}_1 \cup \mathcal{D}_4$ and $\mathcal{B} = \mathcal{D}_2 \cup \mathcal{D}_3$.*

Remark 4.6. In Theorem 4.5 the satisfaction of both (LEE-1) and (LEE-2) are important. To see this, let $I = I_{\mathbf{KD}}$ and $J = I_{\mathbf{RC}}$, which satisfy (EP) (see, Table 1.4 in [1]).

Now, if we let $x = 0.3, y = 0.4$ and $z = 0.5$, then

$$\min(I(x, J(y, z)), J(x, I(y, z))) = 0.8 < 0.85 = \min(I(y, J(x, z)), J(y, I(x, z))),$$

which implies that the pair (I, J) does not satisfy (LEE-1).

On the one hand, since, $I_{\mathbf{KD}} < I_{\mathbf{RC}}$, with usual point-wise ordering (see, Example 1.1.6 in [1]), it follows that $I \vee J = I_{\mathbf{RC}}$ and $I \wedge J = I_{\mathbf{KD}}$ also satisfy (EP).

On the other hand, let us consider the point $\bar{u} = (x, y, z) = (0.2, 0.4, 0.4) \in [0, 1]^3$. From the following

$$I(x, J(y, z)) = 0.8 \neq 0.88 = I(y, J(x, z)) \ ,$$

$$I(x, J(y, z)) = 0.8 \neq 0.92 = J(y, I(x, z)) \ ,$$

it is clear that (I, J) satisfies neither (ME) nor (GEP) at $\bar{u}$. Thus there does not exist any partition $\mathcal{A}, \mathcal{B}$ of $[0, 1]^3$ such that (I, J) satisfies (ME) on $\mathcal{A}$ and (GEP) on $\mathcal{B}$.

Since (ME) and (GEP) play an important role in the characterizations of solutions of Problem 1.5, it is of interest to know the pairs (I, J) of fuzzy implications that do satisfy (ME) or (GEP). We take up this investigation in the following sections.

5. Pairs of fuzzy implications satisfying (ME)

From Theorem 3.1, it follows that (ME) of a pair (I, J) of fuzzy implications satisfying (EP) is a sufficient condition for $I \vee J$ and $I \wedge J$ to satisfy (EP). Due to the variety of fuzzy implications and the complexity of the functional equation, it is not an easy task to investigate the pairs of fuzzy implications that do satisfy (ME). However, in [5], the solutions of (ME), but only for the families of fuzzy implications whose characterizations are well established, have been investigated. In the following, we recall some important results, from [5], that form the solutions of (ME) and thus the solutions of Problem 1.4.

5.1. (S,N)*-implications satisfying* (ME)

In this section, we present the solutions of (ME) for fuzzy implications that come from the family of all (S,N)-implications. Before doing so, we recall the definition of (S,N)-implications.

Definition 5.1 ([1], Definition 2.4.1). A function $I : [0,1]^2 \longrightarrow [0,1]$ is called an (S,N)-*implication* if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x),y), \qquad x,y \in [0,1].$$

Let us denote by

- $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ - the family of (S,N)-implications,
- $\mathbb{I}_{\mathbb{S},\mathcal{N}}$ - the set of (S,N)-implications with trivial range negations N,
- $\mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{C}}$ - the set of (S,N)-implications with continuous negations N.

Clearly, $\mathbb{I}_{\mathbb{S},\mathcal{N}}, \mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{C}} \subsetneq \mathbb{I}_{\mathbb{S},\mathbb{N}} \subsetneq \mathbb{I}$.

Let us begin by considering the (S,N)-implications whose negations are trivial.

Proposition 5.2 ([5], Proposition 4.1)**.** *Let I be an (S,N)-implication whose negation N has trivial range, i.e., $N(x) \in \{0,1\}$ for all $x \in [0,1]$. Then I satisfies* (ME) *with every $J \in \mathbb{I}$.*

From Proposition 5.2, it follows that if at least one of I, J belongs to $\mathbb{I}_{\mathbb{S},\mathcal{N}}$ then the pair (I,J) satisfies (ME) and hence becomes the solution of Problem 1.4.

In the case, if $I, J \in \mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{C}}$ and satisfy (ME), then as the following result suggests the two t-conorms involved in the definition must be the same.

Theorem 5.3 ([8], Theorem 6.7)**.** *Let $I(x,y) = S_1(N_1(x),y)$ and $J(x,y) = S_2(N_2(x),y)$ be two (S,N)-implications such that N_1, N_2 are continuous negations. Then the following statements are equivalent:*

(i) *The pair (I,J) satisfies* (ME)*.*
(ii) $S_1 = S_2$*.*

5.2. R-implications satisfying (ME)

Definition 5.4 ([1], Definition 2.5.1). A function $I : [0,1]^2 \longrightarrow [0,1]$ is called an R- *implication* if there exists a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] | T(x,t) \leq y\}, \qquad x,y \in [0,1].$$

If I is an R-implication generated from a t-norm T, then it is denoted by I_T and $I_T \in \mathbb{I}$.

Since the characterization results are not available for all R-implications, we consider only the R-implications that are obtained from left-continuous t-norms and let us denote this class by $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$.

Theorem 5.5 ([5], Theorem 5.1)**.** *Let $I = I_{T_1}, J = I_{T_2} \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$. The following statements are equivalent:*

(i) *The pair (I,J) satisfies* (ME)*.*
(ii) $I = J$*.*

Before presenting the pairs (I,J) of fuzzy implications that satisfy (ME) from the families of f- and g-implications, we recall two important definitions that will be useful in the sequel.

In [6], the authors have proposed a generating method of fuzzy implications, namely, the $\circledast$-composition.

Definition 5.6 ([6], Definition 7 and [7]). For any $I, J \in \mathbb{I}$, we define $I \circledast J \colon [0,1]^2 \longrightarrow [0,1]$ as

$$(I \circledast J)(x,y) = I(x, J(x,y)), \qquad x,y \in [0,1] .$$

Since the $\circledast$-composition defined as in Definition 5.6 is associative one can define the powers of $I \in \mathbb{I}$ as follows:

Definition 5.7 ([8], Definition 5.1). Let $I \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define the n-th power of I w.r.t. the binary operation $\circledast$ as follows:

For $n = 1, I_{\circledast}^{[n]} = I$,

and for $n \geq 2, I_{\circledast}^{[n]}(x,y) = I\left(x, I_{\circledast}^{[n-1]}(x,y)\right) = I_{\circledast}^{[n-1]}(x, I(x,y))$,

for all $x, y \in [0,1]$.

5.3. f-implications satisfying (ME)

The family of f-implications is proposed by Yager in [9] by using unary monotone functions on $[0,1]$. They are defined as follows:

Definition 5.8 ([1], Definition 3.1.1). Let $f : [0,1] \longrightarrow [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I : [0,1]^2 \longrightarrow [0,1]$ defined by

$$I(x,y) = f^{-1}(x \cdot f(y)) , \qquad x, y \in [0,1] ,$$

with the understanding $0 \cdot \infty = 0$, is called an *f-implication.*

Let us denote the family of f-implications by $\mathbb{I}_{\mathbb{F}}$. Now, the following result establishes the pairs (I,J) of f-implications that do satisfy (ME).

Theorem 5.9 ([5], Theorem 6.6)**.** *Let I, J be two f-implications. Then the following statements are equivalent:*

(i) *The pair (I,J) satisfies* (ME).
(ii) $J = I_{\circledast}^{[n]}$ *for some $n \in \mathbb{N}$.*

5.4. g-implications satisfying (ME)

Like f-implications, the family of g-implications was also proposed by Yager in [9]. They are defined as follows:

Definition 5.10 ([1], Definition 3.2.1). Let $g : [0,1] \longrightarrow [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I : [0,1]^2 \longrightarrow [0,1]$ defined by

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \qquad x, y \in [0,1] ,$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a *g-generated implication*, where the function $g^{(-1)}$ is the pseudo inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0,\ g(1)] , \\ 1, & \text{if } x \in [g(1),\ \infty] . \end{cases}$$

Let us denote the family of g-implications by $\mathbb{I}_{\mathbb{G}}$. As in the case of f-implications, we get the following result.

Theorem 5.11 ([5], Theorem 7.4)**.** *Let I, J be two g-implications. Then the following statements are equivalent:*

(i) *The pair (I,J) satisfies* (ME).
(ii) $J = I_{\circledast}^{[n]}$ *for some $n \in \mathbb{N}$.*

The summary of the results obtained in this section are presented in Table 2, wherein given an I from a particular sub-family $\mathbb{I}_{\mathbb{X}}$ of fuzzy implications, we seek a J - again in $\mathbb{I}_{\mathbb{X}}$ - such that the pair (I,J) satisfies (ME) and hence, from Theorem 3.1, becomes a solution of Problem 1.4. We present also the final form of the implications $I \vee J$ and $I \wedge J$ obtained from them. Note that for some families of fuzzy implications there exist only trivial solutions of Problem 1.4.

Family	$I(x,y)=$	$J(x,y)=$	$(I\vee J)(x,y)=$	$(I\wedge J)(x,y)=$
$\mathbb{I}_{\mathbb{S},\mathcal{N}}$	$\begin{cases}1, & \text{if } x\leq\epsilon,\\ y, & \text{if } x>\epsilon,\end{cases}$	Any J	$\begin{cases}1, & \text{if } x\leq\epsilon,\\ \max(y,J(x,y)) & \text{if } x>\epsilon,\end{cases}$	$\begin{cases}J(x,y), & \text{if } x\leq\epsilon,\\ \min(y,J(x,y)) & \text{if } x>\epsilon,\end{cases}$
$\mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{C}}}$	$S(N_1(x),y)$	$S(N_2(x),y)$	$S(\max(N_1(x),N_2(x)),y)$	$S(\min(N_1(x),N_2(x)),y)$
$\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$	$I(x,y)$	$I(x,y)$	$I(x,y)$	$I(x,y)$
$\mathbb{I}_{\mathbb{F}}$	$I(x,y)$	$I_{\circledast}^{[n]}(x,y)$	$I_{\circledast}^{[n]}(x,y)$	$I(x,y)$
$\mathbb{I}_{\mathbb{G}}$	$I(x,y)$	$I_{\circledast}^{[n]}(x,y)$	$I_{\circledast}^{[n]}(x,y)$	$I(x,y)$

Table 2: Some solutions of Problem 1.5

6. Pairs of fuzzy implications satisfying (GEP)

In this section, we attempt to find the pairs (I,J) of fuzzy implications that do satisfy (GEP). Once again keeping the complexity of the functional equation (GEP) in mind, we restrict ourselves to do so for the families (S,N)-, R-, f- and g- of fuzzy implications.

Note that all of these families of fuzzy implications satisfy the left neutrality property (NP), which is defined as below:

Definition 6.1 (cf. [1], Definition 1.3.1). A fuzzy implication I is said to satisfy the left neutrality property (NP) if

$$I(1,y)=y, \qquad y\in[0,1]\,. \tag{NP}$$

Lemma 6.2. *Let $I,J\in\mathbb{I}$ satisfy* (NP)*. If the pair (I,J) satisfies* (GEP) *then $I=J$.*

Proof. The substitution of $x=1$ in (GEP) and (NP) of $I,J\in\mathbb{I}$ will yield $I=J$. □

From Lemma 6.2, it is clear that if I,J belong to one of the following families of fuzzy implications, viz.,(S,N)-, R-, f-, g- implications, and satisfy (GEP), then $I=J$ and hence it trivially follows that both $I\vee J$ and $I\wedge J$ satisfy (EP).

Remark 6.3. From Lemma 6.2, it does appear that (GEP) is a very strong sufficient condition, especially, for all fuzzy implications that do possess (NP), to satisfy Problem 1.4.

However, there exist fuzzy implications $I,J\in\mathbb{I}_{\mathbf{EP}}$ which do not satisfy (NP) but still satisfy (GEP) and hence both $I\vee J$ and $I\wedge J$ satisfy (EP). For example, let $I,J\in\mathbb{I}$ be defined as follows:

$$I(x,y)=\begin{cases}1, & \text{if } x\leq\epsilon\ ,\\ \varphi(y), & \text{if } x>\epsilon\ ,\end{cases}\quad J(x,y)=\begin{cases}1, & \text{if } x\leq\epsilon\ ,\\ \psi(y), & \text{if } x>\epsilon\ ,\end{cases}$$

for some $\epsilon\in[0,1]$ and $\varphi,\psi\in\Phi$ and neither of φ,ψ is the identity function on $[0,1]$.

In this context, (ME) perhaps is a milder sufficient condition. For instance, from Table 2 we see that for the families of (S,N)-, f- and g- implications, we still have non-trivial solutions of Problem 1.4.

7. Concluding Remarks

In this paper, we have investigated the solutions of an open problem, viz., Problem 3.1 from [3], related to the preservation of the exchange principle (EP) of fuzzy implications under the lattice operations of pointwise meet and join. Our study has shown the significance of two of the generalizations of (CA), viz., (GEP) and (ME) in obtaining the solutions of the problem.

While (GEP) and (ME) are independently sufficient for the lattice operations of fuzzy implications to preserve (EP), these conditions are not necessary. However, the newly proposed pair of equations, namely the Lattice Exchangeable Equations (LEE-1) and (LEE-2) make (GEP) and (ME) also a necessity for a pair of fuzzy implications to be a solution of Problem 1.5.

Since the pairs (I, J) of fuzzy implications satisfying either (GEP) or (ME) are the most general solutions of the problem, known so far, we have investigated them for those families of fuzzy implications whose complete characterizations are known. However, this problem has to be investigated in general. So are the problems dealing with the satisfaction of (GEP) or (ME) of a fixed pair of fuzzy implications (I, J) on different domains $\mathcal{A}, \mathcal{B} \subsetneq [0,1]^3$. Further, the solutions of (LEE) are themselves worthy of study in their own right.

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