Spectrum design with distributed delays using Galerkin approximations

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A Thesis Submitted to

Indian Institute of Technology Hyderabad

In Partial Fulfillment of the Requirements for

The Degree of Master of Technology

भारतीय प्रौद्योगिकी संस्थान हैदराबाद

Indian Institute of Technology Hyderabad

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June 2015

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Acknowledgements

I am extremely grateful to all those who helped me to finish my masters work and submit this thesis.

I am forever thankful for the almighty god for all the opportunities I have been given and the help I had.

I am indebted to my supervisor Dr. V. C. Prakash for his guidance and support he gave me when I worked on completing this thesis. The encouragement and the criticism he gave me can never be forgotten.

Finally, I would like to thank all my friends and colleagues for supporting me so far.

Dedication

This thesis is dedicated to my family
for their love, endless support
and encouragement

Abstract

The problem of designing the spectrum of a linear scalar distributed delay system has been studied. Systems with delays have infinite spectrum. There are many methods for doing this.The generalized stability charts can give a lot of information about the system in question. However using this method can become quite complicated for many types of delay distributed systems. There are other approaches like using the Lambert W function to tackle the problem using numerical methods.

Contents

Chapter 1

Introduction

Delay differential equations are a class of differential equations in which the value of the derivative of the function is dependent on the history of that function. As opposed to ordinary differential equations, they have an infinite dimensional state vector because of their dependence on the history function. In many engineering and design applications, we have to deal with dynamic behavior which is modeled using a system of delay differential equations. Our aim is to design the eigenvalues of the equivalent linear systems of these applications, through which we try to achieve the required dynamic response from the system.

Few applications of delay differential equations are in machining processes, population models, neural networks, vibration absorbers and disease transmission models. In any engineering area where feedback plays a role, signal processing induces time delay. When this delay in the feedback network is considered, the system model will use delay differential equations. The delays present in these problems could take the form of discrete or continuous delays. The aim of studying this area is to determine the conditions for stability of a system. For a system to be considered stable, it should be able to bounce back to it's original state of stability when some disturbance happens. Time delay systems have an infinite spectrum which renders the process of spectrum design challenging. The rightmost eigenvalues in the solution influence the stability of the system the most as they are the solutions with the slowest decay. So our aim is to select the right system parameters and tune the delays appropriately such that the system displays the required dynamic behavior.

When tuning the delays, one of the difficulties that often arises is the limitation on the values of delays obtainable when those values are small in magnitude. To mitigage that problem, we can use distributed delays.

1.1 Literature review

The problem of spectrum design using distributed delays has been the major focus of [1]. Two complementary methods were used to approach this problem. The concept of stability charts is a very helpful tool to tackle this problem. It uses the geometric properties of the curves and the surfaces to design a spectrum or a spectral gap. Another important approach that can be applied is using the Lambert W function.

Lambert W function W(x) is defined as the solution to the equation

$$x = W(x)e^{W(x)}$$

Numerical methods along with the use of lambert W function has been shown to aid in the problem of spectrum design [1]. An immense advantage of the solution using lambert W method is that the eigenvalues were found to follow the same order as the principal branches of the Lambert W function. However this method suffered from numerical convergence issues.

1.2 Problem definition

Take the following model of a system with one delay.

$$\frac{\partial}{\partial t}x(\tau) = \alpha x(\tau) + \beta \int_a^b w(\xi)x(t+\xi)d\xi \tag{1.1}$$

The history function is given by $x(\tau) = \phi(\tau)$ for $-h \leq \tau \leq 0$. The kernel function is according to the condition $\int_{-h}^{0} w(\xi)d\xi = 1$.

The infinite spectrum of this system is $\{\lambda\}_{i=1}^{\infty}$. The solution of this system will be of the form $\sum_{1}^{\infty} C_i e^{\lambda_i \tau}$. The first few eigenvalues dominate and dictate the behavior of the system. The coefficients of the eigenvalues depend on the initial value of the system.

The concept of spectral gap is relevant here. Say we want a number of the rightmost eigenvalues to dominate the behavior of the system. One way of ensuring it to happen is by specifying that the group of these eigenvalues should have a definite gap between them and the rest of the eigenvalues.

Consider k rightmost eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$. For spectrum design, the required condition is that there have to be a spectral gap of G (a positive real number) between the k values and the rest of the eigenvalues in the spectrum.

$$Re(\lambda_y) \leq Re(\lambda_x) - G \quad \forall\ 1 \leq x \leq k < y$$

Now rescaling the design parameters $\tau = ht$ and $\xi = h\theta$ we can rewrite Eq. (1.1) as

$$\dot{x}(t) = ax(t) \ + \ b\int_0^1 w(\theta)x(t+\theta)d\theta \tag{1.2}$$

Where $a = h\alpha$ and $b = h^2\beta$. w is the weight function. w can be created using polynomial functions as

$$w(\theta) = \sum_{i=0}^{n} C_i\theta^i \tag{1.3}$$

The weight function w has to be normalized. $\int_{-1}^{0} w(\theta)d\theta = 1$. Now substituting the trial solution $x(t) = e^{\lambda t}$ into Eq (1.2) and using the definition of w from Eq (1.3) we get the general solution

$$D(\lambda) = \lambda - a - \frac{b}{\lambda^{n+1}} \sum_{i=0}^{n} \left(i!(-1)^i c_i \lambda^{n-i} \Big(1 - e^{-\lambda} \sum_{j=0}^{i} \frac{\lambda^j}{j!}\Big)\right) \tag{1.4}$$

We have infinite eigenvalues λ as the solution for this equation. The expression from Eq. (1.4)

can serve as a generator for the infinite dimensional space of continous functions on [-1,0];

Chapter 2

Generalized stability charts

The stability charts of functions can be made use of while designing the spectral gap in a system. For a linear system, there are two ways by which it can lose stability while changing the values of different parameters. A real eigenvalue could pass zero and cross into the imaginary plane. This is the case of non-oscillatory stability loss. When this is happening, fold bifurcation occurs. Correspondingly we can observe it in the case of fold curves. The other case of stability loss happens when a pair of complex conjugate eigenvalues cross the imaginary axis. This means that hopf bifurcation is occurring and the stability loss is oscillatory. The frequency of this stability loss is given by the imaginary part of the eigenvalues.

The case when $w(\theta) = C_0 = 1$ is considered. So Eg (1.2) on substitution gives

$$\dot{x}(t) = ax(t) \; + \; b\int_0^1 x(t+\theta)d\theta \tag{2.1}$$

Eq (1.4) gives the solution as

$$D(\lambda) = \lambda - a - b\frac{1-e^{-\lambda}}{\lambda} = 0 \tag{2.2}$$

Solving this equation to find the relation between parameter values,

$$a = \gamma + \frac{\gamma - e^{-\gamma}(\gamma\cos\omega - \omega\sin\omega)}{1 - e^{-\gamma}\frac{\omega\cos\omega + \gamma\sin\omega}{\omega}} \tag{2.3}$$

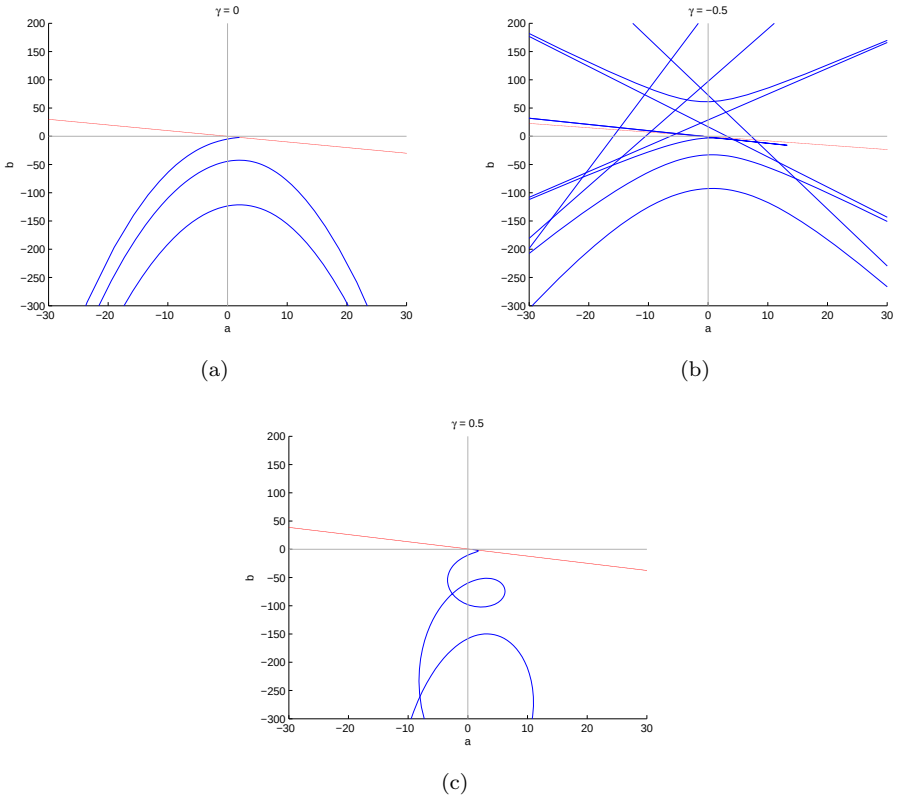
$$b = -\frac{\gamma^2 + \omega^2}{1 - e^{-\gamma}\frac{\omega\cos\omega + \gamma\sin\omega}{\omega}} \tag{2.4}$$

With these equations, we get the hopf curves. When $\omega = 0$ we get

$$b = \frac{\gamma}{1 - e^{-\gamma}}(\gamma - a) \tag{2.5}$$

This equation corresponds to the fold curve.

Stability charts are plotted in Fig. (2.1) for three values of γ, 0, a positive value and a negative value. The red lines corresponds to the fold curves and the blue lines corresponds to the various hopf curves. Stability region in these curves is determined from the intersection of various curves. For the first case, the values of a and b approach ∞ for certain values of ω. The second case where γ



(a) (b) (c)

Figure 2.1: Stability charts for the DDE in Eq. (2.1): (a) is for the case when $\gamma = 0$ (b) is for the case with γ having a negative value of -0.5 and (c) is for the case with a positive value of 0.5

is negative looks similar to the first case. In this case also a and b are approaching very high values for certain values of frequency. However the third case when γ has a positive value, the curve is different. It forms self intersecting loops and does not blow out the value of parameters.

One of the observations noted is that the parameters a and b goes to infinity because the denominators in Eq. (2.3) and Eq. (2.4) goes to zero. The term responsible for this is

$$d(\gamma, \omega) = 1 - e^{-\gamma} \frac{(\omega \cos \omega + \gamma \sin \omega)}{\omega} \tag{2.6}$$

The case when γ = -0.5, the curves are hyperbolic whereas for γ =0 the curves are parabolic. In the former case, it exhibits asymptotic behavior when d from Eq. (2.6) goes to zero as well.

The value of d from Eq. (2.6) is plotted against different values of ω. The red line is for the case when $\gamma = 0$. This red line touches the value of zero in regular intervals when $w \to w_k$, where $w_k = 2k\pi$, k =1, 2, ... The denominator when γ is negative is shown as the green line. It crosses the zero mark and comes back up. During the range of frequencies, when this value crosses zero and comes back, the hopf curve in the stability charts for the function is above the horizontal axis. And for the case when ω is positive, the blue line representing the denominator does not touch zero at all. The corresponding hopf curve simply loops back in the stability chart.

The concept of the stability charts can also be extended for equations with higher orders where the hopf and fold curves become surfaces and hyper surfaces.

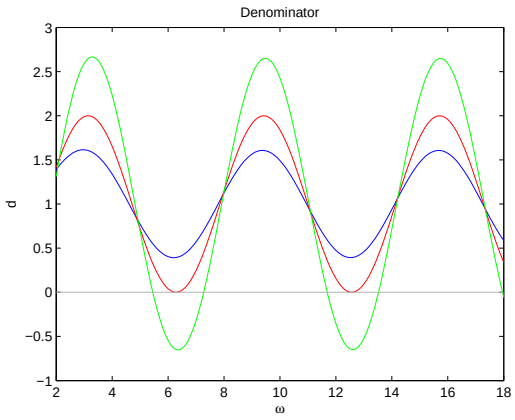


Figure 2.2: Values of denominator plotted against frequency. The red line is the case for $\gamma = 0$, the green line for γ = -0.5 and the blue line for $\gamma = 0.5$

Chapter 3

Bifurcation theory approach

For the system from Eq. (1.1), we consider the case of spectrum design with $w(\theta) = 1$ .Following cases can be considered.

- Designing two real eigenvalues. ($\lambda_1 = -1$ and $\lambda_2 = -3$)
- One real eigenvalue and the real part of a pair of complex conjugate eigenvalues. ($\lambda_1 = -1$ and $\lambda_{2,3} = -3 \pm i\omega$)
- Real parts of two pairs of complex conjugate eigenvalues ($\lambda_{1,2} = -1 \pm i\omega$ and $\lambda_{3,4} = -3 \pm i\tilde{\omega}$)

3.1 Two real eigenvalues

When we specify two real eigenvalues we have fixed the spectral gap. γ_1 =-1 and γ_2 =-3 . To find the value of the parameters a and b, we use Eq. (2.5) which gives,

$$\tilde{a} = \gamma_1 + \frac{G(1 - e^{\gamma_1})(\gamma_1 - G)}{G(1 - e^{\gamma_1}) - \gamma_1(1 - e^{G})} \tag{3.1}$$

and

$$\tilde{b} = \frac{\gamma_1 G e^{\gamma_1}(\gamma_1 - G)}{G(1 - e^{\gamma_1}) - \gamma_1(1 - e^{G})} \tag{3.2}$$

G=$\gamma_1 - \gamma_2 = 2$ is the spectral gap.The curves are plotted in Fig. (3.1). We find that $\tilde{a}$ = -0.26 and $\tilde{b}$ = -0.43 for γ_1 = -1 and γ_2 = -3 . One of the inferences from this case is that for all different values of γ we get different hopf curves which are lines. These lines are not parallel. So it is possible to design a multitude of real values for the two eigenvalues.

3.2 One real eigenvalue and real part of a pair of complex conjugate eigenvalues

In this case $\lambda_1 = \gamma_1$ = -1 is fixed and $\lambda_{2,3} = -3 \pm i\omega$. Plotting the fold curve for γ_1 = -1 and the hopf curve for γ_2 = -3 in Fig. (3.2) the curves are found to intersect at $\tilde{a}$ = -4.97 and $\tilde{b}$ = 2.97 . In

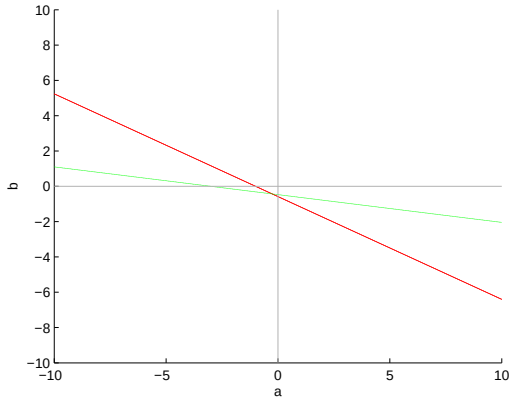


Figure 3.1: Two real eigenvalues have been specified. The red line is the fold curve for γ = -1 and the green line is the fold curve for γ = -3

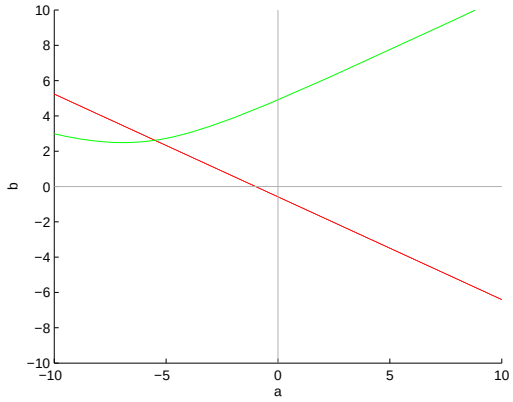


Figure 3.2: One real eigenvalue and the real part of a pair of complex conjugate eigenvalues have been specified. The red line is the fold curve for γ = -1 and the green line is one of the hopf curves for γ = -3

this case also it is proved that the required configuration of spectrum can be designed.

3.3 Real parts of two pairs of complex conjugate eigenvalues

The four rightmost eigenvalues are $\lambda_{1,2}$ = -1 $\pm i\omega$ and $\lambda_{3,4}$ = -3 $\pm i\tilde{\omega}$. In Fig. (3.3), a few hopf curves regarding to both the eigenvalues are plotted. There are multiple intersection points. So the current configuration of spectrum design can also be possible with certain parameter values.

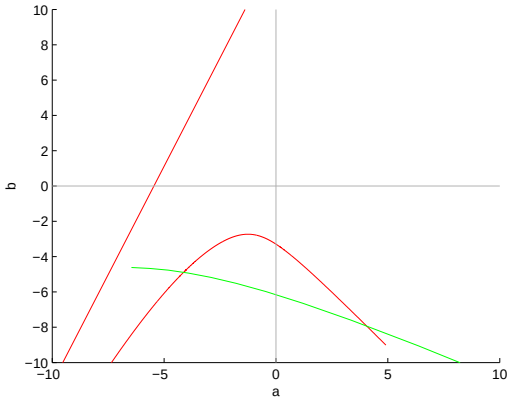


Figure 3.3: Real parts of two pairs of complex conjugate eigenvalues have been specified. The red line is one of the hopf curves for γ = -1 and the green line is one of the hopf curves for γ = -3

Chapter 4

Galerkin approximation method

Stability studies of delay differential equations is a cumbersome task because of the equations being infinite dimensional in nature. In [2], a method has been found to convert delay integro-differential equations (DIDEs) into a system of ODEs. Using certain transformations, the system of DIDEs can be transformed into a set of partial differential equations (PDEs). Plugging in the boundary conditions, the system is transformed to a set of ODEs and then stability analysis is easily carried out.

4.1 Formulation

$$\ddot{x} + c_1(t)\dot{x} + k_1(t)x(t) + c_2(t)\int_{l_1}^{l_2} C(\tau_1)\dot{x}(t-\tau_1)d\tau_1 + k_2(t)\int_{m1}^{m_2} K(\tau_2)x(t-\tau_2)d\tau_2 = 0 \tag{4.1}$$

Here x(t) is the state vector. The initial conditions are

$$x(t) = \alpha(t), -\tau \leq t \leq 0$$

$$x(t) = \alpha(t), -\tau \leq t \leq 0$$

where $\tau = max(l_1, l_2, m_1, m_2)$
To convert Eq (4.1) to a system of PDEs, we use the transformation

$$y(s,t) = x(t+s) \tag{4.2}$$

Differentiating,

$$\frac{\partial y}{\partial t} = \frac{\partial x}{\partial t + s} \tag{4.3}$$

$$\frac{\partial y}{\partial s} = \frac{\partial x}{\partial t + s} \tag{4.4}$$

From Eq. (4.3) and Eq (4.4)

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial s} \tag{4.5}$$

Again differentiating Eq. (4.5) with respect to t,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial t \partial s}, -\tau \leq s \leq 0, t \geq 0 \tag{4.6}$$

We have obtained a second order PDE.The boundary condition is obtained by substituting Eq. (4.2) in Eq. (4.1)

$$\ddot{y}(0,t) + c_1(t)\dot{y}(0,t) + c_2(t)\int_{l_1}^{l_2} C(s)\dot{y}(-s,t)ds + k_2(t)\int_{m1}^{m2} K(s)y(-s,t)ds = 0 \tag{4.7}$$

Now the solution of the PDE in Eq. (4.6) is approximated as

$$y(s,t) \simeq \sum_{k=1}^{N} \phi_k(s)\eta_k(t) = \boldsymbol{\phi}^T \boldsymbol{\eta}(t) \tag{4.8}$$

The basis functions are $\boldsymbol{\phi}(s) = [\phi_1(s), \phi_2(s), ..., \phi_N(s)]^T$ and the $\boldsymbol{\eta}(t) = [\eta_1(t), \eta_2(t), ..., \eta_n(t)]^T$ are the generalized coordinates.
Our choice of basis functions includes shifted Legendre polynomials. So the basis functions are now given by

$$\phi_1(s) = 1$$

$$\phi_2(s) = 1 + \frac{2s}{\tau}$$

$$\phi_n(s) = \frac{(2k-3)\phi_2(s)\phi_{k-1}(s) - (k-2)\phi_{k-2}(s)}{k-1}, k = 3, 4, ..., N$$

Substituting Eq. (4.8) into Eq. (4.6) and pre-multipyling by $\boldsymbol{\phi}(s)$ and integrating with respect to s from $-\tau$ to 0, the system of second order ODEs is obtained.

$$\boldsymbol{M}\ddot{\boldsymbol{\eta}}(t) = \boldsymbol{C}\dot{\boldsymbol{\eta}}(t) \tag{4.9}$$

$\boldsymbol{M}$ and $\boldsymbol{C}$ are given by

$$\boldsymbol{M} = \int_{-\tau}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}(s)^T ds \quad and \quad \boldsymbol{C} = \int_{-\tau}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi'}(s)^T ds \tag{4.10}$$

Since we have used Legendre polynomials as basis functions, we can obtain closed form expressions for $\boldsymbol{M}$ and $\boldsymbol{C}$ as

$$M_{ij} = \frac{\tau}{2i-1}\delta_{ij} \quad i,j = 1,2,...,N \tag{4.11}$$

$$C_{ij} = \begin{cases} 0 & if \quad i \geq j \\ 2 & if \quad i < j \quad and \quad i+j \quad is \quad odd \\ 0 & if \quad i < j \quad and \quad i+j \quad is \quad even \end{cases} \qquad i,j = 1,2,...,N \tag{4.12}$$

Making use of the boundary condition, substituting Eq. (4.8) in Eq. (4.7),

$$\boldsymbol{m}\ddot{\boldsymbol{\eta}}(t) = \boldsymbol{c}\dot{\boldsymbol{\eta}}(t) + \boldsymbol{k}\boldsymbol{\eta}(t) \tag{4.13}$$

where $\boldsymbol{m}$, $\boldsymbol{c}$ and $\boldsymbol{k}$ are

$$\boldsymbol{m} = \boldsymbol{\phi}^T(0) \tag{4.14}$$

$$\boldsymbol{c} = -c_1(t)\boldsymbol{\phi}^T(0) - c_2(t)\int_{l_1}^{l_2} C(s)\boldsymbol{\phi}^T(-s)ds \tag{4.15}$$

$$\boldsymbol{k} = -k_1(t)\boldsymbol{\phi}^T(0) - k_2(t)\int_{m_1}^{m_2} K(s)\boldsymbol{\phi}^T(-s)ds \tag{4.16}$$

The terms $\int_{l_1}^{l_2} C(s)\boldsymbol{\phi}^T(-s)ds$ and $\int_{m_1}^{m_2} K(s)\boldsymbol{\phi}^T(-s)ds$ can be evaluated using methods like trapezoidal quadrature. For solving Eq (4.9), spectral tau method is a good choice. In this method, we incorporate the boundary condition of Eq (4.13) in the last row of Eq. (4.9).

$$\tilde{\boldsymbol{M}}\ddot{\boldsymbol{\eta}}(t) = \tilde{\boldsymbol{C}}\dot{\boldsymbol{\eta}}(t) + \tilde{\boldsymbol{K}}\boldsymbol{\eta}(t) \tag{4.17}$$

Here $\tilde{\boldsymbol{M}} = \begin{pmatrix} \bar{\boldsymbol{M}} \\ \boldsymbol{m} \end{pmatrix}$, $\tilde{\boldsymbol{C}} = \begin{pmatrix} \bar{\boldsymbol{C}} \\ \boldsymbol{c} \end{pmatrix}$ and $\tilde{\boldsymbol{K}} = \begin{pmatrix} \bar{\boldsymbol{0}} \\ \boldsymbol{k} \end{pmatrix}$. $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{C}}$ are matrices of dimension N-1 $\times$ N formed by removing the last row of matrices M and C. If we use $\boldsymbol{z} = [\boldsymbol{\eta}^T, \dot{\boldsymbol{\eta}}^T\ ]$, we can rewrite Eq. (4.17) as

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}(t)\boldsymbol{z}(t) \tag{4.18}$$

where

$$\boldsymbol{A}(t) = \begin{pmatrix} 0 & \boldsymbol{I} \\ \tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{K}} & \tilde{\boldsymbol{M}}^{-1}\tilde{\boldsymbol{C}}. \end{pmatrix} \tag{4.19}$$

$\boldsymbol{I}$ is an identity matrix of dimensions N $\times$ N
The DIDE from Eq. (4.1) has been converted into a set of ODEs as shown in Eq. (4.18). Using the initial conditions,

$$x(s) = \boldsymbol{\phi}^T(s)\boldsymbol{\eta}(0) = \alpha(s) \tag{4.20}$$

$$\dot{x}(s) = \boldsymbol{\phi}^T(s)\dot{\boldsymbol{\eta}}(0) = \beta(s) \tag{4.21}$$

Take $\boldsymbol{\phi}(s)$ and pre-multiply it on Eq. (4.20) and Eq. (4.21) and then integrate over s ϵ $[-\tau, 0]$. Now pre-multiplying with $\boldsymbol{M}^{-1}$, we get

$$\boldsymbol{\eta}(0) = \boldsymbol{M}^{-1} \int_{-\tau}^{0} \alpha(s)\boldsymbol{\phi}(s)ds \tag{4.22}$$

$$\dot{\boldsymbol{\eta}}(0) = \boldsymbol{M}^{-1} \int_{-\tau}^{0} \beta(s)\boldsymbol{\phi}(s)ds \tag{4.23}$$

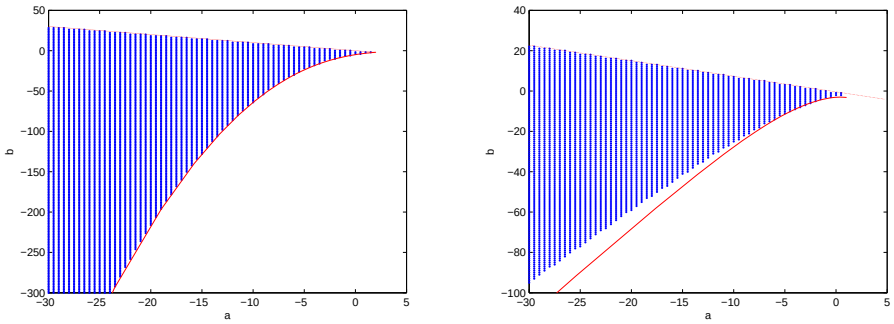
Finally, we can find the functions $x(t)$ and $\dot{x}(t)$ as

$$x(t) = y(0,t) = \boldsymbol{\phi}^T(0)\boldsymbol{\eta}(t) \tag{4.24}$$

$$\dot{x}(t) = \dot{y}(0,t) = \boldsymbol{\phi}^T(0)\dot{\boldsymbol{\eta}}(t) \tag{4.25}$$

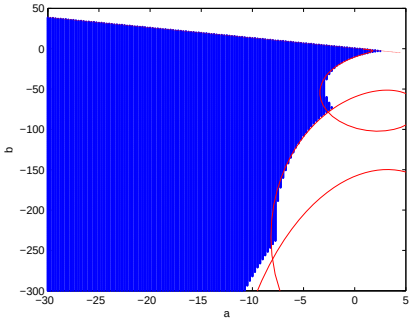
4.2 Results

Galerkin approximation method was successfully employed for the IDDE from Eq. (1.2). Using the approach discussed in the formulation, $\boldsymbol{A}$ from Eq. (4.19) was found. The stability analysis was carried out and the results are shown in Fig. (4.1). The accuracy of roots found out using this method depends on the number of terms taken in Eq. (4.8)



(a)

(b)



(c)

Figure 4.1: Galerkin solutions are shown as blue dots : (a) is for the case when $\gamma = 0$ (b) is for the case with γ having a negative value of -0.5 and (c) is for the case with a positive value of 0.5. The red lines are the boundaries for the stability region as obtained using stability charts. It is found that the results are in agreement with each other.

Chapter 5

Conclusions

The aim of the project was to analyze the problem of spectrum design of a system of delay differential equations. Equations with distributed delays were considered and it was shown that it is indeed possible to design the spectrum for a linear scalar system. The concept of stability charts was used for this problem. Stability charts for a system revealed various properties of the system. It was found out that when stability charts are used, the problem of spectrum design was translated into finding the intersecting regions between various curves and surfaces. Finally the method of Galerkin approximation was studied. It was found that the system of DDEs can be transformed into PDEs and subsequently into a system of ODEs which is a familiar territory for stability analysis. Galerkin method gave good results on the system in question and the results were in agreement with the results achieved using stability charts.

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