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Title: Orthogonality in Banach Space and Almost Constrained Subspaces of Banach Space

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DECLARATION

I declare that this written submission represents my ideas in my own words and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented any idea in my submission. I understand that any violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the source which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

Introduction and Motivation

We know the orthogonality in Hilbert space, which is easy to investigate, because inner product is available there. In this paper we will discuss the orthogonality in a general Banach space and then for a general normed space (the later one may be covered later.) Also we will discuss the existence and properties of elements that are orthogonal to a given closed subspace. Here we will discuss the generalization of Birkhoff orthogonality in a general normed space.In a Hilbert space so-called Birhkoff orthogonality and general orthogonality both are same.But in general its not true.Here Riesz's lemmma comes into the picture, which says that for a given proper closed subspace M of a normed linear space X and for a given $\epsilon > 0$ there is a point $x_\epsilon \in X$ with $\|x_\epsilon\| \equiv 1$ and $dist(x_\epsilon, M) \geq 1$. That is this lemma says that for a given proper closed subspace of a normed space there is always a point in the unit sphere whose distance form M is as closed to 1 as we please.So the natural question is that does there exist a point in the unit sphere which is exactly at the distance 1 from the subspace? In general the answer is "no". But if the normed space is reflexive or finite dimensional. the answer is affirmative.Also if we take $\epsilon = 0$ in Riesz's lemma,then the corresponding vector x_0 is orthogonal to M. In Banach space theory, there are two fundamental theorems, one is James theorem and the next one is Bishops-Phelps theorem.James theorem says that a normed space X is non-reflexive if and only if there is a hyperplane in X such that no point in X is orthogonal to M. This forces us not to take $\epsilon = 0$ in Riesz's lemma. But Bishop-Phelps theorem points a solution in case of non-reflexive space. This theorem says that for a given proper closed subspace of a non-reflexive Banach space, we will be able to find a hyperplane which is as closed to as we wish.

In the last section we discuss an interesting object. Suppose X is a given Banach space and Y be a closed linear subspace fo X. One can ask a natural question whether there is a norm 1 projection from X onto Y. This kind of subspaces are called "constrained subspaces". For a given point $x \in X \setminus Y$, define $Y_x = \overline{span}[Y \cup \{x\}]$. Now if Y is a constrained subspace of X, then it is clear that there is also a norm 1 projection from Y_x onto Y. This kind of spaces are called "almost constrained subspaces". We observe that constrained property of a Banach space implies the almost constrained property of it. Now question is whether the converse is true always. This gives a negative answer. One counterexample is in chapter 2 of reference [5]. Also we discuss the two sets $L(Y, X)$, the set all elements in X which are left orthogonal to Y and $O(Y, X)$, the set of all elements which are right orthogonal to Y. Here the orthogonality is in the sense of Birkhoff only. We characterize all proximinal subspaces in terms of $L(Y, X)$ and all almost constrained subspaces in terms of $O(Y, X)$.

Chapter 1

Banach Spaces

Definition 1.0.1 *Let X be a an additive group.If any two elements x and y of X be combined by an operation,(called addition) to get a new element $x+y$ and any scalar(element of fields F, here we assume field means $\mathbb{R}$, the real number system or* ***C****, the complex number system.) α and any element x of X be combined by an operation (called scalar multiplication) to get a new vector αx such that*

(1) $\alpha(x+y) = \alpha x + \alpha y$.

(2) $(\alpha + \beta)x = \alpha x + \beta x$.

(3) $(\alpha \beta x) = \alpha(\beta x)$.

(4) $1 \cdot x = x$.

Then X is called a linear space or a vector space and any element of X is called a vector.

Example 1.0.2 *The real number system , the complex number system are the easiest examples of vector space.*

Definition 1.0.3 *A function $\| \cdot \| : X \longrightarrow \mathbb{R}_{\geq 0}$ is called norm if it satisfies the following properties:*

(1)$\|x\| \geq 0$. $\forall x \in X$ and equality holds if and only if $x = 0$.
(2)$\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$.
(3)$\|\alpha x\| = |\alpha| \|x\|$, for all $\alpha \in F$ and for all $x \in X$

Definition 1.0.4 *A linear space along with a norm is called a normed linear space.*

Example 1.0.5 *The real number system and the complex number system, where the norm of an element x is given by $\|x\| = |x|$.*

Example 1.0.6 *The linear space $\mathbb{R}^n$ and $\mathbb{C}^n$ of all n-tuples $x = (x_1, x_2, x_3, ..., x_n)$ of real and complex numbers can be made into normed linear spaces in an infinite variety of ways. One such defined norm is given by*

$$\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$$

Example 1.0.7 *The space l_p for $1 \leq p < \infty$, of all bounded sequences $(x_n)_{n=1}^{\infty}$ of scalars such that*

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

Definition 1.0.8 *A complete normed linear space is called a Banach space.*

For example the above three spaces are complete as metric spaces.And so they are Banach spaces.In a linear space,if we change the norm, the space will be changed.For example,the space $\mathbf{C}[0,1]$ of all continuous real valued functions on the closed interval $[0,1]$ is normed linear with respect to the following norms as: $(a)\|f\|_\infty = \sup|f(x)|$ $\quad (b)\|f\|_1 = \int_0^1 |f(x)|dx$.
The space $(C[0,1], \|\cdot\|_\infty)$ is complete whether the space $(C[0,1], \|\cdot\|_1)$ is not.

1.1 Continuous Linear Transformation

Let X and Y be two normed linear spaces with the same scalar field. Let T be a linear transformations from X into Y, called continuous if it is continuous mapping from a metric space X into the metric space Y by means of if a sequence $x_n \longrightarrow x \Rightarrow T(x_n) \longrightarrow T(x)$.

Theorem 1.1.1 *Let X and Y be two normed linear spaces with the same scalar field and T be a linear transformation from X into Y, then the followings are equivalent:*

(a) T is continuous;

(b) T is continuous at the origin;

(c) there exists a non-negative real number $M \geq 0$ such that $\|T(x)\| \leq M\|\|$, for every $x \in X$.

(d) for any closed unit ball $S = \{x \in X : \|x\| \leq 1\}, T(S)$ is bounded.

Proof:

(1) $\Leftrightarrow$ (2):

Assume T is continuous.As $T(0) = 0$, it is obvious that T is continuous at

the origin.So (1) $\Rightarrow$ (2)is fine.
Now assume that T is continuous at the origin.So if $x_n \to x \Leftrightarrow x_n - x \to 0 \Rightarrow T(x_n - x) \to 0$ [because T is continuous at 0]$\Leftrightarrow T(x_n) \to T(x)$[because T is linear]$\Rightarrow T$ is continuous.
(2) $\Leftrightarrow$ (3) :
(3) $\Rightarrow$ (2) is obvious, because if such a M exists, then if $x \to 0$,then obviously $\|T(x)\| \to 0$ which means that $T(x) \to 0$, means T is continuous at the origin.
(2) $\Rightarrow$ (3) : by contrary, we assume that no M exists to satisfy (3).then for each natural number n, there is a vector x_n such that $\|T(x_n)\| > n\|x_n\|$. That is, $\|T(\frac{x_n}{n\|x_n\|})\| > 1$. Put $y_n = \frac{x_n}{n\|x_n\|}$, we get $y_n \to 0$, but $T(y_n) \nrightarrow 0$, means T is not continuous at the origin, a contradiction.
(3) $\Leftrightarrow$ (4):
(3) $\Rightarrow$ (4): we know that a non-empty subset of a normed linear space is bounded if and only if it is contained in a closed sphere centered on the origin. So (3) $\Rightarrow$ (4) is clear.
To show (4) $\Rightarrow$ (3); we assume $T(S)$ is contained in a closed ball of radius K, centered at te origin.If $x = 0$, then$T(x) = 0$ and clearly $\|T((x)\| < K\|x\|$; and if $x \neq 0$, then$\frac{x}{\|x\|} \in S$and hence $\|T(\frac{x}{\|x\|}) \leq K$, so again we have $\|T(x)\| \leq K\|x\|$.

Theorem 1.1.2 *If X and Y be two normed linear spaces, then the set $\mathcal{B}(X,Y)$ of all bounded linear transformations of X into Y is a normed linear space with respect to the operator norm which is defined as*

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}$$

Moreover, Y is complete if and only if $\mathcal{B}(X,Y)$ is a Banach space.

Proof:

We only prove that if Y is complete, then $\mathcal{B}(X,Y)$ is also complete(we left to prove that $\mathcal{B}(X,Y)$ is normed linear as it is easy to check). Let (T_n) be Cauchy sequence in $\mathcal{B}(X,Y)$.Let x be a fixed but arbitrary vector in X, then$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\|x\|$, and this implies that $\|T_n\|$ is Cauchy sequence and since Y is complete there is a vector in Y , say y such that $T_n(x) \to y$. Now define a transformation $T : X \to Y$ by $T(x) = y$.We have to check that T is linear and bounded.Let $T_n(x) \to T(x)$ and $T_n(z) \to T(z)$, then $T_n(x)+T_n(z) \to T(x)+T(z)$ but $T_n(x)+T_n(z) = T_n(x+z) \to T(x+z)$ and due to uniqueness of limit, $T(x + z) = T(x) + T(z)$.Let α be any scalar and x be any vector in X, then $T_n(\alpha x) \to T(\alpha x)$ and $\alpha T_n(x) \to \alpha T(x)$which means that $T(\alpha x) = \alpha T(x)$. Now $T(x) = \|\lim T_n(x)\| = \lim \|T_n(x)\| \leq \sup(\|T_n\|\|x\|) = (\sup \|T_n\|)\|x\|$ shows that T has a bound and hence it is continuous. Now it remains to prove that $\|T_n - T\| \to 0$. Here T_n is Cauchy, so for each $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that $m, n > N_0 \Rightarrow \|T_n - T_m\| < \epsilon$. if

$\|x\| \leq 1$, then$m, n > N_0 \Rightarrow \|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| < \epsilon$. Now keep n fixed and allow m to approach ∞ we get $\|T_n(x)-T_m(x)\| \to \|T_n(x)-T(x)\|$ which means that $\|T_n(x) - T(x)\| \leq \epsilon$ for all $n \geq N_0$ and all x such that $\|x\| \leq 1$. This shows that $\|T_n - T\| \leq \epsilon$ for all $m \geq N_0$. This completes the proof of the theorem.
Now assume that $\mathcal{B}(X, Y)$ is complete.Fix $x_0 \in X$ with $\|x_0\| = 1$. Then by Hahn-Banach theorem, there is $f(x_0) = 1$. Let $\{y_n\}$ be a Cauchy sequence in Y. Define

$$A_n : X \to Y$$

, by

$$A_n(x) = f(x)y_n$$

,then

$$\begin{aligned}
\|A_n - A_m\| &= \sup\{\|(A_n - A_m)x\| : \|x\| = 1\} \\
&= \sup\{\|f(x)y_n - f(x)y_m\| : \|x\| = 1\} \\
&= \sup\{|f(x)|\|y_n - y_m\| : \|x\| = 1\} \\
&= \|y_n - y_m\|
\end{aligned}$$

This shows that $\{A_n\}$ is a Cauchy sequence in $\mathcal{B}(X, Y)$. Since, $\mathcal{B}(X, Y)$ is complete, there is a $A \in \mathcal{B}(X, Y)$ such that $A_n \to A$. Now $A_n(x_0) = f(x_0)y_n = y_n$. So $\lim y_n = A(x_0) \in Y$. Hence Y is complete.

Definition 1.1.3 *Let X be a normed linear space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. We say these two norms are "equivalent" if there is an isomorphism between these two spaces i.e. if there exists two positive real numbers c and C such that*

$$c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2 \text{ , for all } x \in X$$

Note 1.1.4 *For a finite dimensional normed linear space, any two norms are equivalent.*

1.2 New Normed Space from the Old

Let X and Y be two normed linear spaces. We can introduce a new normed linear space $X \oplus Y$ called topological direct sum, consisting of all ordered pairs (x, y) where $x \in X$ and $y \in Y$ with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$.Moreover, if X and Y are Banach spaces, then $X \oplus Y$ is also so.

Definition 1.2.1 *Let Y be closed linear subspace of a normed linear space X . Define a relation " $\sim$ " by for any $x, z \in X, x \sim z \Leftrightarrow x - z \in Y$, then*

for $x \in X$,the coset $\overline{x}$ relative to Y be considered as
$\overline{x} = \{z \in X : (x - z) \in Y\} = \{x + y : y \in Y\}$. *Here we will denote $\overline{x}$ as $x + Y$*
Consider the set $\frac{X}{Y} = \{\overline{x} : x \in X\}$. This set with the operations "addition" and "scalar multiplication" as defined bellow:
$\overline{x} + \overline{y} = \overline{x + y}$ *and* $\lambda\overline{x} = \overline{\lambda x}$ *is a vector space. And it is easy to check that this space with the norm defined as*
$\|x + Y\| = \inf\{\|x + y\| : y \in Y\}$ *, is a normed linear space, called "quotient space".*

Therefore, for a given normed linear space we always be able to get a new normed space.

Theorem 1.2.2 *Let Y be a closed subspace of a Banach space X. Then the space $\frac{X}{Y}$ is a Banach space.*

Proof:

Let $(\overline{x}_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\frac{X}{Y}$.Choose a subsequence $(n(k))_{k=1}^{\infty}$ such that $\|\overline{x}_{n(k)} - \overline{x}_{n(k+1)}\| < 2^{-k}$.Then choose any arbitrary element $x_{n(k)} \in \overline{x}_{n(k)}$ such that $\|x_{n(k)} - x_{n(k+1)}\| < 2^{-k}$. Clearly this is a Cauchy sequence in X and since X is complete, this sequence converges.Let $x_{n(k)} \to x \in X$. But $\|\overline{x}_{n(k)} - \overline{x}\| \leq \|x_{n(k)} - x\|$, immediately it follows that for every k, $\overline{x}_{n(k)} \to \overline{x}$ in $\frac{X}{Y}$.Hence,$\overline{x}_n$ converges to $\overline{x}$ in $\frac{X}{Y}$

Theorem 1.2.3 *In a finite dimensional normed linear spaces, any two norms are equivalent.*

Proof:

Let $\{e_1, e_2, e_3, ..., e_n\}$ be an algebraic basis for a normed linear space X of dimension n.Then any element $x \in X$ can be written as $\sum_{i=1}^{n} \lambda e_i$. Now we would like to introduce a new norm on X, viz. $\|\cdot\|_1$ defined as
for any $x \in X$, $\|x\|_1 = \sum_{j=1}^{n} |\lambda_j|$.To prove this is a norm, we only triangle inequality which is as bellow:
Let $x = \sum_{i=1}^{n} \alpha e_i$ and $y = \sum_{i=1}^{n} \beta e_i$ be two elements of X.Then $\|x + y\|_1 = \sum_{i=1}^{n} |\lambda_i + \beta_i| \leq \sum_{i=1}^{n} |\lambda_i| + \sum_{i-1}^{n} |\beta_i| = \|x\|_1 + \|y\|_1$.Next we will show that an arbitrary norm on X is a Lipschitz function on $(X, \|\cdot\|)$.If $x = \sum_{i-1}^{n} \lambda_i e_i$ and $y = \sum_{i=1}^{n} \beta_i e_i$, then
$\|x - y\| = \sum_{i=1}^{n} |\lambda_i - \beta_i| \|e_i\| \leq \max \|e_i\| \sum_{i=1}^{n} |\lambda_i - \beta_i| = \max \|e_i\| \cdot \|x - y\|_1$.
Therefore,
$|\|x\| - \|y\|| \leq \|x - y\| \leq \max \|e_i\| \cdot \|x - y\|_1$.
Note that $S_1 = \{x \in X : \|x\|_1 = 1\}$ be compact in $(X.\|\cdot\|_1)$. Let $x^k \in S_1$, then $x^k = \sum_{i=1}^{n} \lambda_i^k e_i$, for any $k \in \mathbb{N}$. then we have $\sum_{i=1}^{n} |\lambda_i^k| = 1$.Thus $(\lambda_i^k)_{k=1}^{\infty}$ is a bounded sequence for every i.Let $(k_l)_{l=1}^{\infty}$ be a subsequence such

that $\lambda_{i=1}^{k_l} \to \lambda_i$, as $l \to \infty$ for every i.Then $\sum_{i=1}^{n} |\lambda_i^{k_l} - \lambda_i| \to \infty$ as $l \to \infty$.So we have $x^{k_l} \to x$ as $l \to \infty$, where $x = \sum_{i=1}^{n} \lambda_i e_i$. Since $\sum_{i=1}^{n} |\lambda_i^{k_l}| = 1$, for every l, we have $\sum |\lambda_i| = 1$ and thus $x \in S_1$. Since $\| \cdot \|$ is continuous on the compact set S_1, there exists two non-zero constants c and d such that

$$c \leq \| \frac{x}{\|x\|_1} \| \leq d, \forall x \in X$$

.

that is

$$c\|x\|_1 \leq \|x\| \leq d\|x\|_1, \forall x \in X$$

. Which shows that all norms are equivalent.

Lemma 1.2.4 *Let X be a normed linear space and M be a proper closed subspace of X. Then for each $\epsilon > 0$, there is an element $x_\epsilon \in S_X$ such that $dist(x_\epsilon, X) \geq 1 - \epsilon$.*

Proof: Choose any arbitrary $\overline{z} \in \frac{X}{M}$, with $1 > \|\overline{z}\| > 1 - \epsilon$. Now take any $z \in \overline{z}$ with $\|z\| \leq 1$ and set $x = \frac{z}{\|z\|}$.
Then $dist(x, M) = dist(z, M) \cdot \frac{1}{\|z\|} = \frac{\overline{z}}{\|z\|} \geq \overline{z} > 1 - \epsilon$.

1.3 Four Important Theorems

The following four theorems are the building block of functional analysis:

- Hahn-Banach Theorem;
- Open Mapping Theorem;
- Closed Graph Theorem;
- Uniform Boundedness Theorem.

Let us state and prove the above theorems.

Theorem 1.3.1

The Hahn-Banach Extension Theorem

Theorem A: Let M be a linear subspace of a normed linear space X, and let f be a linear functional defined on M, then f can be extended to a functional g defined on X, such that $\|f\| = \|g\|$.
Proof: Consider the set $\mathbb{F} : \{g : M' \to \mathbf{C} : g|_M = f, M \subseteq M'\}$. Then clearly

$\mathbb{F}$ is partially ordered with respect to the relation:$g_1 \leq g_2 \Leftrightarrow M_1 \subseteq M_2$ and $g_2|_{M_2} = g_1$, where M_i is the domain of $g_i, \forall i$. If X be a normed linear space and x_0 be a given non-zero point in X, then there is a functional $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$. And it is obvious that union of any chain of extensions is again a chain and is an upper bound for the chain. According to Zorn's lemma,there is a maximal extension f_0. We will be done if we can prove that domain of f_0 is the whole X. And in fact , it is. Because otherwise that function could be extended further and would not be maximal, using Zorn's lemma.

Theorem B: If X be a normed linear space and x_0 be a given non-zero point in X, then there is a functional $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

Proof: Let $M = span\{x_0\}$ and define

$$f : M \to \mathbb{C}$$

by

$$f(\alpha x_0) = \alpha \|x_0\|.$$

Clearly, $f(x_0) = \|x_0\|$ and $\|f\| = 1$. So by Hahn-Banach theorem, f can be extended to some functional g on the whole of X, with the required properties.

Theorem C: If M is proper closed subspace of a normed linear space X and $x_0 \in X \setminus M$, then there is a functional $f \in X^*$ such that $f(M) = 0$ and $f(x_0) \neq 0$.

Proof: Since M is closed subspace of X, we get a natural mapping

$$\eta : X \to \frac{X}{M}$$

which is continuous linear.

$\eta(M) = 0$ and $\eta(x_0) = x_0 + M \neq 0$. Now using theorem (B), there is an $f \in \frac{X}{M}$ with

$$f(x_0 + M) \neq 0.$$

Now we can define $f_0 \in X^*$ by $f_0(x) = f(\eta(x))$. This is our required functional.

1.4 The Natural Embedding of X in X^*

If we are given a normed space X. We can think of its conjugate X^* or this is called the *dual* of X, consisting of all linear functionals on X. Now this space X^* is again a normed linear space and again we can think of its conjugate, namely X^{**} or we can call it as second dual of X or dual of dual of X.By this way, we will be getting duals a normed linear space. Now why we are concentrating on the the dual space a normed space is to be discussed.Each vector in $x \in X$ can be imagined as an element of its second dual or its bidual i.e. x can raise to a functional $F_x \in X^{**}$. Let $f \in X^*$ be an arbitrary element.Then F_x be defined by

$$F_x(f) = f(x)$$

Now it is clear that F_x is linear.Because $F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_(x)(f) + \beta F_(x)(g), \forall f, g \in X^*$.
Now we wish to calculate the norm of F_x as:

$$\begin{aligned}\|F_x\| &= \sup\{|F_x(f)| : \|f\| \leq 1\} \\ &= \sup\{|f(x)| : \|f\| \leq 1\} \\ &\leq \sup\{\|f\|\|x\| : \|f\| \leq 1\} \\ &\leq \|x\|.\end{aligned}$$

But by Hahn-Banach theorem, we know that for a given non-zero point x_0 in a normed linear space X,we always be able to find some functional f_0 in its dual such that $f_0(x) = \|x\|$ and $\|f\| = 1$. So from this we can conclude that $\|F_x(f)\| = \|x\|$. , that is the map

$$x \to F_x$$

is norm preserving map of X into X^*. And the functional F_x is called the functional on X^* induced by the vector x. and the mapping

$$x \to F_x$$

is isometrically isomorphism of X into X^*. We can call this as canonical embedding.

Definition 1.4.1 *A normed space is said to be reflexive if the canonical embedding is surjective.*

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Example 1.4.2 *The space l_p, L_p for $1 < p < \infty$ are reflexive. Here* $l_p^* = l_q$ *and* $l_p^{**} = l_q^* = l_p$

Note: Since X^{**} is complete, then if X is reflexive it is necessarily complete.But if X is complete, then it need not be reflexive. A counter-example is given below:
$c_0^* = l_1$ and $c_0^{**} = l_1^* = l_\infty$.

Proposition 1.4.3 *Let X be a compact Hausdorff space, then $\boldsymbol{C}(X)$ is reflexive if and only if X is a finite set.*

An Interesting Fact: We are now ready to go through the following concept, which is *weak** topology on a normed linear space. First we will clarify what we mean by weak topology on a normed space.For normed space X is a metric space,so there is a topology in X induced by the metric in its own right called *strong topology* on X. The weak topology is a topology on the normed space X with respect to which all the functions in X^* are continuous. Clearly this is weaker than the strong topology.
We are now interested about the *weak** topology on X^*. We know that X^* is also a normed linear space, and as a metric space,it has a topology,which is the biggest topology, called $strong - topology$ on X^*. As above we can define weak topology on it. By *weak**-topology on X^* we mean the weakest topology on X^* with respect to which all the $induced\ \ functionals$ are continuous. By canonical embedding, we can think of X as a part of X^{**}. Now we can restate the above definition. The weakest topology on X^* with respect to which all functions in X regarded as a subset of X^{**} are continuous.This is $weak^* - topology$ and clearly it is weaker than the weak topology.
Consider a vector $x \in X$ and its induced functionals $F_x \in X^*$. The $weak^* - topology$ on X^* is the weakest topology on X^* with respect to which all F_x are continuous.Let $f_0 \in X^*$ be arbitrary and $\epsilon > 0$ be given, then

$$\begin{aligned} S(x, f_0, \epsilon) &= \{f : f \in X^* \text{ and } |F_x(f) - F_x(f_0)| < \epsilon\} \\ &=\{f : f \in X^* \text{ and } |f(x) - |f_0(x) < \epsilon\} \end{aligned}$$

This is an open set in the *weak** topology.The family of such kind of open sets defines an open sub-base for the $weak^* - topology$. And all finite intersections of these sets form an open base for the $weak^* - topology$ and the open sets are the unions of these finite intersections.

Remark 1.4.4 *X^* is Hausdorff space with respect to its weak topology. This is because if f and g be two distinct functionals in X^*, then there is a vector $x \in X$ such that $f(x) \neq g(x)$. Choose $\epsilon = \dfrac{f(x) - g(x)}{3}$,then $S(x, f, \epsilon)$ and $S(x, g, \epsilon)$ are two disjoint neighborhoods of f and g.*

Theorem 1.4.5 ***The Open Mapping Theorem:*** *If B and B' are two Banach spaces and T is a continuous linear transformation of B onto B', then T is an open mapping.*

Before going through the proof of this theorem, we will introduce a lemma without proof, which is

Lemma 1.4.6 *If B and B' be two Banach spaces and T is a continuous linear transformation of B onto B'. Then image of an open sphere in B with center at origin contains an open sphere center at the origin in B'.*

Proof: We denote S_r and S'_r to be two open spheres with radius r and centered at the origin in B and B' respectively. Clearly

$$T(S_r) = T(rS_1) = rT(S_1),$$

so we will be done if we can show that $T(S_1)$ contains some S'_r. First we will prove that $\overline{T(S_1)} \supset S'_r$ for some $r > o$.Since T is onto, we see that $B' = \cup_{n=1}^{\infty} T(S_n)$. Since B' is complete, so Baire's category theorem implies that for some n_0 $\overline{T(S_{n_0})}$ has an interior point y_0, we may be assumed to lie in $T(S_{n_0})$. The mapping $y \to y - y_0$ is a homeomorphism of B' onto itself,so $\overline{T(Sn_0)} - y_0$ has the origin as an interior point.Since y_0 is in $T(Sn_0)$ we have $T(Sn_0) - y_0 \subset T(S_{2n_0})$ and from this we obtain interior point. Multiplication by any non-zero scalar is a homeomorphism of B' onto itself, so $\overline{T(S_{2n_0})} = 2n_0T(S_1) = 2n0_0\overline{T(S_1)}$, and it follows from this that origin is also an interior point of $\overline{T(S_1)}$, so $S_\epsilon \subseteq T(S'_\epsilon)$,for some positive ϵ We conclude the proof by showing that $S'_\epsilon \subseteq T(S_3)$which is clearly equivalent to $S'_{\epsilon/3} \subseteq T(S_1)$, for some positive ϵ. Let y be a vector in B' such that $\|y\| < \epsilon$. Since $y \in \overline{T(S_1)}$ there is a vector $x_1 \in B$ such that $\|x_1\| < 1$ and $\|y - y_1\| < \epsilon/2$, where $y_1 = T(x_1)$. We next observe that $S'_{\epsilon/2} \subseteq \overline{T(S_{1/2})}$, so there is a vector $x_2 \in B$ such that $\|x_2\| < 1/2$ and $\|(y - y_1) - y_2\| < \epsilon/4$, where $y_2 = T(x_2)$, continuing this way,we obtain a sequence $\{x_n\}$ in B such that $\|x_n\| < 1/2^{n-1}$ and $\|y - (y_1 + y_2 + ... + y_n)\| < \epsilon/2^n$, where $y_n = T(x_n)$. If we put

$$s_n = x_1 + x_2 + ... + x_n,$$

then it follows from $\|x_n\| < 1/2^{n-1}$ such that $\{s_n\}$ is a Cauchy sequence in B for which

$$s_n \leq \|x_1\| + \|x_2\| + ...\|x_n\| < 1 + 1/2 + ... + 1/2^{n-1} < 2$$

. Since B is complete, there is sequence $x \in B$ such that $s_n \to x$ and $\|x\| = \|\lim s_n\| = \lim \|s_n\| \leq 2 < 3$.This shows that $x \in S_3$and T is also continuous, so

$$T(x) = T(\lim s_n) = \lim T(s_n) = \lim(y_1 + y_2 + ...y_n) = y.$$

This implies that $y \in T(S_3)$.
Proof of Open Mapping Theorem:

We are done if we can show that for an open set G in B, $T(G)$ is also open.Let $y \in T(G)$, then it is sufficient to produce an open sphere centered at origin, which is contained in $T(G)$.Since T is onto, let $x \in G$ such that $T(x) = y$. Let $S_r(x)$ be an open sphere centered at x and radius r, then $S_r(x)$ can be written as $x + S_r$, where by S_r we mean the sphere centered at origin and radius r. And of course, $x + S_r \subset G$. By previous lemma $T(S_r)$ contains some S'_{r_1}.Now it is evident that $y + S'_{r_1} \subseteq T(G)$. So we get $y + S'_{r_1} \subseteq y + T(S_r) = T(x) + T(S_r) = T(x + S_r) \subseteq T(G)$.

Note: In most of the case of application of open mapping theorem the following statement of the open mapping theorem is more useful:
A one-one continuous function of one Banach space onto another is a homeomorphism. In particular, if a one-one linear transformations of one Banach space to itself is continuous, then its inverse is automatically continuous.

Theorem 1.4.7 ***Closed Graph Theorem:*** *If B and B' are two Banach spaces and T is linear transformation of B into B', then T is continuous $\Leftrightarrow$ its graph is closed.*
Proof: *One implication is straight forward, which is if T is continuous, then its graph is closed.Because, if $x_n \to x$ in B, then $T(x_n) \to T(x)$ which means that T is a closed map.So the graph of T is closed. For other implication, let us denote the space B as B_1 after re-norming by*
$\|x\|_1 = \|x\| + \|T(x)\|$. Since

$$\|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|_1.$$

T is a continuous of B_1 into B'.We will be done if we can show that B_1 and B' have the same topology.Since

$$\|x\| \leq \|x\| + \|T(x)\| = \|x\|_1,$$

it is clear that that the identity mapping of B_1 onto B' is continuous.If we can show that B_1 is complete, then by the restatement of the open mapping theorem given above in "Note", it is clear that T is a homeomorphism. To prove, B_1 is complete, let (x_n) be a Cauchy sequence in B_1, then x_n and $(T(x_n))$ are also Cauchy in B and B' respectively. As these two spaces are complete, it follows that for some x and y in B and B' respectively, $\|x_n$-$x\| \to 0$ and $\|T(x_n) - y\| \to 0$. We assumed that the graph of T is closed in (B, B'), so the point (x, y) lies in the graph of T. Which means that $T(x) = y$. Therefore
$\|x_n - x\|_1 \leq \|x_n - x\| + \|T(x_n - x)\| = \|x_n - x\| + \|T(x_n) - T(x)\| = \|T(x_n) - y\| \to 0$.
So B_1 is complete, and hence we are done.

Theorem 1.4.8 Uniform Boundedness Theorem: *Let B be a Banach space. If $\{T_i\}$ be a non-empty set of continuous linear transformations of B*

into X such that $\{T_i(x)\}$ is bounded subset of X , for each $x \in B, \{\|T_i\|\}$ is bounded set of numbers,that is $\{T_i\}$ bounded as a subset of $\mathcal{B}(B, X)$.

Proof: *For each positive integer n, consider the set $F_n = \{x : x \in B \text{ and } \|T_i(x)\| \leq n, \forall i\}$ Clearly this is a closed subset of B. By our assumption,*

$$B = \cup_{n=1}^{\infty} F_n$$

Since B is complete, so Baire's category theorem implies that one of the F_n say,F_{n_0} is a non-empty interior.And so there is $r_0 > 0$ such that for some point x_0, the closed sphere S_0 centered at x_0 and radius r_0 is contained in F_{n_0}. So $\forall i$, each vector in $T_i(S_0)$ has norm at most n_0. i.e. $\|T_i(S_0)\| \leq n_0$. Denote $S = \dfrac{S_0 - x_0}{r_0}$ is the closed unit sphere.Therefore, we have $\|T(S - x_0)\| \leq 2n_0$, which means $\|T(S)\| \leq \dfrac{2n_0}{r_0}, \forall i$. And hence the proof is complete.

Chapter 2

Hilbert Space and Birkhoff Orthogonality

Introduction:
Hilbert space is a special type of Banach space.Because there is some additional structure on the Hilbert space which tells us when two vectors in that space will be orthogonal(or,perpendicular).Besides, there is a natural correspondence between the Hilbert and its conjugate. As we are given a linear space, the first thing that comes into the picture is the linear transformation.So in Hilbert space $\mathcal{H}$,if T is a linear operator, its corresponding conjugate acts on $\mathcal{H}$ (instead of $\mathcal{H}^*$).

Definition 2.0.9 *A Hilbert space is a complex Banach space whose norm comes from the inner product, i.e., in which there is defined a complex function $\langle , \rangle$ on the Hilbert space as*
for any two vectors $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$, the followings hold:
(1)$\langle \alpha x + \beta y, z\rangle = \alpha\langle x + z\rangle + \beta\langle y + z\rangle$
(2)$\overline{\langle x, y\rangle} = \langle y, x\rangle$.
(3) $\langle x, x\rangle = \|x\|^2$.

Example 2.0.10 *The space l_2^n with the inner product of two vectors $x(x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ defined by*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

Example 2.0.11 *The space l_2 with the inner product of two vectors $x = \{x_1, x_2, ..., x_n, ...\}$ and $y = \{y_1, y_2, ..., y_n, ...\}$ defined by*

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y}_i$$

Example 2.0.12 *The space* L_2 *associated with a measure space* X *with measure* μ*, with the inner product of two functions* f *and* g *defined by*

$$\langle f, g\rangle = \int f(x)\overline{g(x)}d\mu(x)$$

Example 2.0.13 *If* x *and* y *be any two vectors in* $\mathcal{H}$*, then it satisfies Schwarz inequality, which is* $|\langle x, y\rangle \leq \|x\|\|y\|$.

Example 2.0.14 *Let* $\mathcal{H} = \{f : f : [0,1] \to \mathbb{C}$ *is absolutely continuous with* $f(0) = 0$ *and* $f' \in L[0,1]\}$. *Define* $\langle f, g\rangle = \int_0^1 f'(t)\overline{g'(t)}dt; \forall f, g \in \mathcal{H}$. *We will see* $\mathcal{H}$ *is a Hilbert space.*

Proof: If $y = 0$, there is nothing to prove. So assume that $y \neq 0.$, then the inequality reduces to $|\langle x, \frac{y}{\|y\|}\rangle| \leq \|x\|$. Without loss of generality we assume that $\|y\| = 1$, so it remains to prove that $|\langle x, y\rangle| \leq \|x\|, \forall x$. This is clear, because
$0 \leq \|x - \langle x, y\rangle y\|^2 = \langle x - \langle x, y\rangle y, \langle x - \langle x, y\rangle y\rangle = \langle x, x\rangle - \langle x, y\rangle\overline{\langle x, y\rangle} - \langle x, y\rangle\overline{\langle x, y\rangle} + \langle x, y\rangle\overline{\langle x, y\rangle}\langle y, y\rangle = \langle x, x\rangle - \langle x, y\rangle\overline{\langle x, y\rangle} = \|x\|^2 - |\langle x, y\rangle|^2$.

Theorem 2.0.15 *A closed convex subset* C *of a Hilbert space* $\mathcal{H}$ *contains a unit vector of the smallest norm.*

Proof: Let $d = \inf\{\|x\| : x \in C\}$ So there is a sequence $\{x_n\}$ of vectors in C such that $\|x_n\| \to d$. Since C is convex, $(x_n + x_m)/2 \in C$ and so $\|(x_n + x_m)/2\| \geq d$, i.e. $\|x_n + x_m\| \geq 2d$.

$$\begin{aligned}\|x_n + x_m\|^2 &= 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \\ &\leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0.\end{aligned}$$

shows that $\{x_n\}$ is Cauchy in C. and since $\mathcal{H}$ is complete,it converges to some point say x.
Now $\|x\| = \|\lim x_n\| = \lim\|x_n\| = d$ shows that x is the vector in with the smallest norm. We are now to prove this x is unique.If not, suppose there is another x' with $\|x'\| = d$. As C is convex, $(x + x')/2$ is also in C. By parallelogram law,

$$\|\frac{x + x'}{2}\|^2 = \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \frac{\|x - x'\|^2}{2} < \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} = d^2$$

. This is a contradiction to the definition of d.

Theorem 2.0.16 ***Parseval's Identity:*** *In a Hilbert space $\mathcal{H}$,for any two vectors $x, y \in \mathcal{H}$*
$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - \|x-iy\|^2.$
holds.This is easy to prove. Just we are to convert the right hand side by inner product.

2.1 Orthogonality in Hilbert Spaces:

Let $x, y \in \mathcal{H}$,are said to be orthogonal if $\langle x, \rangle y = 0$, and we write $x \perp y$. In Hilbert space, orthogonality is symmetric, but in general normed linear spaces it is not true. We will come to this later.
Theorem 3. Let M be closed linear subspace of $\mathcal{H}$ and let $x \in \mathcal{H} \setminus M$, and d be the distance from x to M. There exists a unique vector in M such that $\|x - y_0\| = d$.
Proof:
The set $C = x + M = \{x + m : m \in M\}$ is convex and d is the distance from origin to C. Then by the previous theorem, there is a unique vector $z_0 \in C$ with $\|z_0\| = d$. Set $y_0 = x - z_0 \in M$, moreover, $\|x - y_0\| = \|z_0\| = d$. Now if we prove, such y_0 is unique, we are through. If y_0 is not unique, then there is $y_1 \in M$ with $y_0 \neq y_1$ and $\|x - y_1\| = d$, then $z_1 = x - y_1 \in C$ with $z_1 = z_0$ and $\|z_1\| = d$, a contradiction to the fact that z_0 is unique.

Theorem 2.1.1 *Let $\mathcal{H}$ be a Hilbert space and x and y be any two elements in H, then the following statements are equivalent:*
1. $x \perp y$
2. $\|x + \alpha\, y\| \geq \|x\|, \quad \forall \alpha \in \boldsymbol{C}$
3. $\|x + \alpha y\| = \|x - \alpha y\|, \quad \forall \alpha \in \boldsymbol{C}$
4. $\|x + y\| = \|x - y\|$
5. $\|x + \alpha y\| = \|x + \beta y\|, \quad \forall \alpha, \beta \in \boldsymbol{C}$ *and* $|\alpha| = |\beta|$
6. $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$
7. $\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| = \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|$, *provided* $\|x\| \neq 0$ *and*$\|y\| \neq 0$.

Proof:

(1) $\Leftrightarrow$ (2)
Assume (1) holds.$\|x+\alpha y\|^2 - \|x\|^2 = \langle x+\alpha y, x+\alpha y\rangle - \langle x, x\rangle = |\alpha|^2\|y\|^2 \geq 0$. Now assume (2) holds.$x + \alpha y\|^2 - \|x\|^2 = \langle x + \alpha y, x + \alpha y\rangle - \langle x, x\rangle = |\alpha|^2\|y\|^2 + \alpha\overline{\langle x, y\rangle} + \bar{\alpha}\langle x, y\rangle$.Take $\alpha = \beta\langle x, y\rangle$, where β is any real and we get $\beta|\langle x, y\rangle|^2(\beta\|y\|^2 + 2) \geq 0$. Let $a = |\langle x, y\rangle|^2$and $b = \|y\|^2$.So we have $\beta a(b\beta + 2) = \|x + \alpha y\|^2 - \|x\|^2 \geq 0$. Suppose $a > 0$. So the last inequality fails for a negative realβ whose absolute value is sufficiently small. So $a = 0$ is the only choice, which proves our result.

(1) $\Leftrightarrow$ (3)
Assume (1).$\|x+\alpha y\|^2 - \|x-\alpha y\|^2 = 0$. This obviously follows through inner product.Now suppose (3)holds.In this case we only consider when α; is non-zero complex number,because $\alpha = 0$ is the trivial case. Then$\|x+\alpha y\|^2 - \|x-\alpha y\|^2 = 0 \Rightarrow \alpha\langle y, x\rangle + \bar{\alpha}\langle x, y\rangle = 0$.Which immediate implies $\langle x, y\rangle = 0 = \langle y, x\rangle$. From this our required result follows.
Results (4),(5) and(7) are the immediate consequence of the above result.And simple calculation using inner product will show (1) $\Leftrightarrow$ (6).

Definition 2.1.2 *By an orthonormal set in a Hilbert space, we mean a subset of $\mathcal{H}$ consisting of all mutually orthonormal unit vectors, i.e., if $\{e_i\}$ be a orthonormal set, then*
(1)$i \neq j \Rightarrow e_i \perp e_j$.
(2)$\|e_i\| = 1$.
If $\mathcal{H}$ contains only zero-vector, then $\mathcal{H}$ contains no orthonormal sets.

Theorem 2.1.3 *Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in $\mathcal{H}$, If $x \in \mathcal{H}$, then*

$$\sum_{i-1}^{n} |\langle x, e_i|^2 \leq \|x\|^2;$$

And moreover, $x - \sum_{i=1}^{n}\langle x, e_i\rangle \perp e_j$.

Proof: $0 \leq \|x - \sum_{i=1}^{n}\langle x, e_i\rangle\|^2 = \langle x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i, x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i\rangle = \langle x, x\rangle - \sum_{i=1}^{n}\langle x, e_i\rangle\overline{\langle x, e_i\rangle} - \sum_{j=1}^{n}\langle x, e_j\rangle\overline{\langle x, e_j\rangle} + \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, e_i\rangle\overline{\langle x, e_j\rangle}\langle e_i, e_j\rangle = \|x\|^2 - \sum_{i=1}^{n}|\langle x, e_i\rangle|^2$.
So one part is proved. For the other part we look at the following:
$\langle x - \sum_{i=1}^{n}\langle x, e_i\rangle e_i, e_j\rangle = \langle x, e_i\rangle - \sum_{i=1}^{n}\langle x, e_i\rangle\langle e_i, e_j\rangle = \langle x, e_j\rangle - \langle x, e_j\rangle = 0$.

Theorem 2.1.4 *If $\{e_i\}$ is an orthonormal set in $\mathcal{H}$, if $x \in \mathcal{H}$,then the set $S = \{e_i : \langle x, e_i\rangle \neq 0\}$ is either empty or countable.*

Proof: Consider the set, for each $n \in \mathbb{N}$,

$$S_n = \{e_i : |\langle x, e_i\rangle| > \|x\|^2/n\}$$

By the above inequality, S_n contains at most $n-1$ vectors. But

$$S = \cup_{n=1}^{\infty} S_n$$

And so S is countable.

Theorem 2.1.5 ***Bessel's Inequality:*** *If e_i is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$,*

$$\sum |\langle x, e_i|^2 \leq \|x\|^2;$$

Proof: By Theorem 5. either S is finite or countable. If S is finite, we can define $\sum |\langle x, e_i|^2 = \sum_{i=1}^{n} |\langle x, e_i|^2$, this problem reduces to the theorem 5. So we need to take care when S is countable.If $\sum_{i=1}^{\infty} |\langle x, e_i|^2$ converges, then every series obtained from this also converges with the same sum.Therefore, we can define $\sum |\langle x, e_i|^2$ to be $\sum_{i=1}^{\infty} |\langle x, e_i|^2$; That means our problem reduces to

$$\sum_{i=1}^{\infty} |\langle x, e_i|^2 \leq \|x\|^2;$$

Since no partial sum in the left hand side can exceed $\|x\|^2$, the inequality is in fact true.

The Conjugate Space of $\mathcal{H}$:

Let $y \in \mathcal{H}$ be fixed but arbitrary, and consider the function f_y defined on $\mathcal{H}$ by $f_y(x) = \langle x, \rangle y$. It is easy to see that f_y is linear.We now wish to compute the norm as

$$|f_y(x)| = |\langle x, y\rangle| \leq \|x\|\|y\|.$$

Which shows that

$$\|f_y\| \leq \|y\|.$$

If $y = 0$, then equality is attained. But if $y \neq 0$, then

$$\|f_y\| = \sup\{|f_y(x)\ | : \|x\| = 1\} \geq |f_y(\frac{y}{\|y\|})||\langle \frac{y}{\|y\|}, y\rangle| = \|y\|.$$

So we can say that $y \to f_y$ is a norm preserving mapping of $\mathcal{H}$ into $\mathcal{H}^*$. The next theorem is easy to prove.

Theorem 2.1.6 *Let $f \in \mathcal{H}$ be fixed but arbitrary.Then there exists a unique $y \in \mathcal{H}$ such that*

$$f(x) = \langle x, \rangle y, \forall x \in \mathcal{H}.$$

A short computation will reach one to the proof of the above theorem.

The Adjoint of an Operator

Let $\mathcal{H}$ be a Hilbert space. Let T be an operator on $\mathcal{H}$, then T gives rise to an operator T^* on $\mathcal{H}^*$, where T^* is defined by

$$(T^* f)x = f(Tx).$$

The mapping $T \to T^*$ is an isometrically isomorphism of $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H}^*)$. This map reserves products and preserves identity transformations. By the same argument,T^* gives rise to an operator T^{**} on $\mathcal{H}$. Since $\mathcal{H}$ is reflexive, $T^{**} = T$.

Definition 2.1.7 *Birkhoff Orthogonality: Let X be a normed space and for any two vectors x and y, x is said to be Birkhoff orthogonal to y if $\|x+\alpha y\| \geq \|x\|$, for all $\alpha \in \mathbb{C}$.*

The Birkhoff orthogonality is not symmetric, i.e. $x \perp_B y$ does not imply that $y \perp_x$, for all $x, y \in X$.
In the next chapter we will see the geometric structure of a Banach space in Birkhoff orthogonal sense.

Chapter 3

Reisz's Lemma and Orthogonality in Normed Linear Spaces:

Lemma 5.1. Let X be a normed linear space, $x \in X$, and let $f \in X^*$, then

$$|f(x)| = dist(x, \ker(f)) \cdot \|f\|$$

.

Proof: If $f(x) = 0$, then there is noting to prove. So assume that $f(x) \neq 0$. Let $v \in \ker(f)$. Then $\|x - v\| \cdot \|f\| \geq |f(x - v)| = \|f(x)\|$. This means that $dist(x, \ker(f)) \cdot \|f\| \geq |f(x)|$.
To see the other inequality, let $y \in X \setminus \ker(f)$. Then $y \neq 0$ and

$$\|y\| = |\frac{f(x)}{f(y)}| \cdot \|x - (x - (\frac{f(x)}{f(y)})y\| \geq |\frac{f(x)}{f(y)}| dist(x, \ker(f))$$

,

since $x - (\frac{f(x)}{f(y)})y \in \ker(f)$. Therefore $dist(x, \ker(f))|f(y)| \leq \|y\| \cdot |f(x)|$, for any $y \in X$ and hence the lemma follows.

Lemma 3.0.8 *Let X be a normed linear space and M be a proper closed subspace of X and $x \in X$, there is a $f \in X^*$ such that $\|f\| = 1, M \subset \ker(f)$ and $dist(x, M) = dist(x, \ker(f)) = f(x)$.*

Proof: If $x \in M$, then by Hahn-Banach theorem, there is $f \in X*$ such that $\|f\| = 1$ and $M \subset \ker(f)$. So $dist(x, M) = dist(x, \ker(f) = f(x) = 0)$. So assume that $x \in X \setminus M$. Then Hahn-Banach theorem implies that there is $g \in X^*$ such that $M \subset \ker(g), g(x) = 1$ and $dist(x, M) = \|g\|^{-1}$. Just put $f = \|g\|^{-1} \cdot g$. Therefore, $\|f\| = 1, M \subset \ker(f)$, and $dist(x, \ker(f)) = |f(x)| = f(x) = \|g\|^{-1} = dist(x, M)$, using lemma 1.1.

Proposition 3.0.9 *Let X be a normed space, M be a proper closed subspace of $X, M \neq 0$, and let $\epsilon \in (0,1)$. Then there is a pair $(x,f) \in X \times X^*$ such that $\|x\| = \|f\| = 1, M \subset \ker(f)$ and $f(x) = dist(x,M) = dist(x, \ker(f) = 1 - \epsilon$.*

Proof: Let $y \in X \setminus M$. Then $d \equiv dist(y,M) > 0$. There is a point $z \in M$ such that $d(1 + \frac{\epsilon}{1-\epsilon}) > \|y - z\| \geq d$. Put $x = \frac{(y-z)}{\|y-z\|}$. Therefore $\|x\| = 1$ and for any $w \in M, \|x - w\| \geq 1 - \epsilon$. And this means that $dist(x,M) \geq 1 - \epsilon$. Hence,lemma 1.2. there is a $f \in X^*$ with $\|f\| = 1, M \subset \ker(f)$ and $f(x) = dist(x,M) \geq 1 - \epsilon$. If $dist(x,M) = 1 - \epsilon, (x,f)$ is the desired pair.And if $dist(x,M) > 1 - \epsilon$, we take $x^{'} \in M \setminus 0$ and define $x_t = \frac{[tx + (1-t)x^{'}]}{\|[tx + (1-t)x^{'}]\|}$ for any $t \in [0,1]$. Since $f(x_0) = 0$ and $f(x_1) > 1 - \epsilon, f(x_t) = dist(x_t, M) = 1 - \epsilon$, for some $t \in (0,1)$.

Definition 3.0.10 *Let X be a normed space and M be a proper closed subspace of X. Then for a given point $x \in X$, an element $m \in M$ is said to be an element of best approximation of x if $\|x - m\| = dist(x,M)$.*

Proposition 3.0.11 *Let X be a normed space and M be a proper closed subspace of X and $x \in X \backslash M$. Then we have:*

(a) An element $m \in M$ is an element of best approximation of x if and only if $(x - m) \perp M$.

(b) Let $f \in X^*$.Then $X \perp \ker(f)$ if and only if $f \neq 0$ and $|f(x)| = \|f\| \cdot \|x\|$.

(c) The element is orthogonal to M if and only if there exists $g \in X^*$ such that $g \neq 0, g(x) = \|g\| \cdot \|x\|$ and $M \subset \ker(g)$.

Proof: Assertion (a) is clear from the definition of best approximation. Let $0 \neq f \in X^*$. Let $m \in M$ is a best approximation of x and $f \neq 0$, because $x \in \ker(f)$. Then using lemma 5.1, we get $x \perp \ker(f)$ if and only if $\|x\| = dist(x, \ker(f)) = \frac{|(x)|}{\|f\|}$. and hence the assertion (b) follows.
Now assume that $x \perp M$. Then, by lemma 5.2, there exists a $g \in X^*$ such that $g(x) = dist(x,M) = |g(x)| = \|x\|$. So (c) is proved. Conversely, suppose that there is a $g \in X^*$ with the conditions imposed in (c). So by $M \subset \ker(g)$ and $x \perp M$, it follows that $x \perp M$. To show (d) it is sufficient for the case when $x \neq 0$. Now if $f \in F(x)$ then $f(x) > 0, \|f\| = \|x\|$. Then for very $\alpha \in \mathbf{C}$ and every $y \in \ker(f), \|x\| = \|x\|^{-1} \cdot f(x + \alpha y) \leq \|x + \alpha y\|$.This shows that $x \perp \ker(f)$.
Conversely, assume that $f(x) \geq 0$, and $x \perp \ker(f)$. Then using (b) we get that $f(x) = \|f\| \cdot \|x\|$, and in addition,if $\|f\| = \|x\|$, then we are done.

Theorem 3.0.12 ***James Theorem:***
Let C be closed convex subset of a Banach space X. Then C is $w-$compact if and only if every $f \in X^*$ attains its supremum over C at some point of C.

Corollary 3.0.13 *A Banach space X is reflexive if and only if every $f \in X$ attains its norm.*

Theorem 3.0.14 *Let X be a Banach space. Then the following conditions are equivalent:*
$(a)X$ is reflexive.
(b) For every proper closed subspace M of x, there is an element $x \in X \setminus M$ such that $dist(x, M) = \|x\|$.
(c) For every proper closed subspace M of X there is an element $x \in X \setminus M$ such that $x \perp M$.

Proof: $(a) \Leftrightarrow (b)$: Assume X is reflexive and M be a proper closed subspace of X.Then there is an $f \in X^*$ with $\|f\| = 1$ and $M \subset \ker(f)$. Since X is reflexive, there is $x \in X$ such that $\|x\| = 1$ and $f(x) = 1$. So, for all $v \in M, \|x - v\| \geq f(x - v) = f(x) = 1$ and hence $dist(x, M) = 1$.
$(b) \Leftrightarrow (a)$: Assume (b) holds. Let $f \in X^*$ with $\|f\| = 1$ and we take $M = \ker(f)$. Then M is a closed subspace of co-dimension 1, and so by condition (b), there is an element $x \in X \setminus M$ with $\|x\| = 1$ and $distt(x, M) = 1$. Now applying lemma 5.1, we conclude that $|f(x)| = 1 = \|f\|$, i.e. every $f \in X^*$ achieves its norm .By James theorem, we conclude that X is reflexive.
Using lemma 5.4(a), it is clear that (b) and (c) are equivalent.
Note: We now proceed through an example to understand Theorem 5.6 clearly.
Let us consider the space **C[0,1]** with the *sup* norm. Let $\{x \in \mathbf{C[0,1]} : x(0) = 0\}$. Now define a linear functional f on X by $\int_0^1 x(t)dt = 0$, where the integral is taken in the sense of *Riemann*. Now $\ker(f) = \{x \in X : \int_0^1 x(t)dt = 0\}$. Here $\mathbf{C[0,1]} \cong X$. Then $\|f\| = \sup |f(x)| = 1$, but f does not attain its norm.Because if it does, then there is some $x_0 \in X$ with $\|x_0\| = 1$ such that $f(x_0) = 1$, contradiction to the fact that f is continuous.Which shows that 0 can never be a best approximation to any point on the unit sphere.

3.1 Generalization of Birkhoff Orthogonality:

For every non-reflexive Banach space, there is always a proper closed subspace such that no elements will be orthogonal to that subspace.So now it is interesting whether we can define the orthogonality in a general non-reflexive Banach space or not.With this motivation, we are like to introduce “asymptotic orthogonality” in such spaces. *Bishop – Phelps* orthogonality is one such, which is being introduced below:

Definition 3.1.1 *Let X be a normed linear space and M be a proper closed subspace of X. A sequence (x_n) in X with $\|x_n\| \equiv 1$ is said to be $(BP)-$ orthogonal to M if it satisfies the following conditions:*
(a)There is a sequence (f_n) in X^* with $f_n \in F(x_n)$ for each n.
(b)There is a functional $f \in X^*$ with $\|f\| = 1, M \subset \ker(f)$ and $\lim \|f_n - f\|$.
and we write $(x_n) \perp_{BP} M$.

Remark 3.1.2 *Let M be a proper closed subspace of X and for some $x \in X$ with $\|x\| = 1$, $x \perp_B M$. Then the constant sequence $(x_n) \perp_{BP} M$. Because, using the proposition 5.4(c), we get $g \in X^*$ with $\|g\| = 1, M \subset \ker(g)$ and $g(x) = 1$. Also $dist(x, M) = \|x\| = 1$. We can now apply lemma 5.1 because the constant sequence $g_n = g$ satisfies the conditions (a) and (b);and conditions (a) and (b) imply that $\lim g(x_n) = 1$ and $\lim dist(x_n, \ker(f)) = 1$. This shows that Birkhoff orthogonality is a special case of $BP-$ orthogonality.*

Definition 3.1.3 *Let X be a normed linear space.X is said to be $subreflexive$ if the set of all normed attaining functionals on X is dense in the dual of X. i.e., more precisely,for every $f \in X^*$, and for every $\epsilon > 0$, there is a $g \in X^*$ and $x \in S_X$ such that $|g(x)| = \|g\|$ and $\|f - g\| < \epsilon$.*

Example 3.1.4 *Every Banach space is $subreflexive\dot{B}ut$ a incomplete normed linear space may or may not be $subreflexive$. For example,every dense subspace of a $Hilbert$ space is $subreflexive$.*

Remark 3.1.5 *From the definition of $subreflexivity$ and $Hahn-Banach$ theorem, a consequence is given below:*

Proposition 3.1.6 *A normed linear space X is $subreflexive$ if and only if for every proper closed subspace M of X ,there is a sequence (x_n) of X with $\|x_n\| \equiv 1$ such that $(x_n) \perp_{BP} M$.*

Theorem 3.1.7 *A normed linear space X is $subreflexive$ if and only if for every proper closed subspace M of X, there is a sequence (x_n) of X with $\|x_n\| \equiv 1$ such that $(x_n) \perp_{BP} M$.*

Theorem 3.1.8 *Let X be an arbitrary Banach space.Then for every proper closed subspace M of X, there is a sequence (x_n) in X which is $BP-$ orthogonal to M.*

This is nothing but an easy consequence of Bishop-Phelps theorem.

Lemma 3.1.9 *Let $f \in X^*, \|f\| = 1$ and let (x_n) be a sequence in X with $\|x_n\| \equiv 1$, Then $\lim f(x_n) = 1 \Leftrightarrow w^*(x_n) \subset F^*(f)$, where F^* denotes the duality mapping of X^*.*

Proof: Assume $\lim(x_n) = 1$, and let $\lambda \in w^*(x_n)$, there is a sub-sequence $(x_{n(k)})$ such that $1 = \lim f(x_{n(k)}) = \lambda(f)$. But $\lambda(f) \leq 1$, and so we have, $1 = \lambda(f) \leq \|\lambda\|\|f\| \leq 1$. This means $\lambda \in F^*(f)$. Conversely, assume that $w^*(x_n) \subset F^*(f)$ and $\lim f(x_n) \neq 1$. So there is a subsequence $(x_{n(k)})$ and $\epsilon \in (0, 1)$ such that $f(x_{n(k)}) \leq 1 - \epsilon, \forall k \geq 1$. Let $\lambda w^*(x_{n(k)})$.Then $\lambda(f) \leq 1 - \epsilon$. So $\lambda \in w^*(x_n) \subset F^*(f)$. Therefore $\lambda(f) = 1$. This is a contradiction.

Lemma 3.1.10 *Let $f \in X^*, \|f\| = 1$, and F^* be the duality mapping of X^*. If $\lambda \in F^*(f)$, then $\lambda \perp \ker(f)$, where $\ker(f)$ is understood to be the subspace of X^{**} via the natural embedding of X into X^{**}.*

Proof: Denote $\kappa^* : X^* \to X^{***}$ be a natural embedding and $F^{**} : X^{**} \to X^{****}$ be the duality mapping.Then $\lambda \in F^*(f) \Leftrightarrow \tilde{f} \in F^{**}(\lambda)$,where $\tilde{f} = \kappa^* f$. Hence using proposition 5.3., we can conclude that $\lambda \perp \ker(\tilde{f})$, whenever $\lambda \in F^*(f)$. Therefore, $\lambda \in F^*(f)$ and $\ker(f) \subset \ker(\tilde{f}) \Rightarrow \lambda \perp \ker(f)$ in X^{**}.

3.2 Non-Reflexive Banach Space and BP-Orthogonality

Theorem 3.2.1 *Let X be a Banach space.The followings are equivalent:*

(a) X is non-reflexive.

(b) There exists $f \in X^*$ which does not achieve its norm.

(c) There exists a proper closed subspace M such that none of the elements of X is orthogonal to M

The above theorem is just the restatement of the theorem 5.6.

Theorem 3.2.2 *A Banach space X is non-reflexive if and only if there is a proper closed subspace M of X with co-dimension 1 such that any $BP-$ orthogonal sequence to M does not converge weakly to an element of X.*

Proof: Suppose X be reflexive and M be an arbitrary subspace of X. Let (x_n) be any sequence in x such that $(x_n) \perp_{BP} M$ and let $(f_n) \subset X^*$ be a sequence and $f \in X^*$ satisfies the conditions necessary to define $BP - orthogonality$. Since X is reflexive, then (x_n) contains a subsequence x_{n_k} which converges weakly to some element, say $x \in X$. Therefore we have, for each $k, |f(x)-1| = |f(x)-f_{n_k}(x_{n_k})| \leq f(x-x_{n_k})|+\|f-f_{n_k}\| \to 0$ as $k \to \infty$ and $\|x\| \leq 1$.This proves the "if" part. To prove the "only if" part, let X is non-reflexive and every proper closed subspace M of X admits a $BP - orthogonal$ sequence which weakly converges to some point of X, say,x. Moreover, let $f \in X^*$ and a sequence (f_n)satisfying the conditions necessary to define $BP - orthogonality$.Then surely $M = \ker(f)$, since $M \subset \ker(f)$ and co-dimension of M is 1. Hence by lemma 5.1 with the conditions to define $BP - orthogonality$ imply $dist(x_n, M) \leq f(x_n) \leq 1$. Thus $f(x) = \lim f(x_n)$ and proposition 5.3 ensures that $x \perp M$.Now using theorem 5.6, we can conclude that X is reflexive, a contradiction.

Definition 3.2.3 *Let X be Banach space and M be a proper closed subspace of X. For the subspace M, we define two subspaces $L(M)$ and $R(M)$ by*

$$L(M) = \{x \in X : x \perp v, \forall v \in M\}$$

$$R(M) = \{x \in X : v \perp x, \forall v \in M\}$$

We say that M is $L-$complemented in X if $L(M)$ is a closed linear subspace of X and

$$X = L(M) \oplus M$$

Similarly M is said to be $R-$complemented in X if $R(M)$ is a closed linear subspace of X and

$$X = M \oplus R(M)$$

Chapter 4

Almost Constrained subspaces of Banach spaces

First we will recall certain definitions.

Definition 4.0.4 *Let Y be a subspace of a Banach space X.*

(a) For $y^* \in Y^*$, $HB(y^*) = \{x^* \in X^* : x^*|_Y = y^* \text{ and } \|x^*\| = \|y^*\|\}$

(b) Y ia U-subspace of X if for any $y^* \in Y^*$, $HB(y^*)$ is singleton.
X is said to be Hahn- Banach smooth if X is a U-subspace of X^{**}

(c) The duality mapping $\mathcal{D}$ for X is the set-valued map from S_X to S_{X^*} defined by

$$\mathcal{D}_X(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}, \text{ for all } x \in S_X$$

(d) $x \in S_X$ is called a smooth point of B_X if $\mathcal{D}_X(x)$ is singleton.

Definition 4.0.5 ***Constrained subspace:*** *Let X be a Banach space. A subspace Y (which has to be closed in norm sense) of X is said to be constrained subspace of X if there is a norm 1 projection on X with the range Y.*

Example 4.0.6 *Let $X = l^1$, the space of all absolute summable sequences and $Y = span\{e_1, e_2, ..., e_n\}$, where $e_i = (0, 0, ..., 1, 0, 0, ...), 1$ is in the i^{th} position. Now the map*

$$P : X \to Y$$

defined by

$$P(x) = (x_1, x_2, ..., x_n, 0, 0, ...),$$

where $x = (x_n) \in X$. We see that P is linear and $P(P(x_n)) = P(x_1, x_2, x_3, ...x_n, 0, 0, 0, ...) = (x_1, x_2, x_3, ..., x_n, 0, 0, 0, ...) = P(x_n)$. Which *shows that* $P^2 = P$. *Also* P *is continuous. So* P *is a projection with range* Y. *Now* $\|P\| = \sup\{|Px| : x \in S_X\} = \sup\{|(x_1, x_2, x_3, ..., x_n, 0, 0, 0, ...)| : x = (x_n) \in S_X\}$. *Notice that* $\|(x_1, x_2, x_3, ...)\|_1 \geq \|(x_1, x_2, x_3, ..., x_n, 0, 0, 0, ...)\|_1$. *So* $\|P\| \leq 1$. *As norm of a projection is always greater than or equal to 1, it immediate shows that it's a norm 1 projection.*

Definition 4.0.7 ***Almost Constrained subspace:*** *A closed subspace* Y *of a Banach space* X *is said to be almost constrained if any family of closed balls centered at* Y, *if they intersects in* X, *they also intersect in* Y.

Definition 4.0.8 ***Finite-Infinite-Intersection Property:*** *A Banach space* X *is said to have finite-infinite -intersection property (or in symbol* $IP_{f,\infty}$*) if any family* $\{F_\alpha\}_{\alpha \in \Lambda}$ *of closed balls in* X *with* $\cap_{\alpha \in \Lambda} F\alpha = \Phi$, *then there exists* $\alpha_1, \alpha_2, ..., \alpha_n$ *such that* $\cap_{i=1}^{n} F_{\alpha_i} = \Phi$.
Dual space of every Banach space and their constrained subspaces have $IP_{f,\infty}$.

Note 4.0.9 *Through the thesis we will use* X *is a real Banach space and* Y *is closed subspace of* X. *If somewhere I mention simply* X *and* Y, *it is to be understood that* Y *is closed subspace of the Banach space* X.

Definition 4.0.10 *Let* Y *be a subspace of* X. *Let* $x \in X$ *and* $y^* \in Y^*$. *We define*

$$U(x, y^*) = \inf\{y^*(y) + \|x - y\| : y \in Y\}$$
$$L(x, y^*) = \sup\{y^*(y) - \|x - y\| : y \in Y\}.$$

Lemma 4.0.11 *Let* Y *be a normed closed subspace of a Banach space* X. *Suppose* $x_0 \in X \setminus Y$ *and* $y^* \in Y^*$ *with* $\|y^*\| = 1$. *Then the followings are equivalent:*

(a) $L(x_0, y^*) \leq U(x_0, y^*)$ *and* α *is a real number such that* $L(x_0, y^*) \leq \alpha \leq U(x_0, y^*)$.

(b) There is a Hahn-Banach extension x^* *of* y^* *such that* $x^*(x_0) = \alpha$.

Proof: Assume (a).
Claim: there is a Hahn-Banach extension x^* of y^* such that $x^*(x_0) = \alpha$. By definition,

$$L(x_0, y^*) = \sup\{y^*(y) - \|x_0 - y\| : y \in Y\}. \quad (4.1)$$
$$U(x_0, y^*) = \inf\{y^*(y) + \|x_0 - y\| : y \in Y\}. \quad (4.2)$$
$$= \inf\{-y^*(y) + \|x_0 + y\| : y \in Y\}.\text{(replacing } y \text{ by } -y). \quad (4.3)$$

By given condition,

$$L(x_0, y^*) \leq \alpha \leq U(x_0, y^*)$$

i.e,

$$\sup\{y^*(y) - \|x_0 - y\| : y \in Y\} \leq \alpha \leq \inf\{-y^*(y) + \|x_0 + y\| : y \in Y\}$$

or,

$$y^*(y) - \alpha \leq \|x_0 - y\|$$
$$y^*(y) + \alpha \leq \|x_0 + y\|$$

for all $y \in Y$. Multiply both by any real number $c \in \mathbb{R}$, and take the second one which will become

$$y^*(y) + \alpha c \leq \|\alpha x_0 + y\|$$

(replacing y by αy).
Note: Here we can take the first one also according to the sign of c.
Define a function x^* on the space $X_1 = \text{span}\{Y \cup \{x_0\}\}$ by $x^*(y + cx_0) = y^*(y) + \alpha c$. This is linear and continuous function on X_1, which is an extension of y^* and also $x^*(x_0) = \alpha$. And

$$\|x^*\| = \sup\{x^*(x) : x \in S_{X_1}\} \tag{4.4}$$
$$\geq \sup\{x^*(x) : x \in S_Y\} \tag{4.5}$$
$$= \|y^*\| = 1. \tag{4.6}$$

Also

$$\|x^*\| = \sup\{\|x^*(y + dx_0)\| : y + dx_0 \in S_{X_1}, d \in \mathbb{R}\} \tag{4.7}$$
$$= \sup\{\|y^*(y) + d\alpha\| : y + dx_0 \in S_{X_1}, d \in \mathbb{R}\} \tag{4.8}$$
$$= \sup\{|d|\|y^*(y/d) + \alpha| : y + dx_0 \in S_{X_1}, d \in \mathbb{R}\} \tag{4.9}$$

If $X_1 = Y$, we are done. If this is not, then we can increase the space by one dimension and using Zorn's lemma we will get the Hahn-Banach extension, with $x^*(x_0) = \alpha$.

Remark 4.0.12 *From above lemma, it is clear that for any $x^* \in X_1^*$ and $x \in X$,*

$$L(x, x^*) \leq x^*(x) \leq U(x, x^*)$$

Also every $y^* \in Y_1^*$ has a unique Hahn-Banach extension to X if and only if for all $x \in X$

$$L(x, x^*) = U(x, x^*)$$

Lemma 4.0.13 *Let Y be a subspace of a Banach space and $x_1, x_2 \in X$. Then the followings are equivalent:*

(a) $x_2 \in \cap_{y \in Y} B_X[y, \|x_1 - y\|]$

(b) for all $x^* \in X^*, U(x_2, x^*) \leq U(x_1, x^*)$.

Proof: Assume (a). So $\|x_2 - y\| \leq \|x_1 - y\|, \forall y \in Y$. Or, $\forall y \in Y, x^*(y) + \|x_2 - y\| \leq x^*(y) + \|x_1 - y\| \implies \inf\{x^*(y) + \|x_2 - y\| : y \in Y\} \leq x^*(y) + \|x_1 - y\|$, that is $U(x_2, x^*) \leq x^*(y) + \|x_1 - y\| \implies U(x_2, x^*) \leq \inf\{x^*(y) + \|x_1 - y\| : y \in Y\}$, that is $U(x_2, x^*) \leq U(x_1, x^*)$.
Now assume (b). Suppose (a) is not true. Then $\exists\ y_0 \in Y$ such that $\|x_2 - y_0\| > \|x_1 - y_0\|$. Then $\exists\ \epsilon$ such that $\|x_2 - y_0\| - \epsilon \geq \|x_1 - y_0\|$. Choose $x^* \in X_1^*$ such that

$$\|x_2 - y_0\| \ < x^*(x_2 - y_0) + \epsilon/2$$

(Using the definition of supremum)
or,

$$\|x_1 - y_0\| \leq \|x_2 - y_0\| - \epsilon < x^*(x_2 - y0) - \epsilon/2.$$

Thus

$$U(x_1, x^*) \leq x^*(y_0) + \|x_1 - y_0\| < x^*(x_2) - \epsilon < U(x_2, x^*),$$

a contradiction.

Lemma 4.0.14 *Let Y be a subspace of a Banach space X. For $x_1, x_2 \in X \setminus Y$ and $x^* \in X^*, U(x_1, x^*) - U(x_2, x^*) \leq U(x_1 - x_2, x^*)$.*

Proof: Let $x^* \in X^*$ and $y_1, y_2 \in Y$. Then
$U(x_1, x^*) \leq x^*(y_1 + y_2) + \|x_1 - y_1 - y_2\| = x^*(y_1) + x^*(y_2) + \|x_2 - y_2 + x_1 - x_2 - x_1\| \leq x^*(y_1) + \|(x_1 + x_2) - y_1\| + x^*(y_2) + \|x_2 - y_2\|$.
This happens for all $y_1, y_2 \in Y$. So $U(x_1, x^*) \leq U(x_1 - x_2, x^*) + U(x_2, x^*)$, which follows the proof.

Definition 4.0.15 *Let X be a Banach space and $x \in X$. We define*

(a) $C(x) := \{x^* \in X_1^* : U(x, x^*) = L(x, x^*)\}$

(b) $C := \cap_{x \in X} C(x)$

Proposition 4.0.16 *Let Y be a subspace of a Banach space X. Let $x^* \in X_1^*$ and $x_0 \in X \setminus Y$, then the followings are equivalent:*

(a) $x^* \in C(x_0)$

(b) $\|x^*|_Y\| = 1$ *and every* $x^* \in HB(x^*|_Y)$ *takes the same value at* x_0.

(c) $\|x^*|_Y\| = 1$ *and if* $\{x_\alpha\} \subseteq X_1^*$ *is a net such that* $x_\alpha^*|_Y \to x^*|_Y$ *in the weak*$^*-$ *topology of* Y^*, *then* $\lim_\alpha x_\alpha^*(x_0) = x^*(x_0)$

(d) $\|x^*|_Y\| = 1$ *and if* $\{x_n\}$ *is a sequence such that* $x^*|_Y \to x^*|Y$ *in the weak*$^*-$ *topology of* Y^*, *then* $\lim_n x_n^*(x_0) = x^*(x_0)$

Proof: The theorem will be proved by going $(a) \iff (b), (b) \implies (c), (b) \implies (d)$ and finally $(d) \implies (b)$.
$(a) \iff (b)$: First assume (a). That is $x^* \in C(x_0)$, which implies that $L(x_0, x^*) = U(x_0, x^*)$. Let $\|x^*|_Y\| = \alpha$. Clearly $\alpha \leq \|x^*\| \leq 1$. We will prove that $\alpha = 1$. Let $x_1^* \in HB(x^*|_Y)$. Then $\|x^*\| = \alpha$ and
$|x_1^*(x_0 - y)| \leq \alpha\|x_0 - y\| \leq \|x_0 - y\|$, for all $y \in Y$ $--------(1)$
From (1), we can write
$-\|x_0 - y\| \leq -\alpha\|x_0 - y\| \leq x_1^*(x_0 - y)| \leq \alpha\|x_0 - y\| \leq \|x_0 - y\|$, for all $y \in Y$
Which can be written into two inequalities as:
$x^*(y) - \|x_0 - y\| \leq x^*(y) - \alpha\|x_0 - y\| \leq x^*(x_0)$, for all $y \in Y$ $---------(2)$
$x_1^*(x_0) \leq x^*(y) + \alpha\|x_) - y\| \leq x(y) + \|x_0 + y\|$, for all $y \in Y$ $---------(3)$
Note that here we are using $x_1^*(y) = x^*(y)$ as x_1^* is a Hahn-Banach extension of $x^*|_Y$.
From (2) we get
$L(x_0, x^*) \leq \sup\{x^*(y) - \alpha\|x_0 - y\| : y \in Y\} \leq x_1(x_0)$
and from (3) we get
$x_1^*(x_0) \leq \inf\{x^*(y) + \alpha\|x_0 - y\| : y \in Y\} \leq U(x_0, x^*)$.
So
$L(x_0, x^*) \leq x_1^*(x_0) \leq U(x_0, x^*)$.
By our assumption, equality should hold everywhere.
Therefore $x_1^*(x_0) =$ some constant.
And this is true for any $x_1^* \in HB(x^*|_Y)$.
The only remaining thing to prove is that $\alpha = 1$. By the contrary assume $\alpha < 1$. Let $0 < \delta < d(x_0, Y)$ and $0 < \epsilon < (1 - \alpha)\delta$.
Then $(1 - \alpha)\|x_0 - y\| > \epsilon$ for all $y \in Y$.
So $y^*(y) - \|x_0 - y\| + \epsilon < y^*(y) - \alpha\|x_0 - y\|$ which implies that the inequality in (2) must be strict and that will be a contradiction to the fact that $L(x_0, x^*) = U(x_0, x^*)$. This shows $\alpha \geq 1$ and hence $\alpha = 1$.
The converse can be proved just going by reverse.
$(b) \implies (c)$: Let $\{x_\alpha^*\} \subseteq X_1^*$ be a net such that $\lim_\alpha x_\alpha^*(y) = x^*(y)$, for all $y \in Y$. By Banach-Alaoglu theorem this net has a convergent subnet, let's say $\{x_{\alpha_\beta}^*\}$ be such that $x_{\alpha_\beta}^* \to x^*$ the weak-star topology of X^*. This shows that weak-star cluster point of $\{x_\alpha^*\}$ i.e. $x^* \in HB(x^*|_Y)$. By (b), $\lim_\alpha x_\alpha^*(x_0) = x^*(x_0)$.
$(c) \implies (d)$: Same proof as above.
$(d) \implies (b)$: On the contrary, let us assume there is a $x_1^* \in HB(x^*|_Y)$ such that $x^*(x_0) \neq x_1^*(x_0)$. Take the constant sequence $(x_n^*) = (x_1^*)$. Then

$\lim_n x_n^*(y) = x_1^*(y) = x^*(y)$, for all $y \in Y$. But $\{x_1^*(x_0)\}$ can't converge to $x^*(x_0)$, a contradiction.

Proposition 4.0.17 *Let Y be a subspace of a Banach space X. Let $x^* \in X_1^*$ then the followings are equivalent:*

(a) $x^* \in C = \cap_{x \in X} C(x)$

(b) $\|x^*|_Y\| = 1$ *and every* $x^* \in HB(x^*|_Y) = \{x\}$.

(c) $\|x^*|_Y\| = 1$ *and if* $\{x_\alpha\} \subseteq X_1^*$ *is a net such that* $x_\alpha^*|_Y \to x^*|_Y$ *in the weak*$^*-$ *topology of* Y^*, *then* $x_\alpha \to x^*$ *in the weak-star topology of* X^*.

(d) $\|x^*|_Y\| = 1$ *and if* $\{x_n\} \subseteq X_1^*$ *is a net such that* $x_n^*|_Y \to x^*|_Y$ *in the weak*$^*-$ *topology of* Y^*, *then* $x_n^* \to x^*$ *in the weak-star topology of* X^*.

Proof: Same proof followed by the above proposition.

4.1 Some Characterizations:

Definition 4.1.1 *Let Y be a subspace of a Banach space X.*
For any $x \in X$, $\mathfrak{P}(x) = \cap_{y \in Y} B_Y[y, \|x-y\|]$.

It is obvious that if $y \in Y$, $\mathfrak{P}(y) = \{y\}$.
Also note that Y is an AC-subspace of X if and only if $\mathfrak{P}(x) \neq \Phi$, for all $x \in X$.
Note:

(a) $\mathfrak{P}(\lambda x) = \lambda \mathfrak{P}(x)$, for all $x \in X, \lambda \in \mathbb{R}$

(b) $\mathfrak{P}(x+y) = \mathfrak{P}(x) + y$, for all $x \in X, y \in Y$.

Proof of (a): $\mathfrak{P}(\lambda x) = \cap_{y \in Y} B_Y[y, \|\lambda x - y\|]$.
Let $z \in \mathfrak{P}(\lambda x) \implies z \in B_Y[y, \|\lambda x - y\|]$, for all $y \in Y \implies \|z-y\| \leq \|\lambda x - y\|$, for all $y \in Y - - - - - - -(1)$.
claim: $z \in \lambda \mathfrak{P}(x) = \lambda \cap_{y \in Y} B_Y[y, \|x-y\|]$
i.e. $z \in \lambda B_Y[y, \|x-y\|]$, for all $y \in Y$
i.e. $z \in B_Y[y, \lambda\|x-y\|]$, for all $y \in Y$.
i.e. $\|z-y\| \leq \lambda\|x-y\| = \|\lambda x - \lambda y\|$, for all $y \in Y$.
Therefore, $\|z-y\| \leq \|\lambda x - y\| \leq \|\lambda x - \lambda y\|$, (Using (1))
And this is if and only if condition. Therefore, the proof is clear.
Proof of (b): Let $z \in \mathfrak{P}(x+y) = \cap_{u \in Y} B_Y[u, \|x+y-u\|]$
$\implies z \in B_Y[u, \|x+y-u\|]$, for all $u \in Y$
$\implies \|z-y\| \leq \|x+y-u\|$, for all $u \in Y - - - - - - - - - - - - - -(2)$.
Claim: $z \in \mathfrak{P}(x) + y = \cap_{u \in Y} B_Y[u, \|x-u\|] + y$
$\implies z \in B_Y[u, \|x-u\|] + y$, for all $u \in Y$,
i.e. $\|z-(u+y)\| \leq \|x-y\|$, for all $u \in Y$.

In (2), just replace u by $u + y$, we'll get the result.

Definition 4.1.2 *Let Y be a closed subspace of a Banach space X. The ortho-complement [Notation: $O(Y, X)$] of Y in X is defined as:*

$$O(Y, X) := \{x \in X : \|x - y\| \geq \|y\|, \forall y \in Y\}$$

which is same as:

$$O(Y, X) := \{x \in X : Y \perp_B x\}$$

where the orthogonality is taken in the sense of Birkhoff.

4.2 Characterization of AC-subspaces:

Proposition 4.2.1 *Let Y be a closed subspace of a Banach space X. Then the following statements are equivalent:*

(a) Y is an AC-subspace of X.

(b) For all $x \in X$, there is a $y \in Y$ and $z \in O(Y, X)$, such that $x = y + z$

(c) For every subspace Z with $Y \subseteq Z \subseteq X$ and $\dim(Z/Y) = 1$, Y is constrained in Z.

Proof:
$(a) \implies (b)$: Assume (a).
Let $x_0 \in X$. Then $\exists\ y_0 \in \mathfrak{P}(x_0) = \cap_{y \in Y}[y, \|x_0 - y\|]$
i.e. $y_0 \in B_Y[y, \|x_0 - y\|]$, for all $y \in Y$.
$\|y_0 - y\| \leq \|x_0 - y\|$, for all $y \in Y$
Put $u = y_0 - y$, i.e. $y = y_0 - u$,
We get
$\|u\| \leq \|x_0 - y_0 + u\|$, for all $u \in Y$
$\implies Y \perp_B (x_0 - y_0)$.
Take $z_0 = x_0 - y_0$,
Then $x_0 - y_0 = z_0 \in O(Y, X)$ and $x_0 = y_0 + (x_0 - y_0) = y_0 + z_0$.
$(b) \implies (c)$: Let $x_0 \in X$ and Z as in (c). Any Z of the above type can be written as $Z = \overline{span}[Y \cup \{x_0\}]$
By (b) there is $y_0 \in Y$ and $z_0 \in O(Y, X)$ such that $x_0 = y_0 + z_0$.
Therefore it follows that $Z = Y \oplus \mathbb{R}z_0$.
Define a map

$$P : Z \to Y$$

by

$$P(\alpha z_0 + y) = y$$

for all $y \in Y$ *and for all* $\alpha \in \mathbb{R}$.
Obviously P *is continuous and*

$$P^2(\alpha z_0 + y) = P(y) = y = P(\alpha z_0 + y)$$

for all $y \in Y$ *and for all* $\alpha \in \mathbb{R}$.
Also

$$\|P(z)\| = \|P(\alpha z_0 + y)\| = \|y\| \leq \|\alpha z_0 + y\|$$

$$\implies \|P\| = \sup\{\|P(z)\| : z \in S_Z\} \leq 1.$$

Therefore P *is a norm 1 projection on* Z *with the range* Y. *This shows that* Y *is constrained in* Z *by* P.
$(c) \implies (a)$: *By* (c), *for every* $x \in X$, *there is a norm 1 projection from* $Z_x = \overline{span}[Y \cup \{x\}]$ *onto* Y. *Let* P_x *be the projection by which* Y *is constrained in* Z_x. *Then*

$$\|P_x\| = 1$$

and

$$\|P_x(x - y)\| \leq \|x - y\|,$$

for all $y \in Y$.

$$\implies \|P_x(x) - y\| \leq \|x - y\|,$$

for all $y \in Y$.

$$\implies P_x(x) \in B_Y[y, \|x - y\|],$$

for all $y \in Y$.

$$\implies P_x(x) \in \cap_{y \in Y} B_Y[y, \|x - y\|] = \mathfrak{P}(x).$$

Therefore Y *is an* AC*-subspace of* X.

• ***Sufficient Conditions for an AC-subspace to be Constrained by a Unique Norm 1 Projection:***

Corollary 4.2.2 *A subspace* Y *of a Banach space* X *is an* AC*-subspace if and only if there exists a (not necessarily linear) onto map*

$$P : X \to Y$$

such that

(a) $P^2 = P$

(b) for all $x \in X, \lambda \in \mathbb{R},\ P(\lambda x) = \lambda P(x)$

(c) *for all* $x \in X, y \in Y$ $P(x+y) = P(x) + y$

(d) *for all* $x \in X$, $\|P(x)\| \leq \|x\|$

Proof: *If we take* P *is as above. Then for any* $x \in X$,

$$P(x) \in \mathfrak{P}(x)$$

This proves that Y *is an* AC*-subspace of* X.
Conversely, let Y *be an* AC*-subspace of* X. *For any* $z \in O(Y, X)$, *define* $Y_z = Y \oplus \mathbb{R}z$ *and*

$$P_z : Y_z \to Y$$

be a norm 1 projection with the range Y. *We ca see that for any* $z_1, z_2 \in O(Y, X)$
either $Y_{z_1} = Y_{z_2}$ *or* $Y_{z_1} \cap Y_{z_2}$.
By the above proposition

$$\cup_{z \in O(Y,X)} Y_z = X.$$

Now define a map

$$P : X \to Y$$

by $P(x) = P_z(x)$ *for* $x \in Y_z$.
Then the map P *is well-defined and satisfies all the above properties.*

4.3 Some Sufficient Conditions for an AC-Subspace to Be Constrained:

Let Y be a closed subspace of a Banach space X. Then the following statements are equivalent:

(a) Y is AC-subspace of X and $O(Y,X)$ is a closed subspace of X.

(b) Y is AC-subspace of X and $O(Y,X)$ is a linear subspace of X.

(c) Y is constrained in X by a norm 1 projection and for all $x \in X, \mathfrak{P}(x)$ is singleton. Moreover that projection is unique.

Proof: $(a) \implies (b)$ is trivial.
$(b) \implies (c)$: Since Y is an AC-subspace of X. Then every $x \in X \setminus Y$, can be written as $x = y + z$, for some $y \in Y$ and $z \in O(Y,X)$, by the above proposition. Since Y and $O(Y,X)$ are linear subspaces and $Y \cap O(Y,X) = \{0\}$, the representation of x as above is unique.
Define a map

$$P : X \to Y$$

by

$$P(x) = y$$

for every $x \in X$.
We'll show that this is a norm 1 projection from X onto Y and also this is unique.

$$\|P(x)\| = \|y\| \leq \|y - x - y\| = \|x\|$$
$$\implies \|P\| \leq 1.$$

Also this is continuous and $P^2 = P$.
This shows that Y is a constrained subspace of X by the norm 1 projection P.
Moreover, since for every $x \in X, \exists!\ y \in Y$ and $z \in O(Y,X)$ such that $x = y + z$. Since for every $x \in X, y(= P(x)) \in \mathfrak{P}(x)$ i.e. $\mathfrak{P}(x)$ is single-valued.
$(c) \implies (a)$: Let Y be constrained in X by a norm 1 projection P and for all $x \in X$, let $\mathfrak{P}(x)$ is singleton. So Y is an AC-subspace of X and for all $x \in X$,

$$\mathfrak{P}(x) = \{P(x)\}$$

The only thing we have to prove that $O(Y,X)$ is a closed subspace of X.
We are claiming that

$$O(Y,X) = \ker(P)$$

Let $x \in \ker(P)$. Then

$$\|x - y\| \geq \|P(x - y)\| = \|P(x) - P(y)\| = \|y\|$$

for all $y \in Y$.
This implies that

$$x \in O(Y, X)$$

Therefore

$$\ker(P) \subseteq O(Y, X)$$

Let $x \in O(Y, X)$, Then for all $y \in Y$

$$\|x - y\| \geq \|y\|$$

$$\implies 0 \in \mathfrak{P}(x)$$

Also for all $y \in Y$

$$\|x - y\| \geq \|P(x - y)\| = \|P(x) - y\|$$

$$\implies P(x) \in \mathfrak{P}(x).$$

And as we know for every

$$x \in X,\ \mathfrak{P}(x) = \{P(x)\}.$$

This shows that for all $x \in X$

$$P(x) = 0$$

This proves that

$$x \in \ker(P).$$

Therefore

$$O(Y, X) \subseteq \ker(P)$$

Hence

$$O(Y, X) = \ker(P)$$

Therefore $O(Y, X)$ is a closed subspace of X.

Proposition 4.3.1 *Let Y be a subspace of X. Let $x_1, x_2 \in X$ be such that $x_1 \in \cap y \in Y B_X[y, \|x_2 - y\|]$, then for any $x^* \in C(x_2)$, $x^*(x_1 - x_2) = 0$.*

Proof: We have $x_1, x_2 \ X$ with $x_1 \in \cap y \in YB_X[y, \|x_2 - y\|]$, then by the lemma 4.0.13 for all $x^* \in X_1^*$

$$L(x_2, x^*) \leq L(x_1, x^*) \leq U(x_1, x^*) \leq L(x_2, x^*) - - - - - - - - - - -(1)$$

Now

$$x^* \in C(x_2) \implies L(x_2, x^*) = U(x_2, x^*)$$

So for any $x^* \in C(x_2)$, equality should hold everywhere in (1).
Therefore, for any $x^* \in C(x_2)$,

$$L(x_1, x^*) = U(x_1, x^*) = x^*(x_1)$$

$$L(x_2, x^*) = U(x_2, x^*) = x^*(x_2)$$

But (1) should be equality. Therefore for any $x^* \in C(x_2)$,

$$x^*(x_1) = x^*(x_2)$$

$$\implies x^*(x_1 - x_2) = 0.$$

Corollary 4.3.2 *Let Y be a subspace of a Banach space X. If $x_1, x_2 \in X$ be such that for all $y \in Y$, $\|x_1 - y\| \leq \|x_2 - y\|$, then for all $y \in S_Y$, that is a smooth point of X, $\mathcal{D}_X(y) = \{x\}$, we have $x^*(x_1 - x_2) = 0$.*

Proof: Let $y \in Y$ be a smooth point of of X. Then $\exists! x^* \in X_1^*$ such that

$$x^*(y) = \|x^*\| = 1$$

Then by Hahn-Banach theorem, there is a functional y^* such that

$$y^*(y) = \|y\| = 1.$$

Therefore for all $y \in Y$

$$x^*(y) = \|x^*\| = 1 = \|y\| = y^*(y)$$

So x^* is norm attaining at y which is a Hahn-Banach extension of y^*. But x^* is unique, since y is a smooth point of X.
This shows that $HB(y^*) = \{x^*\}$
By the proposition 4.0.20

$$x^* \in C = \cap_{x \in X} C(x)$$

$$x^* \in C(x_2)$$

Now by the proposition 4.3.1

$$x^*(x_1 - x_2) = 0.$$

Definition 4.3.3 *Let Y be a closed subspace of a Banach space X. Then $NA(Y) := \{y^* \in Y^* : \exists\ y \in B_Y$ such that $y^*(y) = \|y^*\|\}$*

Definition 4.3.4 *Let Y be a closed subspace of a Banach space X. Y is said to be weakly U-subspace of X if for all $y \in Y,\ HB(y^*)$ is singleton.*

Definition 4.3.5 *Let X be Banach space. A set $A \subseteq B_X^*$ is said to separate points of X if for all $x_1, x_2 \in X$ with $x_1 \neq x_2,\ \exists\ x^* \in A$ such that $x^*(x_1) \neq x^*(x_2)$.*

Theorem 4.3.6 *Let Y be a subspace of X. For every $x_1, x_2 \in X,\ C(x_1) \cap C(x_2)$ separates the points of Y. If Y is an AC-subspace of X, then Y is constrained in X. Moreover the projection is unique and $O(Y, X)$ is a closed subspace of X.*

Proof: Since Y is AC-subspace of a Banach space X. Then $\mathcal{P}(x) \neq \Phi$, for all $x \in X$.
By proposition 0.3.1, for all $x \in X$,
$x^*(x - y) = 0$ for any $x^* \in C(x),\ y \in \mathfrak{P}(x)$.
Now if $y_1, y_2 \in \mathfrak{P}(x)$, then for any $x^* \in C(x)$

$$x^*(x - y_1) = 0 = x^*(x - y_2)$$

Therefore

$$x^*(y_1 - y_2) = 0$$

Since $C(x)$ separates the points of Y, we have

$$y_1 - y_2 = 0$$

i.e.

$$y_1 = y_2$$

This implies $\mathfrak{P}(x)$ is singleton.
Let $\mathfrak{P}(x) = \{P(x)\}$
We can see that P satisfies the conditions of the corollary 0.2.2. So the only thing we have to check is that P is additively linear.
Let $x_1, x_2 \in X$. If $x^* \in C(x_1) \cap C(x_2)$. Then by the proposition 4.0.16 we see that

$$x_n^*(x_1) \to x^*(x_1)$$

and

$$x_n^*(x_2) \to x^*(x_2)$$

Getting together,

$$x_n^*(x_1+x_2) \to x^*(x_1+x_2)$$

$$\implies x^* \in C(x_1+x_2)$$

Also

$$x^*(x_1 - P(x_1)) = x^*(x_2 - P(x_2)) = x^*((x_1+x_2) - P(x_1+x_2)) = 0.$$

Therefore

$$x^*(P(x_1+x_2) - P(x_1) - P(x_2)) = 0$$
$$\implies P(x_1+x_2) = P(x_1) + P(x_2)$$

$\implies P$ is additive linear.

Proposition 4.3.7 *Every unit vector $y \in Y$ is a smooth point of X_1 if and only if every subspace of Y is a weakly U-subspace of X. In particular X is smooth if and only if every subspace of X is weakly U-subspace of X*

Proof: Suppose every unit vector $y \in Y$ is a smooth point of X_1.
Let Z be any subspace of Y.
Let $z_0 \in Z$ with $\|z_0\| = 1$ attains its norm at $z^* \in Z_1^*$(Here $\|z^*\| = 1$ and by Hahn- Banach theorem $z^*(z_0) = \|z_0\| = 1$).
Here $z^* \in HB(z^*)$.
Since z_0 is a smooth point of X_1, we have $z^* \in \mathcal{D}_Z(z_0) = \{z^*\}$ This shows that

$$HB(z^*) \subseteq \mathcal{D}_Z(z_0) = \{z^*\}$$
$$\implies HB(z^*) = \{z^*\}$$

This implies that Z is weakly U-subspace of X.
Conversely, assume that every subspace of Y is a U-subspace of X.
Claim: every unit vector $y \in Y_1$ is a smooth point of X_1.
On the contrary, suppose this is not true.
Then there is $y_0 \in Y_1$ such that

$$\{x_1^*, x_2^*\} \subseteq \mathcal{D}_X(y_0) \text{ with } x_1^* \neq x_2^*$$

Let

$$Z = \{x \in Y : x_1^*(x) = x_2^*(x)\} \subset Y$$

Here $Z \neq \Phi$ because $y_0 \in Z$.
Therefore $\|x_1^*\| = \|x_2^*\| = 1$
Taking $z^* = x_1^*|_Z$, we have z^* attains its norm at $y_0 \in Z$.
But $\{x_1^*, x_2^*\} \subset HB(z^*)$.
This is a contradiction, because Z being a subspace of Y is not at all a weakly U-subspace of X.

Corollary 4.3.8 *Let Y be an AC-subspace of X. In each of the following cases, Y is constrained in X by a unique norm 1 projection.*

(a) *C separates points of Y.*

(b) *Y is a U-subspace of X.*

(c) *Y is weakly U-subspace of X.*

(d) *Every unit vector in Y is a smooth point of X.*

Proof: (a) follows from the previous theorem. And also $(b) \implies (c) \implies (a)$. From the last proposition we have $(d) \implies (a)$.

4.4 Proximinality

Definition 4.4.1 *For a closed set K in X and $x \in X$, we denote the distance function of K at x by $d(x, K) = \inf\{\|x - k\| : k \in K\}$. The metric projection of x onto K is $P_K(x) = \{k \in K : \|x - k\| = d(x, K)\}$.*

Definition 4.4.2 *The set K is called proximinal in X if for every $x \in X$, $P_K(x)$ is nonempty.*

Definition 4.4.3 *The set K is called antiproximinal in X if for every $x \in X \setminus K$, $P_K(x) = \emptyset$.*
Or,
Consider the set

$$E(K) = \{x \in X : P(x) = P_K(x) \neq \emptyset.\}$$

The set K is called proximinal in X if

$$E(K) = X$$

and called antiproximinal in x if

$$E(K) = K.$$

4.4.1 Characterization of Proximinal Subspaces:

Theorem 4.4.4 *A subspace Y of X is proximinal if and only if $X = Y + L(Y, X)$.*

Proof: Y is proximinal in X implies $P_Y(x) \neq \emptyset$, $\forall\ x \in X$. Choose $y \in P_Y(x)$ then $\|x - y\| = d(x, Y)$ or in other words $\|x - y\| \leq \|x - z\|$, $\forall$ $z \in Y$, that is $\|(x - y) + (y - z)\| \geq \|x - y\|, \forall z \in Y$ that is $x - y \perp_B Y$. So $x = y + (x - y)$, where $x - y \in L(Y, X)$.

Conversely, let $X = Y + L(Y, X)$, to show $P_Y(x) \neq \emptyset$, $\forall\ x \in X$. Now $x_0 = y_0 + z_0$, with $y_0 \in Y$ and $z_0 \in L(Y, X)$. Now for all $y \in Y$,

$$d(x_0, Y) \leq \|x_0 - y_0\| = \|z_0\| \leq \|z_0 + y\| = \|x_0 - y_0 + y\|,$$

since $z_0 \in L(Y, X)$. Taking infimum over $y \in Y$, we have the result.

Theorem 4.4.5 *A subspace Y is antiproximinal in X if and only if $L(Y, X) = \{0\}$*

Proof: From the above proof, it follows that for $x \in X$, $y \in P_Y(x)$ if and only if $x - y \in L(Y, X)$. The result is now immediate.

The set $O(X, X^{**})$ appears in the works of Godefroy in relation to the study of nicely smooth spaces and the spaces with the finite-infinite intersection property $(IP_{f,\infty})$. In some equivalent formulation.

(a) X is nicely smooth if and only if $O(X, X^{**}) = \{0\}$.

(b) X has the $IP_{f,\infty}$ if and only if $X^{**} = X + O(X, X^{**})$.

Both of these notions have been subsequently generalized in the subspace context :

(c) A subspace Y is a very non-constrained (VN) subspace of X if $O(Y, X) = \{0\}$.

(d) A subspace Y is an almost constrained (AC) subspace of X if $X = Y + O(Y, X)$.

It is easy to see that if there is a norm one projection $P : X \to Y$, then Y is an AC subspace of X. The converse is in general false, but it remains an open question if X has the $IP_{f,\infty}$ implies X is 1-complemented in X^{**}.

Definition 4.4.6 *We say that X is proxbid if X is proximinal in its bidual, X^{**}.*

Chapter 5

Conclusion

The main problem we propose to study in the thesis is the geometry of proxbid spaces. We should point out that proximinality of a subspace is most well-understood for subspaces of finite codimension and otherwise very little is known except in special cases. However, if X is of finite codimension in X^{**}, then X already has some similarity with reflexive spaces. Thus, our study may require somewhat different techniques. Since X is proxbid if and only if $X^{**} = X + L(X, X^{**})$, we feel that in this study, characterizations of the set $L(X, X^{**})$ would be very useful.

One special class of proxbid spaces that are better understood are spaces that are M-ideals in their biduals. In this case, many isometric and isomorphic properties of such spaces are known. So this can very well be our staring point.

The literature on proxbid spaces is rather small. It is known that $C(K)$ spaces are proxbid. It has been proved that if K is compact, Hausdorff and X is uniformly convex then any $C(K)$ module $M \subseteq C(K, X)$ is proximinal in any higher dual of even order of $C(K, X)$, so in particular M is proxbid and hence $C(K, X)$ is also becomes proxbid. The case for X is LUR or MLUR can be investigated.

Bibliography

[1] G.F.Simmons.*Introduction to Topology and Modern Analysis.*

[2] Balmohan V Limaye.*Functional Analysis, 3rd edition.*

[3] (Graduate Texts in Mathematics) John B. Conway-A Course in Functional Analysis,2nd Edition

[4] P. Bandyopadhyay, S. Basu, S. Dutta and B.L. Lin, *Very non constrained subspaces of Banach spaces*, Extracta Math., **18** (2003), 161-185.

[5] Pradipta Bandyopadhayay and S. Dutta, *Almost constrained subspaces of Banach spaces*, Proc. Amer. Math. Soc., **132** (2004), 107–115.

[6] Pradipta Bandyopadhyay and Sudeshna Basu, *On nicely smooth Banach spaces*, Extracta Math., **16** (2001), 27-45.

[7] Kenneth R Davidson, *Banach spaces antiproximinal in their biduals*, J. Approx. Theory, **47**(1986), 203-213.

[8] G. Godefroy, *Nicely smooth Banach spaces*, Longhron Notes, The University of Texas at Austin, Functional Analysis Seminar (1984-85), 117-124.

[9] G. Godefroy, *Existence and uniqueness of isometric preduals: a survey*, Contemp. Math., **85**(1989), 131-193.

[10] G. Godefroy, N.J. Kalton, *The ball topology and its application*, Contemp. Math. **85**(1989), 195-237.

[11] G. Godefroy, P. D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. **32** (1988), 672-695.

[12] Kazuo Hashimoto, Gen Nakamura and Shinnosuke Oharu, *Riesz lemma and orthogonality in normed spaces*, Hiroshima Math.J. **16**(1986), 279-304.

[13] H. Harmand, D. Werner and W. Werner, *M-Ideals in Banach spaces and Banach algebras* , Lecture Notes in Mathematics 1547, Springer-Verlag, Berlin, 1993.

[14] R. B. Holmes and J. D. Ward, *An approximative property of spaces of continuous functions*, Glasgow Math. J., **15** (1974), 48–53.

[15] Robert C James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., **61** (1947), 265-292.

[16] TSSRK Rao, *Some remarks on proximinality in higher dual spaces*, J. Math. Anal. Appl., **328** (2007), 1173-1177.