Bandler-Kohout Subproduct with Yager's Families of Fuzzy Implications: A Comprehensive Study

Sayantan Mandal

A Thesis Submitted to
Indian Institute of Technology Hyderabad
In Partial Fulfillment of the Requirements for
The Degree of Doctor of Philosophy

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Department of Mathematics

May 2014

Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Sayantan Mandal.
05-01-2015
(Signature)

SAYANTAN MANDAL
(- Student Name -)

MA10P002
(Roll No.)

Approval Sheet

This Thesis entitled Bandler-Kohout Subproduct with Yager's Families of Fuzzy Implications: A Comprehensive Study by Sayantan Mandal is approved for the degree of Doctor of Philosophy from IIT Hyderabad

MPBiswal 5/1/2015

(Prof. Mahendra Prasad Biswal) Examiner
Dept. of Mathematics
IIT Kharagpur

5/1/15

(Dr. Challa Subrahmanya Sastry) Examiner
Dept. of Mathematics
IIT Hyderabad

5/1/15

(Dr. Balasubramaniam Jayaram) Adviser
Dept. of Mathematics
IIT Hyderabad

5/1/2015

(Dr. Naveen Sivadasan) Chairman
Dept. of Computer Science and Engineering
IIT Hyderabad

Acknowledgements

"One mark of a great educator is the ability to lead students out to new places where even the educator has never been" – Thomas Groome .

Foremost, I would like to express my sincere gratitude to my supervisor Dr. Balasubramaniam Jayaram for introducing me to such a nice field of research. I would also like to thank him for the continuous support of my Ph.D study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better supervisor and mentor for my Ph.D study.

Besides my advisor, I would like to thank the Department of Mathematics, IIT Hyderabad and Dr. C. S Sastry, head of the department, for giving me such a nice department to be a part of.

My sincere thanks also goes to doctoral committe members of my thesis: Dr. P. A. Lakshmi Narayana , Dr. G. Ramesh, Dr. Naveen Sivadasan and Dr. Sri Rama Murty Kodukula, for their constant encouragement and insightful comments.

I would like to specially thank my roommate cum colleague Nageswara Rao Vemuri for his constant support in every form, whether it is related to research or outside of it.

I thank my special friends Tanima Biswas and Abhishek Sarkar for their support and sending me each and every research article, whatever I did not have access.

I would like to thank my home tutor cum mentor cum elder brother cum friend Mr. Muktaram Palit for his encouragement not only throughout the study but also in many aspects of my life.

Last but not the least, I would like to thank my family: my parents Lt. Biswanath Mandal and Krishna Mandal, my elder brother Abhishek Mandal, my elder sister Nabanita Mandal for being with me everytime at the first place and supporting me spiritually throughout my life.

Dedication

To my beloved father.

Abstract

Approximate reasoning schemes involving fuzzy sets are one of the best known applications of fuzzy logic in the wider sense. Fuzzy Inference Systems (FIS) or Fuzzy Inference Mechanisms (FIM) have many degrees of freedom, viz., the underlying fuzzy partition of the input and output spaces, the fuzzy logic operations employed, the fuzzification and defuzzification mechanism used, etc. This freedom gives rise to a variety of FIS with differing capabilities.

Desirable Properties and Types of FIS: Fuzzy inference systems are expected to possess many desirable properties as explicated below.

- **Interpolativity:** When an antecedent of a rule is given as the input then the corresponding consequent should be the inferred output.

- **Continuity:** If the input is close to some antecedent of the rule base then the output also should be close to the corresponding consequent.

- **Universal Approximation Capability:** Whether it can approximate any continuous function over a compact set to arbitrary accuracy. In other words, does there exist an FIS whose system function g is such that for any continuous function $f\colon [a,b] \to R$ over a closed interval $[a,b]$ and an arbitrary given $\epsilon > 0$,
$$\max_{x\in[a,b]} |f(x) - g(x)| \leq \epsilon.$$

- **Monotonicity:** Whether for a monotone rule base and monotonic inputs we obtain monotonic outputs. In other words, whether for two inputs x' and x'', such that $x' \leq x''$ the corresponding outputs y', y'' are such that $y' = g(x') \leq g(x'') = y''$.

- **Robustness:** Robustness deals with how errors in the premises affect the conclusions. Maximum possible robustness is achieved by reducing the sensitivity of the inference mechanism to input variations to a satisfactory level, i.e., the output should not be too sensitive to *unwanted* variations of the input.

These properties can be treated as some parameters for assessing the goodness/ quality of an inference mechanism. This means an inference mechanism which possesses some or all of these properties are suitable to be used in applications.

In the literature, the most commonly used fuzzy inference mechanisms are Fuzzy Relational Inference Mechanism (FRI), Similarity Based Reasoning (SBR) and Takagi-Sugeno-Kang (TSK) Fuzzy Systems.

Types of FRI: In this work, our main focus is on FRIs. Two of the well-known fuzzy relational inference mechanisms are the Compositional Rule of Inference (CRI) proposed by Zadeh and the Bandler-Kohout Subproduct (BKS) proposed by Pedrycz based on the earlier works of Bandler and Kohout.

Motivation: In these FRIs, the mainly used operations are either a left-continuous t-norm or a fuzzy implication obtained as a residual of left-continuous t-norms. In all the previous works on FRI dealing with desirable properties, the employed fuzzy implication comes from a residuated lattice. It can be seen that many of these desirable properties of an FRI are due to the rich underlying structure, viz., the residuated algebra. The question that arises naturally is the following:

What happens to an FRI if the fuzzy implications employed in it do not come from a residuated structure on $[0, 1]$*?*

This forms the main motivation for the work contained in this thesis. We investigate an FRI which employs fuzzy implications that are not known to come from a residuated algebra. One such class of fuzzy implication is the Yager's families of fuzzy implications which does not come from a residuated lattice structure.

Work Presented in this Thesis:

- In this work, we discuss the BKS relational inference system with the fuzzy implication interpreted as the well known Yager's families of fuzzy implications, which do not form a residuated structure on $[0, 1]$.

- We show that all of the desirable properties, viz., interpolativity, continuity, robustness, universal approximation and monotonicity that are known for BKS with residuated implications are also available under this framework, thus expanding the choice of operations available to practitioners.

- While studying the properties like interpolativity and continuity we have proposed some extended class of fuzzy implications which are defined on $[0, \infty]$ instead of $[0, 1]$. These play an important role in giving crisp expressions to many results and properties.

- Moreover, it can be shown that the results on monotonicity and universal approximation capability are not only true for Yager's families of fuzzy implications, but also for more general classes of fuzzy implications.

Uniqueness of Our Work:

- To the best of the authors' knowledge, this is the first attempt at studying the suitability of an FRI where the operations do not come from a residuated structure.

- Some of the obtained results are valid for more general classes of fuzzy implications.

- We have proven the existence of monotonicity for FRIs by only imposing conditions on the underlying partition and the operations but without modifying the given rule base, as is common in the literature.

Publications from this Thesis

Journal Papers

(i) Sayantan Mandal and Balasubramaniam Jayaram, ”Bandler-Kohout Subproduct with Yager's Classes of Fuzzy Implications”, *IEEE Transactions on Fuzzy Systems*, vol.22, no.3, pp.469-482,2014 (doi:10.1109/ tfuzz.2013.2260551).

(ii) Sayantan Mandal and Balasubramaniam Jayaram, ” SISO Fuzzy Relational Inference systems based on Fuzzy Implications are Universal Approximators”, *Fuzzy Sets and Systems*, ***Accepted***, (doi:10.1016/j.fss.2014.10.003).

Conference Papers

(i) Sayantan Mandal and Balasubramaniam Jayaram, ” Monotonicity of SISO Fuzzy Relational Inference Mechanisms with Yager's class of Fuzzy Implications”, 5th International Conference on Pattern Recognition and Machine Intelligence, PReMi 2013, Kolkata, India, December 10 -14, 2013.

(ii) Sayantan Mandal and Balasubramaniam Jayaram, ” Approximation Capability of SISO Fuzzy Relational Inference systems based on Fuzzy Implications”, 2013 International Conference on Fuzzy Systems, Fuzz-IEEE 2013, Hyderabad, India, July 7-10, 2013.

(iii) Sayantan Mandal and Balasubramaniam Jayaram, , ”Bandler-Kohout Subproduct with Yager's classes of fuzzy implications ”, 13th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2012, Catania, Italy, July 9-13, 2012.

Outline of the Thesis

This thesis can be split into two logical parts. **Part I**, consisting of Chapters 1 – 3, introduces Fuzzy Relational Inference (FRI) mechanisms, presents their desirable properties and discusses the suitability of existing types of FRIs and leads up to the main focus of this work, the Bandler-Kohout Subproduct FRI with Yager's families of fuzzy implications. **Part II**, consisting of Chapters 4 – 7, investigates the conditions under which the Bandler-Kohout Subproduct FRI with Yager's families of fuzzy implications possess the desirable properties, thus establishing their suitability in practical applications.

- In Chapter 1, after presenting some important definitions both from fuzzy set theory and fuzzy logic connectives, we introduce different types of fuzzy rule bases, and go on to describe the inference procedure in a fuzzy inference mechanism.

- Following this, in Chapter 2, we begin by presenting a detailed discussion on fuzzy relational inference (FRI) mechanisms. After reviewing some desirable properties and related results on the suitability of existing FRIs, we set the motivation for the work contained in this thesis and also define clearly the scope of this thesis.

- In Chapter 3, after presenting the definitions and basic properties of the Yager's families of fuzzy implications, we propose two modified versions of the well known Bandler-Kohout Subproduct (BKS) inference mechanism, called the BKS-f and BKS-g inference mechanisms, the study of which will be the main focus of this thesis.

- The Chapter 4 is devoted to investigating the interpolativity and continuity of the proposed modified BKS inference mechanism. Here in this chapter, we begin by proposing an extension of the well known Goguen implication (see Table 1.5) and discuss some of its properties, which play an important role in this chapter. The main results of this chapter pertain to finding necessary and sufficient conditions for the BKS-f and BKS-g inference mechanisms to have interpolativity and continuity.

- The Chapter 5 deals with the robustness of the BKS-f and BKS-g inference mechanisms.

- In Chapter 6 we present a short survey on the works and results related to universal approximation of fuzzy relational inference systems. Then relaxing the often insisted coherence of an implicative model suitably to the context of function approximation, we show that FRIs employing a rather large class of fuzzy implications - which include the BKS-f and BKS-g inference mechanisms - are universal approximators. We illustrate the investigations and analysis related to universal approximation with some examples.

- In Chapter 7, we investigate the monotonicity of FRIs, and along the lines of Chapter 6, show that FRIs employing a rather large class of fuzzy implications - which include the BKS-f and BKS-g inference mechanisms - are monotonic. Once again, we illustrate the investigations and analysis related to monotonicity with some examples.

- Lastly, in Chapter 8 some concluding remarks and pointers to possible extensions and approaches from this thesis are given.

Contents

Part I

Bandler-Kohout Subproduct with Yager's Families of Fuzzy Implications

Chapter 1

Fuzzy Inference Mechanism

Evolution is an inference from thousands of independent sources, the only conceptual structure that can make unified sense of all this disparate information.

–Stephen Jay Gould (1995–2002)

Fuzzy sets and fuzzy logic connectives are the basic units of a fuzzy inference mechanism. Fuzzy sets are a generalization of classical sets, where as fuzzy logic connectives are a generalization of classical logic operations. In this chapter, we begin by recalling the basic definitions from fuzzy set theory and fuzzy logic connectives, following which we introduce the notion of fuzzy rule bases and fuzzy inference mechanisms in detail.

1.1 Fuzzy Set Theory

Let X be the universe of discourse. In this work we only consider $X \subseteq \mathbb{R}$ to be a non-empty, closed and bounded interval and hence X is totally ordered, linear and compact w.r.to the usual topology on $\mathbb{R}$. However, many of the concepts below are applicable to more general sets and hence the definitions are given accordingly.

Let $A \subseteq X$. In classical set theory every element in X, either belongs to A or does not belong to A. This can be represented equivalently by the characteristic function $\chi_A : X \longrightarrow \{0, 1\}$,

$$\chi_A(x) = \begin{cases} 1, & \text{iff } x \in A, \\ 0, & \text{iff } x \notin A. \end{cases}$$

The degree of belongingness of an element to a classical set is either 0 or 1. Fuzzy set is nothing but a generalization of a characteristic function, where the degree of belongingness of an element can be any value between 0 and 1.

1.1.1 Fuzzy Sets

Fuzzy sets are a generalization of classical sets. Fuzzy sets were first introduced by Lotfi A. Zadeh [54] in 1965.

Definition 1.1.1. *A fuzzy set A on X is a mapping from X to $[0, 1]$, i.e, $A : X \to [0, 1]$.*

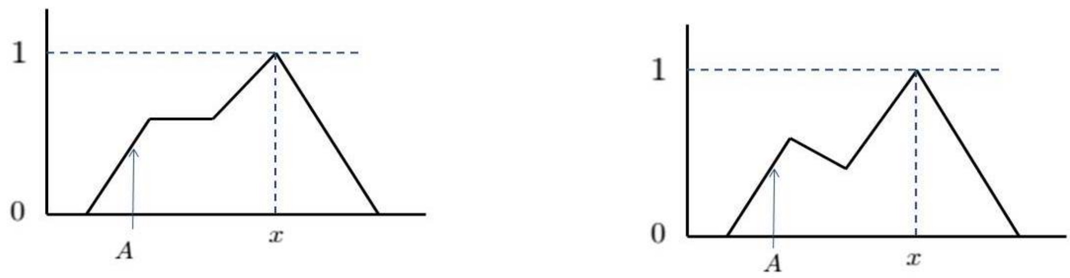
$A(x)$ is the membership function that assigns each element of X to a membership degree between 0 and 1. We denote the fuzzy power set of X as $\mathcal{F}(X)$, i.e., $\mathcal{F}(X) = \{A|A : X \to [0, 1]\}$.

Definition 1.1.2. *A fuzzy set A is said to be*

- normal *if there exists an $x \in X$ such that $A(x) = 1$,*
- convex *if X is a compcat subset of a linear space and for any $\lambda \in [0, 1]$, $x, y \in X$,*

$$A(\lambda x + (1 - \lambda)y) \geq \min\{A(x), A(y)\} .$$

For example in Figure 1.1(a), A is a fuzzy set which is normal and convex whereas in Figure 1.1(b), A is normal but not convex.



(a) Normal Convex Fuzzy set

(b) Normal Non-convex Fuzzy set

Figure 1.1: Fuzzy sets which are normal and (a) Convex, (b) Non-convex.

Definition 1.1.3. *For an $A \in \mathcal{F}(X)$, the* Support, Height, Kernel, Ceiling *and* α-cut *for an $\alpha \in (0, 1]$ are, respectively, defined as:*

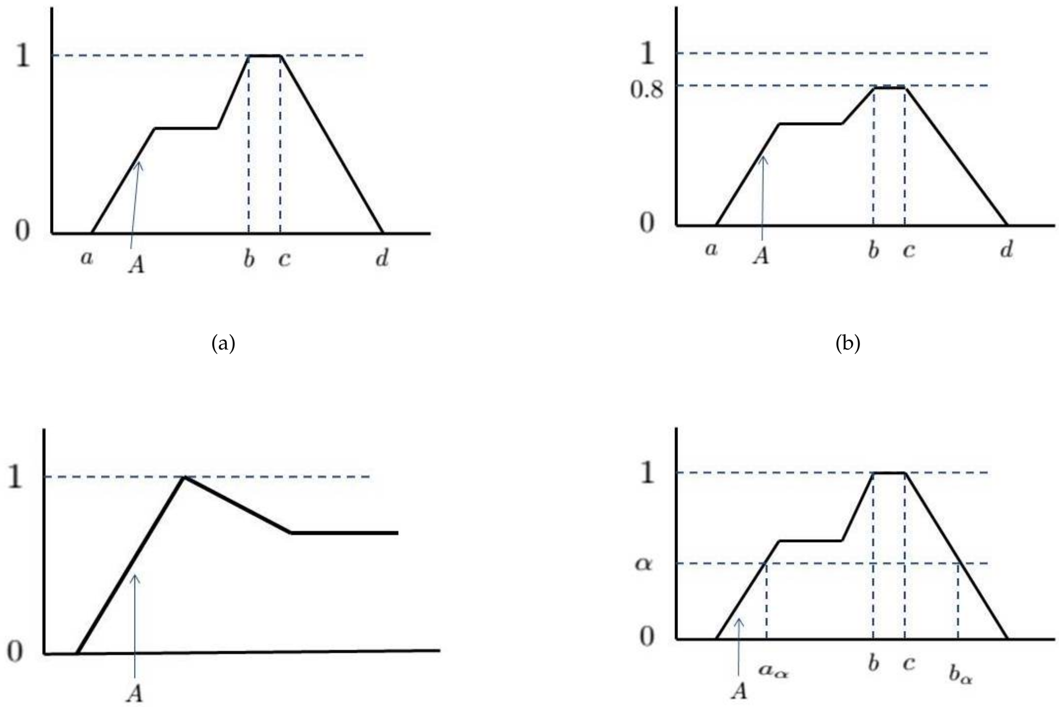
$$\begin{aligned} \text{Supp}(A) &= \{x \in X|A(x) > 0\} , \\ \text{Hgt}(A) &= \sup\{A(x)|x \in X\} , \\ \text{Ker}(A) &= \{x \in X|A(x) = 1\} , \\ \text{Ceil}(A) &= \{x \in X|A(x) = \mathit{Hgt}(A)\} , \\ [A]_\alpha &= \{x \in X|A(x) \geq \alpha\} . \end{aligned}$$

Definition 1.1.4. *A fuzzy set A is said to be* bounded *if* Supp(A) *is a bounded set.*

Note that for a normal fuzzy set Ker(A) = Ceil(A) and Hgt$(A) = 1$. For example in Figure 1.2(a), A is a fuzzy set which is again normal and convex with Supp$(A) = [a, d]$, Hgt$(A) = 1$ and

Ker(A) = Ceil$(A) = [b, c]$, but in Figure 1.2(b), A is convex but not normal, with Supp$(A) = [a, d]$, Hgt$(A) = 0.8$, Ker$(A) = \emptyset$ and Ceil$(A) = [b, c]$. In Figure 1.2(a) and Figure 1.2(b) both the fuzzy sets are bounded fuzzy sets as their support is bounded, whereas the fuzzy set in Figure 1.2(c) has unbounded support, hence it is an unbounded fuzzy set.

Many of the above concepts can be specified in terms of α-cuts. From Figure 1.2(d) we can see that the α-cut for an $\alpha \in (0, 1]$ is $[A]_\alpha = [a_\alpha, b_\alpha]$. Note that the support of a fuzzy set can be defined in terms of α-cuts as Supp$(A) = \bigcup_{\alpha \in (0,1]} [A]_\alpha$. For an $\alpha_0 \in (0, 1]$, if Hgt$(A) = \alpha_0$, then Ceil$(A) = [A]_{\alpha_0}$ and for a normal fuzzy set Ker$(A) = [A]_1$.



(a) (b) (c) (d)

Figure 1.2: Support, Height, Kernel, Ceiling, α-cut of Fuzzy sets

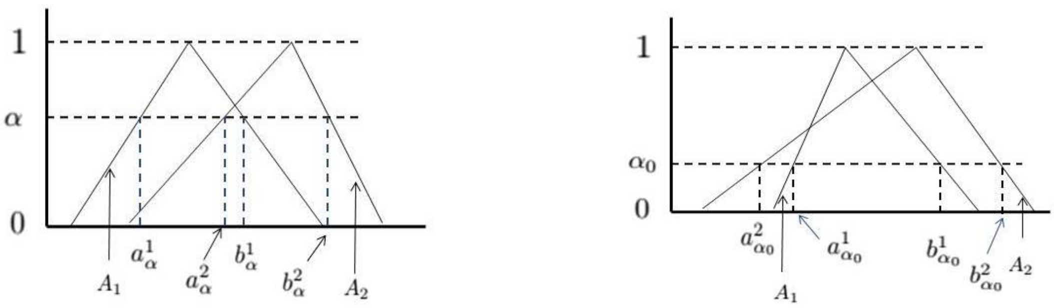
1.1.2 Ordering on Fuzzy Sets

Ordering between two fuzzy sets is a vital concept. There exist different types of orderings on fuzzy sets. In the following, we present one such ordering that will play an important role in this thesis.

Definition 1.1.5 ([38], Definition 3)**.** *For two convex fuzzy sets A_1 and A_2, we say that $A_1 \prec A_2$ if for any $\alpha \in (0, 1]$ it holds that*

$$\inf[A_1]_\alpha \leq \inf[A_2]_\alpha \textit{ and } \sup[A_1]_\alpha \leq \sup[A_2]_\alpha .$$

For example, in the Figure 1.3(a), note that, $[A_1]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[A_2]_\alpha = [a^2_\alpha, b^2_\alpha]$, for every $\alpha \in (0, 1]$. We can see that for any $\alpha \in (0, 1]$, $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$, whereas in the Figure 1.3(b) there exists an $\alpha_0 \in (0, 1]$ such that $a^1_{\alpha_0} > a^2_{\alpha_0}$ and $b^1_{\alpha_0} \leq b^2_{\alpha_0}$. Thus, $A_1 \prec A_2$ holds in Figure 1.3(a) and $A_1 \not\prec A_2$ in Figure 1.3(b).



(a) A_1 and A_2 are ordered

(b) A_1 and A_2 are not ordered

Figure 1.3: Ordering of Fuzzy sets

1.1.3 Operations on Fuzzy Sets

Similar to union, intersection and complement of classical sets, one can define these operations for fuzzy sets. Some relevant fuzzy logic operations like fuzzy union, fuzzy intersection and fuzzy complement are given below.

Definition 1.1.6. *For any two fuzzy sets $A_1, A_2 \in \mathcal{F}(X)$, their union and intersection denoted by $A_1 \cup A_2$ and $A_1 \cap A_2$ are defined as*

$$A_1 \cup A_2(x) = A_1(x) \vee A_2(x) = \max\{A_1(x), A_2(x)\},$$

$$A_1 \cap A_2(x) = A_1(x) \wedge A_2(x) = \min\{A_1(x), A_2(x)\}.$$

Definition 1.1.7. *Complement of a fuzzy set denoted as $\overline{A}(x)$ is defined as*

$$\overline{A}(x) = 1 - A(x).$$

For example, in Figure 1.4(a), Figure 1.4(b) and Figure 1.4(c) the union, intersection and complement of fuzzy sets are shown by the dotted lines, respectively.

Definition 1.1.8. *Two fuzzy sets $A_1, A_2 \in \mathcal{F}(X)$ are equal if $A_1(x) = A_2(x)$ for every $x \in X$.*

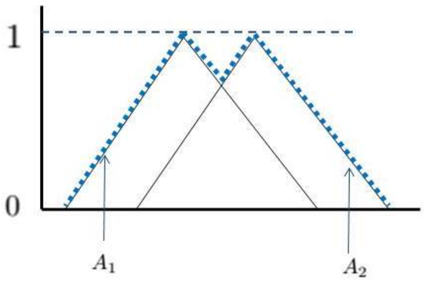
1.1.4 Fuzzy Partition

In classical set theory, partition of a set X is defined as follows:

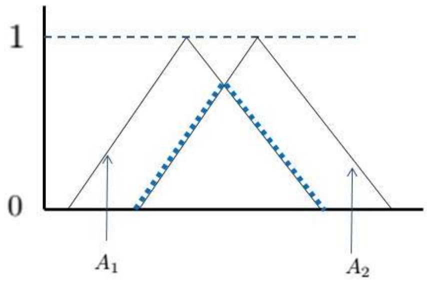
Definition 1.1.9. *A collection of subsets of X, say, $\mathbb{P}$ is said to form a partition of X iff the following holds:*

(i) $\bigcup_{A \in \mathbb{P}} A = X,$

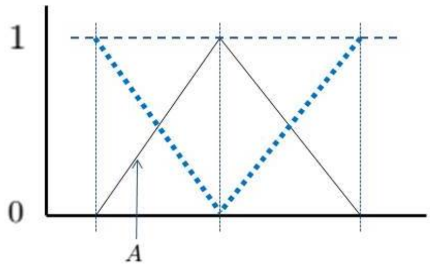
(ii) if $A, B \in \mathbb{P}$ and $A \neq B$ then $A \cap B = \emptyset$,



(a) $A_1 \cup A_2$.



(b) $A_1 \cap A_2$.



(c) $\overline{A}$.

Figure 1.4: Operations on fuzzy sets (a) Union, (b) Intersection and (c) Complement.

In fuzzy set theory, there are several approaches to and definitions of a fuzzy partition, see for instance, [9, 13, 19, 27, 30]. In the following, we present a definition which is considered to be quite general and hence is often used in the literature.

Definition 1.1.10. *Let $\mathcal{P}$ be a finite collection of fuzzy sets on X, i.e, $\mathcal{P} = \{A_k\}_{k=1}^n \subseteq \mathcal{F}(X)$. $\mathcal{P}$ is said to form a* fuzzy partition *on X if*

$$X \subseteq \bigcup_{k=1}^{n} \mathrm{Supp}(A_k) \ .$$

For instance, in Figure 1.5, $\{A_k\}_{k=1}^5$ forms a fuzzy partition on $X = [a_1, b_5]$ since,

$$X = \bigcup_{k=1}^{5} [a_k, b_k] = \bigcup_{k=1}^{5} \mathrm{Supp}(A_k) \ .$$

In essence, a fuzzy partition is a collection of fuzzy sets $\{A_k\}_{k=1}^n$ such that, for every $x \in X$ there exists $k \in \{1, 2, \ldots, n\}$ such that $A_k(x) > 0$.

In the literature, a partition $\mathcal{P}$ of X as defined above is also called a **complete** partition.

Definition 1.1.11. *A fuzzy partition $\mathcal{P} = \{A_k\}_{k=1}^n \subseteq \mathcal{F}(X)$ is said to be*

- ***consistent*** *if whenever for some k, $A_k(x) = 1$ then $A_j(x) = 0$ for all $j \neq k$,*
- *a **Ruspini Partition** if $\displaystyle\sum_{k=1}^{n} A_k(x) = 1$ for every $x \in X$.*

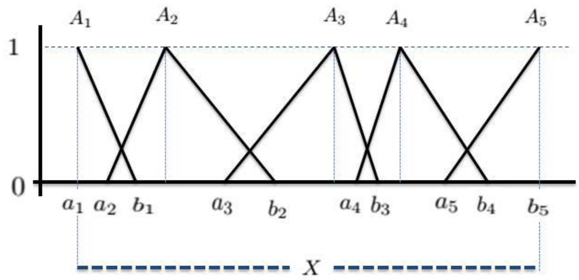


Figure 1.5: $\{A_k\}_{k=1}^5$ forms a fuzzy partition on X.

In Figure 1.6, $\mathcal{P} = \{A_k\}_{k=1}^6 \subseteq \mathcal{F}(X)$ forms a Ruspini partition. For instance, it can be verified that for the $x_0 \in X$ in Figure 1.6,

$$\sum_{k=1}^{6} A_k(x_0) = A_2(x_0) + A_3(x_0) = 0.65 + 0.35 = 1 \ .$$

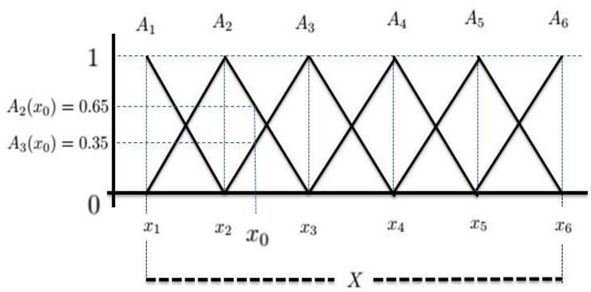


Figure 1.6: $\{A_k\}_{k=1}^6$ forms a Ruspini partition on X.

1.1.5 Defuzzification

Often there is a need to convert a fuzzy set to a crisp value, a process which is called *Defuzzification*. This process of defuzzification can be seen as a mapping $d : \mathcal{F}(X) \longrightarrow X$. There are many types of defuzzification techniques available in the literature, see [40] for a good overview. Here, we recall some of the defuzzifiers that will be relevent in our work.

Example 1.1.12. *For a fuzzy set $A \in \mathcal{F}(X)$, with bounded ceiling* Ceil(A), *the* Mean of Maxima (MOM) *defuzzifier gives as output the mean of all those values in X with the highest membership value, which can be mathematically expressed as*

$$\text{MOM}(A) = \frac{\displaystyle\int_{\text{Ceil}(A)} x dx}{\displaystyle\int_{\text{Ceil}(A)} 1 dx} \ , \quad \textit{if} \quad \int_{\text{Ceil}(A)} 1 dx \neq 0. \tag{1.1}$$

The Smallest of Maxima (SOM) *and* Largest of Maxima (LOM) *defuzzifiers can be mathematically* *expressed as*

$$\mathrm{SOM}(A) = \min\left\{x|A(x) = \mathrm{Hgt}(A)\right\} = \min\left\{x|x \in \mathrm{Ceil}(A)\right\}, \text{ and} \tag{1.2}$$

$$\mathrm{LOM}(A) = \max\left\{x|A(x) = \mathrm{Hgt}(A)\right\} = \max\left\{x|x \in \mathrm{Ceil}(A)\right\}. \tag{1.3}$$

In Figure 1.7, for the given fuzzy set A, $\mathrm{SOM}(A) = b$, $\mathrm{MOM}(A) = x^* = \dfrac{b+c}{2}$ *and* $\mathrm{LOM}(A) = c$.

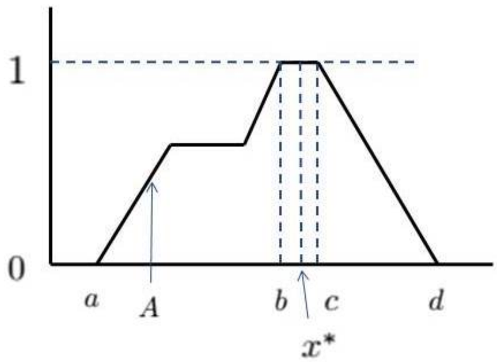


Figure 1.7: Smallest of Maxima (SOM), Mean of Maxima (MOM) and the Largest of Maxima (LOM) defuzzification techniques.

Example 1.1.13. *For a fuzzy set $A \in \mathcal{F}(X)$, with bounded ceiling* $\mathrm{Ceil}(A)$*, the* Center of Gravity (COG) *defuzzifier can be mathematically expressed as*

$$\mathrm{COG}(A) = \frac{\displaystyle\int_{\mathrm{Supp}(A)} A(x)dx}{\displaystyle\int_{\mathrm{Supp}(A)} 1dx}, \quad \text{if} \quad \int_{\mathrm{Supp}(A)} 1dx \neq 0. \tag{1.4}$$

The Bisector (BIS) *defuzzifier can be mathematically expressed as*

$$\mathrm{BIS}(A) = \left\{x^{**} \;\middle|\; \int_{x=\inf \mathrm{Supp}(A)}^{x^{**}} A(x)dx = \int_{x^{**}}^{x=\sup \mathrm{Supp}(A)} A(x)dx\right\}. \tag{1.5}$$

In Figure 1.8, for the given fuzzy set A, $\mathrm{COG}(A) = x^*$ *and* $\mathrm{BIS}(A) = x^{**}$.

1.2 Fuzzy logic Connectives

Fuzzy logic connectives like t-norms, t-conorms and fuzzy implications are a generalization of the classical logic connectives like classical conjunctions ($\wedge$), classical disjunctions ($\vee$) and classical implications ($\rightarrow$) respectively, whose truth table is given in Table 1.1.

Note that in this work, we use the term *decreasing* and *increasing* in a non-strict sense. In other words, we call a function $t_1 : \mathbb{R} \rightarrow \mathbb{R}$ decreasing or non-increasing if $t_1(x) \geq t_1(y)$ whenever $x \leq y$.

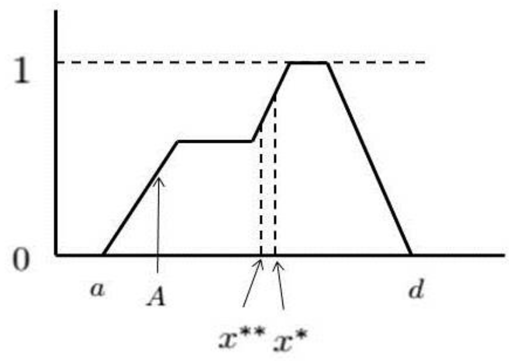


Figure 1.8: Centroid(COG) and Bisector(BIS) defuzzification techniques.

p	q	$p \vee q$	$p \wedge q$	$p \to q$
0	0	0	0	1
0	1	1	0	1
1	0	1	0	0
1	1	1	1	1

Table 1.1: Truth table for the classical logic connectives

Similarly, we call a function $t_2 : \mathbb{R} \to \mathbb{R}$ increasing or non-decreasing if $t_2(x) \leq t_2(y)$ whenever $x \leq y$.

1.2.1 Triangular Norms

Triangular norms, introduced by Menger [29], are a generalization of classical conjunctions and are defined as follows.

Definition 1.2.1 ([22], Definition 1.1)**.** *A function $T : [0,1]^2 \to [0,1]$ is called a* t-norm *, if it is increasing in both variables, commutative, associative and has 1 as the neutral element.*

A t-norm T can be classified depending on various properties, but among them we define only the following class, which is relevant in this thesis.

Definition 1.2.2 ([2], Definition 2.1.2)**.** *A t-norm T is called positive if,*

$$\textit{whenever } T(x,y) = 0 \textit{ then either } x = 0 \textit{ or } y = 0\,. \qquad \text{(T-POS)}$$

Example 1.2.3 (cf. [2])**.** *Table 1.2 lists a few of the basic t-norms along with their classification.*

The algebraic product t-norm $T_{\mathbf{P}}$ plays a very important role in our work and often will be denoted by '$\cdot$' in the infix notation.

1.2.2 Fuzzy Negations

Fuzzy negations are a generalization of classical negation and are defined as follows.

Name	Formula	(T-POS)
minimum	$T_{\mathbf{M}}(x,y) = \min(x,y)$	✓
algebraic product	$T_{\mathbf{P}}(x,y) = xy$	✓
Łukasiewicz	$T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$	×
drastic product	$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$	×
nilpotent minimum	$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$	×

Table 1.2: Basic t-norms

Definition 1.2.4 ([2], Definition 1.4.1)**.** *A function $N : [0,1] \longrightarrow [0,1]$ is called a* fuzzy negation *if $N(0) = 1, N(1) = 0$ and N is decreasing.*

Example 1.2.5 (cf. [2])**.** *Table 1.3 lists out some basic fuzzy negations.*

Name	Formula
Standard	$N_{\mathbf{C}}(x) = 1 - x$
Gödel	$N_{\mathbf{D1}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$
Gödel	$N_{\mathbf{D2}}(x) = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } x < 1 \end{cases}$

Table 1.3: Basic fuzzy negations

1.2.3 Triangular Conorms

Triangular conorms are a generalization of classical disjunctions and are defined as follows.

Definition 1.2.6 ([22], Definition 1.1)**.** *A function $S : [0,1]^2 \to [0,1]$ is called a* t-conorm *, if it is increasing in both variables, commutative, associative and has 0 as the neutral element.*

It can be noted that every t-conorm S can be represented as the $N-$dual of a t-norm T, as $S(x,y) = 1 - T(1-x, 1-y)$, where the fuzzy negation $N_{\mathbf{C}}(x) = 1 - x$ is employed.

Example 1.2.7 (cf. [2])**.** *Table 1.4 lists a few of the common t-conorms. They are the counterparts of the t-norms in Example 1.2.3.*

1.2.4 Fuzzy Implications

Fuzzy implications play a major role in the context of fuzzy inference mechanism. For a large part of this thesis, fuzzy implications remain the focus of our investigations. Fuzzy implications are a generalization of the classical implication as defined below.

Name	Formula
maximum	$S_{\mathbf{M}}(x,y) = \max(x,y)$
probabilistic sum	$S_{\mathbf{P}}(x,y) = x + y - xy$
Łukasiewicz	$S_{\mathbf{LK}}(x,y) = \min(x+y,1)$
drastic sum	$S_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1] \\ \max(x,y), & \text{otherwise} \end{cases}$
nilpotent maximum	$S_{\mathbf{nM}}(x,y) = \begin{cases} 1, & \text{if } x+y \geq 1 \\ \max(x,y), & \text{otherwise} \end{cases}$

Table 1.4: Basic t-conorms

Definition 1.2.8 ([2], Definition 1.1.1). *A function* $I\colon [0,1]^2 \to [0,1]$ *is called a* fuzzy implication *if it satisfies, for all* $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, *the following conditions:*

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1,y) \geq I(x_2,y) \text{ , i.e., } I(\,\cdot\,,y) \text{ is decreasing ,}$$
$$\text{if } y_1 \leq y_2, \text{ then } I(x,y_1) \leq I(x,y_2) \text{ , i.e., } I(x,\,\cdot\,) \text{ is increasing ,}$$
$$I(0,0) = 1 \text{ , } I(1,1) = 1 \text{ , } I(1,0) = 0 \text{ .}$$

The set of all fuzzy implications will be denoted by $\mathbb{I}$.

Example 1.2.9 (cf. [2]). *Table 1.5 lists a few of the common fuzzy implications.*

Name	Formula
Łukasiewicz	$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$
Gödel	$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x,y) = 1 - x + xy$
Kleene-Dienes	$I_{\mathbf{KD}}(x,y) = \max(1-x, y)$
Goguen	$I_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$
Rescher	$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$
Yager	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$
Weber	$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Fodor	$I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x,y), & \text{if } x > y \end{cases}$

Table 1.5: Examples of basic fuzzy implications

Many families of fuzzy implications have been proposed in the literature. One of the important families to have been defined is the family of residuated implication, named as R-implication.

Definition 1.2.10 ([2], Definition 2.5.1). *A function* $I\colon [0,1]^2 \to [0,1]$ *is called an* R-implication *if there* *exists a t-norm T such that*

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\}\ , \qquad x,y \in [0,1]\ .$$

An R-implication I generated from a t-norm T is often denoted by I_T.

Example 1.2.11 (cf. [2]). *Table 1.6 lists few of the well-known R-implications along with the t-norms from which they have been obtained.*

t-norm T	R-implication I_T
$T_{\mathbf{M}}$	$I_{\mathbf{GD}}$
$T_{\mathbf{P}}$	$I_{\mathbf{G}}$
$T_{\mathbf{LK}}$	$I_{\mathbf{LK}}$
$T_{\mathbf{D}}$	$I_{\mathbf{WB}}$
$T_{\mathbf{nM}}$	$I_{\mathbf{FD}}$

Table 1.6: Examples of R-implications

1.2.5 Desirable properties of Fuzzy implications

A fuzzy implication can be classified depending on various properties. In the following definitions we describe some important properties of fuzzy implications which are relevant in this thesis.

Definition 1.2.12 ([2]). *A fuzzy implication $I\colon [0,1]^2 \to [0,1]$ is said to*

- *satisfy the* left neutrality property, *if*

$$I(1,y) = y\ , \qquad y \in [0,1]\ . \tag{NP}$$

- *satisfy the* ordering property, *if*

$$I(x,y) = 1 \Longleftrightarrow x \le y\ , \qquad x,y \in [0,1]\ . \tag{OP}$$

- *be a* positive fuzzy implication *if*

$$I(x,y) > 0, \text{ for all } x,y \in (0,1]\ . \tag{I-POS}$$

- *be a* strict fuzzy implication *if I is strict in both the variables i.e, for $x,y,x_1,x_2,y_1,y_2 \in [0,1]$,*

$$\begin{aligned} &\text{if } x_1 < x_2, \text{ then } I(x_1,y) > I(x_2,y)\ ,\\ &\text{if } y_1 < y_2, \text{ then } I(x,y_1) < I(x,y_2)\ . \end{aligned} \tag{ST}$$

- *the* Law Of Importation, *if for a t-norm T the following holds:*

$$I(x,I(y,z)) = I(T(x,y),z) = I(T(y,x),z)\ , \qquad x,y,z \in [0,1]\ . \tag{LI}$$

Proposition 1.2.13. *Every strict fuzzy implication is a positive fuzzy Implication, i.e., if I satisfies* (ST), *then it satisfies* (I-POS).

Remark 1.2.14. *Not all positive fuzzy implications are strict fuzzy implications, see Table 1.7.*

Example 1.2.15. *Table 1.7 lists the basic fuzzy implications along with their desirable properties.*

Fuzzy implication	(NP)	(OP)	(I-POS)	(ST)
$I_{\mathbf{LK}}$	✓	✓	✓	×
$I_{\mathbf{GD}}$	✓	✓	✓	×
$I_{\mathbf{RC}}$	✓	×	✓	✓
$I_{\mathbf{KD}}$	✓	×	✓	✓
$I_{\mathbf{G}}$	✓	✓	✓	×
$I_{\mathbf{RS}}$	×	✓	×	×
$I_{\mathbf{YG}}$	✓	×	✓	✓
$I_{\mathbf{WB}}$	✓	×	✓	×
$I_{\mathbf{FD}}$	✓	✓	✓	×

Table 1.7: Basic fuzzy implications and their main properties

Example 1.2.16 (cf. [2])**.** *Table 1.8 contain all basic fuzzy implications along with the t-norms for which they satisfy the law of importation* (LI)*. We see that the Rescher implication $I_{\mathbf{RS}}$ does not satisfy* (LI) *with respect to any t-norm T.*

Implication	t-norm
$I_{\mathbf{LK}}$	$T_{\mathbf{LK}}$
$I_{\mathbf{GD}}$	$T_{\mathbf{M}}$
$I_{\mathbf{RC}}$	$T_{\mathbf{P}}$
$I_{\mathbf{KD}}$	$T_{\mathbf{M}}$
$I_{\mathbf{G}}$	$T_{\mathbf{P}}$
$I_{\mathbf{RS}}$	×
$I_{\mathbf{YG}}$	$T_{\mathbf{P}}$
$I_{\mathbf{WB}}$	any T
$I_{\mathbf{FD}}$	$T_{\mathbf{nM}}$

Table 1.8: Basic fuzzy implications and the law of importation

Definition 1.2.17 ([2], Definition 1.4.14)**.** *Let $I \in \mathbb{I}$ be any fuzzy implication. The function $N_I : [0,1] \longrightarrow [0,1]$ defined by $N_I(x) = I(x,0)$ is a fuzzy negation and is called the* natural negation *of I.*

Example 1.2.18 (cf. [2])**.** *Table 1.9 lists the natural negations of all basic fuzzy implications.*

Fuzzy implication I	Natural negation N_I
$I_{\mathbf{LK}}$	$N_{\mathbf{C}}$
$I_{\mathbf{GD}}$	$N_{\mathbf{D1}}$
$I_{\mathbf{RC}}$	$N_{\mathbf{C}}$
$I_{\mathbf{KD}}$	$N_{\mathbf{C}}$
$I_{\mathbf{G}}$	$N_{\mathbf{D1}}$
$I_{\mathbf{RS}}$	$N_{\mathbf{D1}}$
$I_{\mathbf{YG}}$	$N_{\mathbf{D1}}$
$I_{\mathbf{WB}}$	$N_{\mathbf{D2}}$
$I_{\mathbf{FD}}$	$N_{\mathbf{C}}$

Table 1.9: Basic fuzzy implications and their natural negations

Proposition 1.2.19. *Let $\mathcal{I}$ be any finite index set and $\longrightarrow$ denote any fuzzy implication. Then*

$$x \longrightarrow \bigwedge_{i\in\mathcal{I}} (y_i) = \bigwedge_{i\in\mathcal{I}} (x \longrightarrow y_i)\ , \tag{1.6}$$

$$x \longrightarrow \bigvee_{i\in\mathcal{I}} (y_i) \geq \bigvee_{i\in\mathcal{I}} (x \longrightarrow y_i)\ , \tag{1.7}$$

$$\bigvee_{i\in\mathcal{I}} (x_i) \longrightarrow y = \bigwedge_{i\in\mathcal{I}} (x_i \longrightarrow y)\ , \tag{1.8}$$

$$\bigwedge_{i\in\mathcal{I}} (x_i) \longrightarrow y \geq \bigvee_{i\in\mathcal{I}} (x_i \longrightarrow y)\ . \tag{1.9}$$

Proof. Proof follows from the following facts and noting that $\mathcal{I}$ is a finite index set:

- For any non-decreasing function h,

$$h(\min(x,y)) = \min(h(x), h(y)),$$
$$h(\max(x,y)) = \max(h(x), h(y)),$$

- For any non-increasing function h^*,

$$h^*(\min(x,y)) = \max(h^*(x), h^*(y)),$$
$$h^*(\max(x,y)) = \min(h^*(x), h^*(y)),$$

- Any fuzzy implication $\longrightarrow$ is non-increasing in the first variable and non-decreasing in the second variable.

□

1.3 Fuzzy Rule Bases

A fuzzy rule base is a way of representing the knowledge about a system under consideration. It is with the help of this rule base and a given current input one draws inferences in a fuzzy infernce system. Thus, fuzzy rule bases form one of the basic building blocks of a fuzzy inference mechanism. In this section, we look into both a single fuzzy rule and a set of if-then fuzzy rules, which is called a fuzzy rule base, in detail.

1.3.1 Fuzzy Rule

Let $X, Y \subseteq \mathbb{R}$ be two nonempty sets and $\mathcal{F}(X)$, $\mathcal{F}(Y)$ be the set of all fuzzy sets on X and Y, respectively. A fuzzy IF-THEN rule is of the form

$$\mathcal{R}(A, B) : \text{IF } \tilde{x} \text{ is } A \text{ THEN } \tilde{y} \text{ is } B\ , \tag{1.10}$$

where $\tilde{x}, \tilde{y}$ are linguistic variables and $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ are linguistic expressions/values assumed by the linguistic variables over X, Y. A and B are also known as the antecedent and the consequent fuzzy sets of the rule base. For instance, consider the following rule,

$$\textbf{IF } Weather \text{ is } Hot \textbf{ THEN } Fanspeed \text{ is } High\ .$$

Here $Weather$ and $Fanspeed$ are the linguistic variables and Hot, $High$ are the linguistic values taken by the linguistic variables in a suitable domain, e.g. , $X = [15, 50]$ (degrees in Centigrade) and $Y = [300, 1000]$ (rpm).

1.3.2 Fuzzy Rule Base

Given two non-empty sets $X, Y \subseteq \mathbb{R}$, a Single-Input Single-Output (SISO) fuzzy IF-THEN rule base consists of rules of the form:

$$\mathcal{R}(A_i, C_j) : \textbf{IF } \tilde{x} \text{ is } A_i \textbf{ THEN } \tilde{y} \text{ is } C_j\ , \tag{1.11}$$

where $\tilde{x}$, $\tilde{y}$ are the linguistic variables, $A_i, i = 1, 2, \ldots n$ and $C_j, j = 1, 2, \ldots m$ are the linguistic values taken by the linguistic variables. These linguistic values are represented by fuzzy sets in their corresponding domains, i.e., $A_i \in \mathcal{F}(X), C_j \in \mathcal{F}(Y)$.

As an example,

$$\begin{aligned}
&\textbf{IF } Weather \text{ is } Hot \textbf{ THEN } Fanspeed \text{ is } High,\\
&\textbf{IF } Weather \text{ is } Cold \textbf{ THEN } Fanspeed \text{ is } Low,\\
&\textbf{IF } Weather \text{ is } Very\ Cold \textbf{ THEN } Fanspeed \text{ is } Very\ Low.
\end{aligned}$$

Here $Temperature$ and $Fanspeed$ are the linguistic variables. Hot, $Cold$ and $Very\ Cold$ are the linguistic values taken by the linguistic variable $Weather$ in a suitable domain. $High$, Low and $Very\ Low$ are the linguistic values taken by the linguistic variable $Fanspeed$ in a suitable domain.

Definition 1.3.1. *A fuzzy rule base is said to be a complete rule base if for any $x \in X$, there exists an $i \in \{1, 2, \ldots, n\}$ such that $A_i(x) > 0$.*

For a complete fuzzy rule base the antecedent fuzzy sets $\{A_i\}_{i=1}^n$ form a fuzzy partition of X. Let us denote this fuzzy partition of X as $\mathcal{P}_X$. Let the consequent fuzzy sets $\{C_j\}_{j=1}^m$ form a partition of Y. Let us denote this partition of Y as $\mathcal{P}_Y$. Note that the cardinalities of $\mathcal{P}_X$ and $\mathcal{P}_Y$ may not be same but still we can represent every complete rule base in the form of (1.12). For example, consider a complete rule base of the form:

$$
\begin{aligned}
&\textbf{IF } \tilde{x} \text{ is } A_1 \textbf{ THEN } \tilde{y} \text{ is } C_1 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_2 \textbf{ THEN } \tilde{y} \text{ is } C_2 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_3 \textbf{ THEN } \tilde{y} \text{ is } C_2 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_4 \textbf{ THEN } \tilde{y} \text{ is } C_3 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_5 \textbf{ THEN } \tilde{y} \text{ is } C_3 \ .
\end{aligned}
$$

Clearly $\mathcal{P}_X = \{A_1, A_2, A_3, A_4, A_5\}$ and $\mathcal{P}_Y = \{C_1, C_2, C_3\}$. We rename the consequent fuzzy sets such that the rule base takes the form:

$$
\begin{aligned}
&\textbf{IF } \tilde{x} \text{ is } A_1 \textbf{ THEN } \tilde{y} \text{ is } B_1 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_2 \textbf{ THEN } \tilde{y} \text{ is } B_2 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_3 \textbf{ THEN } \tilde{y} \text{ is } B_3 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_4 \textbf{ THEN } \tilde{y} \text{ is } B_4 \ , \\
&\textbf{IF } \tilde{x} \text{ is } A_5 \textbf{ THEN } \tilde{y} \text{ is } B_5 \ ,
\end{aligned}
$$

where $B_2 = B_3$ and $B_4 = B_5$.

Thus we can consider a fuzzy IF-THEN rule base as consisting of rules of the form:

$$\mathcal{R}(A_i, B_i) : \textbf{IF } \tilde{x} \text{ is } A_i \textbf{ THEN } \tilde{y} \text{ is } B_i, \ i = 1, 2, \ldots n, \tag{1.12}$$

where $A_i \in \mathcal{P}_X$, i=1, 2, ...n, form a partition on X and $B_i \in \mathcal{P}_Y$, i=1, 2, ...n, form a partition on Y, respectively. Note that not all B_i's may be distinct.

1.3.3 Implicative and Conjunctive Rule Bases

A fuzzy rule base of the form (1.12) can be viewed in two different ways, as explained in [15, 16]. When each of the rules is viewed as a *constraint*, i.e., when the rules in (1.12) are combined together as

$$
\begin{gathered}
\textbf{IF } \tilde{x} \text{ is } A_1 \textbf{ THEN } \tilde{y} \text{ is } B_1, \\
\vdots \\
\textbf{AND} \\
\vdots \\
\textbf{IF } \tilde{x} \text{ is } A_n \textbf{ THEN } \tilde{y} \text{ is } B_n,
\end{gathered} \tag{1.13}
$$

we have the conditional form (IF-THEN) of the rules. On the other hand, each of the rules can also be viewed as just pieces of data giving possible configurations or positive information, in which case they are combined as follows:

$$\begin{aligned}
\tilde{x} \text{ is } A_1 \textbf{ AND } \tilde{y} \text{ is } B_1&,\\
&\vdots\\
&\textbf{OR}\\
&\vdots\\
\tilde{x} \text{ is } A_n \textbf{ AND } \tilde{y} \text{ is } B_n&. \qquad (1.14)
\end{aligned}$$

Throughout this thesis we will consider conditional rule bases of the type presented in (1.13).

1.3.4 Monotone Rule Base

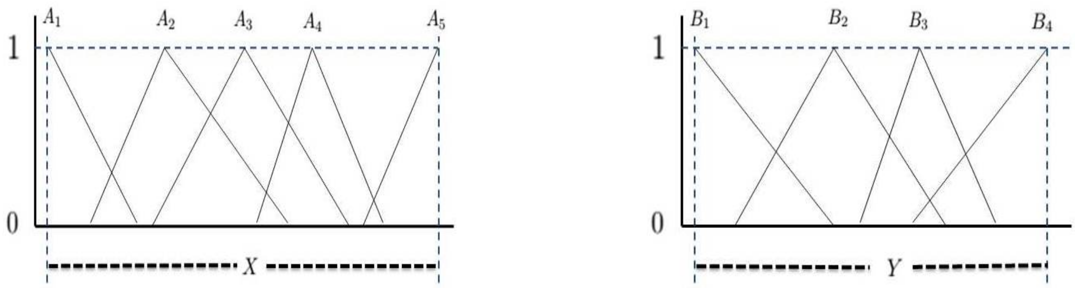
One can classify fuzzy rule bases into different types. For instance, in the above sections, we have seen two types of categorisation: (i) complete or incomplete and (ii) implicative or conjunctive. In this thesis, the following category of rule bases also play an important role.

Definition 1.3.2 ([46]). *A fuzzy rule base* (1.12) *is called* monotone *if for any two rules*

$$\textbf{\textit{IF}}\ \tilde{x} \textit{ is } A_i\ \textbf{\textit{THEN}}\ \tilde{y} \textit{ is } B_i\ ,$$
$$\textbf{\textit{IF}}\ \tilde{x} \textit{ is } A_j\ \textbf{\textit{THEN}}\ \tilde{y} \textit{ is } B_j\ ,$$

such that $A_i \prec A_j$, it also holds that $B_i \prec B_j$, where $\prec$ is as defined in Definition 1.1.5.

Let us consider the fuzzy sets $\{A_1, A_2, A_3, A_4, A_5\}$ as shown in Figure 1.9(a) and the the fuzzy sets $\{B_1, B_2, B_3, B_4\}$ as shown in Figure 1.9(b). It can be noted that $A_1 \prec A_2 \prec A_3 \prec A_4 \prec A_5$ and $B_1 \prec B_2 \prec B_3 \prec B_4$.



(a) Antecedent Fuzzy Sets (b) Consequent Fuzzy Sets

Figure 1.9: Antecedents and Consequents of Rule Base

Thus we see that the rule base of the type,

$$\textbf{IF } \tilde{x} \text{ is } A_1 \textbf{ THEN } \tilde{y} \text{ is } B_1\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_2 \textbf{ THEN } \tilde{y} \text{ is } B_2\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_3 \textbf{ THEN } \tilde{y} \text{ is } B_3\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_4 \textbf{ THEN } \tilde{y} \text{ is } B_3\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_5 \textbf{ THEN } \tilde{y} \text{ is } B_4\ ,$$

is a monotone rule base since $A_1 \prec A_2 \prec A_3 \prec A_4 \prec A_5$ and $B_1 \prec B_2 \prec B_3 \prec B_4$, where as the rule base of the type,

$$\textbf{IF } \tilde{x} \text{ is } A_1 \textbf{ THEN } \tilde{y} \text{ is } B_1\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_2 \textbf{ THEN } \tilde{y} \text{ is } B_3\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_3 \textbf{ THEN } \tilde{y} \text{ is } B_2\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_4 \textbf{ THEN } \tilde{y} \text{ is } B_1\ ,$$
$$\textbf{IF } \tilde{x} \text{ is } A_5 \textbf{ THEN } \tilde{y} \text{ is } B_4$$

is not a monotone rule base since the antecedents of the rules 1 and 2 are ordered as $A_2 \prec A_3$ but their corresponding consequents are ordered differently, $B_3 \not\prec B_2$.

1.4 Fuzzy Inference Mechanism

Approximate reasoning, as introduced by Zadeh in his early papers on fuzzy logic [55, 56], has been paraphrased thus by Hellendoorn [18]: "*Approximate reasoning in its broadest sense is a collection of techniques for dealing with inference under uncertainty in which the underlying logic is approximate or probabilistic rather than exact or deterministic.*"

An Inference mechanism in approximate reasoning can be seen as a function which derives a meaningful output from imprecise inputs. Approximate reasoning schemes involving fuzzy sets are one of the best known applications of fuzzy logic in the wider sense. Given a rule base of the form (1.12) and an input "$\tilde{x}$ is A' ", the main objective of a fuzzy inference mechanism is to find a *meaningful* B' such that " $\tilde{y}$ is B' ". The mechanism of the inference is shown in Figure 1.10.

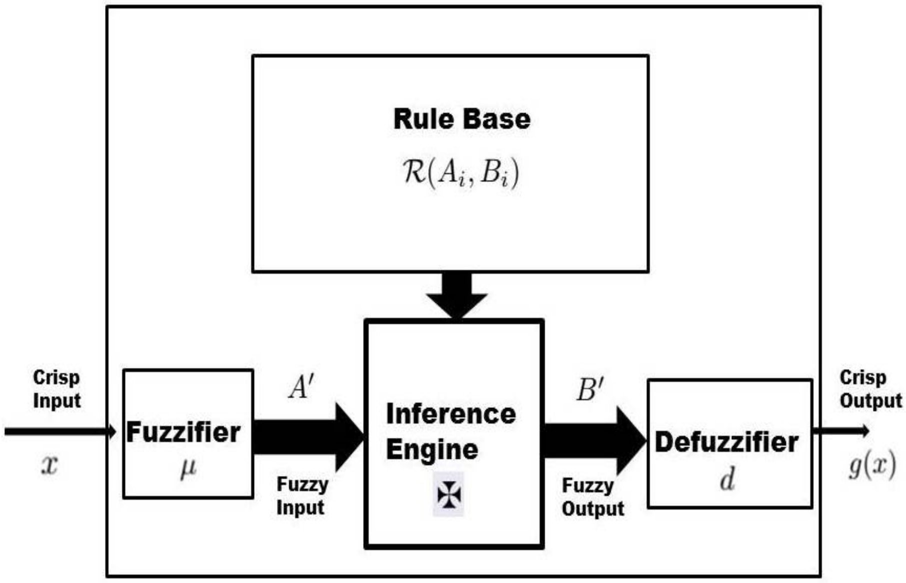


Figure 1.10: Fuzzy Inference Mechanism

A Fuzzy Inference Mechanism (FIM) can be represented in the following form:

$$\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \maltese, \mathcal{R}\big(A_i, B_i\big), \mu, d\Big)\ , \tag{1.15}$$

where

- $\mathcal{R}\big(A_i, B_i\big)$ represents the rule base (1.12) ,
- $A_i \in \mathcal{P}_X$, $B_i \in \mathcal{P}_Y$, i=1, 2, ... n, are the corresponding antecedents and consequents of the rule base which form fuzzy partitions on the spaces X and Y, respectively,
- $\maltese$ consists of some operations that form the core of the inference engine,
- $\mu : X \longrightarrow \mathcal{F}(X)$ is a fuzzifier that converts a crisp input into a fuzzy input, and
- $d : \mathcal{F}(Y) \longrightarrow Y$ is the defuzzifier which converts the fuzzy output into a crisp output.

1.4.1 Fuzzy Inference Mechanism: A Functional View

A fuzzy inference mechanism can be seen as a function from the space of fuzzy sets $\mathcal{F}(X)$ to the space of fuzzy sets $\mathcal{F}(Y)$ as follows:

$$\maltese : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y).$$

Alternatively, it can also be seen as a function from the space X to the space Y as follows:

$$g : X \xrightarrow{\mu} \mathcal{F}(X) \xrightarrow{\maltese} \mathcal{F}(Y) \xrightarrow{d} Y\ .$$

In the literature, g is known as the *system function* of the fuzzy inference mechanism.

Chapter 2

Fuzzy Relational Inference Mechanism

Beyond a doubt truth bears the same relation to falsehood as light to darkness.
–Leonardo da Vinci (1452–1519)

As was noted in the earlier chapter, an inference mechanism in approximate reasoning can be seen as a function which derives a meaningful output from imprecise inputs. Many kinds of inference mechanisms using fuzzy set theory and their logical connectives have been studied in the literature [7, 14, 25, 56]. Fuzzy relational inferences, which use fuzzy relations to model the given rule base, occupy a central position in approximate reasoning using fuzzy sets.

In this chapter, after reviewing fuzzy relations and discussing how they are often used to model the fuzzy rule bases, we discuss fuzzy relational inference (FRI) mechanisms in detail, including two of the well-known FRIs, namely, the Compositional Rule of Inference (CRI) proposed by Zadeh [26, 56], and the Bandler-Kohout Subproduct (BKS) proposed by Pedrycz [34] based on the earlier work of Bandler and Kohout [7]. After presenting a small survey on the suitability of these FRIs, we finally state our motivation and specify the scope of the work contained in this thesis.

2.1 Fuzzy Relations

A crisp binary relation among classical sets X, Y is a subset of $X \times Y$ denoted as $\rho(X, Y)$, i.e, $\rho(X, Y) \subseteq X \times Y$. Each classical relation can be defined by its characteristic function $\rho(X, Y) : X \times Y \to \{0, 1\}$ as:

$$\rho(x, y) = \begin{cases} 1, & \text{iff } (x, y) \in \rho \, , \\ 0, & \text{iff } (x, y) \notin \rho \, . \end{cases}$$

A fuzzy relation on $X \times Y$ is a generalization of the crisp relation.

Definition 2.1.1. *A fuzzy binary relation R is a mapping from $X \times Y$ to $[0, 1]$, i.e, $R : X \times Y \to [0, 1]$.*

In essence a fuzzy relation can be viewed as a fuzzy set defined on the Cartesian product of crisp sets X, Y.

2.2 Fuzzy Relations and Fuzzy Rulebases

Let us consider the rule bases of the type (1.13) and (1.14).

In fuzzy relational inference mechanisms, fuzzy relations $R : X \times Y \to [0,1]$ are employed to represent the rule base (1.13) and (1.14). Two of the commonly employed fuzzy relations are the following: For any $x \in X, y \in Y$,

$$\hat{R}_{\to}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow B_i(y)) \,, \tag{2.1}$$

$$\check{R}_{\star}(x,y) = \bigvee_{i=1}^{n} (A_i(x) \star B_i(y)) \,, \tag{2.2}$$

where $\longrightarrow$ is taken as a fuzzy implication and $\star$ as a t-norm.

Note that the fuzzy relation $\hat{R}_{\to}$ captures the conditional form (1.13) of the given rules, while the relation $\check{R}_{\star}$ captures the Cartesian product form (1.14) of the rules. For more on the semantics of $\check{R}_{\star}$ and $\hat{R}_{\to}$ we refer the readers to [15, 16].

2.3 Fuzzy Relational Inference

Many types of fuzzy inference mechanisms have been proposed in the literature, see for instance, [7, 14, 25, 56], etc. We restrict this study to fuzzy relation based inference mechanisms.

Let X, Y be two nonempty sets. Let us also consider the the rule base $\mathcal{R}(A_i, B_i)$ as in (1.12) and let the fuzzy relation $R : X \times Y \to [0,1]$, i.e, $R \in \mathcal{F}(X \times Y)$ representing or modeling the rule base $\mathcal{R}(A_i, B_i)$ be given. For a given input $A' \in \mathcal{F}(X)$, the output $B' \in \mathcal{F}(Y)$ can be obtained by a fuzzy relational inference mechanism, which can be expressed as follows:

$$B' = f_R^{@}(A') = A' @ R \,, \tag{FRI-R}$$

where $@$ is called the *composition operator*, which is a mapping $@ \colon \mathcal{F}(X) \times \mathcal{F}(X \times Y) \to \mathcal{F}(Y)$. An FRI can be represented by the following pentuple:

$$\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, @, R, d) \,, \tag{2.3}$$

where $\mathcal{P}_X, \mathcal{P}_Y$ are fuzzy partitions on X and Y, respectively, R and $@$ are as mentioned above, and $d : \mathcal{F}(Y) \longrightarrow Y$ is the defuzzification function which converts the output fuzzy set to a crisp value.

On comparing (2.3) with (1.15), we see that one can identify ✠ with the function $f_R^{@} : \mathcal{F}(X) \to \mathcal{F}(Y)$ associated with the given FRI (2.3). Hence, in the sequel, we often refer to $f_R^{@}$ also as the inference function associated with the FRI $\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, @, R, d)$. In the literature, one employs the term FRI to mean both the system $\mathbb{F}$ and its associated inference function $f_R^{@}$, a practice we follow in this thesis too. Note also that the fuzzifier μ is not explicitly specified since we deal with fuzzy inputs $A' \in \mathcal{F}(X)$.

2.3.1 Compositional Rule of Inference

One of the two main FRIs is the Compositional Rule of Inference (CRI) proposed by Zadeh [56] and is given as follows:

$$\begin{aligned} B'(y) &= f_R^{\circ}(A') = A'(x) \circ R(x,y) \\ &= \bigvee_{x \in X} \left[A'(x) \star R(x,y)\right] , \qquad y \in Y , \end{aligned} \tag{CRI-R}$$

where $\star$ is a t-norm. The operator $\circ$ is also known as the $\sup -T$ composition where T is a t-norm. Note that $f_R^{\circ}(A')$ is also known as the *direct image* of A' over R [48]. Thus the structure of CRI can be represented as

$$\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \circ, R, d) .$$

2.3.2 Bandler-Kohout Subproduct

Pedrycz [34] proposed another FRI mechanism based on the Bandler-Kohout Subproduct composition which is given as follows:

$$\begin{aligned} B'(y) &= f_R^{\lhd}(A') = A'(x) \lhd R(x,y) \\ &= \bigwedge_{x \in X} \left[A'(x) \longrightarrow R(x,y)\right] , \qquad y \in Y , \end{aligned} \tag{BKS-R}$$

with $\longrightarrow$ interpreted as a fuzzy implication. The operator $\lhd$ is also known as the $\inf -I$ composition where I is a fuzzy implication. Note that $f_R^{\lhd}(A')$ is also known as the *sub-direct image* of A' over R [48]. Thus the structure of BKS can be represented as

$$\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \lhd, R, d) .$$

2.3.3 Singleton Inputs and FRIs with Redicible Composition

Often one needs to deal with crisp inputs, viz., an $x_0 \in X$. In such a case, it is suitably *fuzzified*, i.e., a fuzzy set $A' \in \mathcal{F}(X)$ is suitably constructed from x_0. Commonly, the following *singleton* fuzzifier $\mu_s : X \longrightarrow \mathcal{F}(X)$ is employed to obtain a fuzzy input $A' \in \mathcal{F}(X)$. For any $x_0 \in X$,

$$\mu_s(x_0) = A'(x) = \begin{cases} 1, & x = x_0 , \\ 0, & x \neq x_0 . \end{cases}$$

With the above input A', the FRI mechanisms (CRI-R) and (BKS-R) reduce to

$$B'(y) = R(x_0, y) , \qquad y \in Y , \tag{FRI-R-Singleton}$$

for any t-norm $\star$ in case of (CRI-R) and any implication I satisfying (NP) in case of (BKS-R). Thus, in the case of a singleton input, both the (CRI-R) and (BKS-R) are essentially the same (provided $\longrightarrow$ in (BKS-R) satisfies (NP)) and the output is fully dependent on the model of the rule base R. In other words, $f_R^{\circ} \equiv f_R^{\lhd} \equiv f_R$ and hence the composition $\circ$ or $\lhd$ - when the I in $\lhd = \inf -I$

composition satisfies (NP) - does not play any role.

In this thesis, we call such an FRI $\mathbb{F}$ to be an *FRI with reducible composition*, i.e., those FRIs whose output, in the case of singleton inputs with singleton fuzzification μ, does not depend on the underlying composition operation and hence $f_R^{@} \equiv f_R$.

In all the subsequent analysis contained in the rest of this thesis, we either consider inputs to be fuzzy sets, or if the input is crisp, we use a singleton fuzzifier μ_s to obtain a fuzzy input. Thus for enhanced readability and shorter notation, the role of the fuzzifier μ will not be highlighted.

2.3.4 The System Function g of an FRI

From the above, it is clear that an FRI can be seen as mapping $f_R^{@} : \mathcal{F}(X) \to \mathcal{F}(Y)$. When the given input is a fuzzy set $A' \in \mathcal{F}(X)$ we obtain a fuzzy output $B' = f_R^{@}(A') \in \mathcal{F}(Y)$. One can also apply the defuzzifier on B' to obtain a final value $y = d(B') \in Y$.

In the case of singleton inputs, the overall inference of an FRI $\mathbb{F}$ can be seen as a function $g : X \to Y$ as follows:

$$g(x') = d(B'(\cdot)) = d(f_R^{@}(A'(x'))) \ , \ x' \in X \ . \tag{2.4}$$

Further, in the case of an FRI $\mathbb{F}$ with *reducible composition*, the overall inference reduces to the even simpler function

$$g(x') = d(B'(\cdot)) = d(R(x', \cdot)) \ , \ x' \in X \ . \tag{2.5}$$

In the literature, g is also known as the system function of a given $\mathbb{F}$, see for instance, [23, 24].

2.4 Desirable Properties of an FRI

A fuzzy inference mechanism has many degrees of freedom, viz., the underlying fuzzy partition of the input and output spaces, the fuzzy logic operations employed, the fuzzification and defuzzification techniques used, etc. This freedom gives rise to a variety of FIMs with differing capabilities.

While dealing with a fuzzy relational inference mechanism (FRI), the operators employed in it can be picked from a plethora of choices. However, the question that arises is whether an FRI with a particular choice of operators is good. Once again, the 'goodness' of an FRI itself can be measured against different parameters. In the literature, some measures of goodness have been proposed which we discuss in brief below, specifically with respect to FRIs. Note that these measures are applicable, in general, to any FIM. However, since in this thesis we restrict our study only to FRIs, the following discussion is given in the context of FRIs.

Interpolativity: Interpolativity is one of the most fundamental properties of an inference mechanism. An FRI is said to be *interpolative* if the following is valid: Whenever an antecedent of a rule A_i is given as the input, the corresponding consequent B_i should be the inferred output, i.e.,

$$B_i = f_R^{@}(A_i) = A_i @ R, \qquad i = 1, 2 \ldots n., \ A_i \in \mathcal{F}(X), \ R \in \mathcal{F}(X \times Y).$$

In the case of FRIs, interpolativity pertains to the solvability of the fuzzy relational equations corresponding to the system, which will be dealt with in detail in Chapter 4.

Continuity: In the literature, the continuity of an FRI is discussed at the level of $f_R^@$, i.e., when the inputs are fuzzy sets. In this context, the continuity of an FRI can be seen as follows: If the given fuzzy input is close to some antecedent of the rule base then continuity insists that the obtained fuzzy output is also close to the corresponding consequent fuzzy set.

- Similar to the classical definition of continuity, we say that an FRI $\mathbb{F}$ or its associated inference function $f_R^@$ is continuous at A_i if, for any given $\epsilon > 0$, we have a $\delta > 0$ such that, $D_X(A_i, A) < \delta \Longrightarrow D_Y\left(B_i, f_R^@(A)\right) < \epsilon$, where D_X and D_Y are suitable metrics on $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, respectively.
- We say that an FRI $\mathbb{F}$ or its associated inference function $f_R^@$ is continuous at $\mathcal{P}_X = \{A_i\}_{i=1}^n$, if it is continuous at A_i for every $i \in \{1, 2, \ldots, n\}$.

Robustness: Robustness deals with how errors in the premises affect the conclusions. It is different from continuity in that, we expect that even when the actual input fuzzy set is not close to the intended fuzzy set but somehow are equivalent - in a certain predefined sense based on the equality relations on the underlying set - the output of the actual fuzzy set is also close to the corresponding intended output. Maximum possible robustness is achieved by reducing the sensitivity of the inference mechanism to input variations to a satisfactory level, i.e., the output should not be too sensitive to *unwanted* variations in the input.

Universal Approximation: In certain contexts, like in control systems, a fuzzy inference system is essentially a function approximator. Thus it is imperative to discuss its approximation capabilities, i.e., whether it can approximate any continuous function over a compact set to arbitrary accuracy. In other words, the question we try to answer here is the following:

Let an $\epsilon > 0$ and a continuous function $h\colon [a, b] \to \mathbb{R}$ over a closed interval $[a, b]$ be given. Does there exist

(i) a fuzzy partition $\mathcal{P}_X$ on $X = [a, b]$,

(ii) a fuzzy partition $\mathcal{P}_Y$ on $Y = h([a, b])$,

(iii) an appropriate rule base $\mathcal{R}(A_i, B_i)$ where $A_i \in \mathcal{P}_X, B_i \in \mathcal{P}_Y$ for $i = 1, 2, \ldots n$,

(iv) a suitable fuzzy relation R that models the rulebase $\mathcal{R}(A_i, B_i)$,

(v) a defuzzifier d,

so that the FRI $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, @, R, d\right)$ with system function g, as defined in (2.4) or (2.5), is such that

$$\max_{x \in [a,b]} |h(x) - g(x)| \le \epsilon \, .$$

Monotonicity: Monotonicity of an FRI refers to whether given a monotone rule base and monotonic inputs we obtain monotonic outputs. Let us be given a monotone rule base, i.e., a rule base $\mathcal{R}(A_i, B_i)$ where the antecedents $A_i \in \mathcal{P}_X$ and consequents $B_i \in \mathcal{P}_Y$ are such that they maintain the same ordering as explained in Section 1.3.4.

The question now is the following:

Does there exist an FRI $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, @, R, d\Big)$ with a suitable fuzzy relation R that models the rule base $\mathcal{R}(A_i, B_i)$, and a defuzzifier d, such that for any two crisp inputs x' and x'' with $x' \leq x''$, the corresponding outputs are such that $g(x') = y' \leq y'' = g(x'')$?

2.5 Works on the Suitability of FRIs

There have been many works that have dealt with the suitability of FRIs vis-á-vis the desirable properties listed above. In this section we present a brief review of some of these works, with our focus being able to draw a thread that will eventually lead up to the main motivation and objectives of the work contained in this thesis.

2.5.1 Interpolativity, Continuity and Robustness of FRIs

Interpolativity of an FRI essentially relates to the solvability of the underlying fuzzy relational equations. The works of Di Nola *et al.* [31] and Sanchez [41] could be seen as the earliest works dealing with interpolativity of CRI, while that of Nosková [32] clearly deals with the interpolativity of BKS inference mechanism.

It was Perfilieva and Lehmke [36] who discussed the conditions under which the CRI mechanism is continuous. In fact, they showed that under these conditions, continuity and interpolativity are equivalent. The robustness of CRI was dealt with by Klawonn and Castro [21]. Later on Štěpnička and Jayaram [49] have undertaken a similar study for the BKS inference mechanism.

2.5.2 Approximation Properties of FRIs

As can be seen from Section 2.4 above, both the universal approximation capability and monotonicity of an FRI are discussed in the context of singleton or crisp inputs (with singleton fuzzification). Since CRI and BKS are FRIs with reducible composition (in this context), the composition operator $@$ does not a play a role (see Section 2.3.3). Hence most of the works that deal with universal approximation, do not distinguish between CRI and BKS. Note that most of these studies consider the fuzzy implication I in the $\inf -I$ composition of BKS to come from the family of R-implications (Definition 1.2.10) which possesses (NP), and hence such an assumption is justified.

Thus, once the partitions $\mathcal{P}_X, \mathcal{P}_Y$ and the rule base $\mathcal{R}(A_i, B_i)$ are formed, an investigation into the approximation properties of an FRI boils down to investigating the fuzzy relation R that models the rule base and the defuzzifier d.

It should also be mentioned that, while many studies have appeared on this topic, most of them deal with FRIs where the rules are interpreted in a non-conditional way or as just aggregation of possibile configurations of the data (see Sections 1.3 and 2.1 for details). When an implicative or a conditional interpretation of the rules are considered, there are only a few works dealing with their approximation properties.

The earliest works to study the approximation capabilities of FRIs can be traced to the works of Wang [50] and Zeng and Singh [57], where the fuzzy relation used to model the rule base is $R = \check{R}_\star$ (as in (2.2)) and hence can be considered to have assumed a Cartesian product interpretation of the fuzzy rules. For some recent works on this topic, please see [35], [53] and the references therein.

When considering the implicative model of the rule base, one of the earliest studies on this topic was that of Castro [10], [11]. Interestingly, it was later on shown by Li *et al.* [24], that some of the operations considered by Castro led to vacuous outputs. Li *et al.* further went on to present their own results with the fuzzy relation $R = \hat{R}_{\rightarrow}$ modeling the rule base (see Section 2.1). While their scope does encompass a few families of fuzzy implications, explicit proofs are given only for the family of R-implications obtained from left-continuous t-norms.

Recently, Štěpnička *et al.* [48] have discussed the same in a slightly more general setting. Specifically, they have considered the following fuzzy relation to model the rule base: $R = R^{\otimes}_{\rightarrow_*}$ where $\otimes$ is the Łukasiewicz t-norm $T_{\mathbf{LK}}$ as in Table 1.2.and $\rightarrow_*$ is any residuated implication obtained from a left-continuous t-norm $*$, which can be different from $T_{\mathbf{LK}}$. Once again, the assumptions they make on some components of the FRIs are not desirable - for instance, the requirements on the input partition make it non-Ruspini.

2.5.3 Monotonicity of FRIs

That a system function g of an FRI $\mathbb{F}$ may not be monotonic is, in itself, a very recently discovered phenomenon. It was Broekhoven and De Baets [43], who studied this interesting aspect by discussing the monotonicity of FRIs, where the rules are interpreted in a non-conditional way with d = MOM, the Mean Of Maxima defuzzification. In a follow-up work, the authors have proven the existence of monotonicity with d = COG, the center of gravity defuzzification in [44]. However, the results have been presented for some specific operators and hence lack in generality. Very recently, Stepnicka and De Baets [46, 47] have taken up the the monotonocity problem where implicative or a conditional interpretation of the rules are considered. It should be mentioned, once again, that the fuzzy implication $\longrightarrow$ employed in the fuzzy relation R modeling the rule base is taken to be the residual of a left-continuous t-norm.

2.6 Motivation behind the work in this thesis

In all the existing works on FRI dealing with desirable properties, the fuzzy logic operations employed, either as part of the underlying composition @ or in the fuzzy relation R modeling the rule base, come from a residuated lattice structure. For instance, the t-norms considered are usually left-continuous on $[0,1]^2$ and the fuzzy implications are, almost always, residuals of such t-norms, i.e., come from the family of R-implications.

From the proofs of the results in the above works, it can be seen that many of these desirable properties of FRI are due to the rich underlying structure, viz., the residuated algebra, that lends its operations very many nice properties. In fact, the (t-norm, fuzzy implication) pair of (T, I) form an adjoint couple [17] and hence are immediately endowed with a long list of useful properties.

The question that naturally arises now is the following:

Let us consider an FRI where the operations employed, either in the underlying composition @ or in the fuzzy relation R modeling the rule base, do not come from a residuated structure on $[0,1]$. Does such an FRI $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, @, R, d\right)$ possess the above desirable properties ? If not, what are the conditions under which these properties are satisfied?

This forms the main motivation for the work contained in this thesis.

2.7 Scope of the work in this thesis

In line with our stated motivation, we investigate the Bandler-Kohout Subproduct inference mechanism, where the fuzzy implication $\longrightarrow$ used in the composition $\lhd$ (BKS-R) is not obtained as the residual of a t-norm, i.e. the fuzzy implication I employed in the $\inf -I$ composition is not an R-implication as defined in Definition 1.2.10.

While there exist many families of fuzzy implications, we restrict the scope of this work to considering the Yager's families of fuzzy implications, viz., f- and g-implications proposed by Yager [52]. The choice of these families of fuzzy implications is two fold : On the one hand, these families have been well-established in the literature over the last decade [1, 20, 28, 39, 45, 51], and on the other hand, the intersection between the families of R-implications and Yager's implications is almost neglible. In fact, it is well known that no f-implication is an R-implication [1, 2] and the only R-implication that is also a g-implication is the Goguen implication $I_{\mathbf{G}}$ [1, 2] .

Further, in this thesis we deal only with the conditional form of rulebases, i.e., of the type presented in (1.13). In essence, we discuss the desirable properties of an FRI of the form:

$$\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \lhd_{\mathcal{Y}}, R, d) \ ,$$

where $\lhd_{\mathcal{Y}}$ denotes that the fuzzy implications I considered in the $\inf -I$ composition comes from the Yager's families of fuzzy implications.

It should also be pointed out that both the f- and g-implications satisfy (NP), and hence when considering singleton inputs with singleton fuzzification, the FRI $\mathbb{F}$ does become one with reducible composition, thus making itself amenable for further analysis.

Finally, note that we deal with only Single Input Single Output (SISO) rule bases and hence the whole study can be seen as dealing with SISO fuzzy inference systems.

Chapter 3

Bandler-Kohout Subproduct with Yager's Families of Fuzzy Implications

Every new body of discovery is mathematical in form, because there is no other guidance we can have.

– Charles Darwin (1809–1882)

Yager [52] introduced two families of fuzzy implications named f- and g-generated implications based on strictly monotonic functions on $[0,1]$. As noted in Chapter 2, we consider these families of fuzzy implications instead of the usual R-implications in the BKS inference mechanism.

In this chapter we recall the definitions of these two families of fuzzy implications, in Sections 3.1 and 3.2, and also discuss some of their properties, which are relevant in this thesis. Following this, we propose the Bandler-Kohout Subproduct with the Yager's families of fuzzy implications, where we employ these two families of fuzzy implications. Finally, in Section 3.4, we specify the fuzzy relation R that we will consider to model the given implicative rule base.

3.1 Yager's Family of f-implications

In this section, we introduce the Yager's family of f-implications. It is well known that no f-implication can be obtained as a residual of a left-continuous t-norm and hence this family of fuzzy implications do not impose a residuated structure on $[0,1]$ with any t-norm T. We also present some relevant properties and results involving this family, which will play a crucial role in the rest of this thesis.

Definition 3.1.1 ([2], Definition 3.1.1)**.** *Let* $f\colon [0,1] \to [0,\infty]$ *be a strictly decreasing and continuous* *function with $f(1) = 0$. The function $I_f \colon [0,1]^2 \to [0,1]$ defined by*

$$I_f(x, y) = f^{-1}\left(x \cdot f(y)\right) , \qquad x, y \in [0,1] , \tag{3.1}$$

*with the understanding $0 \cdot \infty = 0$, is a fuzzy implication and called an f-*implication. *The function f itself is called an f-generator of the I_f generated as in* (3.1).

We will often write $\longrightarrow_f$ instead of I_f. It is worth mentioning that f can also be seen as an additive generator of some continuous Archimedean t-norm [22].

Example 3.1.2 (cf. [2]). *Table 3.1 lists few of the f-implications along with their generators from which they have been obtained.*

f-generator f	f-implication I_f
$f(x) = -\ln x$	$I_{\mathbf{YG}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ and } y > 0 \end{cases}$
$f(x) = 1 - x$	$I_{\mathbf{RC}}(x, y) = 1 - x + xy$
$f_{\mathbf{c}}(x) = \cos(\frac{\pi}{2}x)$	$I_{f_{\mathbf{c}}}(x, y) = \frac{2}{\pi}\cos^{-1}\left(x \cdot \cos\left(\frac{\pi}{2}y\right)\right)$
$f^s(x) = -\ln\left(\frac{s^x - 1}{s - 1}\right)$, $s > 0$, $s \neq 1$	$I_{f^s}(x, y) = \log_s\left(1 + (s-1)^{1-x}(s^y - 1)^x\right)$
$f^\lambda(x) = (1 - x)^\lambda$, where $\lambda \in (0, \infty)$	$I_{f^\lambda}(x, y) = 1 - x^{\frac{1}{\lambda}}(1 - y)$

Table 3.1: Examples of f-implications

Remark 3.1.3. *Note that other basic fuzzy implications $I_{\mathbf{LK}}$, $I_{\mathbf{GD}}$, $I_{\mathbf{KD}}$, $I_{\mathbf{GG}}$, $I_{\mathbf{RS}}$, $I_{\mathbf{WB}}$ and $I_{\mathbf{FD}}$ from Table 1.5 are not f-implications.*

Proposition 3.1.4 (cf. [2], Theorem 3.1.4). *Let $f_1, f_2 \colon [0,1] \to [0, \infty]$ be any two f-generators. Then the following statements are equivalent:*

(i) $I_{f_1} = I_{f_2}$.

(ii) There exists a constant $c \in (0, \infty)$ such that $f_2(x) = c \cdot f_1(x)$ for all $x \in [0,1]$.

Remark 3.1.5 (cf. [2], Remark 3.1.5). *From the above result it follows that if f is an f-generator such that $f(0) < \infty$, then the function $f_1 \colon [0,1] \to [0,1]$ defined by*

$$f_1(x) = \frac{f(x)}{f(0)} , \qquad x \in [0,1] , \tag{3.2}$$

is a well defined f-generator such that $I_f = I_{f_1}$ and $f_1(0) = 1$. In other words, it is enough to consider only decreasing generators for which $f(0) = \infty$ or $f(0) = 1$.

Let $\mathbb{I}_{\mathbb{F}}$ denote the set of all f-implications proposed by Yager [52]. Further, let us denote by

- $\mathbb{I}_{\mathbb{F},\infty} \subsetneq \mathbb{I}_{\mathbb{F}}$ - the set of f-implications that are generated from generators such that $f(0) = \infty$,
- $\mathbb{I}_{\mathbb{F},1} \subsetneq \mathbb{I}_{\mathbb{F}}$ - the set of f-implications that are generated from generators such that $f(0) = 1$.

Clearly $\mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F},\infty} \bigcup \mathbb{I}_{\mathbb{F},1}$.

Proposition 3.1.6 ([2], Proposition 3.1.6). *Let f be an f-generator.*

(i) If $I_f \in \mathbb{I}_{\mathbb{F},\infty}$, then the natural negation N_{I_f} is the Gödel negation $N_{\mathbf{D1}}$.

(ii) If $I_f \in \mathbb{I}_{\mathbb{F},1}$, then the natural negation N_{I_f} is a strict negation.

Proposition 3.1.7 (cf. [2], Theorem 3.1.7)**.** *If f is an f-generator, then*

(i) I_f satisfies (NP).

(ii) $I_f(x,y) = 1$ if and only if $x = 0$ or $y = 1$, i.e., I_f does not satisfy (OP).

(iii) I_f satisfies the law of importation (LI) *with the product t-norm, $T_{\mathbf{P}}(x,y) = xy$.*

Proposition 3.1.8. *The equation* (1.8) *is valid for an arbitrary index set $\mathcal{I}$ when $\longrightarrow$ is any f-implication.*

Proof.

$$\begin{aligned}
\text{L. H. S. of (1.8)} &= \bigvee_{i\in\mathcal{I}} (x_i) \longrightarrow_f y = f^{-1}\left(\bigvee_{i\in\mathcal{I}} (x_i)\cdot f(y)\right) \\
&= f^{-1}\left(\bigvee_{i\in\mathcal{I}} (x_i \cdot f(y))\right) = \bigwedge_{i\in\mathcal{I}} f^{-1}\left((x_i \cdot f(y))\right) \\
&= \bigwedge_{i\in\mathcal{I}} (x_i \longrightarrow_f y) = \text{R. H. S. of (1.8)}\ .
\end{aligned}$$

□

3.2 Yager's Family of g-implications

In this section, we introduce the second of the Yager's families of fuzzy implications, viz., the g-implications. The Goguen implication $I_{\mathbf{G}}$ is the only g-implication which can also be obtained as a residual of the continuous t-norm $T_{\mathbf{P}}$. Thus, no other member of this family of fuzzy implications imposes a residuated structure on $[0,1]$ with any t-norm T. We also present some relevant properties and results involving this family, which will play a crucial role in the rest of this thesis.

Definition 3.2.1 ([2], Definition 3.2.1)**.** *Let $g\colon [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I_g\colon [0,1]^2 \to [0,1]$ defined by*

$$I_g(x,y) = g^{(-1)}\left(\frac{1}{x}\cdot g(y)\right), \qquad x,y \in [0,1], \tag{3.3}$$

*with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is a fuzzy implication and called a g-*implication, *where the function $g^{(-1)}$ in* (3.3), *called the pseudo-inverse of g is given by,*

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0, g(1)]\ , \\ 1, & \text{if } x \in [g(1), \infty] \end{cases} = g^{-1}\left(\min\left(\frac{1}{x}\cdot g(y), g(1)\right)\right).$$

The function g itself is called a g-generator of the I generated as in (3.3). We will often write $\longrightarrow_g$ instead of I_g.

Example 3.2.2 (cf. [2])**.** *Table 3.2 lists few of the g-implications along with their generators from which they have been obtained.*

g-generator g	g-implication I_g
$g(x) = -\ln(1-x)$	$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ 1-(1-y)^{\frac{1}{x}}, & \text{otherwise} \end{cases}$
$g(x) = x$	$I_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \dfrac{y}{x}, & \text{if } x > y \end{cases}$
$g(x) = -\dfrac{1}{\ln x}$	$I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ and } y > 0 \end{cases}$
$g_{\mathbf{t}}(x) = \tan\left(\frac{\pi}{2}x\right)$	$I_{g_{\mathbf{t}}}(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{x}\cdot\tan\left(\dfrac{\pi}{2}y\right)\right)$
$g^s(x) =$ $-\ln\left(\frac{s^{1-x}-1}{s-1}\right)$, $s > 0$, $s \neq 1$	$I_{g^s}(x,y) =$ $1 - \log_s\left(1 + (s-1)^{\frac{x-1}{x}}(s^{1-y}-1)^{\frac{1}{x}}\right)$

Table 3.2: Examples of g-implications

Remark 3.2.3. *Note that $I_{\mathbf{LK}}$, $I_{\mathbf{GD}}$, $I_{\mathbf{RC}}$, $I_{\mathbf{KD}}$, $I_{\mathbf{RS}}$, $I_{\mathbf{WB}}$ and $I_{\mathbf{FD}}$ from Table 1.5 are not g-implications.*

Theorem 3.2.4 ([2], Theorem 3.2.5)**.** *Let $g_1, g_2 \colon [0,1] \to [0,\infty]$ be any two g-generators. Then the following statements are equivalent:*

(i) $I_{g_1} = I_{g_2}$.

(ii) There exists a constant $c \in (0,\infty)$ such that $g_2(x) = c \cdot g_1(x)$ for all $x \in [0,1]$.

Remark 3.2.5 ([2], Remark 3.2.6)**.** *From the above result it follows that, if g is a g-generator such that $g(1) < \infty$, then the function $g_1 \colon [0,1] \to [0,1]$ defined by*

$$g_1(x) = \frac{g(x)}{g(1)}\,, \qquad x \in [0,1]\,, \tag{3.4}$$

is a well defined g-generator such that $I_g = I_{g_1}$ and $g_1(1) = 1$. In other words, it is enough to consider only increasing generators for which $g(1) = \infty$ or $g(1) = 1$.

Let $\mathbb{I}_{\mathbb{G}}$ denote the set of all g-implications. Further, let us denote by

- $\mathbb{I}_{\mathbb{G},\infty} \subsetneq \mathbb{I}_{\mathbb{G}}$ - the set of g-implications that are generated from generators such that $g(1) = \infty$,
- $\mathbb{I}_{\mathbb{G},1} \subsetneq \mathbb{I}_{\mathbb{G}}$ - the set of g-implications that are generated from generators such that $g(1) = 1$.

Clearly $\mathbb{I}_{\mathbb{G}} = \mathbb{I}_{\mathbb{G},\infty} \bigcup \mathbb{I}_{\mathbb{G},1}$.

Proposition 3.2.6 ([2], Proposition 3.2.7)**.** *Let g be a g-generator. If $I_g \in \mathbb{I}_{\mathbb{G}}$ then the natural negation N_{I_g} is the Gödel negation $N_{\mathbf{D1}}$.*

Proposition 3.2.7 ([2], Theorem 3.2.8)**.** *Let g be a g-generator.*

(i) I_g satisfies (NP).

(ii) If $g(1) = \infty$, then $I_g(x,y) = 1 \iff x = 0$ or $y = 1$, i.e., I_g does not satisfy (OP) *when $g(1) = \infty$.*

(iii) I_g satisfies the law of importation (LI) *with the product t-norm, $T_{\mathbf{P}}(x,y) = xy$.*

Proposition 3.2.8 ([2], Theorem 3.2.9). *If g is a g-generator, then the following statements are equivalent:*

(i) I_g *satisfies* (OP).

(ii) $g(1) < \infty$ *and there exists a constant* $c \in (0, \infty)$ *such that* $g(x) = c \cdot x$ *for all* $x \in [0, 1]$.

(iii) I_g *is the Goguen implication* $I_{\mathbf{G}}$.

3.3 Bandler-Kohout Subproduct with Yager's families of fuzzy implications

In this work, we consider the BKS inference mechanism, where the fuzzy implication is one of the Yager's families of implications. Essentially, we interpret the $\longrightarrow$ in (BKS-R) as an f- or g-implication and denote the modified BKS inference mechanism as $\triangleleft_f$ and $\triangleleft_g$, where $\triangleleft_f = \inf -I_f$ and $\triangleleft_g = \inf -I_g$ respectively. Specifically,

$$B'(y) = (A' \triangleleft_f R)(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow_f R(x, y)], \qquad y \in Y, \tag{BKS-f}$$

$$B'(y) = (A' \triangleleft_g R)(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow_g R(x, y)], \qquad y \in Y\ . \tag{BKS-g}$$

From Equation (FRI-R) we see that the above FRIs, viz., (BKS-f) and (BKS-g), can be denoted as $f_R^{\triangleleft_f}$ and $f_R^{\triangleleft_g}$, respectively.

The FRIs (BKS-f) and (BKS-g) can be represented as a pentuple:

$$\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_f, R, d)\ , \tag{3.5}$$

$$\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_g, R, d)\ , \tag{3.6}$$

where $\triangleleft_f$ and $\triangleleft_g$ are as mentioned above.

Note also that, due to Propositions 3.1.7(i) and 3.2.7(i), we see that both the f- and g-implications satisfy the neutrality property (NP) and hence in the case of singleton inputs with singleton fuzzification, we see that the FRIs (3.5) and (3.6) are both FRIs with reducible composition, i.e., the composition $\triangleleft_f$ and $\triangleleft_g$ do not play any role (see Section 2.3.3).

3.4 The Fuzzy Relation R modeling the rule base

As stated before, we limit our study to the implicative form of rules and, once again, with the implication relating the antecedents and consequents being an f- or g-implication. In specific terms, the fuzzy relation R representing the rule base is given as:

$$\hat{R}_f(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_f B_i(y)), \qquad x \in X, y \in Y\ , \tag{Imp-$\hat{R}_f$}$$

$$\hat{R}_g(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_g B_i(y)), \qquad x \in X, y \in Y\ . \tag{Imp-$\hat{R}_g$}$$

In summary, in this thesis, our main objects of study are the following FRIs:

$$\mathbb{F}_{\rightarrow_f} = (\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_f, \hat{R}_f, d) \ , \tag{3.7}$$

$$\mathbb{F}_{\rightarrow_g} = (\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_g, \hat{R}_g, d) \ . \tag{3.8}$$

For notational convenience, in this thesis, we often refer to both BKS-f and BKS-g inference mechanisms as BKS-$\mathcal{Y}$ inference mechanisms. Similarly, we use the notation

$$\mathbb{F}_{\rightarrow_\mathcal{Y}} = (\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_\mathcal{Y}, \hat{R}_\mathcal{Y}, d)$$

to refer to both $\mathbb{F}_{\rightarrow_f}, \mathbb{F}_{\rightarrow_g}$.

Part II

Suitability of BKS-$\mathcal{Y}$ Fuzzy Relational Inference Mechanism

Chapter 4

Interpolativity and Continuity of Bandler-Kohout Subproduct with Yager's Families of Fuzzy Implications

Mathematics is the art of giving the same name to different things.
– J. H. Poincare (1854–1912)

In this chapter, we study first of the two of the desirable properties, viz., interpolativity and continuity for the BKS-f and BKS-g inference mechanisms. Note that the conditions underwhich these desirable properties hold are known for BKS with residuated implications.

We recall that we deal only with the implicative form of the rule base, i.e., the antecedents of the rules are related to their consequents using a fuzzy implication and hence fix $R = \hat{R}_f$ and $\hat{R}_g$ in the sequel. Thus this work deals with FRIs of the form given in (3.7) and (3.8): $\mathbb{F}_{\to_f} = \left(\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_f, \hat{R}_f\right)$ and $\mathbb{F}_{\to_g} = \left(\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_g, \hat{R}_g\right)$.

The chapter is structured in the following way. Firstly, in Section 4.1, we define an extension of the well-known Goguen implication and discuss some of its properties, which will prove useful in the sequel. Following this we derive some necessary and sufficient conditions for interpolativity for the BKS-f and BKS-g inference mechanism in Section 4.2. In Section 4.3 after defining continuity suitably, we have shown that continuity is equivalent to interpolativity for both the modified inference mechanisms. Our study shows that interpolativity and continuity are available for both the modified inference mechanisms, thus adding more choice of operations under the BKS scheme.

4.1 Goguen implication and its Extension

In this section we recall the definition of the Goguen implication and its bi-implication and also present some of the properties it enjoys being a residual implication. Following this, we propose an extension of the Goguen implication and discuss some of its properties.

4.1.1 Goguen implication

Here we define Goguen implication and present some important properties possessed by it, which will be useful later for proving our results.

Definition 4.1.1. *(i) The Goguen implication, the residual of the product t-norm, $I_{\mathbf{G}} : [0,1]^2 \to [0,1]$ is defined as*

$$I_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \dfrac{y}{x}, & \text{if } x > y \end{cases}, \qquad x, y \in [0,1]\ .$$

We denote $I_{\mathbf{G}}$ by $\longrightarrow_{\mathbf{G}}$ for simplicity.

(ii) The bi-implication ([33], Equation 2.24, pg. 27) obtained from $I_{\mathbf{G}}$ is defined and denoted as follows:

$$x \longleftrightarrow_{\mathbf{G}} y = (x \longrightarrow_{\mathbf{G}} y) \wedge (y \longrightarrow_{\mathbf{G}} x) \ = \min\left\{1, \frac{x}{y}, \frac{y}{x}\right\}, \quad x, y \in [0,1]\ ,$$

with the understanding that $\frac{1}{0} = \infty$, $0 \cdot \infty = \infty$ and $\frac{0}{0} = \infty$.

Proposition 4.1.2 (cf. [33], Lemma 2.7). *For $a, b, c \in [0,1]$ and '$\longleftrightarrow_{\mathbf{G}}$' being a Goguen bi-implication and '$\cdot$' being the product t-norm, we have,*

$$(a \longleftrightarrow_{\mathbf{G}} b) \cdot (b \longleftrightarrow_{\mathbf{G}} c) \le (a \longleftrightarrow_{\mathbf{G}} c)\ . \tag{4.1}$$

Proposition 4.1.3 (cf. [33], Lemma 2.7). *Let $a_i, b_i \in [0,1]$ and $i \in \mathcal{I}$, an index set. Then for '$\longleftrightarrow_{\mathbf{G}}$' being a Goguen bi-implication the following inequalities are true:*

$$\left(\bigvee_{i\in\mathcal{I}} a_i\right) \longleftrightarrow_{\mathbf{G}} \left(\bigvee_{i\in\mathcal{I}} b_i\right) \ge \bigwedge_{i\in\mathcal{I}} (a_i \longleftrightarrow_{\mathbf{G}} b_i)\ , \tag{4.2}$$

$$\left(\bigwedge_{i\in\mathcal{I}} a_i\right) \longleftrightarrow_{\mathbf{G}} \left(\bigwedge_{i\in\mathcal{I}} b_i\right) \ge \bigwedge_{i\in\mathcal{I}} (a_i \longleftrightarrow_{\mathbf{G}} b_i)\ . \tag{4.3}$$

Remark 4.1.4. *In fact, Proposition 4.1.3 is true for any '$\longleftrightarrow$' coming from a residuated lattice structure, see [33], for more details.*

4.1.2 Extended Goguen Implication

In this section we modify the Goguen implication, by extending it as a map from $[0,1]^2 \to [0,1]$ to a map $[0,\infty]^2 \to [0,1]$, leaving the formula unchanged and call it the *Extended Goguen implication*. In the sequel, this function plays an important role in giving crisp expressions to many results and properties and hence we define it here and present some of its important properties.

Definition 4.1.5. *(i) The function* $I^*_{\mathbf{G}} : [0,\infty]^2 \to [0,1]$ *defined as*

$$I^*_{\mathbf{G}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \dfrac{y}{x}, & \text{if } x > y \end{cases} \quad , \qquad x, y \in [0,\infty] \ ,$$

is called the Extended Goguen *implication. We will also denote* $I^*_{\mathbf{G}}$ *by* $\xrightarrow{*}_{\mathbf{G}}$ *for better readability in proofs.*

(ii) The bi-implication [33] obtained from $I^*_{\mathbf{G}}$ *is defined and denoted as follows:*

$$x \xleftrightarrow{*}_{\mathbf{G}} y = (x \xrightarrow{*}_{\mathbf{G}} y) \wedge (y \xrightarrow{*}_{\mathbf{G}} x) = \min\left\{1, \frac{x}{y}, \frac{y}{x}\right\} \ , \ x, y \in [0,\infty] \ ,$$

with the understanding that $\frac{1}{0} = \infty$ *and* $0 \cdot \infty = \infty$ *and* $\frac{0}{0} = \infty$.

4.1.3 Properties of Extended Goguen Implication

In this section we present only the relevant properties which will be needed later for proving our results.

Proposition 4.1.6. *For* $a, b, c \in [0,\infty]$ *and* ' $\xleftrightarrow{*}_{\mathbf{G}}$ ' *being an* Extended Goguen bi-implication *and* '$\cdot$' *being the product t-norm, we have,*

$$\left(a \xleftrightarrow{*}_{\mathbf{G}} b\right) \cdot \left(b \xleftrightarrow{*}_{\mathbf{G}} c\right) \le \left(a \xleftrightarrow{*}_{\mathbf{G}} c\right) .$$

Proof. We have to prove that for any $a, b, c \in [0,\infty]$

$$\min\left\{1, \frac{a}{b}, \frac{b}{a}\right\} \cdot \min\left\{1, \frac{b}{c}, \frac{c}{b}\right\} \le \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} .$$

Let $\alpha = \min\left\{1, \frac{a}{b}, \frac{b}{a}\right\}$, $\beta = \min\left\{1, \frac{b}{c}, \frac{c}{b}\right\}$ and $\gamma = \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\}$. Now we have to prove that, $\alpha \cdot \beta \le \gamma$. Note that both α and β contain three terms each. So there will be nine possible values of $\alpha \cdot \beta$. Here we discuss all the cases:

Case-1: $(\alpha = 1, \beta = 1)$: Now we have,

$$\begin{aligned} \alpha = 1 &\iff a = b, \\ \beta = 1 &\iff b = c \\ &\implies a = b = c. \end{aligned}$$

This implies, $\alpha \cdot \beta = 1 = \min\left\{1, \frac{a}{c}, \frac{c}{a}\right\} = \gamma$.

Case-2: $\left(\alpha = 1, \beta = \dfrac{b}{c}\right)$: Now we have,

$$\begin{aligned} \alpha = 1 &\iff a = b, \\ \beta = \frac{b}{c} &\iff b \le c \\ &\implies a = b \le c. \end{aligned}$$

This implies, $\gamma = \min\left\{1, \dfrac{a}{c}, \dfrac{c}{a}\right\} = \min\left\{1, \dfrac{b}{c}, \dfrac{c}{b}\right\} = \dfrac{b}{c} = \alpha \cdot \beta.$

Case-3: $\left(\alpha = 1, \beta = \dfrac{c}{b}\right)$: Same as Case-2.

Case-4: $\left(\alpha = \dfrac{a}{b}, \beta = 1\right)$: Same as Case-2.

Case-5: $\left(\alpha = \dfrac{a}{b}, \beta = \dfrac{b}{c}\right)$: Now we have,

$$\begin{aligned} \alpha = \frac{a}{b} &\iff a \le b, \\ \beta = \frac{b}{c} &\iff b \le c \\ &\implies a \le b \le c. \end{aligned}$$

This implies, $\gamma = \min\left\{1, \dfrac{a}{c}, \dfrac{c}{a}\right\} = \dfrac{a}{c} = \dfrac{a}{b} \cdot \dfrac{b}{c} = \alpha \cdot \beta.$

Case-6: $\left(\alpha = \dfrac{a}{b}, \beta = \dfrac{c}{b}\right)$: Now we have,

$$\begin{aligned} &\alpha = \frac{a}{b} \iff a \le b, \\ &\beta = \frac{c}{b} \iff c \le b \\ \implies &\alpha \cdot \beta = \frac{a}{b} \cdot \frac{c}{b} \le \frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a}, \\ &\alpha \cdot \beta = \frac{a}{b} \cdot \frac{c}{b} \le \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} \text{ and} \\ &\alpha \cdot \beta \le 1. \end{aligned}$$

This implies, $\alpha \cdot \beta \le \min\left\{1, \dfrac{a}{c}, \dfrac{c}{a}\right\} = \gamma.$

Case-7: $\left(\alpha = \dfrac{b}{a}, \beta = 1\right)$: Same as Case-2.

Case-8: $\left(\alpha = \dfrac{b}{a}, \beta = \dfrac{c}{b}\right)$: Same as Case-5.

Case-9: $\left(\alpha = \dfrac{b}{a}, \beta = \dfrac{b}{c}\right)$: Same as Case-6.

Combining all the above nine cases we have $\alpha \cdot \beta \le \gamma$. Hence the proposition. $\square$

Proposition 4.1.7. *Let $a, b, c, d \in [0, \infty]$ and $'\cdot'$ be the product t-norm. Then, the following inequality is true:*

$$\left(a \overset{*}{\longleftrightarrow}_{\mathbf{G}} b\right) \cdot \left(c \overset{*}{\longleftrightarrow}_{\mathbf{G}} d\right) \leq (a \cdot c) \overset{*}{\longleftrightarrow}_{\mathbf{G}} (b \cdot d) \,. \tag{4.4}$$

Proof. The proof is similar to the proof of Proposition 4.1.6 . □

Proposition 4.1.8. *Let $a_i, b_i \in [0, \infty]$ and $i \in \mathcal{I}$, a finite index set. Then the inequality* (4.2) *is true for the extended Goguen bi-implication $\overset{*}{\longleftrightarrow}_{\mathbf{G}}$.*

Proof. We prove this result for $\mathcal{I} = \{1, 2\}$. The result then follows by induction. Thus we need to prove the following:

$$\left(\bigvee_{i \in \{1,2\}} a_i\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} \left(\bigvee_{i \in \{1,2\}} b_i\right) \geq \bigwedge_{i \in \{1,2\}} \left(a_i \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_i\right) \,,$$

or equivalently,

$$(a_1 \vee a_2) \overset{*}{\longleftrightarrow}_{\mathbf{G}} (b_1 \vee b_2) \geq \left(a_1 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_1\right) \wedge \left(a_2 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_2\right) \,.$$

Now, by using the monotonicity of $\overset{*}{\longrightarrow}_{\mathbf{G}}$ and its distributivity over $\vee$, $\wedge$, we have

$$\begin{aligned}
\text{L.H.S.} &= (a_1 \vee a_2) \overset{*}{\longleftrightarrow}_{\mathbf{G}} (b_1 \vee b_2) \\
&= \left[(a_1 \vee a_2) \overset{*}{\longrightarrow}_{\mathbf{G}} (b_1 \vee b_2)\right] \wedge \left[(b_1 \vee b_2) \overset{*}{\longrightarrow}_{\mathbf{G}} (a_1 \vee a_2)\right] \\
&= \left[a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} (b_1 \vee b_2)\right] \wedge \left[a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} (b_1 \vee b_2)\right] \wedge \left[b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} (a_1 \vee a_2)\right] \wedge \left[b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} (a_1 \vee a_2)\right] \\
&\geq \left[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \vee (a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2)\right] \wedge \left[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \vee (a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2)\right] \\
&\qquad \wedge \left[(b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1) \vee (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right] \wedge \left[(b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1) \vee (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right] \qquad \text{(by (1.7))} \\
&\geq \left\{\left[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1)\right] \vee \left[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right]\right\} \\
&\qquad \wedge \left\{\left[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1)\right] \vee \left[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right]\right\} \\
&\qquad\qquad (\because (x \vee y) \wedge (w \vee z) \geq (x \wedge w) \vee (y \wedge z)) \\
&\geq \left\{\left[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1)\right] \wedge \left[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right]\right\} \\
&\qquad \vee \left\{\left[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right] \wedge \left[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1)\right]\right\} \\
&\qquad\qquad (\because (x \vee y) \wedge (w \vee z) \geq (x \wedge w) \vee (y \wedge z)) \\
&\geq \left\{\left[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1)\right] \wedge \left[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\right]\right\} \\
&= \left(a_1 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_1\right) \wedge \left(a_2 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_2\right) = \text{R.H.S.}
\end{aligned}$$

Hence proved. □

Proposition 4.1.9. *Let $a_i, b_i \in [0,\infty]$ and $i \in \mathcal{I}$, a finite index set. Then the inequality* (4.3) *is valid for the extended Goguen bi-implication* $\overset{*}{\longleftrightarrow}_{\mathbf{G}}$.

Proof. Once again we prove this result for $\mathcal{I} = \{1,2\}$ and the result then follows by induction. Thus we need to prove the following:

$$\left(\bigwedge_{i\in\{1,2\}} a_i \right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} \left(\bigwedge_{i\in\{1,2\}} b_i \right) \geq \bigwedge_{i\in\{1,2\}} \left(a_i \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_i \right) ,$$

or equivalently,

$$(a_1 \wedge a_2) \overset{*}{\longleftrightarrow}_{\mathbf{G}} (b_1 \wedge b_2) \geq (a_1 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_1) \wedge (a_2 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_2).$$

Now, by using the monotonicity of $\overset{*}{\longrightarrow}_{\mathbf{G}}$ and its distributivity over $\vee$, $\wedge$, we have

$$\begin{aligned}
\text{L.H.S.} &= (a_1 \wedge a_2) \overset{*}{\longleftrightarrow}_{\mathbf{G}} (b_1 \wedge b_2) \\
&= \Big[(a_1 \wedge a_2) \overset{*}{\longrightarrow}_{\mathbf{G}} (b_1 \wedge b_2)\Big] \wedge \Big[(b_1 \wedge b_2) \overset{*}{\longrightarrow}_{\mathbf{G}} (a_1 \wedge a_2)\Big] \\
&= \Big\{\Big[(a_1 \wedge a_2) \overset{*}{\longrightarrow}_{\mathbf{G}} b_1\Big] \wedge \Big[(a_1 \wedge a_2) \overset{*}{\longrightarrow}_{\mathbf{G}} b_2\Big]\Big\} \\
&\qquad\qquad \wedge \Big\{\Big[(b_1 \wedge b_2) \overset{*}{\longrightarrow}_{\mathbf{G}} a_1\Big] \wedge \Big[(b_1 \wedge b_2) \overset{*}{\longrightarrow}_{\mathbf{G}} a_2\Big]\Big\} \\
&\geq (a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2) \\
&= \Big[(a_1 \overset{*}{\longrightarrow}_{\mathbf{G}} b_1) \wedge (b_1 \overset{*}{\longrightarrow}_{\mathbf{G}} a_1)\Big] \wedge \Big[(a_2 \overset{*}{\longrightarrow}_{\mathbf{G}} b_2) \wedge (b_2 \overset{*}{\longrightarrow}_{\mathbf{G}} a_2)\Big] \\
&= (a_1 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_1) \wedge (a_2 \overset{*}{\longleftrightarrow}_{\mathbf{G}} b_2) = \text{R.H.S.}
\end{aligned}$$

□

4.2 Interpolativity of BKS-$\mathcal{Y}$ Inference Mechanisms

By interpolativity we mean the following: when an antecedent of a rule is given as the input then the corresponding consequent should be the inferred output.

Definition 4.2.1. *An FRI is said to be* interpolative *if the following is valid: Whenever an antecedent of a rule A_i is given as the input, the corresponding consequent B_i should be the inferred output, i.e.,*

$$B_i = f_R^{@}(A_i) = A_i@R, \qquad i = 1, 2 \ldots, n., \ A_i \in \mathcal{F}(X), \ R \in \mathcal{F}(X \times Y).$$

Investigating the interpolativity of an FRI leads to the problem of solving the fuzzy relational equation $B_i = A_i@R$ for R. This leads us to the question " Can R be any fuzzy relation in $\mathcal{F}(X \times Y)$ that solves the fuzzy relational equation $B_i = A_i@R$??"

Definition 4.2.2. *A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a correct model of the given rule base $\mathcal{R}(A_i, B_i)$ for the composition @ if $A_i@R = B_i$ holds for all $i = 1, 2, \ldots, n$.*

Hence, in the case of FRIs, interpolativity pertains to the solvability of the above fuzzy relational equations corresponding to the system.

4.2.1 Interpolativity of BKS with f-implications

The following result gives a necessary and sufficient condition for the interpolativity of the BKS inference mechanism with an f-implication.

Theorem 4.2.3. *Let us consider the fuzzy IF-THEN rulebase $\mathcal{R}(A_i, B_i)$ as in* (1.12). *Consider the BKS inference mechanism of the form, $\mathbb{F}_{\to_f} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, \hat{R}_f\right)$, where $\hat{R}_f \in \mathcal{F}(X \times Y)$ as given in* (Imp-$\hat{R}_f$) *in Section 3.4 models the rule base $\mathcal{R}(A_i, B_i)$. Moreover, let A_i for $i = 1, 2, \ldots, n$ be normal. A necessary and sufficient condition for $\hat{R}_f$ to be a solution to $B_i = A_i \lhd_f R$ is as follows: For any $i, j \in \{1, 2, \ldots, n\}$,*

$$\bigvee_{x \in X} (A_i(x) \cdot A_j(x)) \le \bigwedge_{y \in Y} \left(f\left(B_i(y)\right) \xleftrightarrow{*}_{\mathbf{G}} f\left(B_j(y)\right) \right) , \tag{4.5}$$

where " $\xleftrightarrow{*}_{\mathbf{G}}$ " is the extended Goguen bi-implication and f is the generator function of the corresponding f-implication.

Proof. ($\Longrightarrow$): Let the system have interpolativity. Then we have, for any $y \in Y$, $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned}
&\left(A_i \lhd_f \hat{R}_f\right)(y) = B_i(y), \\
\Longrightarrow &\bigwedge_{x \in X} \left(A_i(x) \longrightarrow_f \bigwedge_j \left(A_j(x) \longrightarrow_f B_j(y)\right) \right) = B_i(y), \\
\Longrightarrow &A_i(x) \longrightarrow_f \left(A_j(x) \longrightarrow_f B_j(y)\right) \ge B_i(y), && (\forall j, \forall x), \\
\Longrightarrow &\left(A_i(x) \cdot A_j(x)\right) \longrightarrow_f B_j(y) \ge B_i(y), && (\forall j, \forall x),\ (\textit{by}\ \text{(LI)}), \\
\Longrightarrow &f^{-1}\left(A_i(x) \cdot A_j(x) \cdot f\left(B_j(y)\right)\right) \ge B_i(y), && (\forall j, \forall x), \\
\Longrightarrow &A_i(x) \cdot A_j(x) \cdot f\left(B_j(y)\right) \le f\left(B_i(y)\right), && (\forall j, \forall x), \\
\Longrightarrow &A_i(x) \cdot A_j(x) \le \frac{f(B_i(y))}{f(B_j(y))}, && (\forall j, \forall x).
\end{aligned}$$

Since i, j are arbitrary, interchanging them in the above inequality, we have,

$$A_j(x) \cdot A_i(x) \le \frac{f(B_j(y))}{f(B_i(y))} .$$

Also trivially we have,

$$A_j(x) \cdot A_i(x) \le 1 .$$

Now from the above inequalities we see,

$$\begin{aligned}
&A_i(x) \cdot A_j(x) \le \min\left\{1, \frac{f(B_i(y))}{f(B_j(y))}, \frac{f(B_j(y))}{f(B_i(y))}\right\} && (\forall i, j)(\forall x, y), \\
\Longrightarrow &\bigvee_{x \in X} (A_i(x) \cdot A_j(x)) \le \bigwedge_{y \in Y} \min\left\{1, \frac{f(B_i(y))}{f(B_j(y))}, \frac{f(B_j(y))}{f(B_i(y))}\right\} && (\forall i, j),
\end{aligned}$$

which is the same as (4.5).

($\Longleftarrow$): Now let us assume that (4.5) holds. Firstly, note that the following is always valid:

$$\left(A_i \lhd_f \hat{R}_f\right)(y) \leq B_i(y)\,, \qquad (\forall i, \forall y). \tag{4.6}$$

The validity of inequality (4.6) can be seen from the following inequalities:

$$\begin{aligned}
\left(A_i \lhd_f \hat{R}_f\right)(y) &= \bigwedge_{x\in X}\left(A_i(x) \longrightarrow_f \bigwedge_j \left(A_j(x) \longrightarrow_f B_j(y)\right)\right) \\
&\leq \left(A_i(x_0) \longrightarrow_f \bigwedge_j \left(A_j(x_0) \longrightarrow_f B_j(y)\right)\right) \\
&\qquad\qquad\qquad\qquad\qquad \text{(Assuming } A_i \text{ attains normality at } x_0\text{)} \\
&= \bigwedge_j \left(A_j(x_0) \longrightarrow_f B_j(y)\right) \qquad \text{(Using (NP))} \\
&\leq A_i(x_0) \longrightarrow_f B_i(y) = B_i(y) \qquad \text{(Again Using (NP))}.
\end{aligned}$$

Thus it only remains to show that

$$\left(A_i \lhd_f \hat{R}_f\right)(y) \geq B_i(y), \qquad (\forall i, \forall y). \tag{4.7}$$

We have from (4.5),

$$\begin{aligned}
& A_i(x)\cdot A_j(x) \leq \min\left\{1, \frac{f(B_i(y))}{f(B_j(y))}, \frac{f(B_j(y))}{f(B_i(y))}\right\}, && (\forall i, j)(\forall x, y), \\
\Longrightarrow\, & A_i(x)\cdot A_j(x) \leq \frac{f(B_i(y))}{f(B_j(y))}, && (\forall i, j, \forall x, y), \\
\Longrightarrow\, & A_i(x)\cdot A_j(x)\cdot f\left(B_j(y)\right) \leq f\left(B_i(y)\right), && (\forall j, \forall x), \\
\Longrightarrow\, & f^{-1}\left(A_i(x)\cdot A_j(x)\cdot f\left(B_j(y)\right)\right) \geq B_i(y), && (\forall i, j, \forall x, y), \\
\Longrightarrow\, & \left(A_i(x)\cdot A_j(x)\right) \longrightarrow_f B_j(y) \geq B_i(y), && (\forall i, j, \forall x, y), \\
\Longrightarrow\, & A_i(x) \longrightarrow_f \left(A_j(x) \longrightarrow_f B_j(y)\right) \geq B_i(y), && (\forall i, j, \forall x, y),\ (\textit{by}\ \text{(LI)}), \\
\Longrightarrow\, & \bigwedge_j \left(A_i(x) \longrightarrow_f \left(A_j(x) \longrightarrow_f B_j(y)\right)\right) \geq B_i(y), && (\forall i, \forall x, y), \\
\Longrightarrow\, & \left(A_i(x) \longrightarrow_f \bigwedge_j \left(A_j(x) \longrightarrow_f B_j(y)\right)\right) \geq B_i(y), && (\forall i, \forall x, y),\ (\textit{by}\ \text{(1.6)}), \\
\Longrightarrow\, & \bigwedge_{x\in X}\left(A_i(x) \longrightarrow_f \bigwedge_j \left(A_j(x) \longrightarrow_f B_j(y)\right)\right) \geq B_i(y), && (\forall i, \forall y), \\
\Longrightarrow\, & \left(A_i \lhd_f \hat{R}_f\right)(y) \geq B_i(y), && (\forall i, \forall y).
\end{aligned}$$

Now from (4.6) and (4.7) it follows that $\left(A_i \lhd_f \hat{R}_f\right)(y) = B_i(y)$. □

4.2.2 Interpolativity of BKS with g-implications

The following result gives a necessary and sufficient condition for the interpolativity of the BKS inference mechanism with a g-implication.

Theorem 4.2.4. *Let us consider the fuzzy IF-THEN rulebase $\mathcal{R}(A_i, B_i)$ as in* (1.12). *Consider the BKS inference mechanism of the form, $\mathbb{F}_{\to_g} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, \hat{R}_g\right)$, where $\hat{R}_g \in \mathcal{F}(X \times Y)$ as given in* (Imp-$\hat{R}_g$) *in Section 3.4 models the rule base $\mathcal{R}(A_i, B_i)$. Moreover, let A_i for $i = 1, 2, \ldots, n$ be normal. A necessary and sufficient condition for $\hat{R}_g$ to be a solution to $B_i = A_i \lhd_g R$ is as follows: For any $i, j \in \{1, 2, \ldots, n\}$,*

$$\bigvee_{x \in X} (A_i(x) \cdot A_j(x)) \leq \bigwedge_{y \in Y} (g(B_i(y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g(B_j(y))) \,, \tag{4.8}$$

where " $\overset{*}{\longleftrightarrow}_{\mathbf{G}}$ " is the extended Goguen bi-implication and g is the generator function of the corresponding g-implication.

Proof. ($\Longrightarrow$): Let the system have interpolativity. Then, for any $y \in Y$, $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned}
&\left(A_i \lhd_g \hat{R}_g\right)(y) = B_i(y) \\
&\Longrightarrow \bigwedge_{x \in X} \left(A_i(x) \longrightarrow_g \bigwedge_j (A_j(x) \longrightarrow_g B_j(y))\right) = B_i(y), \\
&\Longrightarrow A_i(x) \longrightarrow_g (A_j(x) \longrightarrow_g B_j(y)) \geq B_i(y), && (\forall j, \forall x), \\
&\Longrightarrow (A_i(x) \cdot (A_j(x)) \longrightarrow_g B_j(y) \geq B_i(y), && (\forall j, \forall x) \text{ (by (LI))}, \\
&\Longrightarrow g^{(-1)}\left(\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right)\right) \geq B_i(y), && (\forall j, \forall x)\ , \\
&\Longrightarrow g^{-1}\left(\min\left\{\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\}\right) \geq B_i(y), && (\forall j, \forall x), \\
&\Longrightarrow \min\left\{\frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\} \geq g\left(B_i(y)\right), && (\forall j, \forall x), \\
&\Longrightarrow \frac{1}{A_i(x) \cdot A_j(x)} \cdot g\left(B_j(y))\right) \geq g\left(B_i(y)\right), && (\forall j, \forall x), \\
&\Longrightarrow A_i(x) \cdot A_j(x) \leq \frac{g\left(B_j(y)\right)}{g\left(B_i(y)\right)}, && (\forall j, \forall x).
\end{aligned}$$

Since i, j are arbitrary, interchanging them in the above inequality, we have,

$$A_j(x) \cdot A_i(x) \leq \frac{g(B_i(y))}{g(B_j(y))} \,.$$

We also trivially have,

$$A_j(x) \cdot A_i(x) \leq 1 \,.$$

Now from the above inequalities we see,

$$\begin{aligned}
&A_i(x) \cdot A_j(x) \leq \min\left\{1, \frac{g(B_i(y))}{g(B_j(y))}, \frac{g(B_j(y))}{g(B_i(y))}\right\} && (\forall i,j)(\forall x,y),\\
\Longrightarrow &\bigvee_{x\in X} (A_i(x)\cdot A_j(x)) \leq \bigwedge_{y\in Y} \min\left\{1, \frac{g(B_i(y))}{g(B_j(y))}, \frac{g(B_j(y))}{g(B_i(y))}\right\} && (\forall i,j),
\end{aligned}$$

which is the same as (4.8).

($\Longleftarrow$): Now let us assume that (4.8) holds. Then,

$$\begin{aligned}
&A_i(x) \cdot A_j(x) \leq \min\left\{1, \frac{g(B_i(y))}{g(B_j(y))}, \frac{g(B_j(y))}{g(B_i(y))}\right\}, && (\forall i,j)(\forall x,y),\\
\Longrightarrow &A_i(x) \cdot A_j(x) \leq \frac{g(B_j(y))}{g(B_i(y))}, && (\forall i,j,\forall x,y),\\
\Longrightarrow &\frac{1}{A_i(x)\cdot A_j(x)} \cdot g\left(B_j(y)\right)) \geq g\left(B_i(y)\right), && (\forall i,j,\forall x,y),\\
\Longrightarrow &\min\left\{\frac{1}{A_i(x)\cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\} \geq g\left(B_i(y)\right), && (\forall i,j,\forall x,y),\\
\Longrightarrow &g^{-1}\left(\min\left\{\frac{1}{A_i(x)\cdot A_j(x)} \cdot g\left(B_j(y)\right), g(1)\right\}\right) \geq B_i(y), && (\forall i,j,\forall x,y),\\
\Longrightarrow &g^{(-1)}\left(\frac{1}{A_i(x)\cdot A_j(x)} \cdot g\left(B_j(y)\right)\right) \geq B_i(y), && (\forall i,j,\forall x,y),\\
\Longrightarrow &(A_i(x) \cdot (A_j(x)) \longrightarrow_g B_j(y) \geq B_i(y), && (\forall i,j,\forall x,y),\\
\Longrightarrow &A_i(x) \longrightarrow_g (A_j(x) \longrightarrow_g B_j(y)) \geq B_i(y), && (\forall i,j,\forall x,y), (by\ \text{(LI)}),\\
\Longrightarrow &\bigwedge_{x\in X}\left(A_i(x) \longrightarrow_g \bigwedge_j (A_j(x) \longrightarrow_g B_j(y))\right) \geq B_i(y), && (\forall i,\forall y),\\
\Longrightarrow &\left(A_i \lhd_g \hat{R}_g\right)(y) \geq B_i(y), && (\forall i,\forall y).
\end{aligned}$$

So we have the following:

$$\left(A_i \lhd_g \hat{R}_g\right)(y) \geq B_i(y), \qquad (\forall i,\forall y). \tag{4.9}$$

Once again, the following inequality is always true, the proof of which is very much along the lines as that given for (4.6):

$$\left(A_i \lhd_g \hat{R}_g\right)(y) \leq B_i(y), \qquad (\forall i,\forall y). \tag{4.10}$$

Now from (4.9) and (4.10) it follows that $\left(A_i \lhd_g \hat{R}_g\right)(y) = B_i(y)$, for all $y \in Y$. □

4.3 Continuity of BKS-$\mathcal{Y}$ Inference Mechanisms

In [36], [37] Perfilieva *et al*. discussed the *continuity* of a CRI inference mechanism, once again when the underlying operators were from a residuated lattice. Further, the author has defined the *correctness* of a model in terms of its *interpolativity*. Later on Štěpnička and Jayaram [49] have dealt with the continuity of the BKS inference mechanism with the operations coming from a residuated lattice. Since we are dealing with operations that come from a non-residuated lattice structure we define continuity suitably and show that, once again, continuity is equivalent to the correctness of the model.

4.3.1 BKS with f-Implications: Continuity $\equiv$ Interpolativity

Definition 4.3.1. *Let us consider the fuzzy IF-THEN rulebase $\mathcal{R}(A_i, B_i)$ as in* (1.12). *The fuzzy relation $R \in \mathcal{F}(X \times Y)$ which models the rule base $\mathcal{R}(A_i, B_i)$, is said to be a **continuous** model of $\mathcal{R}(A_i, B_i)$ in the BKS inference mechanism of the form, $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_f, R\right)$, if for each $i \in \{1, 2, \ldots, n\}$ and for each $A \in \mathcal{F}(X)$, the following inequality holds:*

$$\bigwedge_{y \in Y} \left[f\left(B_i(y)\right) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A \triangleleft_f R\right)(y)\right)\right] \geq \bigwedge_{x \in X} \left[A_i(x) \stackrel{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right] , \tag{4.11}$$

where " $\stackrel{*}{\longleftrightarrow}_{\mathbf{G}}$ " is the extended Goguen bi-implication and f is the generator function of the corresponding f-implication.

Remark 4.3.2 (Why is (4.11) **Continuity??).** *(i) Note that in the Definition 4.3.1 above, the bi-implication on the right side of the inequality $\stackrel{*}{\longleftrightarrow}_{\mathbf{G}}$ is equivalent to $\longleftrightarrow_{\mathbf{G}}$, since $A_i(x), A(x) \in [0, 1]$. However, for notational consistency, we have retained the above form.*

(ii) Note that if we consider f-implications with $f(0) = 1$ i.e., $\longrightarrow_f \in \mathbb{I}_{\mathbb{F},1}$ then (4.11) *reduces to the following where $\longleftrightarrow_{\mathbf{G}}$ is the Goguen bi-implication:*

$$\bigwedge_{y \in Y} \left[f\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} f\left(\left(A \triangleleft_f R\right)(y)\right)\right] \geq \bigwedge_{x \in X} \left[A_i(x) \longleftrightarrow_{\mathbf{G}} A(x)\right] . \tag{4.12}$$

(iii) Further, from [22], Example 11.7(ii), we see that $\longleftrightarrow_{\mathbf{G}}$ can be represented as

$$x \longleftrightarrow_{\mathbf{G}} y = t^{(-1)}(|t(x) - t(y)|) ,$$

where $t : [0, 1] \longrightarrow [0, \infty]$ is any additive generator of the product t-norm and hence we have $t(0) = \infty$.

Still considering f-generators with $f(0) = 1$, let us define a $D_X : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}^{\geq 0}$ and a $D_Y : \mathcal{F}(Y) \times \mathcal{F}(Y) \to \mathbb{R}^{\geq 0}$ as follows:

$$D_X\left(A_1, A_2\right) = \bigvee_{x \in X} \left(\left|t\left(A_1(x)\right) - t\left(A_2(x)\right)\right|\right) , \qquad A_1, A_2 \in \mathcal{F}(X)$$
$$D_Y\left(B_1, B_2\right) = \bigvee_{y \in Y} \left(\left|\left(t \circ f\right)\left(B_1(y)\right) - \left(t \circ f\right)\left(B_2(y)\right)\right|\right) \qquad B_1, B_2 \in \mathcal{F}(Y) .$$

It can be easily shown that $(\mathcal{F}(X), D_X)$ and $(\mathcal{F}(Y), D_Y)$ are metric spaces.

The following equivalences, along the lines of the proof of Theorem 1 in [36], demonstrate why (4.12) *can be considered as an expression capturing the continuity:*

$$\begin{aligned}
&\bigwedge_{y\in Y}\left[f\left(B_i(y)\right)\longleftrightarrow_{\mathbf{G}} f\left(\left(A\lhd_f R\right)(y)\right)\right]\geq\bigwedge_{x\in X}\left[A_i(x)\longleftrightarrow_{\mathbf{G}} A(x)\right]\\
\Longleftrightarrow&\bigwedge_{y\in Y} t^{-1}\left(\left|t\left(f\left(B_i(y)\right)\right)-t\left(f\left(\left(A\lhd_f R\right)(y)\right)\right)\right|\right)\geq\bigwedge_{x\in X} t^{-1}\left(\left|t\left(A_i(x)\right)-t\left(A(x)\right)\right|\right)\\
\Longleftrightarrow&\bigvee_{y\in Y}\left(\left|t(f(B_i(y)))-t\left(f\left(\left(A\lhd_f R\right)(y)\right)\right)\right|\right)\leq\bigvee_{x\in X}\left(\left|t(A_i(x))-t(A(x))\right|\right)\\
\Longleftrightarrow&\bigvee_{y\in Y}\left(\left|\left(t\circ f\right)\left(B_i(y)\right)-\left(t\circ f\right)\left(\left(A\lhd_f R\right)(y)\right)\right|\right)\leq\bigvee_{x\in X}\left(\left|t\left(A_i(x)\right)-t\left(A(x)\right)\right|\right)\\
\Longleftrightarrow& D_Y\left(B_i,\left(A\lhd_f R\right)\right)\leq D_X\left(A_i,A\right)\ .
\end{aligned}$$

From the classical definition of continuity, we see that for any given $\epsilon>0$, we have a $\delta>0$ such that whenever $D_X\left(A_i,A\right)<\delta$ for any $i\in\{1,2,\ldots,n\}$, we have that $D_Y\left(B_i,\left(A\lhd_f R\right)\right)<\epsilon$. Clearly, in our case $\delta=\epsilon$ is one possibility. Thus from the above we see that R is a continuous model of $\mathcal{R}(A_i,B_i)$ if and only if the inference function $f_R^{\lhd_f}:\mathcal{F}(X)\longrightarrow\mathcal{F}(Y)$ associated with $\mathbb{F}=\left(\mathcal{P}_X,\mathcal{P}_Y,\lhd_f,R\right)$ is continuous on $\mathcal{P}_X$.

Theorem 4.3.3. *Let us consider the BKS inference mechanism of the form $\mathbb{F}=\left(\mathcal{P}_X,\mathcal{P}_Y,\lhd_f,R\right)$ over finite non-empty sets X and Y. The fuzzy relation $R\in\mathcal{F}(X\times Y)$ is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

Proof. Let R be a continuous model of the fuzzy rules (1.12). By Definition 4.3.1, the inequality (4.11) is valid for all $i=1,2,\ldots,n$ and an arbitrary $A\in\mathcal{F}(X)$. Now putting $A=A_i$ in (4.11), we have by the strictness of f,

$$\begin{aligned}
&\bigwedge_{y\in Y}\left[f\left(B_i(y)\right)\overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A_i\lhd_f R\right)(y)\right)\right]\geq 1, && (\forall i),\\
&\Longrightarrow f\left(B_i(y)\right)\overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A_i\lhd_f R\right)(y)\right)=1, && (\forall i,y),\\
&\Longrightarrow f\left(B_i(y)\right)=f\left(\left(A_i\lhd_f R\right)(y)\right), && (\forall i,y),\\
&\Longrightarrow \left(A_i\lhd_f R\right)(y)=B_i(y), && (\forall i,y).
\end{aligned}$$

Thus we have interpolativity starting from continuity.

Now let us assume that the model has interpolativity. Towards proving (4.11), for arbitrary $y\in Y$, note that the following is true for any $i=1,2,\ldots,n$:

$$\begin{aligned}
& f\left(\left(A\lhd_f R\right)(y)\right)\overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(B_i(y)\right)\\
&\qquad = f\left(\left(A\lhd_f R\right)(y)\right)\overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\left(A_i\lhd_f R\right)(y)\right), \qquad \left(\because \left(A_i\lhd_f R\right)(y)=B_i(y)\right)\\
&\qquad = f\left(\bigwedge_{x\in X}\left[A(x)\longrightarrow_f R(x,y)\right]\right)\overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left(\bigwedge_{x\in X}\left[A_i(x)\longrightarrow_f R(x,y)\right]\right)\\
&\qquad = f\left\{\bigwedge_{x\in X} f^{-1}\left[A(x)\cdot f\left(R(x,y)\right)\right]\right\}\overset{*}{\longleftrightarrow}_{\mathbf{G}} f\left\{\bigwedge_{x\in X} f^{-1}\left[A_i(x)\cdot f\left(R(x,y)\right)\right]\right\}
\end{aligned}$$

$$= \bigvee_{x\in X} [A(x)\cdot f(R(x,y))] \overset{*}{\longleftrightarrow}_{\mathbf{G}} \bigvee_{x\in X} [A_i(x)\cdot f(R(x,y))]$$

$$(\because f(\wedge_{i=1}^n a_i) = \wedge_{i=1}^n f(a_i))$$

$$\geq \bigwedge_{x\in X} \Big[A(x)\cdot f(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A_i(x)\cdot f(R(x,y))\Big] \qquad \text{(Using Proposition 4.1.8)}$$

$$\geq \bigwedge_{x\in X} \Big(\Big[A(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A_i(x)\Big]\cdot\Big[f(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} f(R(x,y))\Big]\Big) \qquad \text{(Using(4.4))}$$

$$= \bigwedge_{x\in X} \Big[A(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A_i(x)\Big] \ ,$$

from which we obtain (4.11). □

Corollary 4.3.4. *Let us consider the BKS inference mechanism of the form $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, \hat{R}_f\Big)$ over finite non-empty sets X and Y. The fuzzy relation $\hat{R}_f \in \mathcal{F}(X\times Y)$ is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

The following result shows that if we consider f-generators with $f(0)=1$ then the finiteness of the sets X, Y can be dispensed with and the above result still remains valid.

Theorem 4.3.5. *Let us consider the BKS inference mechanism of the form $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, R\Big)$ with $f(0)=1$, i.e., $\longrightarrow_f \in \mathbb{I}_{\mathbb{F},1}$. The fuzzy relation $R \in \mathcal{F}(X\times Y)$, where X, Y are any non-empty domains, is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

Proof. Note that since $f(0)=1$, the continuity equation (4.11) reduces to (4.12). Since $\longleftrightarrow_{\mathbf{G}}$ satisfies (4.1) and (4.2) even for an infinite index set $\mathcal{I}$, the proof follows immediately along the lines of the proof of Theorem 4.3.3. □

Corollary 4.3.6. *Let us consider the BKS inference mechanism of the form $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, \hat{R}_f\Big)$ with $f(0)=1$, i.e., $\longrightarrow_f \in \mathbb{I}_{\mathbb{F},1}$. The fuzzy relation $\hat{R}_f \in \mathcal{F}(X\times Y)$, where X, Y are any non-empty domains, is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

The above study clearly demonstrates that, as in the case of BKS and CRI, $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, \hat{R}_f\Big)$ also possesses both the desirable properties of interpolativity and continuity .

4.3.2 BKS with g-Implications: Continuity $\equiv$ Interpolativity

Along the similar lines of continuity of a fuzzy relation R in the BKS inference mechanism of the form $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, R\Big)$, we propose the following definition of continuity of R in $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, R\Big)$.

Definition 4.3.7. *Let us consider the fuzzy IF-THEN rulebase $\mathcal{R}(A_i, B_i)$ as in* (1.12)*. The fuzzy relation $R \in \mathcal{F}(X\times Y)$ which models the rule base $\mathcal{R}(A_i, B_i)$, is said to be a* ***continuous*** *model of $\mathcal{R}(A_i, B_i)$ in the BKS inference mechanism of the form $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, R\Big)$, if for each $i \in \{1, 2, \ldots, n\}$ and for each $A \in \mathcal{F}(X)$, the following inequality holds:*

$$\bigwedge_{y\in Y} \Big[g(B_i(y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g((A \lhd_g R)(y))\Big] \geq \bigwedge_{x\in X} \Big[A_i(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\Big], \tag{4.13}$$

where " $\overset{*}{\longleftrightarrow}_{\mathbf{G}}$ *" is the extended Goguen bi-implication and g is the generator function of the corresponding g-implication.*

Remark 4.3.8. *(i) Note that if we consider g-implications with $g(1) = 1$, i.e., $\longrightarrow_g \in \mathbb{I}_{\mathbb{G},1}$then* (4.13) *reduces to the following where $\longrightarrow_{\mathbf{G}}$ is the Goguen bi-implication:*

$$\bigwedge_{y\in Y} [g\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} g\left((A \lhd_g R)(y)\right)] \geq \bigwedge_{x\in X} [A_i(x) \longleftrightarrow_{\mathbf{G}} A(x)] \ . \tag{4.14}$$

(ii) A similar explanation as in Remark 4.3.2 can be given as to why (4.14) *can be considered as an expression capturing the continuity.*

Theorem 4.3.9. *Let us consider the BKS inference mechanism of the form $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, R\right)$ over finite non-empty sets X and Y. The fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

Proof. Let R be a continuous model of the fuzzy rules (1.12). By Definition 4.3.7, we have,

$$\bigwedge_{y\in Y} \left[g\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A \lhd_g R\right)(y)\right)\right] \geq \bigwedge_{x\in X} \left[A_i(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right] \ ,$$

for all $i = 1, 2, \ldots n$ and an arbitrary $A \in \mathcal{F}(X)$. Letting $A = A_i$ in the above inequality, we get by the strictness of g,

$$\begin{aligned}
&\bigwedge_{y\in Y} \left[g\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A \lhd_g R\right)(y)\right)\right] \geq 1, && (\forall i),\\
&\Longrightarrow g\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A_i \lhd_g R\right)(y)\right) = 1, && (\forall i, y),\\
&\Longrightarrow g\left(B_i(y)\right) = g\left(\left(A_i \lhd_g R\right)(y)\right), && (\forall i, y),\\
&\Longrightarrow \left(A_i \lhd_g R\right)(y) = B_i(y). && (\forall i, y).
\end{aligned}$$

So we have interpolativity starting from continuity.

Now let us assume that the model has interpolativity. Towards proving (4.13), for arbitrary $y \in Y$, note that the following is true for any $i = 1, 2, \ldots, n$:

$$\begin{aligned}
&g\left(B_i(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A \lhd_g R\right)(y)\right)\\
&= g\left(\left(A \lhd_g R\right)(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\left(A_i \lhd_g R\right)(y)\right) \qquad (\because (A_i \lhd_g R)(y) = B_i(y))\\
&= g\left(\bigwedge_{x\in X} [A(x) \longrightarrow_g R(x,y)]\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left(\bigwedge_{x\in X} [A_i(x) \longrightarrow_g R(x,y)]\right)\\
&= g\left\{\bigwedge_{x\in X} g^{(-1)}\left[\frac{1}{A(x)} \cdot g(R(x,y))\right]\right\} \overset{*}{\longleftrightarrow}_{\mathbf{G}} g\left\{\bigwedge_{x\in X} g^{(-1)}\left[\frac{1}{A_i(x)} \cdot g(R(x,y))\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= g\left\{\bigwedge_{x\in X} g^{-1}\left[\min\left\{\frac{1}{A(x)}\cdot g(R(x,y)), g(1)\right\}\right]\right\} \\
&\qquad\qquad\qquad \overset{*}{\longleftrightarrow}_{\mathbf{G}}\ g\left\{\bigwedge_{x\in X} g^{-1}\left[\min\left\{\frac{1}{A_i(x)}\cdot g(R(x,y)), g(1)\right\}\right]\right\} \\
&= \bigwedge_{x\in X}\left[\min\left\{\frac{1}{A(x)}\cdot g(R(x,y)), g(1)\right\}\right] \overset{*}{\longleftrightarrow}_{\mathbf{G}} \bigwedge_{x\in X}\left[\min\left\{\frac{1}{A_i(x)}\cdot g(R(x,y)), g(1)\right\}\right] \\
&\geq \bigwedge_{x\in X}\left[\min\left\{\frac{1}{A(x)}\cdot g(R(x,y)), g(1)\right\} \overset{*}{\longleftrightarrow}_{\mathbf{G}} \min\left\{\frac{1}{A_i(x)}\cdot g(R(x,y)), g(1)\right\}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 4.1.9)} \\
&\geq \bigwedge_{x\in X}\left[\left\{\frac{1}{A(x)}\cdot g(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)}\cdot g(R(x,y))\right\} \wedge \left\{g(1) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g(1)\right\}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 4.1.9)} \\
&= \bigwedge_{x\in X}\left[\frac{1}{A(x)}\cdot g(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)}\cdot g(R(x,y))\right] \\
&\geq \bigwedge_{x\in X}\left[\frac{1}{A(x)} \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)}\right]\cdot\left[g(R(x,y)) \overset{*}{\longleftrightarrow}_{\mathbf{G}} g(R(x,y))\right] \qquad \text{(Using(4.4))} \\
&= \bigwedge_{x\in X}\left[\frac{1}{A(x)} \overset{*}{\longleftrightarrow}_{\mathbf{G}} \frac{1}{A_i(x)}\right] = \bigwedge_{x\in X}\left[A_i(x) \overset{*}{\longleftrightarrow}_{\mathbf{G}} A(x)\right]\ ,
\end{aligned}
$$

from whence we have (4.13). □

Corollary 4.3.10. *Let us consider the BKS inference mechanism of the form* $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, \hat{R}_g\right)$ *over finite non-empty sets* X *and* Y. *The fuzzy relation* $\hat{R}_g \in \mathcal{F}(X\times Y)$ *is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

The following result shows that if we consider g-generators with $g(1) = 1$ then the finiteness of the sets X, Y can be dispensed with and the above result still remains valid.

Theorem 4.3.11. *Let us consider the BKS inference mechanism of the form* $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, R\right)$ *with* $g(1) = 1$, *i.e.,* $\longrightarrow_g \in \mathbb{I}_{\mathbb{G},1}$. *The fuzzy relation* $R \in \mathcal{F}(X\times Y)$, *where* X, Y *are any non-empty domains, is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

Proof. Note that since $g(1) = 1$, the continuity equation (4.13) reduces to (4.14). Since $\longleftrightarrow_{\mathbf{G}}$ satisfies (4.1) and (4.2) even for an infinite index set $\mathcal{I}$, the proof follows immediately along the lines of the proof of Theorem 4.3.9. □

Theorem 4.3.12. *Let us consider the BKS inference mechanism of the form* $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, \hat{R}_g\right)$ *with* $g(1) = 1$, *i.e.,* $\longrightarrow_g \in \mathbb{I}_{\mathbb{G},1}$. *The fuzzy relation* $\hat{R}_g \in \mathcal{F}(X\times Y)$, *where* X, Y *are any non-empty domains, is a correct model of fuzzy rules* (1.12) *if and only if it is a continuous model of these rules.*

The above study clearly demonstrates that, as in the case of BKS and CRI, $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, \hat{R}_g\right)$ also possesses both the desirable properties of interpolativity and continuity .

Chapter 5

Robustness of Bandler-Kohout Subproduct with Yager's Families of Fuzzy Implications

Before God we are all equally wise - and equally foolish.
– Albert Einstein (1879 – 1955)

Robustness is an essential property of an inference mechanism. Robustness of an FRI $f_R^{@}$ deals with how variations in the intended input affect the conclusions. It is different from continuity in that, we expect that even when the actual input fuzzy set is not equal to the intended fuzzy set but both are *equivalent* - in a certain predefined sense based on the equality relations on the underlying set - the output fuzzy set should be equal to the corresponding intended output. In other words, the FRI $f_R^{@}$ respects the order and equivalence present in the underlying universe of discourse.

Perhaps a small example will illustrate the concept of robustness. Let us consider the following scenario: Suppose a person goes to a stationary shop to buy a pen which usually costs Rs. 10. However, on reaching there he comes to know that the cost of the pen has gone up to Rs. 11. Now the question is whether he will buy the pen or not? The answer normally is 'YES'. Interestingly, if the cost of the pen was Rs. 20 instead of Rs. 10, then for the same question, the answer would probably be 'NO'. So for Rs. 10 and Rs. 11, the person takes the same decision whereas for Rs. 10 and Rs. 20, the decisions are different.

Now, consider that the same person goes to a garment shop to buy a shirt which usually costs, say, Rs. 500. Once again, on reaching there he comes to know that cost is Rs. 510. Clearly, the answer to the question as to whether he will buy it will be 'YES'.

Thus we see that a difference of Rs. 10 is negligible in one case, while it is not in another case. Or in other words, $10 \not\approx 20$ in one context, while in another context $500 \approx 510$.

Thus robustness can be thought of capturing the extent to which any variation in the price of a pen or a shirt does not affect the original decision. In the context of a fuzzy inference mechanism, robustness can be seen as the ability of the inference mechansim to capture this underlying equality without affecting the output.

That both the CRI and BKS with residuated implications enjoy robustness is well-known. In this chapter, we discuss the BKS-f and BKS-g relational inference systems and show that robustness is also available under these settings, thus expanding the choice of operations available to practitioners.

Similar to interpolativity and continuity, robustness is also discussed at the level of fuzzy sets. So, the defuzzification function d does not play a role here and hence, we deal with FRI of the form $\mathbb{F} = (\mathcal{P}_X, \mathcal{P}_Y, \triangleleft, R)$.

We recall that we deal only with the implicative form of the rule base, i.e., the antecedents of the rules are related to their consequents using a fuzzy implication and hence fix $R = \hat{R}_f$ and $\hat{R}_g$ in the sequel. Thus, we deal with $\mathbb{F}_{\to_f} = \left(\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_f, \hat{R}_f\right)$ and $\mathbb{F}_{\to_g} = \left(\mathcal{P}_X, \mathcal{P}_Y, \triangleleft_g, \hat{R}_g\right)$.

The chapter is structured in the following way: After recalling some definitions and results on fuzzy equivalence relation and extensionality in Section 5.1, we introduce the notion of robustness of an FRI in Section 5.2 and present the main results in Section 5.3.

5.1 Fuzzy Equivalence Relation and Extensionality

Similar to the equivalence relation in classical set theory, similarity relation or fuzzy equivalence relation has been proposed. Similarity relations have been used to characterize the inherent indistinguishability in a fuzzy system [21].

Definition 5.1.1 ([21], Definition 2.5). *A fuzzy equivalence relation $E\colon X \times X \to [0,1]$ with respect to the t-norm $\star$ on X is a fuzzy relation over $X \times X$ which satisfies the following for all $x, y, z \in X$:*

- $E(x,x) = 1.$ *(Reflexivity)*
- $E(x,y) = E(y,x).$ *(Symmetry)*
- $E(x,y) \star E(y,z) \leq E(x,z).$ *($\star$ -Transitivity)*

We denote a fuzzy equivalence relation by the pair $(E, \star)$.

Definition 5.1.2 ([21], Definition 2.7). *A fuzzy set $A \in \mathcal{F}(X)$ is called extensional with respect to a fuzzy equivalence relation $(E, \star)$ on X if, for every $x, y \in X$*

$$A(x) \star E(x,y) \leq A(y).$$

Definition 5.1.3 ([21], Definition 2.8). *Let $A \in \mathcal{F}(X)$ and $(E, \star)$ be a fuzzy equivalence relation on X. The fuzzy set,*

$$\hat{A} = \bigwedge \{C : A \leq C \text{ and } C \text{ is extensional w.r.to } (E, \star)\}, \tag{5.1}$$

is called the extensional hull of A. By $A \leq C$ we mean that for all $x \in X$, $A(x) \leq C(x)$, i.e, ordering in the sense of inclusion.

Interestingly, the extensional hull $\hat{A}$ of fuzzy set A, has a simpler way of calculation as given below.

Proposition 5.1.4 ([21], Proposition 2.9). *Let $A \in \mathcal{F}(X)$ and $(E, \star)$ be a fuzzy equivalence relation on X. Then the extensional hull A can be obtained by*

$$\hat{A}(x) = \bigvee \{A(y) \star E(x, y) \mid y \in X\} .$$

5.2 Robustness of an FRI

We define extensionality of a fuzzy partition in terms of extensionality of fuzzy sets as in Definition 5.1.2. Based on this notion of extensionality, we then define the notion of robustness for fuzzy relational inference mechanism w.r. to $(E, \star)$.

Definition 5.2.1. *A fuzzy partition $\mathcal{P}_X \subseteq \mathcal{F}(X)$ is called extensional with respect to a fuzzy equivalence relation $(E, \star)$ on X if, for all $A \in \mathcal{P}_X$, for every $x, y \in X$*

$$A(x) \star E(x, y) \leq A(y).$$

Definition 5.2.2. *Let $A \in \mathcal{F}(X)$ and $(E, \star)$ be a fuzzy equivalence relation on X. A fuzzy relational inference mechanism is said to be **robust** w.r.to $(E, \star)$ if the associated inference function $f_R^{@}$ is such that*

$$f_R^{@}(A) = f_R^{@}(\hat{A}) , \tag{5.2}$$

where $\hat{A}$ is the extensional hull of the fuzzy set A as defined in (5.1).

In other words, robustness of an inference mechanism is achieved by controlling the sensitivity of the inference mechanism to input variations to a satisfactory level, i.e., the output should only be as sensitive to input variations as is allowed or acceptable with respect to the underlying equality, as specified by the fuzzy equivalence relation defined over the domain.

5.3 Robustness of BKS-$\mathcal{Y}$ Inference Mechanisms

The study of robustness in an FRI using implicative form of rules has largely been confined to operations that come from a residuated lattice. The robustness of CRI was dealt with by Klawonn and Castro [21]. Later on Štěpnička and Jayaram [49] have undertaken a similar study for BKS inference mechanism with R-implications. Both the above works show that, when combined with appropriate models of fuzzy rules, CRI and BKS are robust inference mechanisms. In the following two subsections, we show a similar result which ensures the robustness of BKS-f and BKS-g inference mechanisms when the rules are modeled by the relations $\hat{R}_f$ and $\hat{R}_g$, respectively.

5.3.1 Robustness of BKS-f Inference Mechanism

The following result shows that BKS with f-implications is a robust inference mechanisms.

Theorem 5.3.1. *Let us consider the fuzzy IF-THEN rulebase $\mathcal{R}(A_i, B_i)$ as in* (1.12). *Consider the BKS inference mechanism of the form $\mathbb{F}_{\to_f} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_f, \hat{R}_f\right)$, where $\hat{R}_f \in \mathcal{F}(X \times Y)$ models the rule base $\mathcal{R}(A_i, B_i)$. Moreover let $(E, \cdot)$, where '$\cdot$' is the product t-norm, be a fuzzy equivalence relation on X, such that $\mathcal{P}_X$ is extensional w.r.to $(E, \cdot)$, i.e, every A_i, $i = 1, 2, \ldots n$ is extensional w.r.to $(E, \cdot)$. Then $\mathbb{F}_{\to_f}$ is robust w.r.to $(E, \cdot)$, i.e., $f_{\hat{R}_f}^{\lhd_f}$ satisfies* (5.2) *for any fuzzy set $A' \in \mathcal{F}(X)$, i.e., $A' \lhd_f \hat{R}_f = \hat{A}' \lhd_f \hat{R}_f$.*

Proof. Clearly, by the definition of $\hat{A}'$ we have the following:

$$\hat{A}' \geq A' \Longrightarrow \hat{A}' \longrightarrow_f \hat{R}_f \leq A' \longrightarrow_f \hat{R}_f \Longrightarrow \hat{A}' \lhd_f \hat{R}_f \leq A' \lhd_f \hat{R}_f \ .$$

Since $\hat{R}_f$ is given by (Imp-$\hat{R}_f$), we have

$$\left(\hat{A}' \lhd_f \hat{R}_f\right)(y) = \bigwedge_{x \in X} \left[\hat{A}'(x) \longrightarrow_f \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_f B_i(y))\right], \qquad y \in Y.$$

Since every A_i is extensional with respect to $(E, \cdot)$, for any $x, x' \in X$ and for any $i = 1, 2, \ldots n$,

$$\begin{aligned} &A_i(x') \geq A_i(x) \cdot E(x, x') \\ &\qquad \Longrightarrow A_i(x') \longrightarrow_f B_i(y) \leq [A_i(x) \cdot E(x, x')] \longrightarrow_f B_i(y), \qquad y \in Y. \end{aligned} \tag{5.3}$$

Now for any $x \in X$ and $y \in Y$, we have

$$\begin{aligned}
&\hat{A}'(x) \longrightarrow_f \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_f B_i(y)) \\
&= \left(\bigvee_{x' \in X} [A'(x') \cdot E(x, x')]\right) \longrightarrow_f \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_f B_i(y)), && \text{(using Proposition 5.1.4)} \\
&= \bigwedge_{x' \in X} \left([A'(x') \cdot E(x, x')] \longrightarrow_f \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_f B_i(y))\right), && \text{(using Proposition 3.1.8)} \\
&= \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} ([A'(x') \cdot E(x, x')] \longrightarrow_f (A_i(x) \longrightarrow_f B_i(y))), && \text{(using (1.6))} \\
&= \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} (A'(x') \longrightarrow_f [E(x, x') \longrightarrow_f (A_i(x) \longrightarrow_f B_i(y))]), && \text{(by (LI))} \\
&= \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} (A'(x') \longrightarrow_f [(E(x, x') \cdot A_i(x)) \longrightarrow_f B_i(y)]), && \text{(by (LI))} \\
&\geq \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} (A'(x') \longrightarrow_f [A_i(x') \longrightarrow_f B_i(y)]), && \text{(using (5.3))} \\
&= (A' \lhd_f \hat{R}_f)(y) \ .
\end{aligned}$$

Thus $\hat{A}' \lhd_f \hat{R}_f \geq A' \lhd_f \hat{R}_f$ and the result follows. □

5.3.2 Robustness of BKS-g Inference Mechanism

Theorem 5.3.2. *Let us consider the fuzzy IF-THEN rulebase $\mathcal{R}(A_i, B_i)$ as in* (1.12)*. Consider the BKS inference mechanism of the form $\mathbb{F}_{\rightarrow_g} = \left(\mathcal{P}_X, \mathcal{P}_Y, \lhd_g, \hat{R}_g\right)$, where $\hat{R}_g \in \mathcal{F}(X \times Y)$ models the rule base $\mathcal{R}(A_i, B_i)$. Moreover let $(E, \cdot)$, where '$\cdot$' is the product t-norm, be a fuzzy equivalence relation on X, such that $\mathcal{P}_X$ is extensional w.r.to $(E, \cdot)$, i.e, every A_i, $i = 1, 2, \ldots n$ is extensional w.r.to $(E, \cdot)$. Then $\mathbb{F}_{\rightarrow_g}$ is robust w.r.to $(E, \cdot)$, i.e., $f^{\lhd_g}_{\hat{R}_g}$ satisfies* (5.2) *for any fuzzy set $A' \in \mathcal{F}(X)$, i.e., $A' \lhd_g \hat{R}_g = \hat{A}' \lhd_g \hat{R}_g$.*

Proof. Clearly, by definition of $\hat{A}'$ we have the following:

$$\hat{A}' \geq A' \Longrightarrow \hat{A}' \longrightarrow_g \hat{R}_g \leq A' \longrightarrow_g \hat{R}_g \Longrightarrow \hat{A}' \lhd_g \hat{R}_g \leq A' \lhd_g \hat{R}_g \ .$$

Since $\hat{R}_g$ is given by (Imp-$\hat{R}_g$), we have

$$\left(\hat{A}' \lhd_g \hat{R}_g\right)(y) = \bigwedge_{x \in X} \left[\hat{A}'(x) \longrightarrow_g \bigwedge_{i=1}^{n} \left(A_i(x) \longrightarrow_g B_i(y)\right)\right], \qquad y \in Y.$$

Since every A_i is extensional with respect to $(E, \cdot)$, for any $x, x' \in X$ and for any $i = 1, 2, \ldots n$,

$$\begin{aligned} & A_i(x') \geq A_i(x) \cdot E(x, x') \\ & \qquad \Longrightarrow A_i(x') \longrightarrow_g B_i(y) \leq [A_i(x) \cdot E(x, x')] \longrightarrow_g B_i(y), \qquad y \in Y. \end{aligned} \tag{5.4}$$

Now for any $x \in X$ and $y \in Y$,

$$\begin{aligned} & \hat{A}'(x) \longrightarrow_g \bigwedge_{i=1}^{n} \left(A_i(x) \longrightarrow_g B_i(y)\right) \\ & = \left(\bigvee_{x' \in X} [A'(x') \cdot E(x, x')]\right) \longrightarrow_g \bigwedge_{i=1}^{n} \left(A_i(x) \longrightarrow_g B_i(y)\right), && \text{(using Proposition 5.1.4)} \\ & = \bigwedge_{x' \in X} \left([A'(x') \cdot E(x, x')] \longrightarrow_g \bigwedge_{i=1}^{n} \left(A_i(x) \longrightarrow_g B_i(y)\right)\right), && \text{(using Proposition 3.1.8)} \\ & = \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left([A'(x') \cdot E(x, x')] \longrightarrow_g \left(A_i(x) \longrightarrow_g B_i(y)\right)\right), && \text{(using (1.6))} \\ & = \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(A'(x') \longrightarrow_g [E(x, x') \longrightarrow_g \left(A_i(x) \longrightarrow_g B_i(y)\right)]\right), && \text{(by (LI))} \\ & = \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(A'(x') \longrightarrow_g [(E(x, x') \cdot A_i(x)) \longrightarrow_g B_i(y)]\right), && \text{(by (LI))} \\ & \geq \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(A'(x') \longrightarrow_g [A_i(x') \longrightarrow_g B_i(y)]\right), && \text{(Using (5.4))} \\ & = (A' \lhd_g \hat{R}_g)(y) \ . \end{aligned}$$

Thus $\hat{A}' \lhd_g \hat{R}_g \geq A' \lhd_g \hat{R}_g$ and the result follows. □

The above study clearly demonstrates that, as in the case of BKS and CRI, the (BKS-f) and (BKS-g) inference mechanisms also possess robustness.

Chapter 6

Universal Approximation Capability of SISO Fuzzy Relational Inference Mechanisms based on Fuzzy Implications

Truth is much too complicated to allow anything but approximations.
– John von Neumann (1903–1957)

One of the important factors considered while employing an fuzzy inference mechanism is its approximation capability. While many studies have appeared on this topic, most of them deal with FRIs where the rules are interpreted in a non-conditional way or as just aggregation of possibile configurations of the data (recall the discussion in Sections 1.3 and 2.1 for details). When an implicative or a conditional interpretation of the rules is considered, there are only a few works that deal with their approximation properties.

In this chapter, we present the results that fuzzy relational inference mechanisms with implicative interpretation of the rule base are universal approximators under suitable choice of operations for the other components of the FRI. The presented proofs make no assumption on the form or representations of the considered fuzzy implications and hence show that a much larger class of fuzzy implications other than what is typically considered in the literature can be employed meaningfully in FRIs based on implicative models. We present the results of non residuated implications like Yager's families of fuzzy implications as a corollary. A concept of *weak coherence* is proposed, which plays an important role in enlarging the class of fuzzy implications that can be considered.

In Section 6.1 we present a short survey on the works and results related to universal approximation of fuzzy relational inference systems. Further, the results in this chapter are valid for a much larger class of fuzzy implications than the Yager's families of fuzzy implications. This extended scope of this chapter is clearly specified in Section 6.2. Relaxing the often insisted coherence of an implicative model suitably to the context of function approximation, Section 6.3 investigates the class of fuzzy implications that can be used in FRIs to ensure this form of weak coherence. This section also presents some well-known families of fuzzy implications that belong to the above admissible class of fuzzy implications. Finally, Sections 6.4 and 6.5 contain the main results of this chapter, which shows that FRIs employing a rather large class of fuzzy implications - which include the R-implications and Yager's families of fuzzy implications - are universal approximators. Section 6.7 presents some examples that illustrate the investigations and analysis of the previous sections.

6.1 FRIs as Universal Approximators

In this subsection, we only briefly recall some of the important works dealing with the approximation properties of FRIs and refer the readers to the excellent review of Tikk *et al.* [42] for more details and the other recent works, for instance [35], [53] and the references therein.

The earliest works to appear on this topic dealt with FRIs where $R = \check{R}_\star$ and hence can be considered to have assumed a Cartesian product interpretation of the fuzzy rules as given in Section 2.1, see Wang [50] and Zeng and Singh [57].

It was Castro [10], who was the first to deal with the approximation properties of FRIs that employed $\longrightarrow$ to model the rulebase. However, as was already pointed out by Li *et al. Remark 2.4*, [24], Castro has considered an FRI as given below:

$$B'(y) = \bigvee_j (B'_j(y)) = \bigvee_j (A_j(x_0) \longrightarrow B_j(y)) \ ,$$

which is clearly not an appropriate model to work with, since under most practical settings for any given $x_0 \in X$ there will always exist a rule with an antecedent A_{i_0} such that $A_{i_0}(x_0) = 0$ (for instance, when A_i's are of finite support) and since $0 \longrightarrow b = 1$ for any $b \in [0,1]$, when the maximum t-conorm is used to aggregate the individual outputs one always obtains that $B'(y) = 1$ for all $y \in Y$. Note that this is the case when the input partitions are of the Ruspini type - a property that is normally both practical and desirable.

In the same work, after pointing out the above, Li *et al.* (see *Theorem 3.4*, [24]) have given a constructive proof of the approximation capability of an FRI with $R = \hat{R}_\rightarrow$. However, the scope of their work is restricted to the following three families of fuzzy implications, namely, R-implications from left-continuous t-norms, (S,N)- and QL-implications. Further, many of the results are without complete proofs, thus making a deeper understanding of the approach difficult.

Perfilieva and Kreinovich [35] have discussed approximation capability of fuzzy systems that reflect the CNF-DNF duality. However, the considered / constructed partitions are not 'fuzzy' and hence the constructed rule base contains antecedents and consequents that are crisp sets. Further, they make an implicit assumption that the considered implications can be written as a generalization of the classical material implication, which in the context of fuzzy logic connectives is equivalent to assuming that the considered implication is an (S,N)-implication [2, 3]. While this assumption is valid in their context, since the relations R_{CNF} [35] are crisp and hence only need to deal with $\{0,1\}$ values, in general, this is not true when we consider truth-values over the entire $[0,1]$ interval.

Recently, Štěpnička *et al.* [48] considered an FRI with $R = R^{\otimes}_{\rightarrow_*}$ where $\otimes$ is the Łukasiewicz t-norm $T_{\mathbf{LK}}(x, y) = \max(0, x + y - 1)$ and $\rightarrow_*$ is any residuated implication obtained from a left-continuous t-norm $*$, which can be different from $T_{\mathbf{LK}}$. They have shown that the FRIs $\mathbb{F}^{\otimes}_{\rightarrow_*} = \mathbb{F}^{T_{\mathbf{LK}}}_{\rightarrow_*} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^{T_{\mathbf{LK}}}_{\rightarrow_*}, \text{MOM}\right)$ are universal approximators. Their result is true for any continuous function f but the A_i's do not form a Ruspini partition which is normal and desirable in practical settings.

6.2 Enlarged Scope of this chapter

From Section 2.3.4 we recall that in the case of singleton inputs overall inference of an FRI $\mathbb{F}$ can be seen as a function $g : X \to Y$ as follows:

$$g(x') = d(B'(\cdot)) = d(f^{@}_R(A'(x'))) \ , \ x' \in X \ .$$

Further, in the case of an FRI $\mathbb{F}$ with *reducible composition*, the overall inference reduces to the even simpler function

$$g(x') = d(B'(\cdot)) = d(R(x', \cdot)) \ , \ x' \in X \ . \tag{6.1}$$

In this work, we show that FRIs with *reducible composition* of the form $\mathbb{F}^{T}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^{T}_{\rightarrow}, d\right)$ are universal approximators, where we consider the following generalised form of $\hat{R}_{\rightarrow}$:

$$R^{T}_{\rightarrow}(x, y) = T^{n}_{i=1}(A_i(x) \longrightarrow B_i(y)) \ , \ , \tag{Imp-$R^{T}_{\rightarrow}$}$$

where T is any t-norm not necessarily the minimum t-norm. A concept of *weak coherence* is proposed, which plays an important role in enlarging the class of fuzzy implications that can be considered. The proof is general enough for a large class of fuzzy implications and is valid for any continuous function, not necessarily monotonic and the partitions used are of the Ruspini type. Thus, we believe that these results are very much applicable in most of the practical and desirable contexts [15, 48, 49].

6.3 Weak Coherence and Implicative Models

Dubois *et al.* [15] defined the concept of coherence for an implicative model $\hat{R}_{\rightarrow}$ (see (2.1)) of a rule base as follows, which is suitably modified to fit into our notation.

Definition 6.3.1 ([12], [15]). *Given an implicative rule base* (1.13)*, a fuzzy relation $R^{T}_{\rightarrow}(x, y)$, as in* (Imp-$R^{T}_{\rightarrow}$) *modelling this rule base, is coherent if for any $x \in X$ there exist $y \in Y$ such that $R^{T}_{\rightarrow}(x, y) = 1$.*

The coherence property states that for any x, the final fuzzy output B' should be normal, i.e., $\text{Ker}(B') \neq \emptyset$. Coherence of an implicative model of a rule base is very much dictated by the semantics involved [15]. Further, it is essential when using defuzzification techniques that are dependent on the kernel to be non-empty.

However, there exist other *reasonable* defuzzification methods that do not depend on the kernel of the output fuzzy set and, further, in the setting of function approximation, as is the case here, perhaps there is an arguable justification to not to insist on this otherwise extremely important property.

6.3.1 A Weaker form of Coherence

Relaxing the coherence property we define the following weaker form of coherence.

Definition 6.3.2. *For a given implicative rule base* (1.13), *a fuzzy relation $R^T_{\to}(x, y)$ is said to be* weakly coherent *if for any $x \in X$ there exist $y \in Y$ such that $R^T_{\to}(x, y) > 0$.*

From (FRI-R-Singleton) and (Imp-$R^T_{\to}$), we have the following:

$$\begin{aligned} B'(y) = R^T_{\to}(x_0, y) &= T^n_{i=1}(A_i(x_0) \longrightarrow B_i(y)) \\ &= T\big(A_1(x_0) \longrightarrow B_1(y), A_2(x_0) \longrightarrow B_2(y), \ldots, A_n(x_0) \longrightarrow B_n(y)\big). \end{aligned}$$

Now if the antecedent fuzzy sets are normal and form a Ruspini partition (See Definition 1.1.11), then x_0 intersects atmost two fuzzy sets say, A_m, A_{m+1}. Then the above reduces to

$$B'(y) = T(A_m(x_0) \longrightarrow B_m(y), A_{m+1}(x_0) \longrightarrow B_{m+1}(y)) = T(B'_m(y), B'_{m+1}(y)) \ ,$$

where B'_m and B'_{m+1} are the fuzzy sets B_m and B_{m+1} modified by the fuzzy implication $\longrightarrow$ with $A_m(x_0)$, $A_{m+1}(x_0)$.

It is clear that for B' to be non-empty the supports of B'_m and B'_{m+1} should intersect, i.e., Supp(B'_m)$\cap$ Supp(B'_{m+1}) $\neq \emptyset$ and also the t-norm T should be positive. It should be mentioned that this positivity condition is sufficient but not necessary - for instance, B' may be non-empty even for T not being positive, if the two consequent fuzzy sets overlap in sufficiently high degrees. While coherence insists that the kernels of B'_m and B'_{m+1} should intersect, the weak coherence defined above relaxes this to a mere intersection of their supports.

It should be noted that while relaxing coherence to weak coherence does expand the set of fuzzy implications that can be considered in $\hat{R}_{\to}$, it still does not encompass the whole set of fuzzy implications $\mathbb{I}$.

In the following, we discuss the class of fuzzy implications that can be considered for an FRI with $R^T_{\to}$ to be at least weakly coherent. This leads us to study the effect of using fuzzy implications to modify fuzzy sets.

6.3.2 Fuzzy Sets modified by Fuzzy Implications

From the above section it is clear that to ensure weak coherence, we need to deal with fuzzy sets that are modified by a fuzzy implication. Thus studying the properties of such modified fuzzy sets is important and we proceed to do this in the section.

Definition 6.3.3. *Let $C \in \mathcal{F}(X)$ and $I \in \mathbb{I}$ be any fuzzy implication. We say that a $C^I_\alpha \in \mathcal{F}(X)$ is the* modification or modified fuzzy set of *C* by *I* at a given *$\alpha \in [0, 1]$ if*

$$C^I_\alpha(x) = I(\alpha, C(x)), \ x \in X. \tag{6.2}$$

Since in this work we consider modification only by a fuzzy implication, we often use the simpler term *modified fuzzy set* without any explicit mention of either I or the $\alpha \in [0,1]$.

The following results show that modification by an $I \in \mathbb{I}$ preserves convexity and also gives some relations between the supports of the original and modified fuzzy sets when an $I \in \mathbb{I}$ is used.

Proposition 6.3.4. *For a convex fuzzy set C, a fuzzy implication I and any $\alpha \in [0,1]$, $C_\alpha^I = I(\alpha, C)$ is also convex.*

Proof. C being a convex fuzzy set, for all $\lambda \in [0,1]$ and $x, y \in X$, we have $C(\lambda x + (1-\lambda)y) \geq C(x) \wedge C(y)$. Now for any $\alpha \in [0,1]$ we have the following:

$$\begin{aligned}
&C(\lambda x + (1-\lambda)y) \geq C(x) \wedge C(y) \\
\Longrightarrow &I(\alpha, C(\lambda x + (1-\lambda)y)) \geq I(\alpha, C(x) \wedge C(y)) \\
\Longrightarrow &I(\alpha, C(\lambda x + (1-\lambda)y)) \geq I(\alpha, C(x)) \wedge I(\alpha, C(y)) \\
\Longrightarrow &C_\alpha^I(\lambda x + (1-\lambda)y) \geq C_\alpha^I(x) \wedge C_\alpha^I(y).
\end{aligned}$$

This proves that the modified fuzzy set $C_\alpha^I = I(\alpha, C)$ is convex. □

Remark 6.3.5. *In fact, the above result is true for any increasing function t. In Proposition 6.3.4, $t(C) = C_\alpha^I = I(\alpha, C)$, where $\alpha \in [0,1]$ is a constant.*

Proposition 6.3.6. *Let C be a bounded, normal, continuous convex fuzzy set, $I \in \mathbb{I}$ and $\alpha \in [0,1]$. Consider the following inclusion relating the supports of C and its modified set C_α^I:*

$$\mathrm{Supp}(C_\alpha^I) \supseteq \mathrm{Supp}(C)\ . \tag{6.3}$$

(i) If I is a non-positive fuzzy implication, then there exists an $\alpha \in [0,1]$ such that (6.3) *is not valid.*

(ii) For a given $I \in \mathbb{I}$, let $A_I = \{x \in [0,1] | I(x,0) = 0\}$ and let $\delta = \inf A_I$.

(a) If $\alpha < \delta$, then (6.3) *is valid always.*

(b) If $\alpha > \delta$, then (6.3) *is valid only if I is positive.*

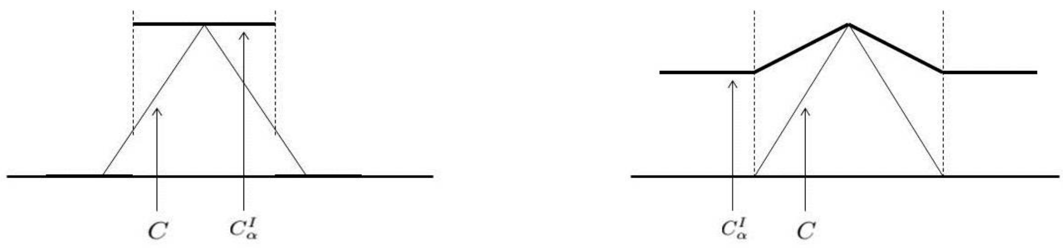
(c) Let $\alpha = \delta$. If $\delta \in A_I$, then (6.3) *is valid only if I is positive, while* (6.3) *holds for any $I \in \mathbb{I}$ if $\delta \notin A_I$.*

Proof. (i) Since I is non-positive, there exists some $x_0, y_0 \in (0,1)$ such that $I(x_0, y_0) = 0$. By the monotonicity of I we have that for $\alpha \in [x_0, 1]$ and $y \in [0, y_0]$, $I(\alpha, y) = 0$. Since C is continuous, normal and convex, there will exist a $U \subseteq X$ such that $C(x) \leq y_0$ on U. If we take $\alpha \in [x_0, 1]$ then $C_\alpha^I(x) = 0$ for all $x \in U$, i.e., $\mathrm{Supp}(C_\alpha^I) \subsetneq \mathrm{Supp}(C)$. *For a graphical illustration see Figure 6.1(a) where $I = I_{\mathbf{RS}}$ as defined in Table 1.5.*

(ii) Let $\delta = \inf A_I = \inf\{x \in [0,1] | I(x,0) = 0\}$. Note that for any $I \in \mathbb{I}$, $I(1,0) = 0$ and hence $\{x \in [0,1] | I(x,0) = 0\} \neq \emptyset$. Consider an $\alpha \in [0,1]$.

(a) Let $\alpha < \delta$. Then $I(\alpha, 0) > 0$ and by the monotonicity of I, we have $I(\alpha, \beta) > 0$ for any $\beta \in [0,1]$.

- On the one hand, if $x \in X \setminus \mathrm{Supp}(C)$, then $C(x) = 0$ and $C_\alpha^I(x) > 0$,



(a) $\text{Supp}(C^I_\alpha) \subsetneq \text{Supp}(C)$

(b) $\text{Supp}(C) \subseteq \text{Supp}(C^I_\alpha)$

Figure 6.1: Inclusions between the supports of the original and modified fuzzy sets, when (a) I is non-positive, (b) I is positive and $N_I \neq N_{\mathbf{D1}}$.

- On the other hand, when $x \in \text{Supp}(C)$, then $C(x) > 0$ and $C^I_\alpha(x) > 0$.

Thus it is clear that $\text{Supp}(C) \subseteq \text{Supp}(C^I_\alpha)$ and (6.3) holds. *For a graphical illustration see Figure 6.1(b) where $I = I_{\mathbf{RC}}$ as defined in Table 1.5.*

(b) Let $\alpha > \delta$. Once again, by the monotonicity of I, we have $I(\alpha, 0) = 0$. If $x \in X \setminus \text{Supp}(C)$, then $C(x) = 0$ and hence $C^I_\alpha(x) = 0$. If $x \in \text{Supp}(C)$, then $C(x) \in (0, 1]$. In fact, by the continuity and normality of C, for any $\beta \in (0, 1)$ there exists an $x \in \text{Supp}(C)$ such that $C(x) = \beta$. Now, if I is not positive, i.e., if there exists a $\beta \in (0, 1)$ such that $I(\alpha, \beta) = 0$ then for all $x \in \text{Supp}(C)$ such that $C(x) \leq \beta$ we have that $C^I_\alpha(x) = 0$. Thus to ensure that (6.3) holds we need an I which is positive.

(c) Let $\alpha = \delta$. If $\delta \in A_I$, then $I(\delta, 0) = I(\alpha, 0) = 0$ and hence it reduces to the case (b) above. If $\delta \notin A_I$, then $I(\delta, 0) = I(\alpha, 0) > 0$ and hence it reduces to the case (a) above.

□

Remark 6.3.7. *Note that, from Proposition 6.3.6 we see that whenever $\delta < 1$, to ensure that* (6.3) *holds we need an $I \in \mathbb{I}$ that is positive. If an $I \in \mathbb{I}$ which is positive and whose $N_I = N_{\mathbf{D1}}$ is used to modify C above, then the supports of C^I_α, C are equal, i.e., $\text{Supp}(C^I_\alpha) = \text{Supp}(C)$, for all $\alpha \in (0, 1]$. For a graphical illustration see Figure 6.2 where $I = I_{\mathbf{GD}}$ as defined in Table 1.5.*

Also, note that when $I \in \mathbb{I}$ is positive but $N_I \neq N_{\mathbf{D1}}$ then the modified fuzzy set may have infinite support, in which case (6.3) *holds trivially (Figure 6.1(b)).*

6.3.3 Classes of Admissible Fuzzy Implications

From Section 6.3.1 above we know that for an $R^T_{\rightarrow}$ to ensure weak coherence, we need the support of the output fuzzy sets B'_m and B'_{m+1} - which are the modified fuzzy sets of B_m, B_{m+1} using a fuzzy implication $I \in \mathbb{I}$ - to intersect. Also, it can be seen from Section 6.3.2 that when we use a non-positive fuzzy implication the supports of these modified fuzzy sets can shrink and hence there is a possibility that the intersection of their supports is *empty*, which is not desirable. Hence to ensure weak coherence at the least, we see that the class of implications I that can be considered should be restricted.

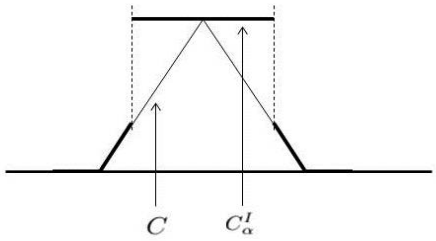


Figure 6.2: Supp(C) $\subseteq$ Supp(C_α^I) - When I is positive and $N_I = N_{\mathbf{D1}}$ - see Remark 6.3.7

Towards this end, let us consider the following subsets of $\mathbb{I}$:

- $\mathbb{I}_{\mathbf{OP}}$ - the set of all fuzzy implications satisfying ordering property (OP),
- $\mathbb{I}^+$ - the set of all fuzzy implications which satisfies positivity (I-POS),
- $\mathbb{I}^+_{N_{\mathbf{D1}}}$ - the set of all fuzzy implications which satisfies positivity (I-POS) and $N_I = N_{\mathbf{D1}}$.

Since in most practical settings we deal only with fuzzy sets that are bounded, continuous, convex and that which often form a Ruspini partition, it is sufficient to consider fuzzy implications $I \in \mathbb{I}$ that either

- satisfy the ordering property (OP), i.e., $I \in \mathbb{I}_{\mathbf{OP}}$, in which case often we can ensure even coherence [48], or
- are positive (I-POS) with $N_I = N_{\mathbf{D1}}$, i.e., $I \in \mathbb{I}^+_{N_{\mathbf{D1}}}$, in which case we can ensure at least a weak coherence.

Thus, in the following sections we will deal with rules modeled by fuzzy relations $R^T_{\rightarrow}$ where the fuzzy implication $\longrightarrow$ either satisfies (OP) or is positive with or without (OP) but whose natural negation $N_I = N_{\mathbf{D1}}$, the Gödel negation as in Table 1.3.

Remark 6.3.8. *Note that the properties* (OP), *positivity* (I-POS) *and $N_I = N_{\mathbf{D1}}$ are not mutually exclusive. Table 6.1 lists some fuzzy implications illustrating the same.*

6.3.4 Some Families of Fuzzy Implications that belong to $\mathbb{I}_{\mathbf{OP}} \cup \mathbb{I}^+_{N_{\mathbf{D1}}}$

In fact, many established families of fuzzy implications fall in either of the above two classes. For the definitions and the properties these families satisfy, please refer to the monograph [2].

- Let $\mathbb{I}_{\mathbb{T}_{\mathrm{BC}}}$ denote the set of all R-implications obtained from border continuous t-norms. Then every $I \in \mathbb{I}_{\mathbb{T}_{\mathrm{BC}}}$ satisfies (OP) ([3], Proposition 5.8). Further, the set of all R-implications obtained from left-continuous t-norms $\mathbb{I}_{\mathbb{T}_{\mathrm{LC}}} \subsetneq \mathbb{I}_{\mathbb{T}_{\mathrm{BC}}}$ and hence we have that

$$\mathbb{I}_{\mathbb{T}_{\mathrm{LC}}} \subsetneq \mathbb{I}_{\mathbb{T}_{\mathrm{BC}}} \subsetneq \mathbb{I}_{\mathbf{OP}} .$$

Implications	(OP)	(I-POS)	$N_I = N_{\mathbf{D1}}$
$I_{\mathbf{G}}(x,y) = \min(1, \frac{y}{x})$	✓	✓	✓
$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$	✓	✓	×
$I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$	✓	×	✓
$I(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0.5, & \text{if } x > y \text{ and } x \in [0, 0.5) \\ 0, & \text{if } (x,y) \in [0.5, 1) \times [0, 0.5) \\ 0.5, & \text{if } x > y \text{ and } y \in [0.5, 1) \end{cases}$	✓	×	×
$I_{\mathbf{YG}}(x,y) = \min(1, y^x)$	×	✓	✓
$I_{\mathbf{RC}}(x,y) = 1 - x + xy$	×	✓	×
$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ or } y < 1 \end{cases}$	×	×	✓
$I(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0.7, 1] \times [0, 0.6] \\ 0.5, & \text{if } (x,y) \in [0.4, 0.7] \times [0, 0.6] \\ 1, & \text{otherwise} \end{cases}$	×	×	×

Table 6.1: Fuzzy Implications that satisfy some or all of the properties of (OP), (I-POS) and $N_I = N_{\mathbf{D1}}$.

- If $\mathbb{I}_{\mathbb{S}}^*$ denotes the set of all $(S,N)-$ implications such that $N = N_S$, the natural negation of S, is a strong negation and the pair (S, N_S) is such that $S(N_S(x), x) = 1, \quad x \in [0,1]$ then every $I \in \mathbb{I}_{\mathbb{S}}^*$ satisfies (OP) ([3], Theorem 4.7). Hence

$$\mathbb{I}_{\mathbb{S}}^* \subsetneq \mathbb{I}_{\mathbf{OP}} .$$

- Let $\mathbb{I}_{\mathbb{QL}}^*$ denote the set of QL-implications obtained from the triplet $(T_{\mathbf{M}}, S, N_S)$ where $T_{\mathbf{M}}(x,y) = \min(x,y)$, S is any t-conorm and N_S, the natural negation of S, is a strong negation and the pair (S, N_S) is such that $S(N_S(x), x) = 1, \quad x \in [0,1]$. Then every $I \in \mathbb{I}_{\mathbb{QL}}^*$ satisfies (OP) ([5], Section 4.4). Hence

$$\mathbb{I}_{\mathbb{QL}}^* \subsetneq \mathbb{I}_{\mathbf{OP}}.$$

- From Section 3.1 we recall that, $\mathbb{I}_{\mathbb{F}}$ is the set of all f-implications and $\mathbb{I}_{\mathbb{F},\infty} \subsetneq \mathbb{I}_{\mathbb{F}}$ is the set of f-implications that are generated from generators such that $f(0) = \infty$. Every $I \in \mathbb{I}_{\mathbb{F},\infty}$ is positive and its natural negation is the Gödel negation (see Proposition 3.1.6), i.e., $N_I = N_{\mathbf{D1}}$. Thus

$$\mathbb{I}_{\mathbb{F},\infty} \subsetneq \mathbb{I}_{N_{\mathbf{D1}}}^{+}.$$

- Recall from Section 3.2 that $\mathbb{I}_{\mathbb{G}}$ denotes the set of all g-implications. Every $I \in \mathbb{I}_{\mathbb{G}}$ is positive and $N_I = N_{\mathbf{D1}}$ (see Proposition 3.2.6). Thus

$$\mathbb{I}_{\mathbb{G}} \subsetneq \mathbb{I}_{N_{\mathbf{D1}}}^{+}.$$

- For examples of fuzzy implications from other well-known families, viz., (U,N)-, RU- implications and the relationships among the properties they satisfy, please see, for instance, [4, 6] or the work of Bustince *et al.* [8].

6.4 $\mathbb{F}^{\boldsymbol{T}}_{\rightarrow_{\mathbf{OP}}}$ are Universal Approximators

Let us denote by $R^T_{\rightarrow_{\mathbf{OP}}}$ the fuzzy relation where the fuzzy implication $\longrightarrow$ is from $\mathbb{I}_{\mathbf{OP}}$ and the corresponding FRI by $\mathbb{F}^T_{\rightarrow_{\mathbf{OP}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^T_{\rightarrow_{\mathbf{OP}}}, d\right)$ where, $\mathcal{P}_X = \{A_i\}_{i=1}^n \subseteq \mathcal{F}(X)$ and $\mathcal{P}_Y = \{B_i\}_{i=1}^n \subseteq \mathcal{F}(Y)$.

Recall from (FRI-R-Singleton), for any $y \in Y$,

$$\begin{aligned} B'(y) &= R^T_{\rightarrow}(x_0, y) = T_{i=1}^n(A_i(x_0) \longrightarrow_{\mathbf{OP}} B_i(y)) \\ &= T\big(A_1(x_0) \longrightarrow_{\mathbf{OP}} B_1(y), A_2(x_0) \longrightarrow_{\mathbf{OP}} B_2(y), \ldots, A_n(x_0) \longrightarrow_{\mathbf{OP}} B_n(y)\big). \end{aligned} \tag{6.4}$$

Note that to get a final crisp output $y' \in Y$, we need to defuzzify the above $B' \in \mathcal{F}(Y)$ using d.

In this section, we show that, FRIs of the type $\mathbb{F}^T_{\rightarrow_{\mathbf{OP}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^T_{\rightarrow_{\mathbf{OP}}}, \text{MOM}\right)$ are universal approximators, i.e., they can approximate any continuous function over a compact set to arbitrary accuracy. Moreover, we show that the approximator function is continuous.

In the following results we take $X = [a, b]$ and $Y = h([a, b])$, but for the sake of readability we retain the same notations.

Theorem 6.4.1. *For any continuous function $h \colon [a, b] \to \mathbb{R}$ over a closed interval and an arbitrary given $\epsilon > 0$, there is an FRI $\mathbb{F}^T_{\rightarrow_{\mathbf{OP}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^T_{\rightarrow_{\mathbf{OP}}}, \text{MOM}\right)$ with $\mathcal{P}_X$ and $\mathcal{P}_Y$ being Ruspini partitions such that*

(i) the system function g as defined in (6.1) *is continuous on $[a, b]$, and*

(ii) $\max\limits_{x \in [a,b]} |h(x) - g(x)| < \epsilon$.

Proof. We prove this result in the following steps.

Step I : Choosing the points of normality

Since h is continuous over a closed interval $[a, b]$, h is uniformly continuous on $[a, b]$.
Thus for a given $\epsilon > 0$ there exists $\delta > 0$ (depending on ϵ) such that, for all $w, w' \in [a, b]$,

$$|w - w'| < \delta \Longrightarrow |h(w) - h(w')| < \frac{\epsilon}{2}\,.$$

***Step I** (a)*: A Coarse Initial Partition

With the δ defined above and taking $l = 1 + \left\lceil \frac{b-a}{\delta} \right\rceil$ we now choose $w_i \in X, i = 1, 2, \ldots l$, such that $|w_i - w_{i+1}| < \delta$.

Let $z_i = h(w_i)$, the value h takes at the above chosen w_i, for $i = 1, 2, \ldots l$. We call these points w_i and z_i the points of normality on the input space and the output space, respectively.

In Figure 6.3, the points $w_1, w_2, \ldots, w_{11}$ and the points $z_1, z_2, \ldots z_8$ (in paranthesis) are the points of normality in the input and the output spaces, respectively.

***Step I** (b)*: Redundancy Removal and Reordering

Let us choose the distinct z_i's from the above and sort them in ascending order. Let $\sigma : \mathbb{N}_l \longrightarrow \mathbb{N}_k$ denote the above permuation map such that $z_i = u_{\sigma(i)}$, for $i = 1, 2, \ldots l$ and $u_j,\ j = 1, 2, \ldots, k$ are in ascending order.

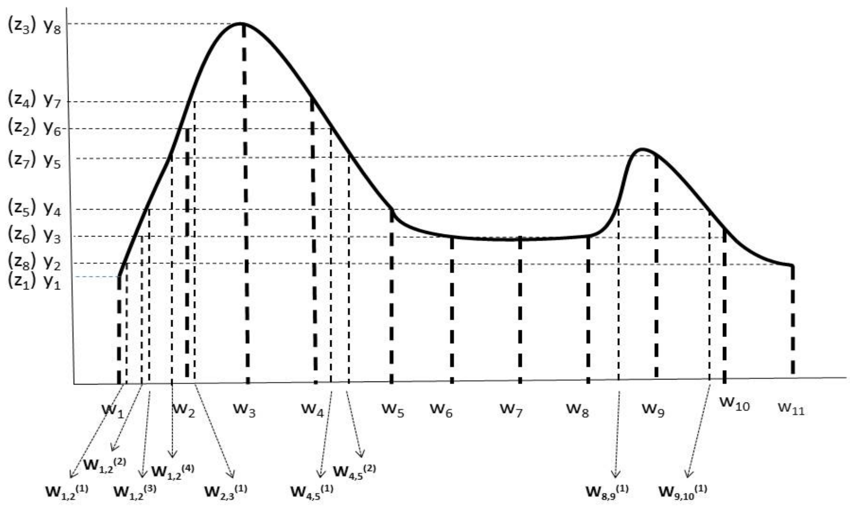


Figure 6.3: An Illustrative Example for **Step I** in the proof of Theorem 6.4.1. The intersection of the thin-dotted and thick-dotted lines with the x-axis give the points w_i and $w_{i,i+1}^{(q)}$, respectively.

Once again, from Figure 6.3, by rearranging the z_i's in ascending order and renaming them we obtain: $u_1 = z_1 < u_2 = z_8 < u_3 = z_6 < u_4 = z_5 < u_5 = z_7 < u_6 = z_2 < u_7 = z_4 < u_8 = z_3$.

***Step I** (c)*: Refinement of the input space partition:

Thus for each $i = 1, 2, \ldots, l$ we have $h(w_i) = z_i = u_{\sigma(i)}$. However, note that consecutive points of normality w_i, w_{i+1} in the input space need not be mapped to consecutive points of normality $u_{\sigma(i)}, u_{\sigma(i)+1}$ or $u_{\sigma(i)}, u_{\sigma(i)-1}$.

In Figure 6.3, $h(w_1) = u_1$ and $h(w_2) = u_6$. Thus for the consecutive points w_1 and w_2 the function values are u_1 and u_6, which are not consecutive.

To ensure the above, we further refine the input space partition. To this end, we refine every sub-interval $[w_i, w_{i+1}]$, for $i = 1, 2, \ldots l-1$ as follows. Note that $h(w_{i+1}) = u_{\sigma(i+1)}$.

Refinement Procedure:

For every $i = 1, 2, \ldots l-1$ do the following:

(i) If $u_{\sigma(i+1)} = u_{\sigma(i)+1}$ or $u_{\sigma(i)-1}$ then we do nothing.

(ii) Let $u_{\sigma(i+1)} = u_{\sigma(i)+p}$, where $p \geq 2$. For every $u \in \{u_{\sigma(i)+1}, u_{\sigma(i)+2}, \ldots, u_{\sigma(i)+p-1}\}$ we find a point $v \in [w_i, w_{i+1}]$ such that $h(v) = u$. Note that the existence of such a $v \in [w_i, w_{i+1}]$ is guaranteed by the continuity - essentially the ontoness - of the function h. If $u = u_{\sigma(i)+q}$, for some $1 \leq q \leq p-1$, then we denote the point v as $w_{i,i+1}^{(q)}$.

(iii) Similarly, let $u_{\sigma(i+1)} = u_{\sigma(i)-p}$, where $p \geq 2$. For every $u \in \{u_{\sigma(i)-1}, u_{\sigma(i)-2}, \ldots, u_{\sigma(i)-p+1}\}$ we find a $v \in [w_i, w_{i+1}]$ such that $h(v) = u$. Once again, if $u = u_{\sigma(i)-q}$, for some $1 \leq q \leq p-1$, then we denote v as $w_{i,i+1}^{(q)}$.

From Figure 6.3, it can be seen that we have inserted points $w_{1,2}^{(1)}, w_{1,2}^{(2)}, w_{1,2}^{(3)}, w_{1,2}^{(4)} \in [w_1, w_2]$. Proceeding similarly, the following sub-intervals, shown in Figure 6.3, have been refined: $[w_2, w_3]$, $[w_4, w_5]$, $[w_8, w_9]$ and $[w_9, w_{10}]$.

***Step I** (d)*: Final Points of Normality:

Once the above process is done, we again rename the points of normality in the input space, viz., w_i's and $w_{i,i+1}^{(q)}$'s as $x_1, x_2, \ldots, x_n (n \geq l)$ and the $u_{\sigma(i)}$'s of the the output space as $y_1, y_2, \ldots y_k$.

Step II : Construction of the Fuzzy Partitions - $\mathcal{P}_X, \mathcal{P}_Y$

In the next step, we construct fuzzy sets on both the input and output spaces with the above obtained x_i's and y_j's as the points of normality, as given below.

***Step II** (a)*: Fuzzy Partition on the input space $\mathcal{P}_X = \{A_i\}_{i=1}^n$.

We construct n fuzzy sets such that

- Supp$(A_i) = (x_{i-1}, x_{i+1})$ for $i = 2, \ldots, n-1$, while Supp$(A_1) = [x_1, x_2)$ and Supp$(A_n) = (x_{n-1}, x_n]$,
- each A_i is normal at x_i, i.e., $A_i(x_i) = 1$,
- each A_i is a continuous convex fuzzy set, strictly increasing on $[x_{i-1}, x_i]$ and strictly decreasing on $[x_i, x_{i+1}]$.
- $\{A_i\}_{i=1}^n$ form a Ruspini partition (See Definition 1.1.11 for instance).

For instance, let each of the A_i's ($i = 2, \ldots, n-1$) be a triangular fuzzy set, (i.e, each A_i is linear and strictly increasing on $[x_{i-1}, x_i]$, each A_i is linear and strictly decreasing on $[x_i, x_{i+1}]$), let A_1 be right-half-triangular, (i.e., A_1 is linear and strictly decreasing on $[x_1, x_2)$) and let A_n be left-half-triangular, (i.e., A_n is linear and strictly increasing on $(x_{n-1}, x_n]$). Further, let all the A_i's attain normality at x_i. Then, clearly, the fuzzy partition $\{A_i\}_{i=1}^n$ of the input space X is a Ruspini partition and each of the A_i's is continuous, convex, of finite support and $A_i(x_i) = 1$.

***Step II** (b)*: Fuzzy Partition on the output space $\mathcal{P}_Y = \{C_j\}_{j=1}^k$.

We construct k fuzzy sets in a similar way as above, such that

- Supp$(C_j) = (y_{j-1}, y_{j+1})$ for $j = 2, \ldots, k-1$, while Supp$(C_1) = [y_1, y_2)$ and Supp$(C_k) = (y_{k-1}, y_k]$,
- each C_j is normal at y_j, i.e., $C_j(y_j) = 1$,
- each C_j is a continuous convex fuzzy set, strictly increasing on $[y_{j-1}, y_j]$ and strictly decreasing on $[y_j, y_{j+1}]$.
- $\{C_j\}_{j=1}^k$ form a Ruspini partition.

Here obviously, $|y_j - y_{j-1}| < \frac{\epsilon}{2}, \quad j = 1, 2, \ldots k$.

Step III: Construction of the *smooth* rule base

We construct the rule base with n rules of the form:

$$\text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } C_j \ , \quad i = 1, 2, \ldots n, \tag{6.5}$$

where the consequent C_j in the i-th rule is chosen such that $y_j = h(x_i)$, where x_i is the point at which A_i attains normality.

Now as in Section 1.3.2, we can rewrite the rule base (6.5) as follows:

$$\text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i \ , \quad i = 1, 2, \ldots n. \tag{6.6}$$

Note that not all B_i's may be distinct.

Note that, since h is continuous, by the above assignment of the rules, we have that rules whose antecedents are adjacent also have adjacent consequents, i.e., for any $i = 1, 2, \ldots n-1$ we have $\text{Supp}(B_i) \cap \text{Supp}(B_{i+1}) \neq \emptyset$. Thus the constructed rule base is smooth [46].

Step IV : Approximation capability of the output

Let $x' \in X$ be the arbitrary given input. Clearly, $x' \in [x_m, x_{m+1}]$ for some $m \leq n-1$. Once again, by our construction, x' belongs to atmost two adjacent A_i's, and they are A_m, A_{m+1}. Thus, from (6.4),

$$\begin{aligned} B'(y) &= T\left[A_m(x') \longrightarrow_{\mathbf{OP}} B_m(y), A_{m+1}(x') \longrightarrow_{\mathbf{OP}} B_{m+1}(y)\right] \\ &= T\left[s_m \longrightarrow_{\mathbf{OP}} B_m(y), s_{m+1} \longrightarrow_{\mathbf{OP}} B_{m+1}(y)\right], \end{aligned}$$

where we introduce the notations $s_m = A_m(x')$ and $s_{m+1} = A_{m+1}(x')$ for better readability in the proofs. Note that since A_i's form a Ruspini partition, we have that $s_m + s_{m+1} = 1$. Further, note that by the construction of $\{A_i, B_j\}$, B_m, B_{m+1} are adjacent fuzzy sets.

Consider the kernel of B'. We choose the defuzzified output y' such that it belongs to $\text{Ker}(B')$. In fact, as we show below, by the construction of $\{A_i, B_j\}$ we see that $\text{Ker}(B')$ is a singleton and this becomes the defuzzified output.

Since T is a t-norm, we know that $T(p, q) = 1$ if and only if $p = 1$ and $q = 1$. Further, note that $p \longrightarrow_{\mathbf{OP}} q = 1$ if and only if $p \leq q$ and $s_m + s_{m+1} = 1$ and hence we have

$$\begin{aligned} \text{Ker}(B') &= \{y | B'(y) = 1\} \\ &= \{y | s_m \longrightarrow_{\mathbf{OP}} B_m(y) = 1\} \bigcap \\ &\qquad\qquad \{y | s_{m+1} \longrightarrow_{\mathbf{OP}} B_{m+1}(y) = 1\} \\ &= \{y | s_m \leq B_m(y)\} \bigcap \{y | s_{m+1} \leq B_{m+1}(y)\} . \end{aligned}$$

Let $\alpha_m = \inf\{\alpha | \ s_m \longrightarrow_{\mathbf{OP}} \alpha = 1\}$ and $\beta_{m+1} = \inf\{\beta | s_{m+1} \longrightarrow_{\mathbf{OP}} \beta = 1\}$. Since $\longrightarrow_{\mathbf{OP}}$ has (OP), clearly $\alpha_m = s_m$ and $\beta_{m+1} = s_{m+1}$.

By the continuity and convexity of B_m, B_{m+1} there exist $a_m, b_m, a_{m+1}, b_{m+1}$ such that $B_m(a_m) = B_m(b_m) = s_m$ and $B_{m+1}(a_{m+1}) = B_{m+1}(b_{m+1}) = s_{m+1}$. By the monotonicity of the implication in the second variable, for every $y \in [a_m, b_m]$ we have that $s_m \to B_m(y) = 1$ and for every $y \in [a_{m+1}, b_{m+1}]$ we have that $s_{m+1} \to B_{m+1}(y) = 1$. Thus,

$$\begin{aligned} \{y | s_m \leq B_m(y)\} &= [a_m, b_m] , \\ \{y | s_{m+1} \leq B_{m+1}(y)\} &= [a_{m+1}, b_{m+1}] , \quad \text{and} \\ \text{Ker}(B') = \{y | B'(y) = 1\} &= [a_m, b_m] \bigcap [a_{m+1}, b_{m+1}]. \end{aligned}$$

Claim: $\text{Ker}(B') = \{a_{m+1}\} = \{b_m\} \neq \emptyset$.

Firstly, note that for any $s_m \in [0, 1]$ by the normality of B_m we have that $B_m(y_m) = 1$ and hence $y_m \in \{y | s_m \leq B_m(y)\} \implies y_m \in [a_m, b_m] \neq \emptyset$. Similarly, $y_{m+1} \in [a_{m+1}, b_{m+1}] \neq \emptyset$. It suffices to show that $a_{m+1} \leq b_m$ from whence $\text{Ker}(B') = [a_{m+1}, b_m]$.

Note that since $m < m+1$ and B_m, B_{m+1} are adjacent fuzzy sets, either $y_m < y_{m+1}$ or $y_m > y_{m+1}$. Without loss of generality, let us assume $y_m < y_{m+1}$. Now, from $a_{m+1} \in \text{Supp}(B_{m+1})$ we have that $y_m \leq a_{m+1} \leq y_{m+1}$. Similarly, $y_m \leq b_m \leq y_{m+1}$. Hence, $y_m \leq a_{m+1}, b_m \leq y_{m+1}$. Since,

$$\begin{aligned} s_m + s_{m+1} = 1 &\Longrightarrow B_{m+1}(a_{m+1}) + B_m(b_m) = 1, \\ &\Longrightarrow B_{m+1}(a_{m+1}) = 1 - B_m(b_m), \\ &\Longrightarrow B_{m+1}(a_{m+1}) = B_{m+1}(b_m), \\ &\Longrightarrow b_m \in [a_{m+1}, b_{m+1}], \\ &\text{i.e., } a_{m+1} \leq b_m \ . \end{aligned}$$

Now, to see that $b_m = a_{m+1}$, note that since $\{B_j\}$ form a Ruspini partition and B_m, B_{m+1} are adjacent fuzzy sets, we have $B_{m+1}(a_{m+1}) = 1 - B_m(a_{m+1})$ and hence

$$B_m(a_{m+1}) = s_m = B_m(b_m). \tag{6.7}$$

Since $b_m, a_{m+1} \in \text{Supp}(B_m) \cap \text{Supp}(B_{m+1})$ on which both B_m, B_{m+1} are strictly monotonic (but of opposite types) we have that $b_m = a_{m+1}$.

Since d is the MOM defuzzification, we get that $y' = g(x') = d(B') = a_{m+1} = b_m \in [y_m, y_{m+1}]$.

Claim: g is continuous on $[a, b]$.

Let us consider an $x' \in [a, b]$. Clearly, $x' \in [x_m, x_{m+1}]$ for some $1 \leq m < n$ and $g(x') = b_m \in [y_m, y_{m+1}]$. Let $A_m(x') = s_m \in [0, 1]$.

To show that g is continuous at x', we need to show that for any given $\epsilon > 0$, we can find a $\delta > 0$ such that, for any $x^* \in [a, b]$, whenever

$$|x^* - x'| < \delta \ \text{ then } \ |g(x^*) - g(x')| < \epsilon. \tag{6.8}$$

Since B_m is strictly decreasing and continuous on $[y_m, y_{m+1}]$, we have that $B_m^{-1} : [0, 1] \longrightarrow [y_m, y_{m+1}]$ exists. Thus from (6.7) we have $b_m = B_m^{-1}(s_m)$.

Further, B_m^{-1} is strictly decreasing and continuous on $[0, 1]$. Hence, for any $\epsilon_1 > 0$ there exists some $\delta_1 > 0$ such that for any $s_m^* \in [0, 1]$,

$$|s_m^* - s_m| < \delta_1 \Longrightarrow |B_m^{-1}(s_m^*) - B_m^{-1}(s_m)| < \epsilon_1. \tag{6.9}$$

Since $A_m : [x_m, x_{m+1}] \longrightarrow [0, 1]$ is continuous, for any $\epsilon_2 > 0$ there exists some $\delta_2 > 0$ such that

$$|x^* - x'| < \delta_2 \Longrightarrow |A_m(x^*) - A_m(x')| < \epsilon_2. \tag{6.10}$$

Let $s_m^* = A_m(x^*)$. Then $g(x^*) = b_m^* \in [y_m, y_{m+1}]$ and

$$|s_m^* - s_m| = |A_m(x^*) - A_m(x')| \ , \text{ and} \tag{6.11}$$
$$|g(x^*) - g(x')| = |b_m^* - b_m| = |B_m^{-1}(s_m^*) - B_m^{-1}(s_m)| \ . \tag{6.12}$$

Now, let us set $\epsilon_1 = \epsilon$ and $\epsilon_2 = \delta_1$. Then, for $\delta = \delta_2$, we have

$$
\begin{aligned}
|x^* - x'| < \delta \Longrightarrow & |A_m(x^*) - A_m(x')| < \epsilon_2 \ , && \text{(using (6.10))} \\
\Longrightarrow & |s_m^* - s_m| < \epsilon_2 = \delta_1 \ , && \text{(using (6.11))} \\
\Longrightarrow & |B_m^{-1}(s_m^*) - B_m^{-1}(s_m)| < \epsilon_1 = \epsilon \ , && \text{(using (6.9))} \\
\Longrightarrow & |b_m^* - b_m| < \epsilon \ , && \\
\Longrightarrow & |g(x^*) - g(x')| < \epsilon \ . && \text{(using (6.12))}
\end{aligned}
$$

Thus for any $\epsilon > 0$, there exists a $\delta > 0$ such that, whenever $|x^* - x'| < \delta$ then $|g(x^*) - g(x')| < \epsilon$, i.e., g is continuous on $[a, b]$.

Clearly, now,

$$
|y_m - g(x')| < \frac{\epsilon}{2} \quad \text{or} \quad |y_{m+1} - g(x')| < \frac{\epsilon}{2} \ .
$$

Without loss of generality, let $|y_m - g(x')| < \frac{\epsilon}{2}$, i.e., $|y_m - y'| < \frac{\epsilon}{2}$.

Further, since $x' \in [x_m, x_{m+1}]$ we have $|h(x') - y_m| < \frac{\epsilon}{2}$. Putting them all together, we have

$$
\begin{aligned}
|g(x') - h(x')| &= |y' - h(x')| \\
&\leq |y' - y_m| + |y_m - h(x')| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}
$$

Since x' is arbitrary we have, $\max_{x \in [a,b]} |h(x) - g(x)| < \epsilon$. □

6.5 $\mathbb{F}^{\boldsymbol{T}}_{\rightarrow_{\mathbf{D1}}}$ are Universal Approximators

While in the previous section, we dealt with fuzzy implications satisfying (OP), this class of fuzzy implications is rather limited. In this section, we consider those positive implications whose natural negations are Gödel negation.

Let us denote by $R^T_{\rightarrow_{\mathbf{D_1}}}$ the fuzzy relation where the fuzzy implication $\longrightarrow$ is from $\mathbb{I}^+_{N_{\mathbf{D1}}}$ and the corresponding FRI by $\mathbb{F}^T_{\rightarrow_{\mathbf{D1}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^T_{\rightarrow_{\mathbf{D_1}}}, d\right)$.

Once again, recall that from (FRI-R-Singleton), with $R = R^T_{\rightarrow_{\mathbf{D_1}}}$ for any $y \in Y$, we have

$$
\begin{aligned}
B'(y) &= R^T_{\rightarrow_{\mathbf{D_1}}}(x_0, y) = T_{i=1}^n (A_i(x_0) \longrightarrow_{\mathbf{D_1}} B_i(y)) \\
&= T\big(A_1(x_0) \longrightarrow_{\mathbf{D_1}} B_1(y), A_2(x_0) \longrightarrow_{\mathbf{D_1}} B_2(y), \ldots, A_n(x_0) \longrightarrow_{\mathbf{D_1}} B_n(y)\big). \qquad (6.13)
\end{aligned}
$$

We now show that the FRIs $\mathbb{F}^T_{\rightarrow_{\mathbf{D1}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^T_{\rightarrow_{\mathbf{D1}}}, \text{MOM}\right)$ are universal approximators, i.e., they can approximate any continuous function over a compact set to arbitrary accuracy.

Theorem 6.5.1. *For any continuous function $h\colon [a, b] \to \mathbb{R}$ over a closed interval and an arbitrary given $\epsilon > 0$, there is an FRI $\mathbb{F}^T_{\rightarrow_{\mathbf{D_1}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, R^T_{\rightarrow_{\mathbf{D_1}}}, \text{MOM}\right)$ with $\mathcal{P}_X$ and $\mathcal{P}_Y$ being Ruspini partitions such that the system function g approximates h uniformly, i.e., $\max_{x \in [a,b]} |h(x) - g(x)| < \epsilon$.*

Proof. Once again the proof is given in many steps. **Steps I–III** dealing with the construction of the input and output partitions and the rule base are done in exactly the same way as in **Steps I–III** of the proof of Theorem 6.4.1.

Step IV: Approximation capability of the output

Once again, let $x' \in X$ be the arbitrary given input. Clearly, $x' \in [x_m, x_{m+1}]$ for some $m \leq l-1$. Once again, by our construction, x' belongs to A_m, A_{m+1}. Thus, from (6.13),

$$\begin{aligned} B'(y) &= T\left[A_m(x') \longrightarrow_{\mathbf{D_1}} B_m(y), A_{m+1}(x') \longrightarrow_{\mathbf{D_1}} B_{m+1}(y)\right] \\ &= T\left[s_m \longrightarrow_{\mathbf{D_1}} B_m(y), s_{m+1} \longrightarrow_{\mathbf{D_1}} B_{m+1}(y)\right] \\ &= T\left[B'_m(y), B'_{m+1}(y)\right] \end{aligned}$$

where $s_m = A_m(x')$ and $s_{m+1} = A_{m+1}(x')$. Note that since A_i's form a Ruspini partition, we have that $s_m + s_{m+1} = 1$.
Now since $\longrightarrow_{\mathbf{D_1}}$ is positive and such that $x \longrightarrow_{\mathbf{D_1}} 0 = 0$ for any $x \in (0,1]$, we have from Remark 6.3.7 that the supports of both the modified fuzzy sets B'_m, B'_{m+1} are the same as those of B_m, B_{m+1}, i.e., Supp(B'_m)=Supp(B_m) and Supp(B'_{m+1}) = Supp(B_{m+1}). Hence,

$$\begin{aligned} \text{Supp}(B') &= \text{Supp}(B'_m) \cap \text{Supp}(B'_{m+1}) \\ &= \text{Supp}(B_m) \cap \text{Supp}(B_{m+1}) \\ &= \text{Supp}(B_m \cap B_{m+1}) = [y_m, y_{m+1}]\,. \end{aligned} \tag{6.14}$$

Now since (6.14) holds we have,

$$\begin{aligned} y' &= g(x') = \text{MOM}(B') \\ &\in \text{Supp}(B_m \cap B_{m+1}) = [y_m, y_{m+1}]. \end{aligned}$$

So $|y_m - g(x')| < \dfrac{\epsilon}{2}$ and $|y_{m+1} - g(x')| < \dfrac{\epsilon}{2}$. Now consider $|y_m - g(x')| < \dfrac{\epsilon}{2}$. Once again, since $x' \in [x_m, x_{m+1}]$ we have $|h(x') - y_m| < \dfrac{\epsilon}{2}$. Putting them all together, we have

$$\begin{aligned} |g(x') - h(x')| &= |y' - h(x')| \\ &\leq |y' - y_m| + |y_m - h(x')| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Since x' is arbitrary we have, $\max\limits_{x \in [a,b]} |h(x) - g(x)| < \epsilon$. $\square$

6.6 Approximation Capability of BKS-$\mathcal{Y}$ Inference Mechanisms

In Chapters 4 and 5, we have seen that the FRIs with Yager's families of fuzzy implications $\mathbb{F}_{\to_\mathcal{Y}}$ possess the following desirable properties, namely, interpolativity, continuity and robustness.

In this section, we show that FRIs with Yager's families of fuzzy implications are also universal approximators. The results are, in fact, some special cases of the above Theorem 6.5.1.

Corollary 6.6.1. *For any continuous function $h\colon [a,b] \to \mathbb{R}$ over a closed interval and an arbitrary given $\epsilon > 0$, there is an FRI $\mathbb{F}_{\to_f} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_f, \mathrm{MOM}\right)$ with $\hat{R}_f$ as defined in* (Imp-$\hat{R}_f$) *and $\longrightarrow_f \in \mathbb{I}_{\mathbb{F},\infty}$ and $\mathcal{P}_X$ and $\mathcal{P}_Y$ being Ruspini partitions such that the system function g approximates h uniformly, i.e.,*

$$\max_{x\in[a,b]} |h(x) - g(x)| < \epsilon.$$

Proof. Every $\longrightarrow_f \in \mathbb{I}_{\mathbb{F},\infty}$ is positive and its natural negation is the Gödel negation. Thus $\longrightarrow_f \in \mathbb{I}_{\mathbb{F},\infty} \subsetneq \mathbb{I}^+_{N_{\mathbf{D1}}}$ and the result follows from Theorem 6.5.1. □

Corollary 6.6.2. *For any continuous function $h\colon [a,b] \to \mathbb{R}$ over a closed interval and an arbitrary given $\epsilon > 0$, there is an FRI $\mathbb{F}_{\to_g} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_g, \mathrm{MOM}\right)$ with $\hat{R}_g$ as defined in* (Imp-$\hat{R}_g$) *and $\longrightarrow_g \in \mathbb{I}_{\mathbb{G}}$ and $\mathcal{P}_X$ and $\mathcal{P}_Y$ being Ruspini partitions such that the system function g approximates h uniformly, i.e.,*

$$\max_{x\in[a,b]} |h(x) - g(x)| < \epsilon.$$

Proof. Every $\longrightarrow_g \in \mathbb{I}_{\mathbb{G}}$ is positive and its natural negation is the Gödel negation. Thus $\longrightarrow_g \in \mathbb{I}_{\mathbb{G}} \subsetneq \mathbb{I}^+_{N_{\mathbf{D1}}}$ and the result follows from Theorem 6.5.1. □

6.7 Illustrative Examples

In this section we illustrate our results through some examples. We consider 3 functions one each from the following three types or classes of functions, viz., those that are (i) purely monotonic, (ii) mixed monotonic and symmetric, and (iii) mixed monotonic and asymmetric. We then approximate these functions by the FRIs $\mathbb{F}^T_{\to_{\mathbf{OP}}}$ or $\mathbb{F}^T_{\to_{\mathbf{D1}}}$ as proposed and constructed in Sections 6.4 and 6.5.

We consider the Mean of Maxima defuzzification and the input and output space partitions are constructed as detailed in Section 6.4. However, we consider different fuzzy implication operators I coming from both the classes, viz., $\mathbb{I}_{\mathbf{OP}}$ and $\mathbb{I}^+_{N_{\mathbf{D1}}}$, in the examples.

In the given figures, the original functions $h(x)$ are shown in thick lines, the approximating system functions $g(x)$ in thin lines and the bounds $h(x) - \epsilon$ and $h(x) + \epsilon$ are plotted using dotted $--$ lines.

Example 6.7.1. *Let us consider the function*

$$h(x) = \ln(x), \quad x \in [2,7]\ ,$$

which is strictly increasing on the interval $[2,7]$ and let $\epsilon = 0.1$. According to the proposed construction we obtain 50 rules, since $\delta = \epsilon = 0.1$. We approximate h using the FRI $\mathbb{F}^T_{\to_{\mathbf{OP}}}$, where the implication operator employed in the relation $R^T_{\to_{\mathbf{OP}}}$ is the Rescher implication $I_{\mathbf{RS}} \in \mathbb{I}_{\mathbf{OP}} \setminus \mathbb{I}^+_{N_{\mathbf{D1}}}$ (see Table 6.1), which satisfies (OP) *and its natural negation $N_{I_{\mathbf{RS}}} = N_{\mathbf{D1}}$, but $I_{\mathbf{RS}}$ is not positive. The function h and its approximation g are shown in Figure 6.4.*

Example 6.7.2. *Let us consider the function*

$$h(x) = \sin(x), \quad x \in [-2\pi, 2\pi]\ ,$$

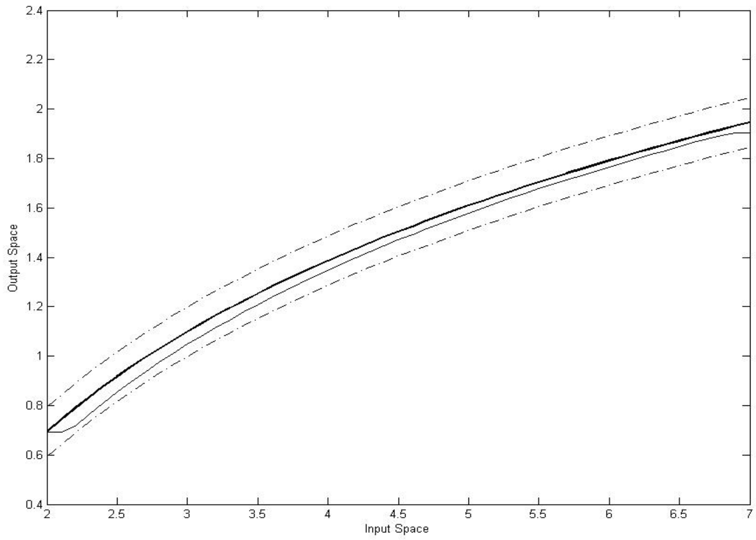


Figure 6.4: The natural logarithm function $h(x) = \ln x$ approximated within $\epsilon = 0.1$ bound over $[2, 7]$

which is mixed monotonic and symmetric on the interval $[-2\pi, 2\pi]$. However, note that it is piecewise strictly increasing or decreasing. Let $\epsilon = 0.1$. According to the proposed construction we obtain 130 rules. We approximate h using the FRI $\mathbb{F}^{T}_{\rightarrow \mathbf{OP}}$, where the implication operator employed in the relation $R^{T}_{\rightarrow \mathbf{OP}}$ is the Łukasiewicz implication $I_{\mathbf{LK}} \in \mathbb{I}_{\mathbf{OP}} \setminus \mathbb{I}^{+}_{N_{\mathbf{D1}}}$ (see Table 6.1) which satisfies (OP) *and is positive, but $N_{I_{\mathbf{LK}}} \neq N_{\mathbf{D1}}$. The function h and its approximation g are shown in Figure 6.5.*

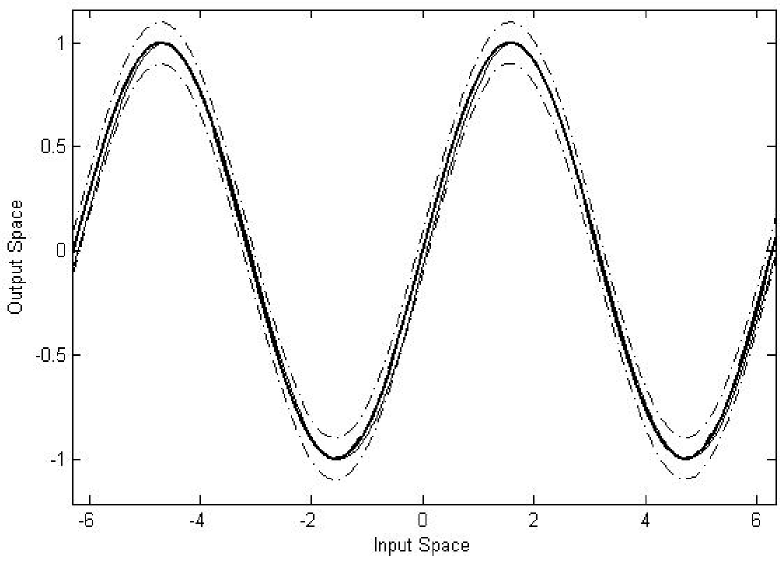
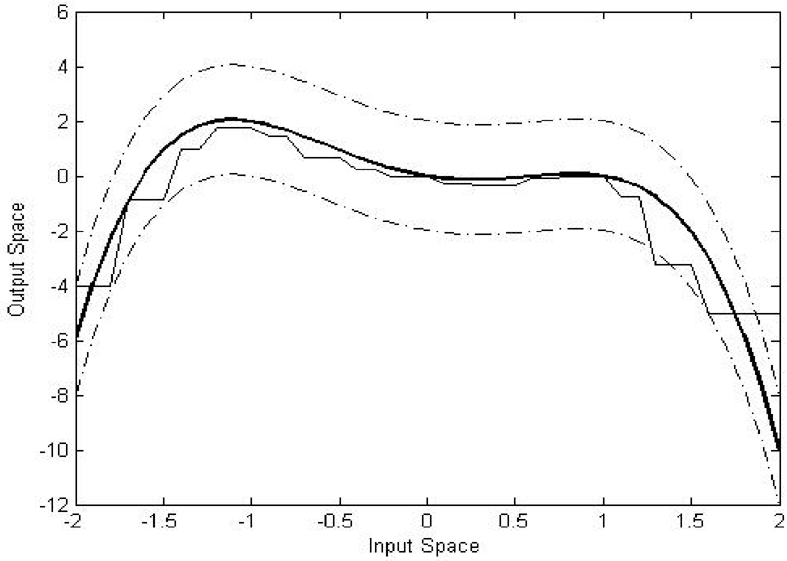
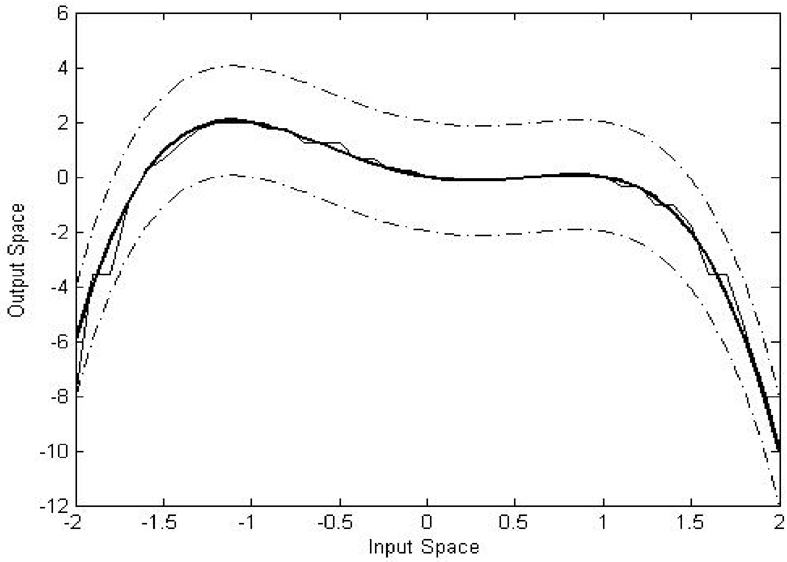


Figure 6.5: The function $h(x) = \sin(x)$ approximated within $\epsilon = 0.1$ bound over $[-2\pi, 2\pi]$.



(a) Using 20 rules obtained from the unrefined partition



(b) Using 38 rules obtained from the refined partition

Figure 6.6: A 4th degree polynomial $h(x) = -x^4 + 2x^2 - x$ approximated over $[-2, 2]$ within $\epsilon = 2$ bound.

Example 6.7.3. *Let us consider the function*

$$h(x) = -x^4 + 2x^2 - x, \quad x \in [-2, 2] \,,$$

which is both mixed monotonic and asymetric on the interval $[-2, 2]$. *Let* $\epsilon = 2$.

It is clear from the proof of Theorem 6.4.1 that the δ *obtained for a given* ϵ *is extremely conservative. Thus, in this case we would get a* $\delta = 0.0025$. *However, we have assumed* $\delta = 0.25$ *and proceeded to verify the approximation capability. According to the proposed construction, with* $\delta = 0.25$, *we obtain* 20 *rules by* ***Step I****(a) of Theorem 6.4.1, which are then refined to* 38 *rules by* ***Step I****(b) and* ***Step I****(c) of Theorem 6.4.1. We approximate* h *using the FRI* $\mathbb{F}^{T}_{\to_{\mathbf{D_1}}}$ *, where the implication operator employed in the relation* $R^{T}_{\to_{\mathbf{D_1}}}$ *is the Yager implication* $I_{\mathbf{YG}} \in \mathbb{I}^{+}_{N_{\mathbf{D1}}} \setminus \mathbb{I}_{\mathbf{OP}}$ *(see Table 6.1) which does not satisfy* (OP)*, but is both positive and* $N_{I_{\mathbf{YG}}} = N_{\mathbf{D1}}$.

The approximated function is shown in Figure 6.6. Figure 6.6(a) gives the plot of the approximator g *that was obtained from the original rule base with* 20 *rules that were obtained before the refinement of the input space, while Figure 6.6(b) gives the plot of the approximator* g *that was obtained by employing the refined rule base with* 38 *rules.*

Remark 6.7.4. *From* ***Step I****(a) of the proof of Theorem 6.4.1, note that the number of rules generated from the proposed construction in Section 6.4 is dependent both on the function and the* ϵ *value given but not on the fuzzy implication employed in* $\mathbb{F}^{T}_{\to}$. *It is also clear from the illustrated examples that even a coarser partition of the input space than what is proposed can still approximate the given function within the bounds, i.e., even with a bigger* δ *we can still get the same* ϵ *approximation. Of course, the* δ *itself can be adapted depending on the prior knowledge of the slope of the given function to be approximated.*

Chapter 7

Monotonicity of SISO Fuzzy Relational Inference Mechanisms based on Fuzzy Implications

The mathematical sciences particularly exhibit order, symmetry and limitations; and these are the greatest forms of the beautiful.
– Aristotle (367 BC–347 BC)

Monotonicity of the system function (see Section 2.3.4) of an inference mechanism is one of the essential properties of an inference mechanism, unavailibility of which leads to an unreliable inference mechanism [43], [44], [46], [47]. Let us be given a monotone rule base of the form (see Definition 1.3.2):

$$\mathcal{R}_M(A_i, B_i) : \textbf{IF } \tilde{x} \text{ is } A_i \textbf{ THEN } \tilde{y} \text{ is } B_i,\ i = 1, 2, \ldots n, \tag{7.1}$$

where the antecedents $A_i \in \mathcal{P}_X$ and consequents $B_i \in \mathcal{P}_Y$ are such that they maintain the same ordering as explained in Section 1.3.4. Given a monotone rule base and monotonic inputs, monotonicity of an FRI refers to whether we obtain monotonic outputs.

Absence of monotonicity of the system function of an inference mechanism is not desirable. For example, let us consider a fuzzy rule base given by an expert for an air conditioner. The rules are typically of the form *"The higher is the room temperature, the higher is the fan speed"'* and hence the rule base is clearly monotone. The non-monotonicity of the resulting system function is not at all desirable, since for higher temperature the fan speed should not decrease.

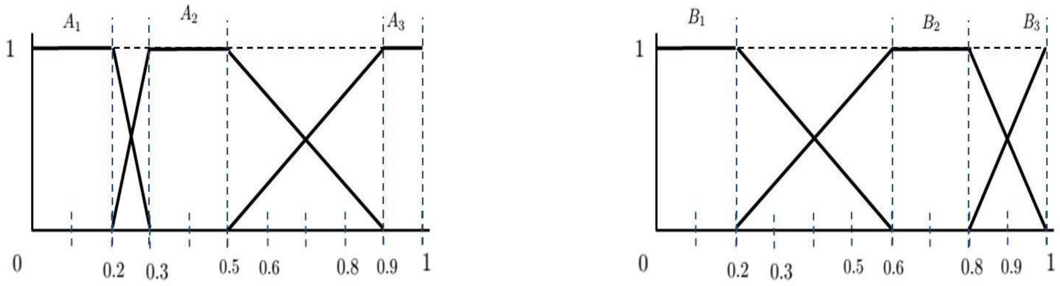
The question now is the following:

For a given monotone rule base $\mathcal{R}_M(A_i, B_i)$, does there exist an FRI $\mathbb{F} = \Big(\mathcal{P}_X, \mathcal{P}_Y, @, R, d\Big)$ with a suitable fuzzy relation R that models the rule base $\mathcal{R}_M(A_i, B_i)$, and a defuzzifier d, such that for any two crisp inputs x' and x'' with $x' \leq x''$, the corresponding outputs are such that $g(x') = y' \leq y'' = g(x'')$?

Let us give an example which shows that a fuzzy relational inference mechanism, may or may not be monotone.

Example 7.0.5. *Let the input and output space be $X = [0,1]$ and $Y = [0,1]$, respectively. Let us consider the fuzzy sets $A_1 = \langle 0,\ 0,\ 0.2,\ 0.3\rangle$, $A_2 = \langle 0.2,\ 0.3,\ 0.5,\ 0.9\rangle$, $A_3 = \langle 0.5,\ 0.9,\ 1,\ 1\rangle$ and $B_1 = \langle 0,\ 0,\ 0.2,\ 0.6\rangle$, $B_2 = \langle 0.2,\ 0.6,\ 0.8,\ 1\rangle$, $B_3 = \langle 0.8,\ 1,\ 1,\ 1\rangle$, as shown in the Figure 7.1, where a quadruple $\langle a,\ b,\ c,\ d\rangle$ represents a trapezoidal fuzzy set A as shown in the Figure 7.2.*

It can be easily verified that $A_1 \prec A_2 \prec A_3$ and $B_1 \prec B_2 \prec B_3$.



(a) The antecedent fuzzy sets A_i's (b) The consequent fuzzy sets B_i's

Figure 7.1: The Fuzzy Sets A_i's and B_i's

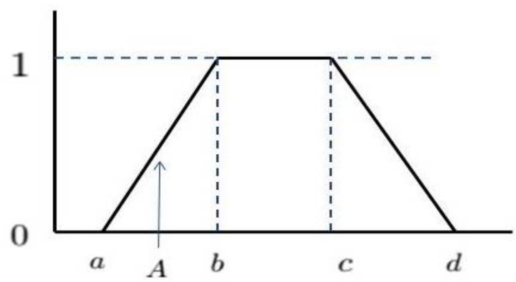


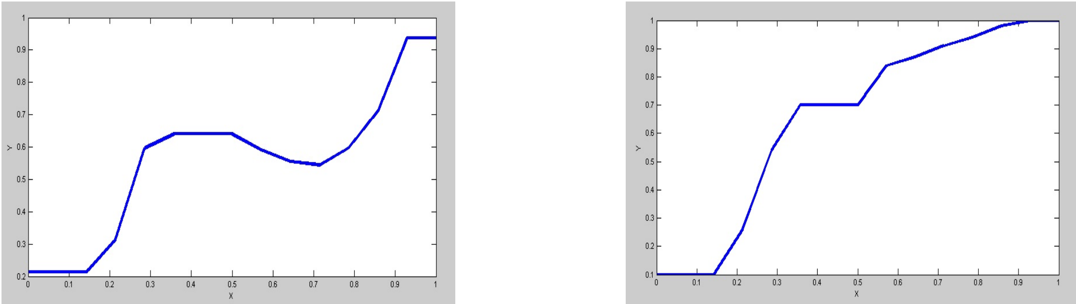
Figure 7.2: The Trapezoidal Fuzzy Set representing $A = \langle a\, b,\ c,\ d\rangle$.

Consider the rule base,

$$\textbf{\textit{IF}}\ \tilde{x} \text{ is } A_i\ \textbf{\textit{THEN}}\ \tilde{y} \text{ is } B_i,\ i = 1,2,3, \tag{7.2}$$

which is monotone, since $A_1 \prec A_2 \prec A_3$ and $B_1 \prec B_2 \prec B_3$.

Let us consider the FRI with reducible composition $\mathbb{F}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, d\right)$, *where* $\longrightarrow = I_{\mathbf{LK}}$. *Then the system function (sometimes called the input-output function) is as shown in Figure 7.3 for two different types of defuzzification methods, viz., (i) Center of Gravity (*COG*) (see* (1.4)*) and (ii) Mean of Maxima (*MOM*) (see* (1.1)*). From the Figure 7.3, it can be noticed that the system function in one of the cases is monotonic and in another case it is not. Hence the FRI* $\mathbb{F}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, \mathrm{MOM}\right)$ *with* $\longrightarrow = I_{\mathbf{LK}}$ *is monotonic, whereas* $\mathbb{F}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, \mathrm{COG}\right)$ *with* $\longrightarrow = I_{\mathbf{LK}}$ *is not.*

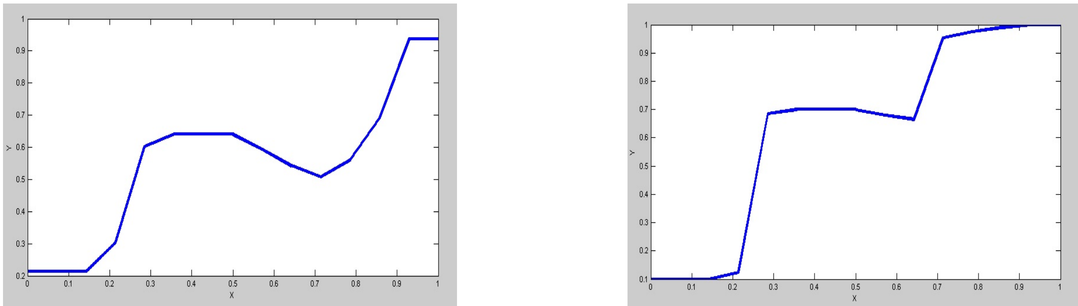


(a) System function with COG defuzzification

(b) System function with MOM defuzzification

Figure 7.3: System function of the FRI $\mathbb{F}_{\rightarrow}$ given in Example 7.0.5 with (a) COG and (b) MOM defuzzifier and $\longrightarrow = I_{\mathbf{LK}}$, the Lukasiewicz implication.

Example 7.0.6. *Let us again consider the same rule base as in Example 7.0.5. Now we consider the FRI with* reducible composition $\mathbb{F}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, d\right)$, *where* $\longrightarrow = I_{\mathbf{KD}}$. *Then the system function is as shown in Figure 7.4 for two different types of defuzzification methods, viz., (i) Center of Gravity* (COG) *(see* (1.4)*) and (ii) Mean of Maxima* (MOM) *(see* (1.1)*). From the Figure 7.4, it can be noticed that the system function in both the cases is not monotonic, hence the FRIs* $\mathbb{F}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, \mathrm{COG}\right)$ *and* $\mathbb{F}_{\rightarrow} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, \mathrm{MOM}\right)$ *with* $\longrightarrow = I_{\mathbf{KD}}$ *are not monotonic.*



(a) System function with COG defuzzification

(b) System function with MOM defuzzification

Figure 7.4: System function of the FRI $\mathbb{F}_{\rightarrow}$ given in Example 7.0.6 with (a) COG and (b) MOM defuzzifier and $\longrightarrow = I_{\mathbf{KD}}$, the Kleene-Dienes implication.

In this chapter, we discuss the monotonicity of FRIs when an implicative model of the rule base is employed, i.e., where the operation between the antecedents and consequents is taken as a fuzzy implication. In Section 7.1, we present a short survey of the works related to monotonicity of fuzzy relational inference systems. In Section 7.2, we define the scope of this chapter by specifying clearly both the admissible class of fuzzy implications, and the class of fuzzy sets that are admissible as antecedents and consequents in a given monotone rule base $\mathcal{R}_M(A_i, B_i)$. Sections 7.3 and 7.4 contain the main contributions of this chapter, which shows that FRIs employing a rather large class of fuzzy implications - which include Yager's families of fuzzy implications - are monotonic.

Section 7.5 presents some examples that illustrate the investigations and analysis of the previous sections.

7.1 Monotonicity of FRIs

In the study of monotonicity of the system function of a fuzzy inference mechanism, most of the works in the literature deal with FRIs where the rules are interpreted in a non-conditional way or as just aggregation of possible configurations of the data (see the discussions in Sections 1.3 and 2.1 for more details). When an implicative or a conditional interpretation of the rules are considered, there are only a few works dealing with monotonicity of the system function.

The earliest works to appear on this topic dealt with FRIs where $R = \check{R}_\star$ (as defined in Section 2.1) and hence can be considered to have assumed a Cartesian product interpretation of the fuzzy rules, see Broekhoven and De Baets [43], [44]. In these two works the authors are specific in terms of choosing the operators in the inference mechanism. For instance, their results are valid only for $R = \check{R}_\star$ where $\star$ a is t-norm specified by $\star = T_{\mathbf{M}}, T_{\mathbf{P}}$ or $T_{\mathbf{LK}}$ as in Table 1.2. They have shown that the FRIs $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \check{R}_\star, \text{MOM}\right)$ and $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \check{R}_\star, \text{COG}\right)$ are monotonic only for the above mentioned t-norms. The results lack generality in terms of choosing the opertaors in the inference mechanism.

Later Štěpnička and De Baets in [46] and [47] considered an FRI with $R = \hat{R}_\to$ where $\longrightarrow$ is any residuated implication obtained from a left-continuous t-norm. The authors have modeled the rule base after modifying the antecedent and consequent fuzzy sets in some specified manner and denoted them as, $\hat{R}^\uparrow_\to$, $\hat{R}^\downarrow_\to$ and $\hat{R}^\updownarrow_\to$. Finally, they have shown that with the modified rule bases, the FRIs $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}^\uparrow_\to, \text{FOM}\right)$, $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}^\downarrow_\to, \text{LOM}\right)$ and $\mathbb{F} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}^\updownarrow_\to, \text{MOM}\right)$ are monotonic.

7.2 Enlarged Scope of This Chapter

In this work, we show that FRIs of the form $\mathbb{F}_\to = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_\to, d\right)$can be made monotonic for suitable choice of operations. The proof is general enough for a large class of fuzzy implications and the partitions formed by the antecedents $A_i \in \mathcal{P}_X$ and consequents $B_i \in \mathcal{P}_Y$ fuzzy sets are of the Ruspini type. Further, we do not modify the antecedent and consequent fuzzy sets of the rule base. However, in the following we clearly specify the type of admissible antecedents and consequents in a given monotone rule base. Further, we also restrict the scope of this chapter to a subclass of fuzzy implications, for which at least weak coherence can be ensured. It should also be mentioned that our results are valid for a large class of fuzzy implications, that also contains the Yager's families of fuzzy implications.

7.2.1 Class of Admissible Fuzzy Sets in the Rule Base

Let $\mathcal{F}^*(X)$ denote the space of fuzzy sets on X which are normal, convex and strict on both sides of the ceiling.

For instance, the fuzzy set A in Figure 7.5 is normal and convex but not strict on both sides of the ceiling.

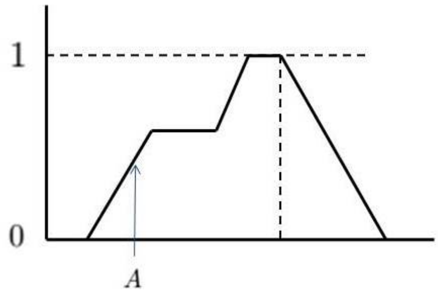
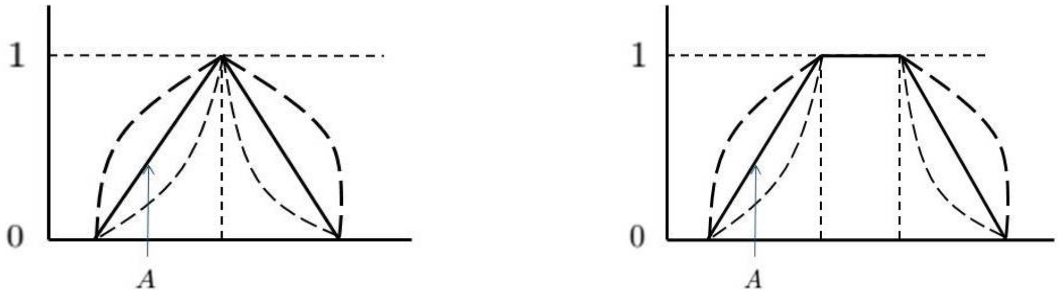


Figure 7.5: Normal, convex but not strict on both sides of the ceiling.



(a) Ceiling is a single point

(b) Ceiling is an interval

Figure 7.6: Normal, convex and strict on both sides of the ceiling.

In Figure 7.6(a), the ceiling of the triangular fuzzy set A is a single point and A is normal, convex and strict on both sides of the ceiling. In Figure 7.6(b), the ceiling of the trapezoidal fuzzy set A is an interval and A is again normal, convex and strict on both ides of the ceiling. Note that both these triangular and trapezoidal fuzzy sets belong to $\mathcal{F}^*(X)$ and further, the graphs of these fuzzy sets on either side of the ceiling are linear functions of $x \in X$. In fact, the graphs of any fuzzy set $A \in \mathcal{F}^*(X)$ over the region $\text{Supp}(A) \setminus \text{Ceil}(A)$ are strictly monotonic, linear or non-linear functions of x - see, for instance, the fuzzy sets in Figure 7.6 in thick and thin dashed lines.

The fuzzy sets $A_k \in \mathcal{F}^*(X), k = 1, 2, \ldots, 6$ in both Figure 7.7 and Figure 7.8 form a Ruspini partition (see Definition 1.1.11).

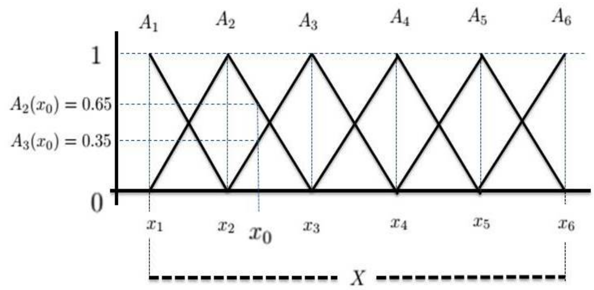


Figure 7.7: $\{A_k\}_{k=1}^{6}$ forms a Ruspini partition on X.

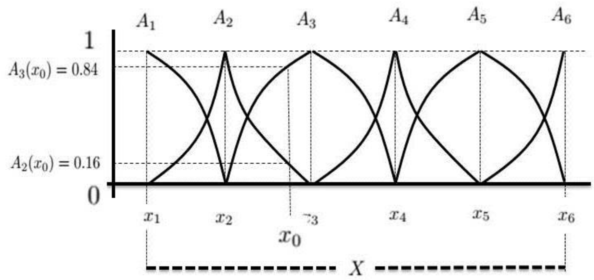


Figure 7.8: $\{A_k\}_{k=1}^{6}$ forms a Ruspini partition on X.

For instance, if we consider the $x_0 \in X$, as in Figure 7.7, then

$$\sum_{k=1}^{6} A_k(x_0) = A_2(x_0) + A_3(x_0) = 0.65 + 0.35 = 1 \, .$$

Again if we consider the $x_0 \in X$, as in Figure 7.8. Then

$$\sum_{k=1}^{6} A_k(x_0) = A_2(x_0) + A_3(x_0) = 0.16 + 0.84 = 1 \, .$$

It should be noted that the collection of fuzzy sets $A_k \in \mathcal{F}^*(X), k = 1, 2, \ldots, 6$ in both Figure 7.7 and Figure 7.8 are normal, convex and strict on both side of their ceilings. The only difference is that, in the first case, both sides of the ceilings of $A_k \in \mathcal{F}^*(X), k = 1, 2, \ldots, 6$ are linear functions of x whereas in the 2nd case they are not.

In this chapter, we only consider monotone rule bases $\mathcal{R}_M(A_i, B_i)$ where $A_i \in \mathcal{F}^*(X), B_i \in \mathcal{F}^*(Y)$ and form Ruspini partitions on the underlying domains X, Y, respectively.

7.2.2 Classes of Admissible Fuzzy Implications

From Section 6.3.2 in Chapter 6, it is clear that for an FRI with *reducible composition*, $\mathbb{F}_{\rightarrow} = \Big(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow}, d\Big)$, to obtain a nonempty output, we at least need to ensure weak coherence (as defined in Definition 6.3.2).

For a given monotone rule base of the form (7.1) and a singleton input $x_0 \in X$, from (2.1) and (FRI-R-Singleton), we have the following:

$$\begin{aligned} B'(y) = R_{\rightarrow}(x_0, y) &= \bigwedge_{i=1}^{n} (A_i(x_0) \longrightarrow B_i(y)) \\ &= (A_1(x_0) \longrightarrow B_1(y)) \wedge (A_2(x_0) \longrightarrow B_2(y)) \wedge \ldots \wedge (A_n(x_0) \longrightarrow B_n(y)). \end{aligned} \tag{7.3}$$

Now if the antecedent fuzzy sets form a Ruspini partition, then x_0 intersects atmost two fuzzy sets say, A_m, A_{m+1}. Then the above reduces to

$$B'(y) = (A_m(x_0) \longrightarrow B_m(y)) \wedge (A_{m+1}(x_0) \longrightarrow B_{m+1}(y)) = B'_m(y) \wedge B'_{m+1}(y) \, ,$$

where B'_m and B'_{m+1} are the fuzzy sets B_m and B_{m+1} modified by the fuzzy implication $\longrightarrow$ with $A_m(x_0)$, $A_{m+1}(x_0)$.

It is clear that for B' to be non-empty the supports of B'_m and B'_{m+1} should intersect, i.e., $\text{Supp}(B'_m) \cap \text{Supp}(B'_{m+1}) \neq \emptyset$. While coherence insists that the kernels of B'_m and B'_{m+1} should intersect, the weak coherence defined in Definition 6.3.2 relaxes this to a mere intersection of their supports.

From Section 6.3.1 we know that for a fuzzy relation $\hat{R}_{\rightarrow}$ to ensure weak coherence at the least, the class of implications I that can be considered should be restricted.

Since in most practical settings we deal only with fuzzy sets that are bounded, continuous, convex and that which often form a Ruspini partition, to ensure weak coherence or non emptiness of the output, it is sufficient to consider fuzzy implications $I \in \mathbb{I}$ that either

- satisfy the ordering property (OP), i.e., $I \in \mathbb{I}_{\mathbf{OP}}$, in which case often we can ensure even coherence [48], or
- are positive i.e., $I \in \mathbb{I}^+$, in which case we can ensure at least a weak coherence.

It is clear from Proposition 6.3.6(i) that if we use a non-positive implication, then the support of B'_m and B'_{m+1} may shrink, giving rise to an empty fuzzy set as B', which is not at all desirable.

Thus, in this chapter, we limit the study of monotonicity to FRIs that employ fuzzy implications that come from the class $\mathbb{I}^+$. Further, among fuzzy implications $I \in \mathbb{I}^+$ we only consider those that are strict (ST) (see Definition 1.2.12) and denote this class by $\mathbb{I}^{\mathbf{st}} \subsetneq \mathbb{I}^+$.

Towards better clarity and readability of the proofs presented later, we partition $\mathbb{I}^{\mathbf{st}}$ into two subclasses, viz., (i) $\mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$, which contain fuzzy implications I that are strict (ST) with $N_I = N_{\mathbf{D1}}$ and (ii) $\mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}}$, which contain fuzzy implications I that are strict (ST) but with N_I different from $N_{\mathbf{D1}}$.

Remark 7.2.1. *Note that $\mathbb{I}_{\mathbf{OP}}$ and $\mathbb{I}^{\mathbf{st}}$ are mutually exclusive. Table 7.1 lists some fuzzy implications illustrating the same.*

Implication $I \in \mathbb{I}$	$I \in \mathbb{I}_{\mathbf{OP}}$	$I \in \mathbb{I}^{\mathbf{st}}$	$I \in \mathbb{I}_{N_{\mathbf{D1}}}$
$\times$	✓	✓	✓
$\times$	✓	✓	$\times$
$I_{\mathbf{G}}(x,y) = \min(1, \frac{y}{x})$	✓	$\times$	✓
$I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$	✓	$\times$	$\times$
$I_{\mathbf{YG}}(x,y) = \min(1, y^x)$	$\times$	✓	✓
$I_{\mathbf{RC}}(x,y) = 1-x+xy$	$\times$	✓	$\times$
$I(x,y) = \begin{cases} 0, & \text{if } x > 0 \text{ and } y = 0 \\ 1, & \text{otherwise} \end{cases}$	$\times$	$\times$	✓
$I(x,y) = \begin{cases} 0, & \text{if}(x,y) \\ & \in [0.7,1] \times [0,0.6] \\ 0.5, & \text{if}(x,y) \\ & \in [0.4,0.7] \times [0,0.6] \\ 1, & \text{otherwise} \end{cases}$	$\times$	$\times$	$\times$

Table 7.1: Fuzzy Implications that satisfy some, all or none of (OP), (ST) and $N_I = N_{\mathbf{D1}}$.

7.2.3 Some families of Fuzzy Implications that belong to $\mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}} \cup \mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}} = \mathbb{I}^{\mathbf{st}}$

In fact, many established families of fuzzy implications fall in either of the above two classes. For the definitions and the properties these families satisfy, please refer to the monograph [2].

- From Section 3.1 we recall that $\mathbb{I}_{\mathbb{F}}$ is the set of all f-implications and $\mathbb{I}_{\mathbb{F},\infty} \subsetneq \mathbb{I}_{\mathbb{F}}$ is the set of f-implications that are generated from generators such that $f(0) = \infty$. Every $I \in \mathbb{I}_{\mathbb{F},\infty}$ is strict and its natural negation is the Gödel negation (see Proposition 3.1.6), i.e., $N_I = N_{\mathbf{D1}}$. Thus

$$\mathbb{I}_{\mathbb{F},\infty} \subsetneq \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}.$$

- Again from Section 3.1 recall that $\mathbb{I}_{\mathbb{F},1} \subsetneq \mathbb{I}_{\mathbb{F}}$ is the set of f-implications that are generated from generators such that $f(0) = 1$. Every $I \in \mathbb{I}_{\mathbb{F},1}$ is strict but their natural negation is a strict negation which is not the Gödel negation (see Proposition 3.1.6), i.e., $N_I \neq N_{\mathbf{D1}}$. Thus

$$\mathbb{I}_{\mathbb{F},1} \subsetneq \mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}}.$$

- If $\mathbb{I}_{\mathbb{G}}$ denotes the set of all g-implications, then every $I \in \mathbb{I}_{\mathbb{G}}$ is positive and $N_I = N_{\mathbf{D1}}$ (see [1], Proposition 4). Thus

$$\mathbb{I}_{\mathbb{G}} \subsetneq \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}.$$

In the following sections we will only deal with rules modeled by fuzzy relations $\hat{R}_{\to}$ where the fuzzy implication $\longrightarrow$ satisfies (ST). However, as noted above, the presented results are valid for the Yager's families of f- and g-implications.

7.3 Monotonicity of FRI $\mathbb{F}_{\to_{\mathbf{ST}}}$

In this section we discuss the monotonicity of the system function of the FRI

$$\mathbb{F}_{\to_{\mathbf{ST}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\to_{\mathbf{ST}}}, \mathrm{MOM}\right), \textit{ where } \hat{R}_{\to_{\mathbf{ST}}} = \bigwedge_{i=1}^{n} (A_i \longrightarrow_{\mathbf{ST}} B_i)\,, \text{ and } \longrightarrow_{\mathbf{ST}} \in \mathbb{I}^{\mathbf{st}}\,.$$

While investigating this FRI we partition the set $\mathbb{I}^{\mathbf{st}}$ into two parts (i) $\mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$ and (ii) $\mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}}$ as given in Section 7.2.2 and investigate the following FRIs for monotonicity:

$$\mathbb{F}_{\to_{\mathbf{D1}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\to_{\mathbf{D1}}}, \mathrm{MOM}\right), \qquad \text{and} \qquad \hat{R}_{\to_{\mathbf{D1}}} = \bigwedge_{i=1}^{n} (A_i \longrightarrow_{\mathbf{D1}} B_i)\,,$$

$$\mathbb{F}_{\to_{\mathbf{D1^c}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\to_{\mathbf{D1^c}}}, \mathrm{MOM}\right), \qquad \text{and} \qquad \hat{R}_{\longrightarrow_{\mathbf{D1^c}}} = \bigwedge_{i=1}^{n} (A_i \longrightarrow_{\mathbf{D1^c}} B_i)\,,$$

where $\longrightarrow_{\mathbf{D1}} \in \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$ and $\longrightarrow_{\mathbf{D1^c}} \in \mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}}$.

In the following two results we propose some sufficient conditions under which the corresponding system functions of $\mathbb{F}_{\to_{\mathbf{D1}}}$ and $\mathbb{F}_{\to_{\mathbf{D1^c}}}$ are monotonic.

Theorem 7.3.1. *Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in* (7.1) *which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \ldots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \ldots n$, form a*

Ruspini partition on Y, respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow_{\mathbf{D1}}}, \mathrm{MOM}\right)$ is monotonic, where $\longrightarrow_{\mathbf{D1}} \in \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$.

Proof. While the proof is valid for any fuzzy sets which are normal, convex and strict on both sides of the ceiling, for better readability we prove this result only for triangular fuzzy sets. For an input $x' \in X$ the fuzzy relational inference mechanism (FRI-R-Singleton) with $R = \hat{R}_{\rightarrow_{\mathbf{D1}}}$ is of the form,

$$B'(y) = \hat{R}_{\rightarrow_{\mathbf{D1}}}(x', y), \quad y \in Y. \tag{7.4}$$

Let the convex fuzzy sets $\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n$ be such that Supp$(A_i) = [x_{i-1}, x_{i+1}]$, Supp$(B_i) = [y_{i-1}, y_{i+1}], i = 2, 3, \ldots, n-1$, Supp$(A_1) = [x_1, x_2]$, Supp$(A_n) = [x_{n-1}, x_n]$, Supp$(B_1) = [y_1, y_2]$, Supp$(B_n) = [y_{n-1}, y_n]$. Further, let $A_i(x_i) = 1$ and $B_i(y_i) = 1$ for $i = 1, 2, \ldots, n$.

Let $x' \in X$ be any given input. Clearly, $x' \in [x_m, x_{m+1}]$ for some $m \in \{1, 2, \ldots, n-1\}$. Since $\{A_i\}_{i=1}^n$ are normal and form a Ruspini partition, $A_j(x') = 0$, for all $j \neq m, m+1$. Then from (7.4), we have

$$B'(y) = [A_m(x') \longrightarrow_{\mathbf{D1}} B_m(y)] \wedge [A_{m+1}(x') \longrightarrow_{\mathbf{D1}} B_{m+1}(y)].$$

Let $A_m(x') = s'_m$ and $A_{m+1}(x') = s'_{m+1}$. Thus,

$$B'(y) = [s'_m \longrightarrow_{\mathbf{D1}} B_m(y)] \wedge [s'_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y)] = B'_m(y) \wedge B'_{m+1}(y) .$$

Clearly, since B_m, B_{m+1} are convex and normal, B'_m, B'_{m+1} are also convex and normal (see Proposition 6.3.4). Hence $B' = B'_m \cap B'_{m+1}$ is also convex. Let $y' = \mathrm{MOM}(B')$.

Since we consider the rule base to be only monotone and not strictly monotone, there can be two possibilities: (i) $B_m \neq B_{m+1}$ and (ii) $B_m = B_{m+1}$.

Case I - $B_m \neq B_{m+1}$:

Claim 1. *If $x' \in [x_m, x_{m+1}]$, then $y' \in [y_m, y_{m+1}]$ for any $m \in \{1, 2, \ldots, n-1\}$.*

For a better understanding of the proof we refer to Figure 7.9 where the implication used is $I_{\mathbf{YG}}(x, y) = \min(1, y^x)$, the Yager's implication. Note that $I_{\mathbf{YG}} \in \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$.

From Remark 6.3.7 we can verify that, since $x \longrightarrow_{\mathbf{D1}} 0 = 0$ for any $x \in (0, 1]$, we have that the supports of both the modified fuzzy sets $B'_m = s_m \longrightarrow_{\mathbf{D1}} B_m$ and $B'_{m+1} = s_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}$ are the same as those of B_m, B_{m+1}, i.e.,

$$\begin{aligned} &\mathrm{Supp}(B'_m) = \mathrm{Supp}(B_m) \text{ and } \mathrm{Supp}(B'_{m+1}) = \mathrm{Supp}(B_{m+1}) . \\ \text{Hence,} \quad &\mathrm{Supp}(B') = \mathrm{Supp}(B'_m) \cap \mathrm{Supp}(B'_{m+1}) = \mathrm{Supp}(B_m) \cap \mathrm{Supp}(B_{m+1}) \\ &\qquad = \mathrm{Supp}(B_m \cap B_{m+1}) = [y_m, y_{m+1}] . \end{aligned} \tag{7.5}$$

Since (7.5) holds we have,

$$y' = g(x') = \mathrm{MOM}(B') \in \mathrm{Supp}(B_m \cap B_{m+1}) = [y_m, y_{m+1}].$$

Thus, $y' \in [y_m, y_{m+1}]$. Hence the **Claim** 1. $\square$

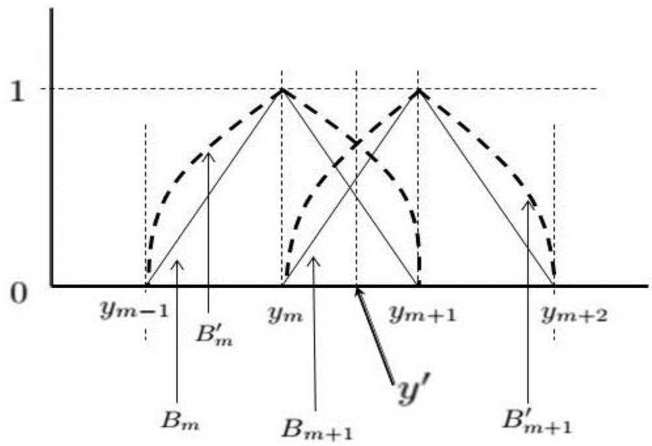


Figure 7.9: The Modified Fuzzy Sets, using $I_{\mathbf{YG}}(x, y) = \min(1, y^x)$

Let $x', x'' \in X$ be the two given inputs such that $x' \leq x''$. By considering the following two subcases, we show that the obtained outputs $y', y'' \in Y$ also are such that, $y' \leq y''$.

Subcase-1: Let $x', x'' \in [x_m, x_{m+1}]$, for some $m \in \{1, 2, \ldots, n-1\}$. By the above claim we obtain, $y', y'' \in [y_m, y_{m+1}]$.

Now, since A_m is strictly decreasing on $[x_m, x_{m+1}]$ and A_{m+1} is strictly increasing on $[x_m, x_{m+1}]$,

$$x' \leq x'' \text{ and } x', x'' \in [x_m, x_{m+1}] \Longrightarrow A_m(x') \geq A_m(x'') \text{ and } A_{m+1}(x') \leq A_{m+1}(x''),$$

$$\text{i.e., } s'_m \geq s''_m \text{ and } s'_{m+1} \leq s''_{m+1}, \tag{7.6}$$

where $s''_m = A_m(x'')$ and $s''_{m+1} = A_{m+1}(x'')$.

For any $y \in [y_m, y_{m+1}]$ we obtain the following inequalities:

$$\begin{aligned} s'_m \geq s''_m &\Longrightarrow s'_m \longrightarrow_{\mathbf{D1}} B_m(y) \leq s''_m \longrightarrow_{\mathbf{D1}} B_m(y), \\ &\Longrightarrow B'_m(y) \leq B''_m(y) \,. \\ \text{Similarly, } s'_{m+1} \leq s''_{m+1} &\Longrightarrow s'_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y) \geq s''_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y), \\ &\Longrightarrow B'_{m+1}(y) \geq B''_{m+1}(y) \,. \end{aligned}$$

Claim 2. $y' = \text{MOM}(B') \in \text{Supp}(B_m \cap B_{m+1}) = [y_m, y_{m+1}]$ *is the point of intersection of* B'_m *and* B'_{m+1}.

On $[y_m, y_{m+1}]$, B'_m is strictly decreasing and B'_{m+1} is strictly increasing. Let B'_m and B'_{m+1} intersect at $y^0 \in [y_m, y_{m+1}]$, i.e,

$$B'_m(y^0) = B'_{m+1}(y^0). \tag{7.7}$$

Now, for $y \in [y_m, y^0)$, it holds that $B'_m(y) > B'_m(y^0) = B'_{m+1}(y^0) > B'_{m+1}(y)$ and we have the following inequality:

$$\begin{aligned} B'(y) &= \min\big(B'_m(y), B'_{m+1}(y)\big) \\ &= B'_{m+1}(y) && (\because B'_m(y) > B'_{m+1}(y)) \\ &< B'_{m+1}(y^0) && (\because B'_{m+1} \text{ is strictly increasing in } [y_m, y_{m+1}]) \\ &= B'_m(y^0) = B'(y^0)\ . && (7.8) \end{aligned}$$

Again for $y \in (y^0, y_{m+1}]$, it holds that, $B'_m(y) < B'_{m+1}(y)$ and we have the following inequality:

$$\begin{aligned} B'(y) &= \min\big(B'_m(y), B'_{m+1}(y)\big) \\ &= B'_m(y) && (\because B'_m(y) < B'_{m+1}(y)) \\ &< B'_m(y^0) && (\because B'_m \text{ is strictly decreasing in } [y_m, y_{m+1}]) \\ &= B'_{m+1}(y^0) = B'(y^0)\ . && (7.9) \end{aligned}$$

From (7.7), (7.8) and (7.9), we have

$$y' = \mathrm{MOM}(B') = y^0\ ,$$

the point of intersection of B'_m and B'_{m+1}. Hence the **Claim** 2. □

Since B'_m and B'_{m+1} are monotonic on $[y_m, y_{m+1}]$, we have that $y', y'' \in [y_m, y_{m+1}]$ are also the points which satisfy the following:

$$B'_m(y') = B'_{m+1}(y') \Longrightarrow s'_m \longrightarrow_{\mathbf{D1}} B_m(y') = s'_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y')\ , \quad (7.10)$$
$$B''_m(y'') = B''_{m+1}(y'') \Longrightarrow s''_m \longrightarrow_{\mathbf{D1}} B_m(y'') = s''_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y'')\ . \quad (7.11)$$

Now, to prove monotonicity, we need to show that $y' \le y''$.

If possible, let us assume to the contrary that $y' > y''$. Since B_m and B_{m+1} are, respectively, strictly decreasing and strictly increasing on $[y_m, y_{m+1}]$,

$$\begin{aligned} & y' > y'' \ \text{ and } \ y', y'' \in [y_m, y_{m+1}] \\ & \qquad \Longrightarrow B_m(y') < B_m(y'') \ \text{ and } \ B_{m+1}(y') > B_{m+1}(y'')\ . \qquad (7.12) \end{aligned}$$

This leads to the following inequalities:

$$\begin{aligned} s'_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y') &> s'_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y'') && (\because \longrightarrow_{\mathbf{D1}} \text{ is strict and using (7.12)}) \\ &\ge s''_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y'') && (\because\ s'_{m+1} \le s''_{m+1} \text{ from (7.6)}) \\ &= s''_m \longrightarrow_{\mathbf{D1}} B_m(y'') && \text{(Using (7.11))} \\ &> s''_m \longrightarrow_{\mathbf{D1}} B_m(y') && \text{(Using (7.12))} \\ &\ge s'_m \longrightarrow_{\mathbf{D1}} B_m(y')\ , && (\because\ s'_m \ge s''_m \text{ from (7.6)}) \end{aligned}$$

i.e., $s'_{m+1} \longrightarrow_{\mathbf{D1}} B_{m+1}(y') > s'_m \longrightarrow_{\mathbf{D1}} B_m(y')$, a contradiction to (7.10). So we have

$$x' \leq x'' \Longrightarrow y' \leq y''.$$

Subcase-2: Let $x' \in [x_m, x_{m+1}]$ and $x'' \in [x_{m+p}, x_{m+p+1}], p \geq 1$. By the **Claim** 1 above, we have $y' \in [y_m, y_{m+1}]$ and $y'' \in [y_{m+p}, y_{m+p+1}], p \geq 1$ and hence $y' \leq y''$.

Thus, the system function g is monotonic.

Case II- $B_m = B_{m+1}$**:**

Claim 3. *If $x' \in [x_m, x_{m+1}]$, then $y' \in [y_m, y_{m+1}]$ for any $m \in \{1, 2, \ldots, n-1\}$.*

From (7.4), we have

$$\begin{aligned} B'(y) &= [A_m(x') \longrightarrow_{\mathbf{D1}} B_m(y)] \wedge [A_{m+1}(x') \longrightarrow_{\mathbf{D1}} B_m(y)] \\ &= [s'_m \longrightarrow_{\mathbf{D1}} B_m(y)] \wedge [s'_{m+1} \longrightarrow_{\mathbf{D1}} B_m(y)] \\ &= [(s'_m \vee s'_{m+1}) \longrightarrow_{\mathbf{D1}} B_m(y)]. \end{aligned}$$

Since $B_m(y_m) = 1$,

$$B'(y_m) = (s'_m \vee s'_{m+1}) \longrightarrow_{\mathbf{D1}} B_m(y_m) = (s'_m \vee s'_{m+1}) \longrightarrow_{\mathbf{D1}} 1 = 1 .$$

From the fact that $\longrightarrow_{\mathbf{D1}}$ is strict, B_m is strictly increasing on $[y_{m-1}, y_m]$ and strictly decreasing on $[y_m, y_{m+1}]$, we have $(s'_m \vee s'_{m+1}) \longrightarrow_{\mathbf{D1}} B_m$ is strictly increasing on $[y_{m-1}, y_m]$ and strictly decreasing on $[y_m, y_{m+1}]$. So B' reaches 1 only at y_m. Hence Ker$(B') = \{y_m\}$, consequently,

$$y' = g(x') = \text{MOM}(B') = y_m.$$

Note that, trivially, $y' \in [y_m, y_{m+1}]$. Hence the **Claim 3**. □

Let $x', x'' \in X$ be the two given inputs such that $x' \leq x''$. By considering the following two subcases, we show that the obtained outputs $y', y'' \in Y$ also are similarly ordered, i.e., $y' \leq y''$.

Subcase-1: Let $x', x'' \in [x_m, x_{m+1}]$, for some $m \in \{1, 2, \ldots, n-1\}$. By the above claim we obtain, $y' = y'' = y_m$. Thus, trivially, we have $x' \leq x'' \Longrightarrow y' \leq y''$.

Subcase-2: Let $x' \in [x_m, x_{m+1}]$ and $x'' \in [x_{m+p}, x_{m+p+1}], p \geq 1$. Since, $B_m = B_{m+1}$, we have $y' = y_m$ and $y'' \in [y_{m+p}, y_{m+p+1}]$. So, $y' \leq y''$.

Thus, the system function g is monotonic. □

Remark 7.3.2. *For better readability the proof of Theorem 7.3.1 has been presented only for triangular fuzzy sets, whereas the proof is valid for any fuzzy sets which are normal, convex and strict on both sides of the ceiling. It should be noted that, the result remains unaffected, when we consider trapezoidal fuzzy sets instead of triangular fuzzy sets, since the only extra case that needs to be considered is when the input x' falls in the kernel of an antecedent fuzzy set A_m. However, in this case, due to the Ruspini partition of* *the antecedent fuzzy sets $\mathcal{P}_X$, it can be easily shown that the output $g(x')$ will fall within the kernel of the corresponding consequent fuzzy set B_m.*

Theorem 7.3.3. *Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in* (7.1) *which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \ldots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \ldots, n$, form a Ruspini partition on Y, respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_{\rightarrow_{\mathbf{D1^c}}}, \mathrm{MOM}\right)$ is monotonic, where $\longrightarrow_{\mathbf{D1^c}} \in \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1^c}}}$.*

Proof. Once again, while the proof is valid for any fuzzy sets which are normal, convex and strict on both sides of the ceiling, for better readability we prove this result only for triangular fuzzy sets.

For an input $x' \in X$ the fuzzy relational inference mechanism (FRI-R-Singleton) with $R = \hat{R}_{\rightarrow_{\mathbf{D1^c}}}$ is of the form,

$$B'(y) = \hat{R}_{\rightarrow_{\mathbf{D1^c}}}(x', y), \quad y \in Y. \tag{7.13}$$

Let the convex fuzzy sets $\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n$ be such that $\mathrm{Supp}(A_i) = [x_{i-1}, x_{i+1}]$, $\mathrm{Supp}(B_i) = [y_{i-1}, y_{i+1}], i = 2, 3, \ldots, n-1$, $\mathrm{Supp}(A_1) = [x_1, x_2]$, $\mathrm{Supp}(A_n) = [x_{n-1}, x_n]$, $\mathrm{Supp}(B_1) = [y_1, y_2]$, $\mathrm{Supp}(B_n) = [y_{n-1}, y_n]$. Further, let $A_i(x_i) = 1$ and $B_i(y_i) = 1$ for $i = 1, 2, \ldots, n$.

Let $x' \in X$ be any given input. Clearly, $x' \in [x_m, x_{m+1}]$ for some $m \in \{1, 2, \ldots, n-1\}$. Since $\{A_i\}_{i=1}^n$ are normal and form a Ruspini partition, $A_j(x') = 0$, for all $j \neq m, m+1$. From (7.13),

$$B'(y) = [A_m(x') \longrightarrow_{\mathbf{D1^c}} B_m(y)] \wedge [A_{m+1}(x') \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y)].$$

Once again, let $A_m(x') = s'_m$ and $A_{m+1}(x') = s'_{m+1}$. Thus,

$$B'(y) = [s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y)] \wedge [s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y)] = B'_m(y) \wedge B'_{m+1}(y) \ .$$

Clearly, since B_m, B_{m+1} are convex and normal, B'_m, B'_{m+1} are also convex and normal (see Proposition 6.3.4). Hence $B' = B'_m \cap B'_{m+1}$ is also convex.

Let $y' = \mathrm{MOM}(B')$. Now, since the rule base is monotone, there can be two possibilities: (i) $B_m \neq B_{m+1}$ and (ii) $B_m = B_{m+1}$.

Firstly we prove the result when $B_m \neq B_{m+1}$.

Case I - $B_m \neq B_{m+1}$:

Claim 4. *If $x' \in [x_m, x_{m+1}]$, then $y' \in [y_m, y_{m+1}]$ for $m \in \{1, 2, \ldots, n-1\}$.*

Here we prove the claim by considering three different subcases:

Subcase-1 ($s'_m > s'_{m+1} > 0$): *For a better understanding of the proof we refer to the Figure 7.10 where the implication used is $I_{\mathbf{RC}}(x, y) = 1 - x + xy$, the Reichenbach implication. Note that $I_{\mathbf{RC}} \in \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1^c}}}$.*

Recall that $B'_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y)$ and $B'_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y)$. We partition the space $Y = [y_1, y_n]$ into the following five sub-domains:

$$Y = \{y | y \le y_{m-1}\} \cup [y_{m-1}, y_m] \cup [y_m, y_{m+1}] \cup [y_{m+1}, y_{m+2}] \cup \{y | y \ge y_{m+2}\} \ , \tag{7.14}$$

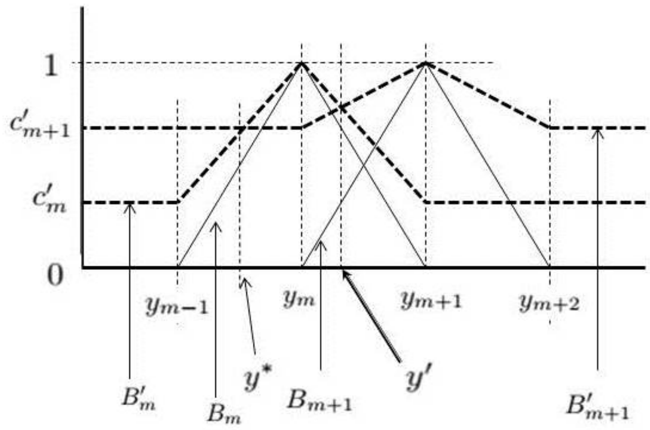


Figure 7.10: The modified fuzzy sets using $I_{\mathbf{RC}}(x, y) = 1 - x + xy$, $0 < s'_{m+1} < s'_m$

and then discuss the behavior of B'_m and B'_{m+1} on these five sub-domains.

Behavior of B'_m and B'_{m+1} :

- **Over $\{y|y \le y_{m-1}\}$**: For $y \le y_{m-1}$, $B_m(y) = B_{m+1}(y) = 0$. Hence,

$$B'_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_m(\text{say}) \;\text{ and}$$
$$B'_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_{m+1}(\text{say}).$$

  Now using the strictness of $\longrightarrow_{\mathbf{D1^c}}$, we have

$$B'_m(y) = c'_m = s'_m \longrightarrow_{\mathbf{D1^c}} 0 < s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = c'_{m+1} = B'_{m+1}(y).$$

  Thus B'_m and B'_{m+1} never intersect in $\{y|y \le y_{m-1}\}$.

- **Over $[y_{m-1}, y_m]$**: For any $y \in [y_{m-1}, y_m]$, $B_{m+1}(y) = 0$ while on this interval B_m is strictly increasing. Since $\longrightarrow_{\mathbf{D1^c}}$ is strict, $B'_m = s'_m \longrightarrow_{\mathbf{D1^c}} B_m$ is strictly increasing. Hence, $B'_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = c'_{m+1}$, which is a constant value. Now, since B'_m is strictly increasing on this interval, and

$$B'_m(y_{m-1}) = c'_m \,,$$
$$B'_m(y_m) = s'_m \longrightarrow_{\mathbf{D^c_1}} B_m(y_m) = s'_m \longrightarrow_{\mathbf{D1^c}} 1 = 1 \,,$$

  the range of B'_m over $[y_{m-1}, y_m]$ is $[c'_m, 1]$. Once again,

$$\begin{aligned} c'_{m+1} &= s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 \\ &> s'_m \longrightarrow_{\mathbf{D1^c}} 0 \\ &= s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_m(y_{m-1}) \\ &= B'_m(y_{m-1}) = c'_m. \end{aligned}$$

  Now, $c'_{m+1} > c'_m$ implies that $c'_{m+1} \in [c'_m, 1]$ and hence, clearly, B'_m and B'_{m+1} intersect at only one point in $[y_{m-1}, y_m]$. Let it be $y^* \in [y_{m-1}, y_m]$.

- **Over $[y_m, y_{m+1}]$**: On the interval $[y_m, y_{m+1}]$, B_m is strictly decreasing while B_{m+1} is strictly increasing. Since $\longrightarrow_{\mathbf{D1^c}}$ is strict, B'_m is strictly decreasing, B'_{m+1} is strictly increasing and thus they intersect exactly at one point. Let it be $y' \in [y_m, y_{m+1}]$.
- **Over $[y_{m+1}, y_{m+2}]$**: For any $y \in [y_{m+1}, y_{m+2}]$, $B_m(y) = 0$, while on this interval B_{m+1} is strictly decreasing. Hence, $B'_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} 0 = c'_m$, which is a constant value and by the strictness of $\longrightarrow_{\mathbf{D1^c}}$, $B'_{m+1} = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}$ is strictly decreasing. Now, since

$$B'_{m+1}(y_{m+1}) = s'_{m+1} \longrightarrow_{\mathbf{D_1^c}} B_{m+1}(y_{m+1}) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 1 = 1 \; ,$$
$$B'_{m+1}(y_{m+2}) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = c'_{m+1} \; ,$$

and B'_{m+1} is strictly decreasing on this interval, the range of B'_{m+1} over the interval $[y_{m+1}, y_{m+2}]$ is $[c'_{m+1}, 1]$. Again, since $c'_{m+1} > c'_m$ and $c'_m \notin [c'_{m+1}, 1]$ clearly, B'_m and B'_{m+1} do not intersect in $[y_{m-1}, y_m]$.
- **Over $\{y | y \geq y_{m+2}\}$**: For $y \geq y_{m+2}$, $B_m(y) = B_{m+1}(y) = 0$. Hence,

$$B'_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_m \quad \text{and}$$
$$B'_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_{m+1}.$$

Now using the strictness of $\longrightarrow_{\mathbf{D1^c}}$, we have

$$B'_m(y) = c'_m = s'_m \longrightarrow_{\mathbf{D1^c}} 0 < s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = c'_{m+1} = B'_{m+1}(y) \; ,$$

and hence B'_m and B'_{m+1} never intersect on $\{y | y \geq y_{m+2}\}$.

The behavior of both B'_m and B'_{m+1} on the partition of Y as given in (7.14) is summarized in the following Table 7.2, where by $\nearrow$ and $\swarrow$, we mean strictly increasing and strictly decreasing, respectively.

	$y \leq y_{m-1}$	$[y_{m-1}, y_m]$	$[y_m, y_{m+1}]$	$[y_{m+1}, y_{m+2}]$	$y \geq y_{m+2}$
B'_m	Constant (c'_m)	$\nearrow$	$\swarrow$	Constant (c'_m)	Constant (c'_m)
B'_{m+1}	Constant (c'_{m+1})	Constant (c'_{m+1})	$\nearrow$	$\swarrow$	Constant (c'_{m+1})

Table 7.2: Behavior of B'_m and B'_{m+1} on the output space Y, when $s'_m > s'_{m+1}$

Hence the only points of intersection between B'_m and B'_{m+1} are $y^* \in [y_{m-1}, y_m]$ and $y' \in [y_m, y_{m+1}]$. From **Claim** 2 of Theorem 7.3.1, we have

$$\begin{aligned} y_0 = g(x') &= \text{MOM}(B') \\ &= \text{MOM}(B'_m \cap B'_{m+1}) \\ &= \text{MOM}\{y \in Y | B'_m(y) = B'_{m+1}(y)\} = \max(y^*, y') \; . \end{aligned}$$

Since B'_{m+1} is constant on $[y^*, y_m]$ and increasing on $[y_m, y_{m+1}]$, $B'_{m+1}(y') > B'_{m+1}(y^*)$ and hence, $B'(y') > B'(y^*)$ and

$$y_0 = y' \in [y_m, y_{m+1}].$$

Hence the **Claim** 4. □

Subcase-2 ($s'_{m+1} > s'_m > 0$): Along similar lines as argued in **Subcase-1**, **Claim** 4 can be proven in this case too.

For a better understanding we refer to Figure 7.11 where the implication used is the same Reichenbach implication.

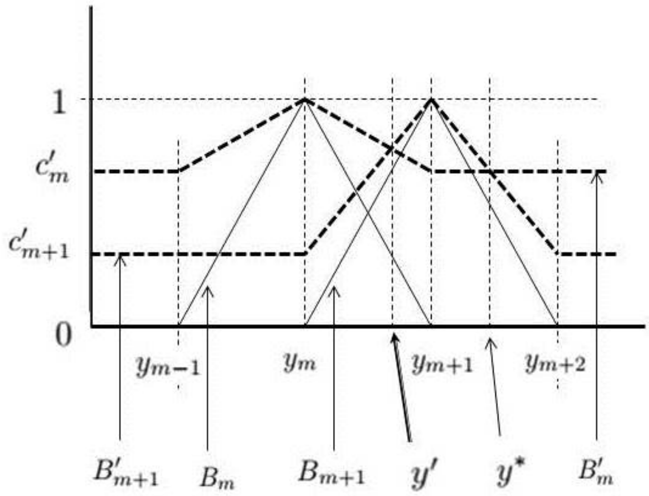


Figure 7.11: The modified fuzzy sets using $I_{\mathbf{RC}}(x, y) = 1 - x + xy$, $s'_{m+1} > s'_m > 0$

The behavior of both B'_m and B'_{m+1} on the partition of Y as given in (7.14) when $s'_{m+1} > s'_m > 0$ is summarized in the following Table 7.3, where, once again, by $\nearrow$ and $\swarrow$, we mean strictly increasing and strictly decreasing, respectively.

	$y \le y_{m-1}$	$[y_{m-1}, y_m]$	$[y_m, y_{m+1}]$	$[y_{m+1}, y_{m+2}]$	$y \ge y_{m+2}$
B'_m	Constant (c'_m)	$\nearrow$	$\swarrow$	Constant (c'_m)	Constant (c'_m)
B'_{m+1}	Constant (c'_{m+1})	Constant (c'_{m+1})	$\nearrow$	$\swarrow$	Constant (c'_{m+1})

Table 7.3: Behavior of B'_m and B'_{m+1} on the output space Y, when $s'_m < s'_{m+1}$

Subcase-3 ($s'_m = s'_{m+1} = \frac{1}{2}$): Similarly, as in **Subcase-1**, we partition the space Y as given in (7.14) and discuss the behavior of B'_m and B'_{m+1} over these five sub-domains.

For a better understanding we refer to the Figure 7.11 where the implication used is once again the same Reichenbach implication.

Behavior of B'_m and B'_{m+1} :

- **Over $\{y|y \le y_{m-1}\}$**: For $y \le y_{m-1}$, $B_m(y) = B_{m+1}(y) = 0$. Hence,

$$B'_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_m(\text{say}) \ \text{ and}$$
$$B'_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_{m+1}(\text{say}).$$

Now since, $s'_m = s'_{m+1} = \frac{1}{2}$, we have,

$$c'_m = B'_m(y) = B'_{m+1}(y) = c'_{m+1}\ ,$$

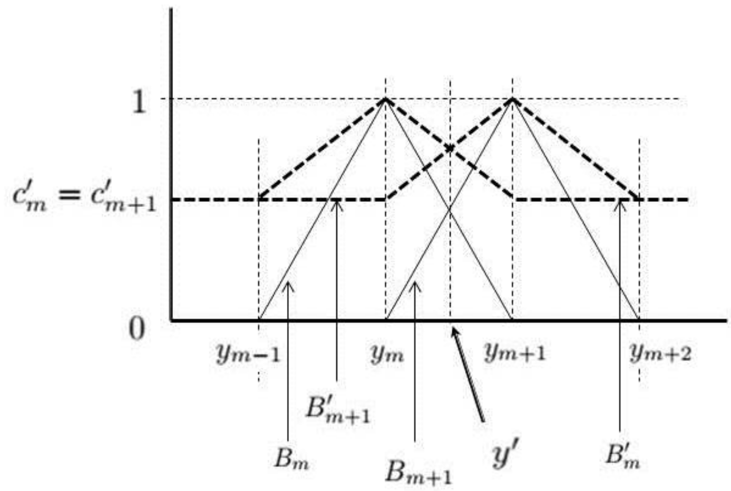


Figure 7.12: The modified fuzzy sets using $I_{\mathbf{RC}}(x, y) = 1 - x + xy$, $s'_{m+1} = s'_m = \frac{1}{2}$

for $y \leq y_{m-1}$ and hence B'_m and B'_{m+1} overlap and are identically equal over the sub-domain $\{y | y \leq y_{m-1}\}$.

- **Over $[y_{m-1}, y_m]$**: On the interval $[y_{m-1}, y_m]$, B_m is strictly increasing while $B_{m+1} \equiv 0$. Since $\longrightarrow_{\mathbf{D1^c}}$ is strict, $B'_m = s'_m \longrightarrow_{\mathbf{D1^c}} B_m = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} B_m$ is strictly increasing, while $B'_{m+1} = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = c'_{m+1}$, which is a constant value. Since B'_m is strictly increasing on this interval,

$$B'_m(y_{m-1}) = c'_m, \text{ and}$$
$$B'_m(y_m) = s'_m \longrightarrow_{\mathbf{D_1^c}} B_m(y_m) = s'_m \longrightarrow_{\mathbf{D1^c}} 1 = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 1 = 1 \,,$$

the range of B'_m on $[y_{m-1}, y_m]$ is $[c'_m, 1]$. Once again,

$$\begin{aligned} c'_{m+1} &= s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 \\ &= s'_m \longrightarrow_{\mathbf{D1^c}} 0 = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y_{m-1}) \\ &= B'_m(y_{m-1}) = c'_m. \end{aligned}$$

Thus, B'_m and B'_{m+1} intersect at only one point in $[y_{m-1}, y_m]$ and that point is y_{m-1}.

- **Over $[y_m, y_{m+1}]$**: Once again, on the interval $[y_m, y_{m+1}]$, B_m is strictly decreasing while B_{m+1} is strictly increasing and by the strictness of $\longrightarrow_{\mathbf{D1^c}}$ we have that B'_m is strictly decreasing, B'_{m+1} is strictly increasing and that they intersect exactly at one point, say, $y' \in [y_m, y_{m+1}]$.

- **Over $[y_{m+1}, y_{m+2}]$**: On the interval $[y_{m+1}, y_{m+2}]$, $B_m \equiv 0$ while B_{m+1} is strictly decreasing. Hence, $B'_m = s'_m \longrightarrow_{\mathbf{D1^c}} 0 = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = c'_m$, which is a constant value, while by the strictness of $\longrightarrow_{\mathbf{D1^c}}$ we see that $B'_{m+1} = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}$ is strictly decreasing. Now, since

$$B'_{m+1}(y_{m+1}) = s'_{m+1} \longrightarrow_{\mathbf{D_1^c}} B_m(y_m) = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 1 = 1\,,$$
$$B'_{m+1}(y_{m+2}) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} 0 = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = c'_{m+1}\,,$$

and B'_{m+1} is strictly decreasing in this interval, the range of B'_{m+1} over the interval $[y_{m+1}, y_{m+2}]$ is $[c'_{m+1}, 1]$. Since $c'_{m+1} = c'_m$, B'_m and B'_{m+1} intersect exactly at one point in $[y_{m+1}, y_{m+2}]$ and that point is y_{m+1}.

- **Over $\{y|y \geq y_{m+2}\}$**: For $y \geq y_{m+2}$, $B_m(y) = B_{m+1}(y) = 0$. Hence,

$$B'_m(y) = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y) = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_m \text{ and}$$
$$B'_{m+1}(y) = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y) = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = \text{constant} = c'_{m+1}.$$

Now since, $s'_m = s'_{m+1} = \frac{1}{2}$, we have, $B'_m(y) = B'_{m+1}(y)$ for $y \geq y_{m+1}$ and thus B'_m and B'_{m+1} overlap and are identically equal over $\{y|y \geq y_{m+2}\}$.

The behavior of both B'_m and B'_{m+1} on the partition of Y as given in (7.14) when $s'_m = s'_{m+1} = \frac{1}{2}$ is summarized in the following Table 7.4.

	$y \leq y_{m-1}$	$[y_{m-1}, y_m]$	$[y_m, y_{m+1}]$	$[y_{m+1}, y_{m+2}]$	$y \geq y_{m+2}$
B'_m	Constant (c'_m)	$\nearrow$	$\swarrow$	Constant (c'_m)	Constant (c'_m)
B'_{m+1}	Constant (c'_{m+1})	Constant (c'_{m+1})	$\nearrow$	$\swarrow$	Constant (c'_{m+1})

Table 7.4: Behavior of B'_m and B'_{m+1} on the output space Y, when $s'_m = s'_{m+1} = \frac{1}{2}$

Thus the set of points over which B'_m and B'_{m+1} intersect can be summarised as follows:

$$\{y \in Y|B'_{m+1}(y) = B'_m(y)\} = \{y|y \leq y_{m-1}\} \cup \{y'\} \cup \{y|y \geq y_{m+2}\}\,.$$

Firstly, note that $B'_{m+1}(y) = B'_m(y) = c'_m$ on the sub-domains $\{y|y \leq y_{m-1}\}$ and $\{y|y \geq y_{m+2}\}$. Further, since $y' \in [y_m, y_{m+1}]$

$$B'_m(y') = s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y') > s'_m \longrightarrow_{\mathbf{D1^c}} B_m(y_{m+1}) = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = c'_m\,,$$
$$\text{while } B'_{m+1}(y') = s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y') > s'_{m+1} \longrightarrow_{\mathbf{D1^c}} B_{m+1}(y_m) = \frac{1}{2} \longrightarrow_{\mathbf{D1^c}} 0 = c'_m\,.$$

Thus, from **Claim** 2 of Theorem 7.3.1, we have

$$\begin{aligned} g(x') &= \text{MOM}(B') \\ &= \text{MOM}(B'_m \cap B'_{m+1}) \\ &= \text{MOM}\{y \in Y|B'_m(y) = B'_{m+1}(y)\} = y'\,. \end{aligned}$$

Hence $x' \in [x_m, x_{m+1}] \Longrightarrow y' \in [y_m, y_{m+1}]$ for $m \in \{1, 2, \ldots, n-1\}$.

Let $x', x'' \in X$ be the two given inputs such that $x' \leq x''$. We consider the following two cases and show that the obtained outputs $y', y'' \in Y$ also are similarly ordered, i.e., $y' \leq y''$.

Subcase-1: Let $x', x'' \in [x_m, x_{m+1}]$, for some $m \in \{1, 2, \ldots, n-1\}$. The argument to show that $y' \leq y''$ proceeds along similar lines of the corresponding case as given in Theorem 7.3.1 above.

Subcase-2: Let $x' \in [x_m, x_{m+1}]$ and $x'' \in [x_{m+p}, x_{m+p+1}], p \geq 1$. By the **Claim** 4 above, we have $y' \in [y_m, y_{m+1}]$ and $y'' \in [y_{m+p}, y_{m+p+1}], p \geq 1$ and hence $y' \leq y''$.

Case II - $B_m = B_{m+1}$**:**

Once again, the proof when $B_m = B_{m+1}$ follows along similar lines of the corresponding case in Theorem 7.3.1 above. □

7.4 Monotonicity of BKS-$\mathcal{Y}$ Inference Mechanisms

In Chapters 4, 5 and 6 we have seen that BKS with Yager's families of fuzzy implications $\mathbb{F}_{\rightarrow_{\mathcal{Y}}}$ possess the following desirable properties, namely, interpolativity, continuity, robustness and universal approximation capability. Here in this section we show that, BKS with Yager's families of fuzzy implications are also monotonic, i.e., the corresponding system functions are monotonic. The results are, in fact, some special cases of the above Theorem 7.3.1 and Theorem 7.3.3

Corollary 7.4.1. *Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in* (7.1) *which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \ldots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \ldots, n$, form a Ruspini partition on Y, respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI $\mathbb{F}_{\rightarrow_f} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_f, \mathrm{MOM}\right)$ is monotonic, where $\hat{R}_f$ is as defined in* (Imp-$\hat{R}_f$) *and $\longrightarrow_f \in \mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F},\infty} \cup \mathbb{I}_{\mathbb{F},1}$.*

Proof. Every $\longrightarrow_f \in \mathbb{I}_{\mathbb{F}}$ is strict. Thus $\longrightarrow_f \in \mathbb{I}_{\mathbb{F}} \subsetneq \mathbb{I}^{\mathbf{st}}$ and the result follows from Theorem 7.3.1 and Theorem 7.3.3. □

Corollary 7.4.2. *Let us be given a fuzzy IF-THEN rule base $\mathcal{R}_M(A_i, B_i)$ as in* (7.1) *which is monotone and $A_i \in \mathcal{P}_X$, $i = 1, 2, \ldots, n$, form a Ruspini partition on X and $B_i \in \mathcal{P}_Y$, $i = 1, 2, \ldots, n$, form a Ruspini partition on Y, respectively. Further, let every element of $\mathcal{P}_X$ and $\mathcal{P}_Y$ be normal, convex and strictly monotone on both sides of the ceiling, i.e., $\mathcal{P}_X \subseteq \mathcal{F}^*(X)$ and $\mathcal{P}_Y \subseteq \mathcal{F}^*(Y)$. Then the system function g of the FRI $\mathbb{F}_{\rightarrow_g} = \left(\mathcal{P}_X, \mathcal{P}_Y, \hat{R}_g, \mathrm{MOM}\right)$ is monotonic, where $\hat{R}_g$ is as defined in* (Imp-$\hat{R}_g$) *and $\longrightarrow_g \in \mathbb{I}_{\mathbb{G}}$.*

Proof. Every $\longrightarrow_g \in \mathbb{I}_{\mathbb{G}}$ is strict and its natural negation is the Gödel negation. Thus $\longrightarrow_g \in \mathbb{I}_{\mathbb{G}} \subsetneq \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}} \subsetneq \mathbb{I}^{\mathbf{st}}$ and the result follows from Theorem 7.3.1. □

7.5 Illustrative Examples

In this section we illustrate the results of the previous section through some examples.

Let us consider the rule base as given in Example 7.0.5 and the FRIs:

(i) $\mathbb{F}_{\rightarrow_{\mathbf{D1}}} = \left(\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3, \hat{R}_{\rightarrow_{\mathbf{D1}}}, \mathrm{MOM}\right)$ and

(ii) $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}} = \left(\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3, \hat{R}_{\rightarrow_{\mathbf{D1^c}}}, \mathrm{MOM}\right)$.

In the examples, we have considered five defuzzification methods and investigated the behaviour of the system function for monotonicity. The defuzzification methods are: (i) MOM (Mean of Maxima), (ii) LOM (Largest of Maxima), (iii) SOM (Smallest of Maxima), (iv) COG (Centroid) and (v) BIS (Bisector). For more details on these defuzzification methods and their formulae, see Section 1.1.5

7.5.1 Illustrative examples for monotonicity of $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}} = \left(\{A_i\}, \{B_i\}, \hat{R}_{\rightarrow_{\mathbf{D1^c}}}, d\right)$

Example 7.5.1. *Let us consider the fuzzy system $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ with the the rule base* (7.2)*, and let the implication operator employed in the relation $\hat{R}_{\rightarrow_{\mathbf{D1^c}}}$ be the Reichenbach implication, which is strict but does not satisfy* (OP) *and also $N_{I_{\mathbf{RC}}} \neq N_{\mathbf{D1}}$, i.e., $I_{\mathbf{RC}} \in \mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}}$ (see Table 7.1).*

The corresponding system functions with different types of defuzzification are shown in Figures 7.13-7.17. From Figure 7.13, we can see that the system function g is monotonic as claimed in Theorem 7.3.3.

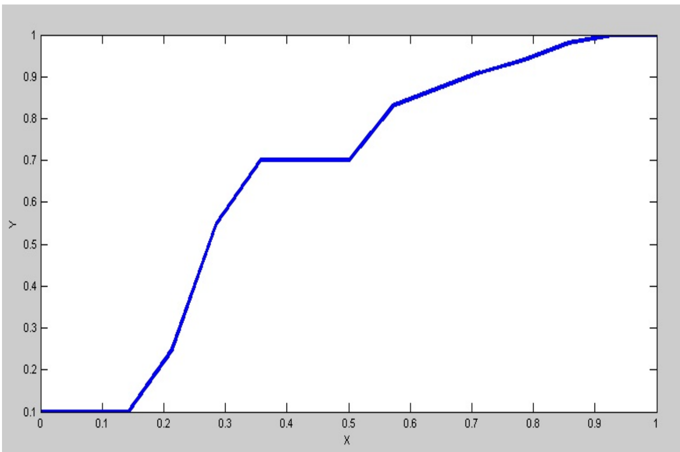


Figure 7.13: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ given in Example 7.5.1 with MOM defuzzifier and $\longrightarrow_{\mathbf{D1^c}} = I_{\mathbf{RC}}$, the Reichenbach implication.

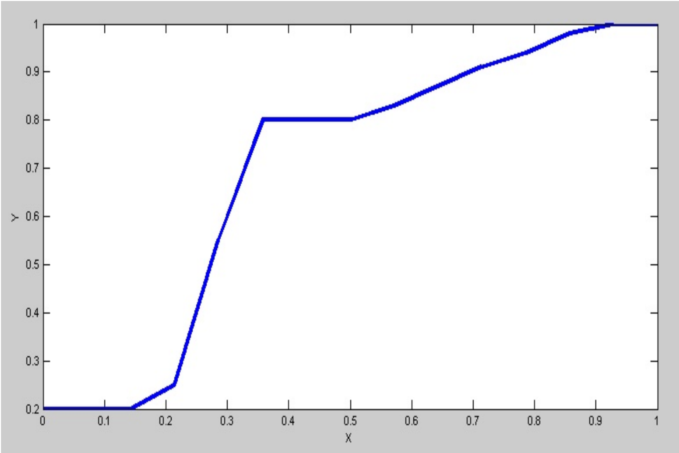


Figure 7.14: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ given in Example 7.5.1 with LOM defuzzifier and $\longrightarrow_{\mathbf{D1^c}} = I_{\mathbf{RC}}$, the Reichenbach implication.

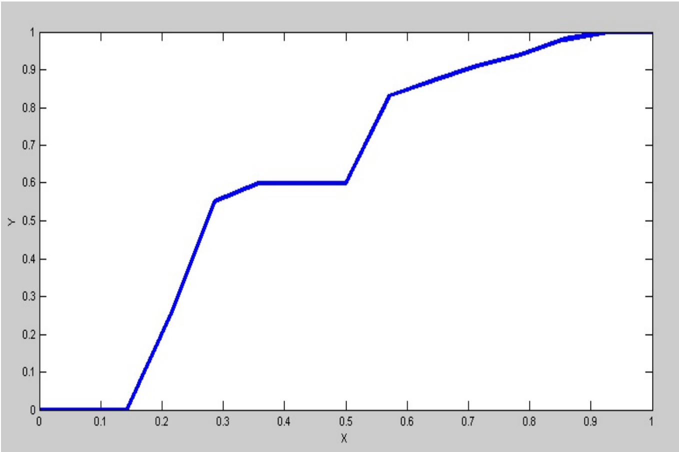


Figure 7.15: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ given in Example 7.5.1 with SOM defuzzifier and $\longrightarrow_{\mathbf{D1^c}} = I_{\mathbf{RC}}$, the Reichenbach implication.

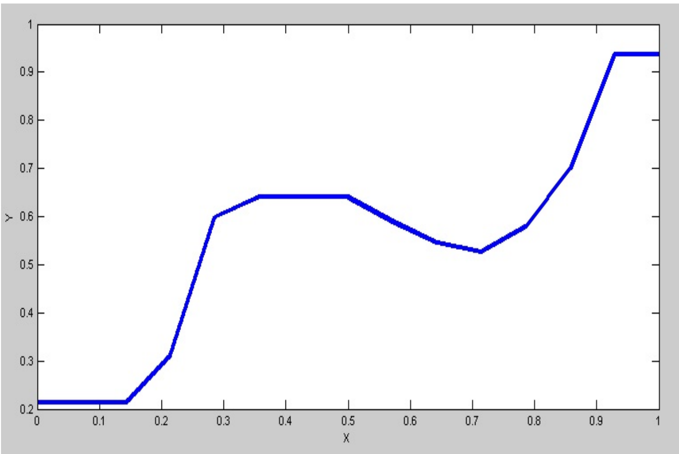


Figure 7.16: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ given in Example 7.5.1 with COG defuzzifier and $\longrightarrow_{\mathbf{D1^c}} = I_{\mathbf{RC}}$, the Reichenbach implication.

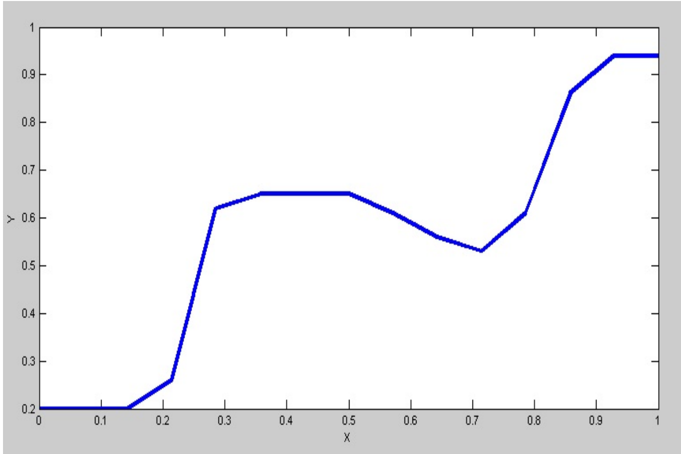


Figure 7.17: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ given in Example 7.5.1 with BIS defuzzifier and $\longrightarrow_{\mathbf{D1^c}} = I_{\mathbf{RC}}$, the Reichenbach implication.

7.5.2 Illustrative examples for monotonicity of $\mathbb{F}_{\rightarrow_{\mathbf{D1}}} = \left(\{A_i\}, \{B_i\}, \hat{R}_{\rightarrow_{\mathbf{D1}}}, d\right)$

Example 7.5.2. *Let us consider the fuzzy system $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ with the the rule base* (7.2)*, and let the implication operator employed in the relation $\hat{R}_{\rightarrow_{\mathbf{D1}}}$ be the Yager's implication, which is strict, $N_{I_{\mathbf{YG}}} = N_{\mathbf{D1}}$ but does not satisfy* (OP)*, i.e., $I_{\mathbf{YG}} \in \mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$ (see Table 7.1).*

The corresponding system functions with different types of defuzzification are shown in Figures. 7.18-7.22. Once again, Figure 7.18, shows that the system function g is monotonic as claimed in Theorem 7.3.1.

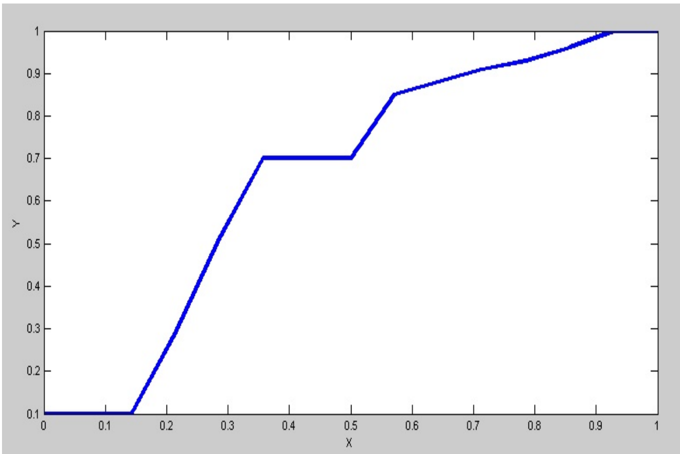


Figure 7.18: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ given in Example 7.5.2 with MOM defuzzifier and $\longrightarrow_{\mathbf{D1}} = I_{\mathbf{YG}}$, the Yager's implication.

Remark 7.5.3. *Note that we have proven our results, viz., Theorems 7.3.1 and 7.3.3 for the FRIs $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ and $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ with* MOM *defuzzification. The examples above illustrate these results, albeit by considering some specific fuzzy implications from each of the classes of $\mathbb{I}^{\mathbf{st}}_{N_{\mathbf{D1}}}$ and $\mathbb{I}^{\mathbf{st}}_{N^c_{\mathbf{D1}}}$.*

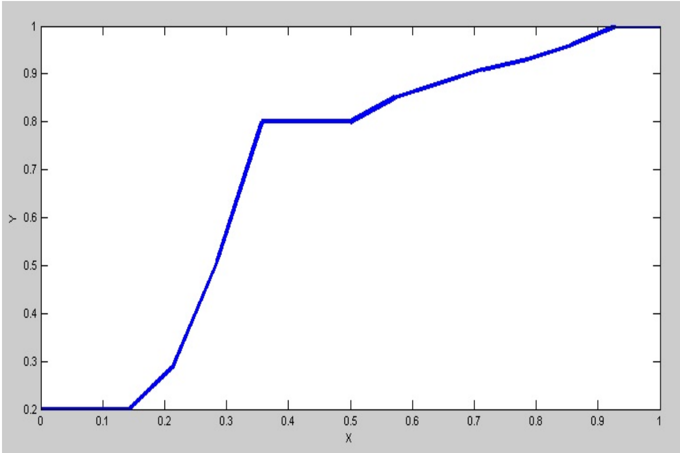


Figure 7.19: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ given in Example 7.5.2 with LOM defuzzifier and $\longrightarrow_{\mathbf{D1}} = I_{\mathbf{YG}}$, the Yager's implication.

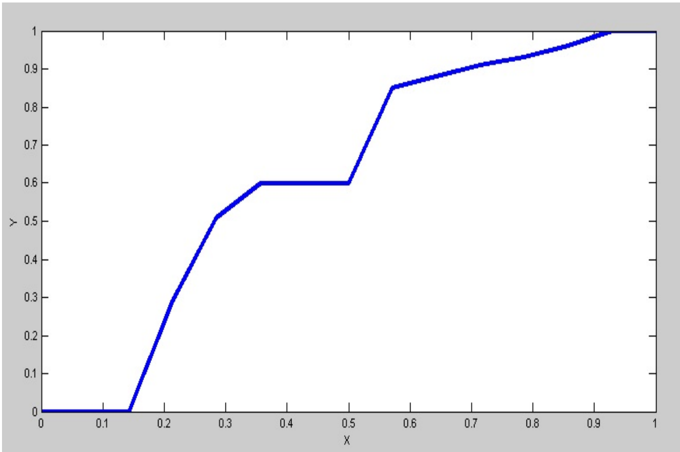


Figure 7.20: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ given in Example 7.5.2 with SOM defuzzifier and $\longrightarrow_{\mathbf{D1}} = I_{\mathbf{YG}}$, the Yager's implication.

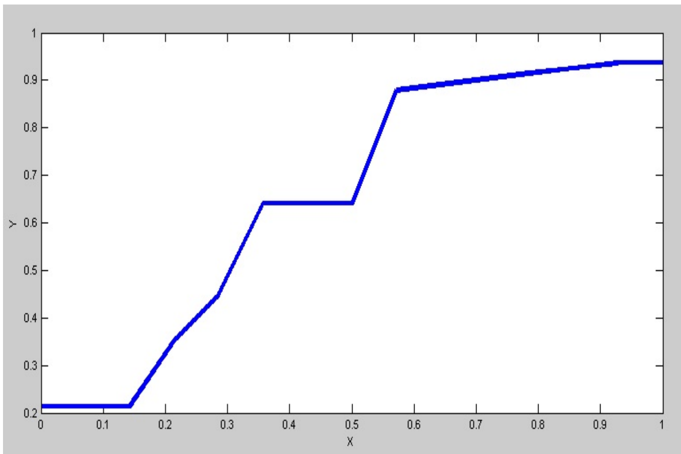


Figure 7.21: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ given in Example 7.5.2 with COG defuzzifier and $\longrightarrow_{\mathbf{D1}} = I_{\mathbf{YG}}$, the Yager's implication.

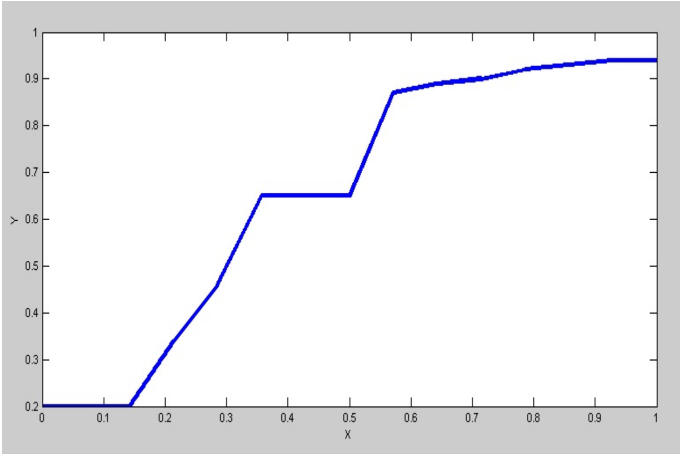


Figure 7.22: System function of the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ given in Example 7.5.2 with BIS defuzzifier and $\longrightarrow_{\mathbf{D1}} = I_{\mathbf{YG}}$, the Yager,s implication.

Note that, among the defuzzification methods considered, the MOM, SOM *and* LOM *methods can be seen as* Ceiling-based *methods, since the defuzzified value depends only on the ceiling of the fuzzy set under consideration. In the case of convex fuzzy sets, these three methods are such that the defuzzified output falls within the ceiling of the fuzzy set, while the same is not true for* COG *and* BIS *methods always.*

However, it is interesting to note the following. On the one hand, in Example 7.5.1, for the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1^c}}}$ with $\rightarrow_{\mathbf{D1^c}} = I_{\mathbf{RC}}$, the corresponding system function is monotonic with MOM *defuzzification as well as with other ceiling-based defuzzification methods (e.g.,* SOM *and* LOM*), whereas it is not monotonic while considering the other two defuzzification methods, viz.,* COG *and* BIS.

On the other hand, in Example 7.5.2, for the FRI $\mathbb{F}_{\rightarrow_{\mathbf{D1}}}$ with $\rightarrow_{\mathbf{D1}} = I_{\mathbf{YG}}$, the corresponding system function is monotonic with all the defuzzification methods considered here.

This seems to point to the fact that Theorems 7.3.1 and 7.3.3 may be valid even when d is taken to be any ceiling based defuzzification method instead of the MOM *defuzzification method.*

Chapter 8

Concluding Remarks and some Open Problems

The future belongs to those who believe in the beauty of their dreams.
– Eleanor Roosevelt (1884 – 1962)

8.1 Summary of the work contained in this thesis

Fuzzy relational inference mechanisms are one of the earliest inference mechanisms to be studied. However, all of the known works have concentrated on FRIs that employ operations that come from a residuated lattice structure. In this work, we have proposed two modified versions of the well known Bandler-Kohout Subproduct (BKS) inference mechanism, by employing the Yager's families of fuzzy implications instead of the usual residual implications. Calling them as BKS-f and BKS-g inference mechanisms, we have shown that many of the desirable properties like interpolativity, continuity, robustness, approximation capability and monotonicity which are available for the BKS inference mechanisms with residuated implications are also available for the proposed modified BKS inference mechanisms when we employ the Yager's families of fuzzy implications, which do not come from a residuated structure.

We have obtained necessary and sufficient conditions for interpolativity of both the BKS-f and BKS-g inference mechanisms. Note that these conditions are similar, but not identical, to those of the inference mechanisms which employ operations from a residuated lattice structure.

Following this, we have shown that both the BKS-f and BKS-g inference mechanisms, in fact, a much larger class of FRIs, are capable of approximating a continuous function over a closed interval. While proving this, we have given a constructive proof and also have shown that the approximator function is also continuous. It should be mentioned that the results are valid for a larger class of fuzzy implications, which includes the family of residuated implications and Yager's families of fuzzy implications.

Finally, we have shown that both the BKS-f and BKS-g inference mechanisms are also monotonic. The results corresponding to monotonicity are once again valid for a larger class of fuzzy implications, which includes the Yager's families of fuzzy implications.

Thus, we believe that these results are very much applicable in most of the practical and desirable contexts and show that a much larger class of fuzzy implications other than what is typically considered in the literature can be employed meaningfully in FRIs based on implicative models.

8.2 Problems for further exploration

While discussing the interpolativity, continuity and robustness of the proposed modified BKS inference mechanism, we have considered only one family of fuzzy implications which is not a residual implication. It can be seen that there are many other families of fuzzy implications other than the Yager's families of fuzzy implications which do not come from a residuated lattice structure. Thus it remains to study these inference mechanisms as well.

Problem 8.2.1. *What are the conditions under which a modified BKS inference mechanism with an implicative rule base is admissible, when the fuzzy implications neither come from a residuted lattice structure nor from Yager's families?*

While discussing approximation capability of BKS (*rather FRIs* with reducible composition) we have considered fuzzy implications, $I \in \mathbb{I}_{\mathbf{OP}} \cup \mathbb{I}^{+}_{N_{\mathbf{D1}}}$. We have seen that when an $I \notin \mathbb{I}_{\mathbf{OP}} \cup \mathbb{I}^{+}_{N_{\mathbf{D1}}}$ is employed, the resulting output can be empty, which is not at all desirable. Of course, note also that there are some assumptions on the other components of the inference mechanisms. Changing these assumptions suitably it may still be possible to get non-empty output when a fuzzy implication $I \notin \mathbb{I}_{\mathbf{OP}} \cup \mathbb{I}^{+}_{N_{\mathbf{D1}}}$ is employed.

Problem 8.2.2. *Do BKS inference mechanisms with fuzzy implications $I \notin \mathbb{I}_{\mathbf{OP}} \cup \mathbb{I}^{+}_{N_{\mathbf{D1}}}$ give reasonable outputs? Is the concept of 'Weak Coherence' proposed in this thesis sufficient, or should it be made even more lenient? Further, are such modified BKS inference mechanisms universal approximators?*

Based on our intuition from some simulation results, we believe that FRIs in which we employ fuzzy implications which are positive but $N_I \neq N_{\mathbf{D1}}$ are capable of universal approximation.

In our proofs exhibiting the monotonicity of an FRI with implicative rules, we have considered a subclass $\mathbb{I}^{\mathbf{st}} \subsetneq \mathbb{I}^{+}$. In the results related to monotonicity of an FRI, we have considered only the MOM defuzzification technique. Noting that the MOM defuzzification takes the value from the core of the output fuzzy set, we may formulate the following question:

Problem 8.2.3. *Are BKS inference mechanisms with fuzzy implications $I \in \mathbb{I} \setminus \mathbb{I}^{\mathbf{st}}$ monotonic? Further, does the monotonicity remain if one were to consider more general defuzzification techniques which take value from the core of the output fuzzy set?*

Based on some numerical simulations, we once again believe that keeping the other parameters fixed, such FRIs are capable of preserving monotonicity.

As a final observation, we have the following to state: In all the existing FRIs, the underlying operations are either t-norms T or a fuzzy implication I. As was stated in the introduction, these are but a generalisation of a conjunction and an implication to multi-valued logic, where the truth values lie in $[0, 1]$ instead of $\{0, 1\}$. Our work in this thesis has shown that the often used or required adjoint relation or the residuation property between the pair (T, I) can be dispensed with, without majorly affecting the properties of the FRIs. One then wonders, if we could also consider a more general (Conjunctor, Implicator) pair in FRIs. Alternatively, the question that naturally arises is the following:

Problem 8.2.4. *What are the conditions and properties required of a (Conjunctor, Implicator) pair ($C, I : [0,1]^2 \to [0,1]$) to make it admissible in an FRI without sacrificing the desirable properties of an FRI?*

Some preliminary results along these lines were recently published in the following work:

- Sayantan Mandal and Balasubramaniam Jayaram, " Suitability of FRIs based on generalised Operators", 12th International Conference on Fuzzy Set Theory and Applications, FSTA 2014, Ján Liptovský, Slovakia, January 26 - 31, 2014.

References

[1] M. Baczyński and B. Jayaram, "Yager's classes of fuzzy implications: some properties and intersections," *Kybernetika*, vol. 43, no. 2, pp. 157–182, 2007.

[2] ——, *Fuzzy Implications*, ser. Studies in Fuzziness and Soft Computing. Springer, 2008, vol. 231.

[3] ——, "(S,N)- and R-implications: A state-of-the-art survey," *Fuzzy Sets and Systems*, vol. 159, no. 14, pp. 1836 – 1859, 2008.

[4] ——, "(U,N)-implications and their characterizations," *Fuzzy Sets and Systems*, vol. 160, no. 14, pp. 2049 – 2062, 2009.

[5] ——, "QL-implications: Some properties and intersections," *Fuzzy Sets and Systems*, vol. 161, no. 2, pp. 158 – 188, 2010.

[6] ——, "Intersections between some families of (U,N)- and RU-implications," *Fuzzy Sets and Systems*, vol. 167, no. 1, pp. 30 – 44, 2011.

[7] W. Bandler and L. J. Kohout, "Semantics of implication operators and fuzzy relational products," *International Journal of Man-Machine Studies*, vol. 12, no. 1, pp. 89 – 116, 1980.

[8] H. Bustince, P. Burillo, and F. Soria, "Automorphisms, negations and implication operators," *Fuzzy Sets and Systems*, vol. 134, no. 2, pp. 209 – 229, 2003.

[9] D. Butnariu, "Additive fuzzy measures and integrals I," *Journal of Mathematical Analysis and Applications*, vol. 93, no. 2, pp. 436 – 452, 1983.

[10] J. Castro, "Fuzzy logic controllers are universal approximators," *IEEE Trans. on Systems, Man and Cybernetics*, vol. 25, no. 4, pp. 629–635, 1995.

[11] J. Castro and M. Delgado, "Fuzzy systems with defuzzification are universal approximators," *IEEE Trans. on Systems, Man, and Cybernetics, Part B: Cybernetics,*, vol. 26, no. 1, pp. 149–152, 1996.

[12] D. Coufal, "Coherence of radial implicative fuzzy systems," in *IEEE International Conference on Fuzzy Systems*, 2006, pp. 307–314.

[13] B. De Baets and R. Mesiar, "T-partitions," *Fuzzy Sets and Systems*, vol. 97, no. 2, pp. 211 – 223, 1998.

[14] D. Driankov, H. Hellendoorn, and M. Reinfrank, *An introduction to fuzzy control (2nd ed.)*. London, UK: Springer-Verlag, 1996.

[15] D. Dubois, H. Prade, and L. Ughetto, "Checking the coherence and redundancy of fuzzy knowledge bases," *IEEE Trans. on Fuzzy Systems*, vol. 5, no. 3, pp. 398–417, 1997.

[16] D. Dubois and H. Prade, "What are fuzzy rules and how to use them," *Fuzzy Sets and Systems*, vol. 84, no. 2, pp. 169 – 185, 1996.

[17] S. Gottwald, "A treatise on many-valued logics," in *Studies in Logic and Computation*. Press, 2001.

[18] H. Hellendoorn, "Fuzzy numbers and approximate reasoning," *Annales Univ. Sci Budapest., Sect. Comp*, vol. 12, pp. 113 – 119, 1991.

[19] U. Höhle, "Many-valued equalities, singletons and fuzzy partitions," *Soft Computing*, vol. 2, no. 3, pp. 134–140, 1998.

[20] B. Jayaram, "Yager's new class of implications J_f and some classical tautologies," *Information Sciences*, vol. 177, no. 3, pp. 930 – 946, 2007.

[21] F. Klawonn and J. L. Castro, "Similarity in fuzzy reasoning," *Mathware & Soft Computing*, vol. 3, pp. 197–228, 1995.

[22] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, ser. Trends in Logic. Dordrecht: Kluwer Academic Publishers, 2000, vol. 8.

[23] Y.-M. Li, Z.-K. Shi, and Z.-H. Li, "Approximation theory of fuzzy systems based upon genuine many-valued implications: MIMO cases," *Fuzzy Sets and Systems.*, vol. 130, pp. 159–174, 2002.

[24] ——, "Approximation theory of fuzzy systems based upon genuine many-valued implications: SISO cases," *Fuzzy Sets and Systems.*, vol. 130, pp. 147–157, 2002.

[25] P. Magrez and P. Smets, "Fuzzy modus ponens: A new model suitable for applications in knowledge-based systems," *International Journal of Intelligent Systems*, vol. 4, no. 2, pp. 181–200, 1989.

[26] E. H. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *International Journal of Human-Computer Studies*, vol. 51, no. 2, pp. 135–147, 1999.

[27] D. Markechová, "The entropy of fuzzy dynamical systems and generators," *Fuzzy Sets and Systems*, vol. 48, no. 3, pp. 351 – 363, 1992.

[28] S. Massanet and J. Torrens, "On the characterization of Yager's implications," *Information Sciences*, vol. 201, pp. 1 – 18, 2012.

[29] K. Menger, "Statistical metrics," *Proceedings of the National Academy of Sciences*, vol. 28, no. 12, pp. 535–537, 1942.

[30] S. Montes, I. Couso, and P. Gil, "Fuzzy δ - ϵ-partitions," *Information Sciences*, vol. 152, pp. 267 – 285, 2003.

[31] A. D. Nola, E. Sanchez, W. Pedrycz, and S. Sessa, *Fuzzy Relation Equations and Their Applications to Knowledge Engineering*. Norwell, MA, USA: Kluwer Academic Publishers, 1989.

[32] L. Nosková, "Systems of fuzzy relation equation with inf-$\longrightarrow$ composition: solvability and solution," *Journal of Electrical Engineering*, vol. 12(s), pp. 69–72, 2005.

[33] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*. Boston: Kluwer Academic Publishers, 1999.

[34] W. Pedrycz, "Applications of fuzzy relational equations for methods of reasoning in presence of fuzzy data," *Fuzzy Sets and Systems.*, vol. 16, no. 2, pp. 163–175, 1985.

[35] I. Perfilieva and V. Kreinovich, "A new universal approximation result for fuzzy systems, which reflects CNF–DNF duality," *International Journal of Intelligent Systems*, vol. 17, no. 12, pp. 1121–1130, 2002.

[36] I. Perfilieva and S. Lehmke, "Correct models of fuzzy IF–THEN rules are continuous," *Fuzzy Sets and Systems*, vol. 157, no. 24, pp. 3188–3197, 2006.

[37] I. Perfilieva and V. Novák, "System of fuzzy relation equations as a continuous model of IF–THEN rules," *Information Sciences*, vol. 177, no. 16, pp. 3218–3227, 2007.

[38] J. Ramík and J. ímánek, "Inequality relation between fuzzy numbers and its use in fuzzy optimization," *Fuzzy Sets and Systems*, vol. 16, no. 2, pp. 123 – 138, 1985.

[39] R. Reiser, B. Bedregal, R. Santiago, and M. Amaral, "Canonical representation of the Yager's classes of fuzzy implications," *Computational and Applied Mathematics*, vol. 32, no. 3, pp. 401–412, 2013.

[40] S. Roychowdhury and W. Pedrycz, "A survey of defuzzification strategies," *International Journal of Intelligent Systems*, vol. 16, no. 6, pp. 679–695, 2001.

[41] E. Sanchez, "Resolution of composite fuzzy relation equations," *Information and Control*, vol. 30, no. 1, pp. 38 – 48, 1976.

[42] D. Tikk, L. T. Kóczy, and T. D. Gedeon, "A survey on universal approximation and its limits in soft computing techniques," *International Journal of Approximate Reasoning*, vol. 33, no. 2, pp. 185–202, 2003.

[43] E. Van Broekhoven and B. De Baets, "Monotone Mamdani–Assilian models under mean of maxima defuzzification," *Fuzzy Sets and Systems*, vol. 159, no. 21, pp. 2819 – 2844, 2008.

[44] ——, "Only smooth rule bases can generate monotone Mamdani–Assilian models under center-of-gravity defuzzification," *IEEE Trans. on Fuzzy Systems*, vol. 17, no. 5, pp. 1157–1174, 2009.

[45] N. R. Vemuri and B. Jayaram, "Representations through a monoid on the set of fuzzy implications," *Fuzzy Sets and Systems*, 2014.

[46] M. Štěpnička and B. De Baets, "Monotonicity of implicative fuzzy models," in *IEEE International Conference on Fuzzy Systems*, 2010, pp. 1–7.

[47] ——, “Implication-based models of monotone fuzzy rule bases,” *Fuzzy Sets and Systems*, vol. 232, pp. 134 – 155, 2013.

[48] M. Štěpnička, B. De Baets, and L. Nosková, “Arithmetic fuzzy models,” *IEEE Trans. on Fuzzy Systems*, vol. 18, no. 6, pp. 1058 –1069, 2010.

[49] M. Štěpnička and B. Jayaram, “On the suitability of the Bandler-Kohout subproduct as an inference mechanism,” *IEEE Trans. on Fuzzy Systems*, vol. 18, no. 2, pp. 285–298, 2010.

[50] L.-X. Wang and J. Mendel, “Fuzzy basis functions, universal approximation, and orthogonal least-squares learning,” *IEEE Trans. on Neural Networks*, vol. 3, no. 5, pp. 807 –814, 1992.

[51] A. Xie and H. Liu, “A generalization of Yager’s f-generated implications,” *International Journal of Approximate Reasoning*, vol. 54, no. 1, pp. 35 – 46, 2013.

[52] R. R. Yager, “On some new classes of implication operators and their role in approximate reasoning,” *Information Sciences*, vol. 167, no. 14, pp. 193 – 216, 2004.

[53] R. R. Yager and V. Kreinovich, “Universal approximation theorem for uninorm-based fuzzy systems modeling,” *Fuzzy Sets and Systems*, vol. 140, no. 2, pp. 331–339, 2003.

[54] L. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338 – 353, 1965.

[55] ——, “Fuzzy logic and approximate reasoning,” *Synthese*, vol. 30, no. 3-4, pp. 407–428, 1975.

[56] L. A. Zadeh, “Outline of a new approach to the analysis of complex systems and decision processes,” *IEEE Trans. on Systems, Man and Cybernetics*, vol. SMC-3, no. 1, pp. 28 –44, 1973.

[57] X.-J. Zeng and M. Singh, “Approximation theory of fuzzy systems-SISO case,” *IEEE Trans. on Fuzzy Systems*, vol. 2, no. 2, pp. 162 –176, 1994.

List of Figures

List of Tables