On an Open Problem of U. Höhle - A Characterization of Conditionally Cancellative T-subnorms

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Abstract. In this work we solve an open problem of U.Höhle [Problem 11, Fuzzy Sets and Systems 145 (2004) 471-479]. We show that the solution gives a characterization of all conditionally cancellative t-subnorms. Further, we give an equivalence condition for a conditionally cancellativite t-subnorm to be a t-norm and hence show that conditionally cancellativite t-subnorms whose natural negations are strong are, in fact, t-norms.

1 Introduction

The paper by Klement et al. [6] is a collection of open problems posed during the 24^{th} Linz Seminar on fuzzy set theory. They deal with unsolved problems (as of then) related to fuzzy aggregation operations, especially t-norms and t-subnorms. Since the publication of [6], some problems mentioned therein have been solved - for instance, Problem **1** was solved by Ouyang et al. [8], Problem **5** was solved by Ouyang and Li [8] while for some other problems partial solutions have been given, see for instance, the papers of Viceník [9], [10], [11] relating to Problem **4**(i).

One of the open problems listed therein was posed by Prof. U. Höhle (Problem **11**) which reads as follows:

*Problem 1 (U.Höhle, [6], Problem **11**).* Characterize all left-continuous t-norms T which satisfy

$$I(x, T(x, y)) = \max(n(x), y), \quad x, y \in [0, 1] \ . \tag{1}$$

where I is the residual operator linked to T, i.e.,

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\}, \quad x, y \in [0, 1] \ , \tag{2}$$

$$n(x) = n_T(x) = I(x, 0) \ \textit{for all} \ x \in [0, 1]. \tag{3}$$

Further, Prof. U.Höhle goes on to remark the following:

Remark 1. "In the class of continuous t-norms, only nilpotent t-norms fulfill the above property."

In this work we deal with two problems. Firstly, we solve the above open problem of U.Höhle and show that the solution gives a characterization of all conditionally cancellative t-subnorms. From the proven result it does follow that the remark of Prof. U.Höhle - Remark 1 - is not always true and give an equivalence condition for it to be true, viz., that the natural negation obtained from the t-norm is strong.

Secondly, this quite naturally leads us to consider conditionally cancellative t-subnorms whose natural negations are involutive. Once again, by proving an equivalence condition for a conditionally cancellative t-subnorm to be a t-norm, we show that conditionally cancellative t-subnorms whose natural negations are involutive, in fact, become t-norms.

2 Preliminaries

Definition 1. *A function* $N\colon [0,1] \to [0,1]$ *is called a* fuzzy negation *if* N *is decreasing and* $N(0) = 1$, $N(1) = 0$.

Definition 2 ([5], Definition 1.7). *A* t-subnorm *is a function* $M\colon [0,1]^2 \to [0,1]$ *such that it is monotonic non-decreasing, associative, commutative and* $M(x,y) \le \min(x,y)$ *for all* $x,y \in [0,1]$.

Note that for a t-subnorm 1 need not be the neutral element, unlike in the case of a t-norm.

Definition 3 (cf. [5], Definition 2.9 (iii)). *A t-subnorm* M *satisfies the* Conditional Cancellation Law *if, for any* $x,y,z \in (0,1]$,

$$M(x,y) = M(x,z) > 0 \text{ implies } y = z \; . \tag{CCL}$$

Alternately, (CCL) implies that on the positive domain of M, i.e., on the set $\{(x,y) \in (0,1]^2 \mid M(x,y) > 0\}$, M is strictly increasing.

Definition 4 (cf. [1], Definition 2.3.1). *Let* M *be any t-subnorm. Its natural negation* n_M *is given by*

$$n_M(x) = \sup\{t \in [0,1] \mid M(x,t) = 0\}, \quad x \in [0,1] \; . \tag{4}$$

Note that though $n_M(0) = 1$, it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. However, we have the following result.

Lemma 1 (cf. [1], Proposition 2.3.4). *Let* M *be any t-subnorm and* n_M *its natural negation. Then we have the following:*

(i) $M(x,y) = 0 \Longrightarrow y \le n_M(x)$.
(ii) $y < n_M(x) \Longrightarrow M(x,y) = 0$.
(iii) If M *is left-continuous then* $y = n_M(x) \Longrightarrow M(x,y) = 0$*, i.e., the reverse implication of (i) also holds.*

3 Solution to the Open Problem of U. Höhle

It should be noted that in the case T is left-continuous - as stated in **Problem 1** - the sup in (2) actually becomes max. It is worth mentioning that the residual can be determined for more generalised conjunctions and the conditions underwhich this residual becomes a fuzzy implication can be found in, for instance, [2], [4]. Hence we further generalise the statement of **Problem 1** by considering a t-subnorm instead of a t-norm and also dropping the condition of left-continuity. As we show below the solution characterizes the set of all conditionally cancellative t-subnorms.

Theorem 1. *Let M be any t-subnorm and I the residual operation linked to M by* (2)*. Then the following are equivalent:*

(i) The pair (I, M) satisfies (1)*.*
(ii) M is a Conditionally Cancellative t-subnorm.

Proof. Let M be any t-subnorm, not necessarily left-continuous. Note that we denote n_M simply by n.

(i) $\Longrightarrow$ (ii): Let the adjoint pair (I, M) satisfy (1). On the contrary, let us assume that there exist $x, y, z \in (0, 1)$ such that $M(x, y) = M(x, z) > 0$ but $y < z$. Then we have that

$$\text{LHS (1)} = I(x, M(x, y)) = \sup\{t \in [0, 1] \mid M(x, t) \le M(x, y)\} \ge z > y\ .$$

However, note that, from Lemma 1 (i) we have that $y \ge n(x)$, since $M(x, y) > 0$. Thus

$$\text{RHS (1)} = \max(n(x), y) = y < \text{LHS (1)}\ ,$$

a contradiction to the fact that the adjoint pair (I, M) satisfies (1). Hence M satisfies (CCL).

(ii) $\Longrightarrow$ (i): Now, let M satisfy (CCL). Consider any arbitrary $x, y \in [0, 1]$. Then either $n(x) > y$ or $n(x) \le y$.
If $n(x) > y$, then by Lemma 1 (ii) we see that $M(x, y) = 0$ and hence

$$\text{LHS (1)} = I(x, M(x, y)) = I(x, 0) = n(x) = \max(n(x), y) = \text{RHS (1)}.$$

If $n(x) \le y$ and $M(x, y) = 0$ then by Lemma 1(i) we have that $n(x) \ge y$ and hence $n(x) = y$ and it reduces to the above case. Hence let $M(x, y) > 0$. Then

$$\text{RHS (1)} = \max(n(x), y) = y\ .$$

We claim now that LHS (1) $= I(x, M(x, y)) = y$. If this were not true, then there exists $1 \ge z > y$ ($z \not< y$ by the monotonicity of M) such that

$$I(x, M(x, y)) = \sup\{t \in [0, 1] \mid M(x, t) \le M(x, y)\} = z.$$

This implies that there exists a $w \in (0,1)$ such that $z > w > y$ and $M(x,w) \leq M(x,y)$, which by the monotonicity of t-subnorm implies that $M(x,w) = M(x,y)$ with $w \gneq y$, a contradiction to the fact that M satisfies (CCL). Thus the adjoint pair (I, M) satisfies (1). ☐

Example 1. Consider the product t-norm $T_{\mathbf{P}}(x,y) = xy$, which is a strict t-norm and hence continuous and Archimedean, whose residual is the Goguen implication given by

$$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \dfrac{y}{x}, & \text{if } x > y. \end{cases}$$

It can be easily verified that the pair $(T_{\mathbf{P}}, I_{\mathbf{GG}})$ does indeed satisfy (1) whereas the natural negation of $T_{\mathbf{P}}$ is the Gödel negation

$$n_{T_{\mathbf{P}}}(x) = I_{\mathbf{GG}}(x,0) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0. \end{cases}$$

This example clearly shows that the remark of U.Höhle, Remark 1, is not always true. In the following we give an equivalence condition under which it is true.

Theorem 2. *Let T be a continuous t-norm that satisfies* (1) *along with its residual. Then the following are equivalent:*

(i) *T is nilpotent.*
(ii) *n_T is strong.*

Proof. (i) $\Longrightarrow$ (ii): Obvious.
(ii) $\Longrightarrow$ (ii): If T is continuous and satisfies (1) along with its residual then, from Theorem 1, T is conditionally cancellative and hence necessarily Archimedean by [5], Proposition 2.15 (ii). Thus T is either nilpotent or strict. If T is continuous with a strong natural negation, clearly, T has zero-divisors and hence T is nilpotent. ☐

4 Conditional Cancellativity and Unit element

From the above remarks we note that when the natural negation of the underlying conjunction (a continuous t-norm, in the above case) is strong the class of conjunctions that satisfy (1) along with its residual gets restricted. Hence we study the class of t-subnorms M that satisfy (1) along with its residual and whose natural negations are strong. In other words, we seek the characterization of the class of conditionally cancellative t-subnorms with strong natural negations.

Let us recall from the remark following Definition 4 that the natural negation of a t-subnorm n_M need not be a fuzzy negation. If a t-subnorm has 1 as its neutral element, i.e., if it is a t-norm, then we have

$$M(1,y) = 0 \Longleftrightarrow y = 0,$$
$$\text{i.e., } y = \sup\{t | M(1,t) = 0\} = n_M(1) = 0.$$

Equivalently, by the monotonicity of M we have that n_M is a fuzzy negation. However, this is only a necessary and not a sufficient condition.

Note that, so far, no general result giving equivalence conditions under which a t-subnorm becomes a t-norm is available. It was Jenei [3] who proposed some suficiency conditions and showed that left-continuous t-subnorms with strong natural negations are t-norms, i.e., 1 does become a neutral element.

In the following we give an equivalence condition for a conditionally cancellative t-subnorm to be a t-norm and show that in the case n_M is a strong negation then M always is a t-norm.

Lemma 2. *Let M be a conditionally cancellative t-subnorm. Let $M(1, y_0) = y_0$, for some $y_0 \in (0, 1]$.*

(i) Then $M(1, y) = y$ for all $y \in [y_0, 1]$.
(ii) Let $y^* = \sup\{t|M(1, t) = 0\} = n_M(1)$. Then $M(1, y) = y$ for all $y \in (y^*, y_0]$.

Proof. Let $M(1, y_0) = y_0$, for some $y_0 \in (0, 1]$.

(i) Let $y_0 < y \leq 1$. Clearly, $y_0 = M(1, y_0) < M(1, y) \leq y$. If $M(1, y) = y' < y$, then by associativity and conditional cancellativity we have

$$\left.\begin{aligned} M(M(1, y_0), y) &= M(y_0, y) \\ M(M(1, y), y_0) &= M(y', y_0) \end{aligned}\right\} \Longrightarrow M(y_0, y) = M(y_0, y') \Longrightarrow y = y' ,$$

i.e., $M(1, y) = y$ for all $y \geq y_0$.

(ii) Let $y^* < y \leq y_0$. Clearly, $y_0 = M(1, y_0) > y \geq M(1, y) = y'$. If $M(1, y) = y' < y$, then, once again, by associativity and conditional cancellativity we have

$$\left.\begin{aligned} M(M(1, y_0), y) &= M(y_0, y) \\ M(M(1, y), y_0) &= M(y', y_0) \end{aligned}\right\} \Longrightarrow M(y_0, y) = M(y_0, y') \Longrightarrow y = y' ,$$

i.e., $M(1, y) = y$ for all $y \in (y^*, y_0]$. □

Based on the above result, we now have the following equivalence condition for a conditionally cancellative t-subnorm to be a t-norm:

Theorem 3. *Let M be any conditionally cancellative t-subnorm. Then the following are equivalent:*

(i) M is a t-norm.
(ii) n_M is a negation and $M(1, y_0) = y_0$, for some $y_0 \in (0, 1]$.

Proof. Sufficiency is obvious. Necessity follows from the fact that if n_M is a negation then $y^* = 0$ in Lemma 2 above. □

The final result of this work shows that in the case n_M is a strong negation then M always is a t-norm.

Theorem 4. *Let M be any conditionally cancellative t-subnorm. If n_M is a strong natural negation then M is a t-norm.*

Proof. Our approach will be to show that $M(1,1) = 1$ and then the result follows easily from Theorem 3. Note also that since n_M is a strong negation, we have that $n_M(x) = 1 \iff x = 0$ and $n_M(x) = 0 \iff x = 1$. Equivalently, $M(1,x) = 0 \iff x = 0$.

On the contrary, let us assume that $M(1,y) < y$ for all $y \in (0,1]$. In particular, $M(1,1) = z$ such that $0 < z < 1$. Since n_M is strong, there exists a $z' \in (0,1)$ such that $z = n_M(z')$. We claim that $z' = 0$ and hence $z = 1$.

If not, then there exists $0 < z'' < z'$ and by the definition of n_M we have that $M(z,z'') = 0$. Also, by our assumption $0 < M(1,z'') = z^* < z''$. Now, by associativity and conditional cancellativity we have

$$\left.\begin{array}{l} M(M(1,1),z'') = M(z,z'') \\ M(M(1,z''),1) = M(z^*,1) \end{array}\right\} \Longrightarrow M(z,z'') = 0 = M(z^*,1)$$

$$\Longrightarrow z^* = 0\ ,$$

a contradiction. Thus $z = 1$ and hence we have the result. □

5 Concluding Remarks

In this work we have solved a more generalised version of an open problem of U.Höhle and shown that the solution gives a characterization of all conditionally cancellative t-subnorms. Further, by proving an equivalence condition for a conditionally cancellative t-subnorm to be a t-norm, we have shown that conditionally cancellative t-subnorms with involutive natural negations are t-norms.

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