VARIATION IN BUILDING HEIGHTS UNDER ZONING REGULATIONS OF BUILDING COVERAGE RATIO AND FLOOR AREA RATIO: THEORETICAL AND EMPIRICAL INVESTIGATION OF DOWNTOWN DISTRICTS IN TOKYO

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ABSTRACT

Building height is one of the most important determinants of streetscapes along with building setbacks. Under zoning regulations in countries such as Japan, building heights are indirectly controlled by the floor-area-ratio (FAR) and the building coverage ratio (BCR) at a district scale. However, it is difficult to understand the relationship between building height and these zoning regulations. Moreover, precise data regarding building heights are unavailable as open data. These problems give us the following research question: how can we estimate the statistical distribution of building heights under zoning regulations at a district scale? To answer this question, we investigated the relationship between the variation in BCR and building heights in downtown districts of the Tokyo metropolitan region by modelling the statistical distribution of BCR as a beta distribution. By applying stochastic variable transformation to a beta function, the probability density function of building heights was derived. This function can be estimated from the mean and variance of BCR in a district and legal FAR. These findings can contribute to investigating the following practical problems: (1) understanding the relationship between the indices regarding zoning regulations (BCR and FAR) and the variation in building heights without precise building height data; and (2) how to control the variation in building heights by considering the lower boundary regarding BCR in order to create harmonious building heights and setbacks.

Keywords: height, building coverage ratio (BCR), floor area ratio (FAR), zoning, plot

INTRODUCTION

Building height is one of the most important determinants of streetscapes along with building setbacks (Bertaud and Brueckner, 2005; Harvey and Aultman-Hall, 2016; Harvey et al., 2017; Marshall, 2011). In European countries, every building's height is directly regulated through form-based codes considering its relation to its front road width and building setbacks in order to create well-arranged and harmonious streetscapes (Berghauser Pont and Haupt, 2009; Marshall, 2005). On the other hand, in the United States of America (USA) and Japan, building heights are indirectly controlled by floor-area-ratio (FAR), defined as the ratio of the total floor area of a building to its plot size (Batty, 2018; Bertaud and Brueckner, 2005). Given that the total floor area of a building is the product of the building coverage area by the number of stories, FAR depends on both the number of stories and the coverage area of a building regulated by the building-coverage-ratio (BCR), defined as the ratio of a building area to its plot area (Schläpfer et al., 2015). The maximum (also called legal) FAR and BCR are the main tools of zoning regulations at a district scale that legally regulate each building's shape and volume.

Although precise data regarding building heights are unavailable as open data, a building's height under zoning regulations can be approximately computed by the product of the average floor height and the ratio of FAR to BCR in a plot. This ratio approximates the number of stories of a building. In theory, if building owners were to choose the maximum FAR and BCR, building heights would be almost identical, because the average floor height tends to be constant (around three metres). In practice, building owners tend to use up the maximum FAR in order to make the floor area of their buildings as large as possible. However, they do not always choose the maximum BCR. This is because building owners do not only want building coverage area but also open space for a back or front yard within their plots.

Furthermore, according to the Building Standard Law of Japan (Japanese building codes), building owners can legally obtain additional FAR at the expense of providing a part of their own plot as open space. As mentioned later, the larger the plot size, the more the area of open space within a plot tends to increase (Asami and Ohtaki, 2000; Kawamura, 2010; Ohba, 1995). Therefore, the height of a building can be regarded as the result of the building owner's choice of BCR by considering their plot size and their preference for the area of open space. This means that unless neither building heights nor setbacks are directly regulated, both will exhibit a statistical distribution at the district scale.

The objective of this paper is to explicitly investigate the relationship between the variation in building heights and BCR assuming that FAR is given as constant legal value. As mentioned above, precise data regarding building heights are unavailable. Hence, it may be useful to estimate the statistical distribution of building heights from indices regarding zoning regulations. These are the substantive motivations behind the following research question: *how can we estimate the statistical distribution of building heights under zoning regulations at a district scale?* To answer this question, we investigated the relationship between the variation in BCR and building heights in downtown districts of the Tokyo metropolitan region by modelling the statistical distribution of BCR as a beta distribution. The findings are expected to provide urban planners with a theoretical basis for discussing the relationship between variations in building heights and zoning regulations (BCR and FAR) and for controlling and harmonizing the former.

ASSUMPTION AND DATA PROCESSING

In this paper, the probability density function of building heights will be derived theoretically with the following assumption: (A1) one plot has one building. This rationale can be justified as follows. According to Japanese building codes, in order to appropriately regulate the size and shape of a building by BCR and FAR in its plot, a plot basically has no more than one building, which is called the rule of one plot for one building. Thus, the number of buildings is equal to that of plots, n. This rule enables us not only to investigate the relationship between building height distribution and zoning regulations (BCR and FAR), but also to overcome data limitations regarding plot shapes. In the literature, it has been proposed that data regarding the shape of area Voronoi cells – whose generators are building polygons – overcome not only the potential lack of data availability regarding plot shapes, but also potential spatial coverage problems between buildings and plot layers (Fleischmann, et al., 2020; Hamaina et al., 2014; Usui and Asami, 2020).

In this paper, to substitute for the set of data regarding the shape of actual plots in a district, I focus on the set of data on the shape of area Voronoi cells $\{V(B_i)\}$, whose generators are building polygons, $\{B_i\}$, and the centre line of road networks, R, in a district (Usui and Asami, 2020). Area Voronoi diagrams represent the extension of ordinary Voronoi diagrams, whose generators are the set of points (Okabe et al., 2000). The generators of area Voronoi diagrams are the set of polylines or polygons that respectively correspond to the centre line of road networks, R, and the edges of building polygons. As substitutes for the set of the shape of actual plots, I use the intersection of $\{V(B_i)\}$ and urban blocks, denoted by T_i. Hereafter, T_i will be called the plot of B_i. For any B_i, the building coverage ratio of B_i, denoted by $\square_i$, is defined as the ratio of the area of B_i, (denoted by s_{Bi}), to the area of T_i, (denoted by s_i). The sources of spatial data regarding buildings, road networks and districts are *Residential Maps* by Zenrin, Co., Ltd, *Mapple 10000 Digital Data* released by Shobunsha Publications, Inc., and *Statistical GIS* provided by the Statistics Bureau, Ministry of Internal Affairs and Communications, Japan. Hereafter, where unnecessary, the suffixes of s_{Bi}, s_i and θ_i are omitted.

As the case study, two downtown districts of the Tokyo metropolitan region are selected, namely Higashi-Mukojima 5 and Ishihara 3 districts in Sumida ward. Figure 1 shows the detailed patterns of buildings and their plots in the two districts selected for the empirical case study. Whereas the northern districts in Sumida ward are marked by irregular patterns of plots, the southern districts have regular patterns because land readjustment projects were implemented after World War II. The plot patterns of the former and the latter contrast with each other.

MODELLING THE STATISTICAL DISTRIBUTION OF BCR AS A BETA DISTRIBUTION

Figure 2 shows the relative frequency distributions of BCR in the two districts. It was found that: (1) each distribution can be regarded as a unimodal distribution; (2) the mode of BCR ranges from 0.6 to 0.7; and (3) the average and standard deviation of BCR are approximately equal to 0.7 and 0.2, respectively. The maximum BCR in Higashi-Mukojima 5 and Ishihara 3 is 0.8 and 0.6, respectively. If a building is located within a fire prevention district and its structure is fireproof, this regulation is relaxed to 1. However, as shown in Figure 2, the maximum BCR is not always used up. Considering these characteristics (unimodal distribution ranging from 0 to 1), the statistical distribution of BCR is modelled as a beta distribution:

$$p(\theta) = \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1},\ \theta \in (0,1), \tag{1}$$

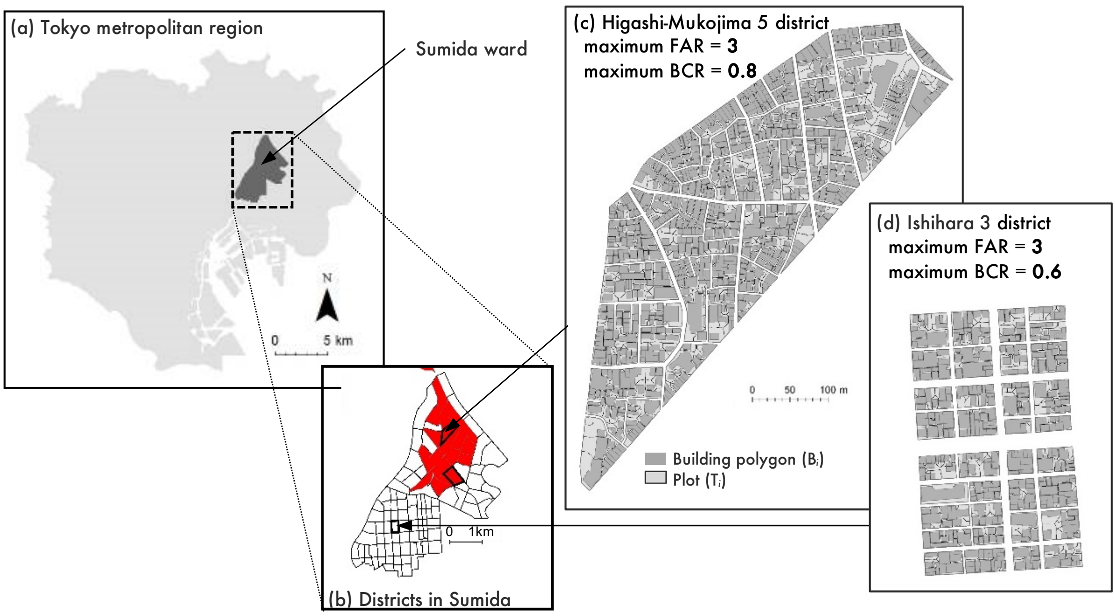

Figure 1. Districts in the downtown area of the Tokyo metropolitan region and the detailed patterns of buildings and their plots in the two districts selected for the empirical case study.

where $B(a,b)$ denotes the beta function of a and b, which can be estimated from the following equations (2) and (3):

$$E[\theta] = \frac{a}{a+b}, \qquad V[\theta] = \frac{ab}{(a+b)^2(a+b+1)}, \tag{2}$$

where $E[\theta]$ and $V[\theta]$ denote the mean and variance of BCR. By solving these equations in terms of *a* and *b*, the estimators of *a* and *b* can be obtained:

$$\hat{a} = \left\{\frac{E[\theta](1-E[\theta])}{V[\theta]} - 1\right\} E[\theta], \qquad \hat{b} = \left\{\frac{E[\theta](1-E[\theta])}{V[\theta]} - 1\right\} (1 - E[\theta]) \tag{3}$$

In Figure 2, $p(\theta)$ as given by Equation (1) is drawn as a solid curve. $E[\theta]$ and $V[\theta]$ are estimated from empirical data (see Figure 2). It was found that $p(\theta)$ fits the observed distributions of BCR. In the next section, the probability density function of building heights will be derived based on Equation (1).

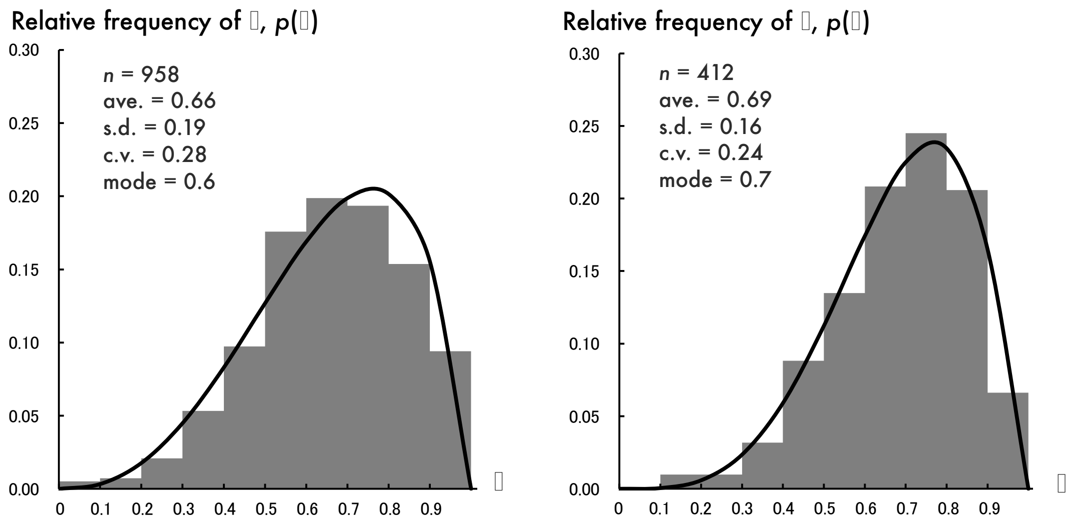

Figure 2. The relative frequency distributions of BCR in Higashi-Mukojima 5 district (left) and Ishihara 3 district (right) in Sumida ward.

THEORETICAL DERIVATION OF PROBABILITY DENSITY FUNCTION OF BUILDING HEIGHTS

In order to investigate the relationship between the variation in building heights and BCR, the probability density function of building heights can be derived based on Equations (1) and (3):

$$p(\theta) = \frac{1}{B(\hat{a},\hat{b})} \theta^{\left\{\frac{E[\theta](1-E[\theta])}{V[\theta]} - 1\right\}E[\theta]-1} (1-\theta)^{\left\{\frac{E[\theta](1-E[\theta])}{V[\theta]} - 1\right\}(1-E[\theta])-1}. \tag{4}$$

Hereafter, the average floor height and legal FAR are denoted by *u* and ϕ, respectively. Given that for any building, building height can be approximately computed from $h = \phi u / \theta$, *h* is the monotonically decreasing function of θ. Hence, the cumulative distribution function of *h* can be formulated as follows:

$$\Pr\{H \le h\} = \Pr\{\Theta \ge \theta\} = 1 - \Pr\{\Theta \le \theta\}, \tag{5}$$

where large capital indicates a stochastic variable. This is equivalent to the following:

$$\int_0^h g(H)dH = 1 - \int_0^\theta p(\Theta)d\Theta. \tag{6}$$

Thus, the probability density function of building heights, denoted by $g(h)$, can be derived as follows:

$$g(h) = -\frac{d\int_0^{\theta} p(\Theta)d\Theta}{d\theta} \cdot \frac{d\theta(h)}{dh} = p\big(\theta(h)\big)\frac{\varphi u}{h^2}$$

$$= \frac{1}{B(\hat{a},\hat{b})} \cdot \frac{\varphi u}{h^2} \cdot \left(\frac{\varphi u}{h}\right)^{\left\{\frac{E[\theta](1-E[\theta])}{V[\theta]}-1\right\}E[\theta]-1} \left(1-\frac{\varphi u}{h}\right)^{\left\{\frac{E[\theta](1-E[\theta])}{V[\theta]}-1\right\}(1-E[\theta])-1}, \qquad (7)$$

where φ*u* < *h*. This condition means that *h* has the lower limit, which is given as the product of the legal FAR and the average floor height. Figure 3 shows the probability density function of building heights, $g(h)$, (ϕ = 1.5, 2 and 3, respectively). $g(h)$ exhibits a right-skewed distribution. This variation in building heights results from the variation in BCR because *u* and ϕ are constant. This tendency becomes more pronounced as ϕ increases. Therefore, as ϕ increases, it is inevitable that the range and the variation in building heights will also increase, making it difficult to control building heights. The mean and variance of *h* can be derived as follows:

$$E[h] = \frac{B(\hat{a}-1,\hat{b})}{B(\hat{a},\hat{b})}\varphi u = \frac{\hat{a}+\hat{b}-1}{\hat{a}-1}\varphi u = \frac{E[\theta](1-E[\theta])}{E[\theta]^2(1-E[\theta])-(1+E[\theta])V[\theta]}\varphi u, \qquad (8)$$

where 0 < $V[\theta] < E[\theta]^2(1-E[\theta])/\,(1+E[\theta])$,

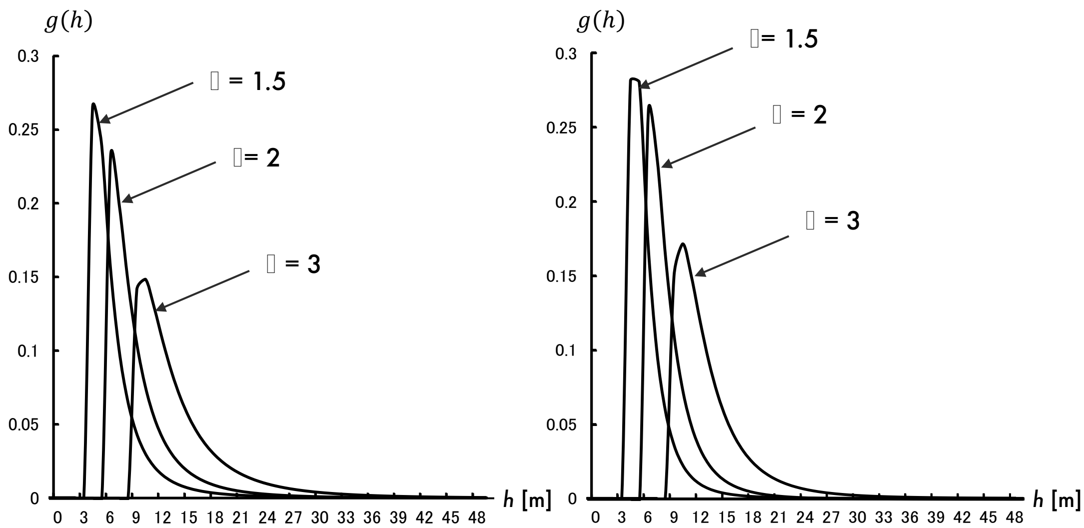

Figure 3. Probability density function of building heights in Higashi-Mukojima 5 district (left) and Ishihara 3 district (right) in Sumida ward.

$$V[h] = \frac{B(\hat{a}-2,\hat{b})B(\hat{a},\hat{b})-B(\hat{a}-1,\hat{b})^2}{B(\hat{a},\hat{b})^2}(\varphi u)^2 = \frac{\hat{b}(\hat{a}+\hat{b}-1)}{(\hat{a}-1)^2(\hat{a}-2)}(\varphi u)^2$$

$$= \frac{\{E[\theta](1-E[\theta])\}\{E[\theta](1-E[\theta])^2-(1-E[\theta])V[\theta]\}V[\theta]}{\{E[\theta]^2(1-E[\theta])-(1+E[\theta])V[\theta]\}^2\{E[\theta]^2(1-E[\theta])-(2+E[\theta])V[\theta]\}}(\varphi u)^2$$

$$= (\varphi u)^2 R(V[\theta]) \qquad (9)$$

where 0 <$V[\theta] < E[\theta]^2(1-E[\theta])/\,(2+E[\theta])$,

$$h^* = \frac{\hat{a}+\hat{b}+2}{\hat{a}+2}\varphi u = \frac{E[\theta](1-E[\theta])-V[\theta]}{\{E[\theta](1-E[\theta])-V[\theta]\}E[\theta]+2V[\theta]}\varphi u,$$

(10)

where 0 $< V[\theta] < E[\theta](1-E[\theta])$. From Equations (8) and (9), the relationship between the variations in both building height and BCR can be investigated.

To control variation in building heights, we focus here on the relationship between the standard deviation of BCR and building heights. In Figure 4, the standard deviation of *h*, $\sqrt{V[h]}$, is drawn by a solid line as the function of $V[\theta]$, while the coefficient of variation, $\sqrt{V[h]}/E[h]$, is drawn by a broken line as the function of $V[\theta]$. It can be observed that (1) $\sqrt{V[h]}$ and $E[h]$ are the monotonically increasing functions of $V[\theta]$; and (2) $\sqrt{V[h]}/E[h]$ is also the monotonically increasing function of $V[\theta]$. Hence, the greater is $V[\theta]$, the greater both $\sqrt{V[h]}$ and $E[h]$ as well. Also, from Equation (9), $\sqrt{V[h]}$ can be regarded as a linear function of □, which indicates that as legal FAR increases, the greater is $\sqrt{V[h]}$. Furthermore, for any $V[\theta]$, it can be seen that: (1) the greater is □, the greater both $\sqrt{V[h]}$ and $E[h]$ as well; and (2) $\sqrt{V[h]}/E[h]$ does not change as □ increases. Hence, to create harmonious streetscapes in terms of *h*, it is necessary to reduce $V[\theta]$ within a certain value. This is the important implication for harmonising the variation in *h*.

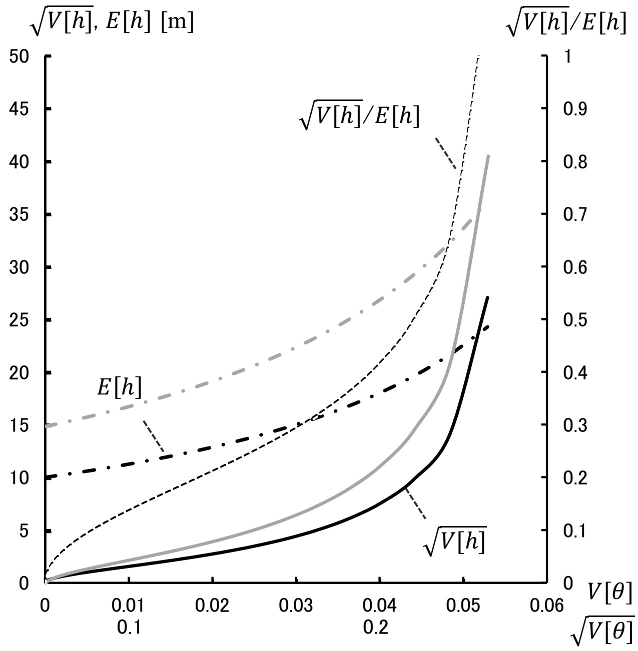

Figure 4. The relationship between $V[\theta]$ and $\sqrt{V[h]}$ (drawn by the solid line) and $E[h]$ (drawn by the dash-dotted line) as well as the relationship between $V[\theta]$ and $\sqrt{V[h]}/E[h]$ (drawn by the broken line) ($E[\theta]$ = 0.6, φ = 3 (grey lines), φ = 2 (black lines), u = 3 [m]).

To harmonise the variation in *h*, we will now try to set the criteria regarding $V[\theta]$ which can determine the range of the variation in building heights. First, from the point of view of harmonious streetscapes, the allowance value, δ, is predetermined if $\sqrt{V[h]}$ is smaller than δ, hence the building heights in a district can be regarded as harmonious. Second, we will determine the range of $V[\theta] < \theta_c$, which corresponds to δ. According to Equation (9), $\sqrt{V[h]}$ is the monotonically increasing function of $V[\theta]$. Thus, θ_c can be uniquely specified as the inverse function of δ, denoted by $\theta_c(\delta)$. Therefore, the allowance of BCR is derived as follows:

$$V[\theta] \leq \theta_c(\delta) \Leftrightarrow -\sqrt{\theta_c(\delta)} \leq \sqrt{V[\theta]} \leq \sqrt{\theta_c(\delta)}$$

$$\Leftrightarrow E[\theta] - \sqrt{\theta_c(\delta)} \le E[\theta] + \sqrt{V[\theta]} \le E[\theta] + \sqrt{\theta_c(\delta)}. \quad (11)$$

Hereafter, this domain will be called *the allowance of the freedom of choosing BCR*. If the BCR of a building satisfies Equation (11), this building contributes to harmonious streetscapes in a district. Otherwise, it does not contribute and some sort of penalty (e.g. a Pigovian tax or control by legal regulations) needs to be imposed.

For example, δ is given as a linear function of $E[h]$, $\delta = \varepsilon E[h]$, where ε corresponds to the coefficient of variation, $\sqrt{V[h]}/E[h]$. This is because the range of variations in *h* depends on the relative scale to $E[h]$. In the following case study, the values of $E[\theta]$ and ε are 0.6 and 0.1, respectively. Thus, $\sqrt{\theta_c(\delta)}$ is equal to 0.07 irrespective of the value of ϕ. Hence, the allowance of the freedom of choosing BCR ranges from 0.53 to 0.67. If the legal BCR is 0.6, the upper and lower boundaries are 0.6 and 0.53, respectively. This indicates the importance of setting the lower boundary regarding BCR in order to create harmonious building heights.

CONCLUSION

To date, the variation in BCR has prevented us from understanding the variation in building heights under zoning regulations. Furthermore, precise data regarding building heights are unavailable. In response, this study has explicitly investigated the relationship between the variation in building heights and BCR by modelling the statistical distribution of BCR as a beta distribution. Subsequently, stochastic variable transformation has been applied for a beta function to derive the probability density function of building heights. This function can contribute to understanding the relationship between the indices regarding zoning regulations (BCR and FAR) and the variation in building heights. As a practical problem, how variations in building heights can be controlled has been discussed and proposed. In future research, how to impose some sort of incentive or penalty (e.g. a Pigovian tax) should be discussed in order to create harmonious streetscapes under zoning regulations without introducing a direct regulation of building heights.

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