Efficient Two-Sided Markets with Limited Information*

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Abstract

Many important practical markets inherently involve the interaction of strategic buyers with strategic sellers. A fundamental impossibility result for such two-sided markets due to Myerson and Satterthwaite [33] establishes that even in the simplest such market, that of bilateral trade, it is impossible to design a mechanism that is individually rational, truthful, (weakly) budget balanced, and efficient. Even worse, it is known that the "second best" mechanism—the mechanism that maximizes social welfare subject to the other constraints—has to be carefully tailored to the Bayesian priors and is extremely complex.

In light of this impossibility result it is very natural to seek "simple" mechanisms that are approximately optimal, and indeed a very active line of recent work has established a broad spectrum of constant-factor approximation guarantees, which apply to settings well beyond those for which (implicit) characterizations of the optimal (second best) mechanism are known.

In this work, we go one step further and show that for many fundamental two-sided markets—e.g., bilateral trade, double auctions, and combinatorial double auctions—it is possible to design near-optimal mechanisms with provable, constant-factor approximation guarantees with just a single sample from the priors! In fact, most of our results in addition to requiring less information also improve upon the best known approximation guarantees for the respective setting.

1 Introduction

Recently, increased attention has turned to the problems that arise in two-sided markets, in which the set of agents is partitioned into *buyers* and *sellers*. In contrast to the one-sided setting (where one could say that the mechanism itself initially holds the items), in the two-sided setting the items are initially held by the sellers, who have valuations over the items they hold, and who are assumed to act rationally and strategically. The mechanism's task is now to decide which buyers and sellers should trade, and at which prices, with the goal of maximizing the *social welfare* of

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the reallocation of the goods. Two-sided markets are usually studied in a Bayesian setting: there is public knowledge of probability distributions, one for each buyer and one for each seller, from which the valuations of the buyers and sellers are drawn.

In two-sided markets, a further important requirement is *strong budget balance (SBB)*, which states that monetary transfers happen only among the agents in the market, i.e., the buyers and sellers are allowed to trade without leaving to the mechanism any share of the payments, and without the mechanism adding money to the market. A weaker version of SBB often considered in the literature is *weak budget balance (WBB)*, which only requires the mechanism not to inject money into the market. However, it is known from the work of [33] that it is generally impossible for an *individually rational (IR)*, *Bayesian incentive compatible (BIC)*, and WBB mechanism to maximise social welfare in such a market, even in the *bilateral trade* setting, i.e., when there is just one seller and one buyer and public knowledge of the distribution of agents' values. This is in sharp contrast to the celebrated optimal results available for one-sided markets [32, 43].

A recent line of research has focused on mechanisms that satisfy IR, SBB, and IC (or failing that, the weaker notion of BIC), and that reallocate the items in such a way that the expected social welfare is within some constant fraction of the optimum, where the expectation is taken over the given probability distributions of the agents' valuations and over the random choices of the mechanism. Such mechanisms circumvent the impossibility result of [33] by weakening the requirement of optimal social welfare to that of *approximately* optimal social welfare.

Important special cases of two-sided markets that have been considered with the aforementioned goal include: *bilateral trade* with one buyer and one seller [8, 24]; *double auctions* in which unit-demand buyers interact with multiple sellers that each hold a single copy of an identical item [13, 19]; and *combinatorial double auctions* in which the buyers have combinatorial (e.g., *fractionally subadditive (XOS)*) valuations and the sellers hold non-identical items [7, 14].

We improve on the existing results in two ways. First of all, we give the first two-sided market mechanisms with limited information. More concretely, we show how to obtain mechanisms for two-sided markets that satisfy IR, BB, IC, and that achieve a constant factor approximation of the optimum social welfare, even when the mechanism only knows a *single sample* from each distribution of the buyers and of the sellers. Secondly, in some cases, we are able to improve over previous bounds obtained with full knowledge of the distributions. Our work is close in spirit to the previous works on one-side mechanism design that obtain approximately optimal revenue with limited information about an existing distribution of bidders' values [12, 18, 23], and to the previous work on prophet inequalities with limited information from the distributions [2, 16, 17, 38].

1.1 Overview of the Results

This paper studies the problem of designing mechanisms for two-sided markets in the Bayesian setting with limited information: that is we consider the case in which both buyers' and sellers' valuations are private information drawn from independent, arbitrary distributions that are known through a limited number of samples. If not specified otherwise, *a single sample* is available from each distribution. We study this problem in various settings, summarized in Table 1.

1.1.1 Bilateral Trade

As a warm up, we consider the bilateral trade setting with one buyer b with valuation $v_b \sim F_b$ and a seller s with valuation $v_s \sim F_s$. We present an IR, IC, SBB mechanism that gives a 2-approximation

Setting	IR+IC	SBB	WBB	Approximation	Samples	Arrivals	Poly
bilateral trade	✓	✓		2	No/Yes	$---$	Yes
double auction matroid	✓	✓		$3+\sqrt{3}$	No/Yes	Off/Off	Yes
double auction k-uniform matroid	✓	✓		$1+3.73(1+o(1))$	No/Yes	OnRa/OnRa	Yes
double auction k-uniform matroid	✓	✓		$1+3.73(1+o(1))$	No/Yes	OnFi/OnRa	Yes
combinatorial XOS	✓		✓	3	No/Yes	Off/Off	No
combinatorial submodular/XOS	✓		✓	$O((\log\log m)^3)$	No/Yes	Off/Off	Yes
combinatorial GS	✓	✓		3	No/Yes	Off/Off	Yes
combinatorial unit demand	✓		✓	$2e$	No/Yes	OnRa/Off	Yes

Table 1: Overview of single-sample mechanisms for two-sided markets that we develop in this paper. The samples column indicates whether we need a single sample from the buyer side and seller side. The arrival column specifies the arrival order of the buyers and sellers. Off stands for offline, OnFi for online fixed order, and OnRa for online random order.

using a single sample from F_s as posted price for the agents. We also show that no deterministic IC mechanism (all IC mechanisms are posted price [13]) that uses just a single sample from F_s or just a single sample from F_b can do better. Our result achieves the same approximation bound obtained *when F_s is known* by posting a price equal to the median of F_s [8]. Our mechanism also matches with just one sample the best possible result that can be obtained by a deterministic mechanism that uses only F_s or only F_b [8]. The work of [8] also gives a randomized mechanism that achieves a $e/(e-1)$-approximation by using full knowledge of F_s. We show a IC, IR, SBB mechanism that gives a $(e/(e-1)+\epsilon)$-approximation using $\frac{16e^2}{\epsilon^2}\log\left(\frac{4}{\epsilon}\right)$ samples from F_s.

1.1.2 Double Auctions

In the double auction setting, there are n unit-demand buyers and m unit-supply sellers *with identical items*. Buyer valuations are drawn from $F_{b_1},\dots,F_{b_n}$, and seller valuations are drawn from $F_{s_1},\dots,F_{s_m}$, independently. We distinguish between offline and online mechanisms. In offline mechanisms, the agents can trade in any order. For online mechanisms, the agents can trade in online fixed order or in online random order. We also consider additional constraints on the set of buyers that can trade simultaneously in the form of downward closed set systems that captures for example matroid settings. We prove the following main theorem (informal version):

Theorem 1. *Denote by α the approximation guarantee of an offline/online one-sided IC, IR, single-sample mechanism for the intersection of a downward closed set system $\mathcal{I}$ with a m-uniform matroid. We give a two-sided single-sample IR, IC, SBB mechanism for double auctions with constraints $\mathcal{I}$ on the buyers that yields in expectation a $(1+1/(2-\sqrt{3})\cdot\alpha)\approx(1+3.73\cdot\alpha)$ approximation* *to the expected optimal social welfare. The mechanism inherits the same online/offline properties of the one-sided mechanism on the buyer side and it is online random order on the seller side.*

For general matroid settings, by applying the optimal offline truthful VCG mechanism on the buyer side, we obtain a $3+\sqrt{3} \approx 4.73$-approximate single-sample two-sided mechanism for double auctions for general matroid constraints on the buyers. This improves over the best known approximation guarantee of 6 due to Colini-Baldeschi et al. [14] for this problem obtained with full knowledge of $F_{b_1}, \ldots, F_{b_n}$ and $F_{s_1}, \ldots, F_{s_m}$!

We obtain two single-sample online mechanisms for the two-sided setting with different information requirements and for different online arrival models by using the truthful $1+O(1/\sqrt{k})$-competitive secretary algorithm for k-uniform matroids due to Kleinberg [25] or the truthful $1+O(1/\sqrt{k})$-competitive single-sample prophet inequality for k-uniform matroids due to Azar et al. [2]. The resulting approximation guarantees of $\approx 1+3.73(1+O(1/\sqrt{\min\{n,m\}}))$ improve for a large spectrum of values of n and m over the best known bound of 6 for this problem [14] obtained with full knowledge of the distributions.

Observe that any future advances on α-approximate secretary algorithms or single-sample prophet inequalities for the intersection of any matroid (partition matroid, graphical matroid, etc.) with a k-uniform matroid constraint would lead to a new $(1+3.73\cdot\alpha)$-approximate single-sample two-sided mechanism with online arrivals.

1.1.3 Combinatorial Double Auctions

We also consider combinatorial double auctions with n buyers having *combinatorial valuations* for sets of items and m unit-supply sellers with non-identical items. Buyer valuation functions $v_b : 2^{\mathcal{S}} \to \mathbb{R}_{\geqslant 0}$ are drawn from $F_{b,S}$, $S \in 2^{\mathcal{S}}$. Seller valuations are drawn from $F_{s_1}, \ldots, F_{s_m}$, independently. We specifically consider fractionally subadditive (XOS) valuations. We prove the following main theorem (informal version):

Theorem 2. *Denote by α the approximation guarantee of any one-sided IR, IC offline/online single-sample mechanism for maximizing social welfare for XOS valuations. We give a two-sided single-sample mechanism for combinatorial double auctions with XOS buyers and unit-supply sellers that is IR, IC, WBB, and provides in expectation a $(2\alpha+\mathbb{1}\{\alpha<1.5\})$ approximation. The two-sided mechanism inherits the offline/online properties of the one-sided mechanism on the buyer side and is offline on the seller side.*

This result implies a (non-computational) 3-approximation for XOS valuations via the VCG mechanism. A $O(\text{poly}(\log\log m))$-approximate poly-time mechanism in the demand oracle model for XOS valuations follows from the recent breakthrough of [1] and a 3-approximate poly-time mechanism for Gross Substitute (GS) valuations follows from the LP-based poly-time algorithm of Nisan and Segal [35] or the poly-time algorithms for convolutions of $M^{\natural}$-valuation functions [30, 31] (also see the excellent survey [36] for the latter).

In the special case of unit-demand buyers and unit supply sellers with non-identical items, we obtain a single sample IR, IC, WBB mechanism that yields a $2e \approx 5.44$-approximation that is online random order on the buyer side and offline on the seller side. The mechanism uses the one-sided online truthful e-approximate secretary matching mechanism [37].

These results compare with the IR, IC, SBB 6-approximation mechanism for XOS valuations and unit supply sellers of [14] with full knowledge of the distributions that is online on the buyer side and offline on the seller side.

Again, any future improvements in the one-sided problem (whether offline or online) will translate into two-sided results through our theorem(s).

1.2 Techniques

Our techniques are very different from those in prior work on revenue-maximizing one-sided mechanisms from samples [12, 18, 23], and also from the prophet inequalities with limited information literature [2, 16, 17, 38].

A first challenge that we encounter, and show how to solve approximately with a single sample, already occurs in the bilateral trade case. The difficulty here is to decide whether any given buyer-seller "couple" (b, s) with valuations v_b and v_s should trade. Ideally, they would trade whenever $v_b \geqslant v_s$. However, as we know from [33] we can't achieve this with a IR, IC, and BB mechanism; and a constant factor loss is unavoidable [13].

Our solution to this problem is simple (but the analysis requires some care!): Simply draw a single sample from the seller distribution and post this a price, and let the buyer and seller trade if the buyer's value is above this price and the seller's value is below. Clearly, this entails some loss, namely whenever the buyer has a higher valuation then the seller but either both are below the price (and so the buyer does not accept) or both are above the price (and so the seller does not accept). However, as we show, the loss is not too bad: posting the seller sample as a price will, in expectation, recover 1/2 of the optimal social welfare.

Our analysis of this bilateral trade case indeed applies to any fixed value v_b, and we exploit this for our double auction results. Here we show that approximately optimal solutions result from combining one-sided mechanisms that (approximately) choose an optimal set of buyers (the k highest buyers in the unconstrained case), and randomly match the tentatively selected buyers to the sellers, maxing the price that the buyers would face in the one-sided mechanism with the single sample from the respective seller's distribution.

The second—and main—challenge arises when going from single parameter to multi-parameter settings with non-identical items, because here we can't just randomly match buyers to sellers (as this would jeopardize the approximation guarantee), but if we don't just match them randomly ensuring truthfulness becomes a very tricky thing!

Our solution to this is to modify any given one-sided mechanism for the buyers by discounting the buyer's valuations for the different items by the respective seller's sample valuations. We show that selecting allocations based on this, proposing these (possibly set-wise) trades to the buyers and sellers, increasing the payments asked from the buyers as determined by the one-sided mechanism by the respective seller samples, and offering the sellers to trade at "their" sample, yields an IR, IC, and WBB mechanism. Moreover, and perhaps surprisingly, this same mechanism also ensures near-optimal social welfare!

1.3 Further Related Work

There are two important precursors to the more recent work on approximately optimal simple mechanisms for two-sided markets: The first studies the non-truthful *buyer's bid mechanism* in a double auction setting with i.i.d. buyers and i.i.d. sellers, and shows convergence to efficiency as the number of sellers and buyers grows to infinity [39–41]. The second is work on trade reduction mechanisms [3, 4, 19, 27], which starts from McAfee's truthful *trade reduction mechanism* for double auctions, which extracts a $(1 - 1/\ell)$ fraction of the maximum social welfare, where ℓ is the number of traders in the optimal solution.

A number of recent works [5, 9, 10, 15, 28, 42] has considered the related objective of optimizing the *gain from trade*, which measures the expected increase in total value that is achievable by applying the mechanism, with respect to the initial allocation to the sellers. Gain from trade is harder to approximate than social welfare, and $O(1)$ approximations of the optimal Bayesian mechanism are only possible in BIC implementations.

Goldner et al. [21] recently suggested an alternative, resource augmentation approach to gains from trades in two-sided markets, in the spirit of the celebrated result of Bulow and Klemperer [11]. They ask how many buyers (resp. sellers) need to be added into the market so that a variant of McAfee's trade reduction mechanism yields a gain from trade superior to the optimal gains from trade in the original market. As a side product they obtain a 4-approximate single-sample mechanism for gains from trade under natural conditions on the distributions.

Another related line of work considers the problem of maximizing revenue in double auction settings, either in static environments [22, 34] or in dynamic environments [6]. A variation where both buyers and sellers arrive dynamically and the mechanism can hold on to items was investigated in [20, 26].

Our work is also related to prior work which has studied how incentive compatible mechanisms can be composed in an incentive compatible manner, such as [29] in the context of one-sided markets and [19] in the context of two-sided markets.

2 Model and Definitions

Two-Sided Markets. In a two-sided market we are given a set of n buyers $\mathcal{B}$ and a set of m sellers $\mathcal{S}$. Each seller s has a single indivisible item for sale. Every seller s has a private valuation $v_s \in \mathbb{R}_{\geqslant 0}$ for the item she sells. Each buyer b has a private valuation function $v_b : 2^{\mathcal{S}} \to \mathbb{R}_{\geqslant 0}$, mapping each set of sellers to a non-negative real. We write $v_{\mathcal{B}}$ and $v_{\mathcal{S}}$ for the vector of buyer valuations and seller valuations, respectively. Buyer and seller valuations are drawn independently from distributions F_b for $b \in \mathcal{B}$ and F_s for $s \in \mathcal{S}$.

In our model, sellers have a single indivisible item for sale. We refer to such sellers as *unit supply* sellers. The valuation functions of the buyers will be constrained to come from some class of functions $\mathcal{V}$. Buyers are *unit demand* if for each buyer b and set of sellers S, $v_b(S) = \max_{s \in S}\{v_b(\{s\})\}$. Buyers have *fractionally subadditive* (or XOS) valuations if for each buyer b and every set of sellers S, $v_b(S) = \max_{a \in A_b} \sum_{s \in S} a(s)$, where A_b is a set of additive valuation functions.

We say that items are *identical* if the valuation function $v_b(S)$ of all buyers b only depends on the cardinality of the set S they receive, i.e., for all b and all $S, S' \subseteq \mathcal{S}$ with $|S| = |S'|$ we have $v_b(S) = v_b(S')$. Otherwise, items are *non-identical.*

We also allow for constraints on which buyers can trade simultaneously. We express these constraints through set systems $\mathcal{I}_{\mathcal{B}} \subseteq 2^{\mathcal{B}}$. We require these set systems $\mathcal{I}_{\mathcal{B}}$ to be downward closed. That is, whenever $X \subseteq Y$ and $Y \in \mathcal{I}_{\mathcal{B}}$, then also $X \in \mathcal{I}_{\mathcal{B}}$. Of particular importance for our work will be *matroids*, i.e., downward-closed set systems that additionally satisfy a natural exchange property. Formally: Whenever $A, B \in \mathcal{I}$ and $|S| > |B|$ then there exists $a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$. A special case are *k-uniform matroids* where $A \in \mathcal{I}$ whenever $|A| \leqslant k$.

An *allocation* is a partition of the sellers $\mathcal{S}$ into n disjoint sets $(S_1, \ldots, S_n)$, i.e., $\bigcup_i S_i \subseteq \mathcal{S}$ and $S_i \cap T_j = \varnothing$ for all $i \neq j$, with the interpretation that buyer b_i for $1 \leqslant i \leqslant n$ receives the items of the sellers in S_i. An allocation $A = (S_1, \dots, S_n)$ is *feasible* if the set of buyers $\mathcal{B}_A = \{b_i \in \mathcal{B} \mid S_i \neq \emptyset\}$ that receive a non-empty allocation is admissible (i.e., $\mathcal{B}_S \in \mathcal{I}_\mathcal{B}$). The *social welfare* of an allocation $A = (S_1, \dots, S_n)$ is given by the sum of the valuations that buyers b_i for $1 \leqslant i \leqslant n$ have for the items of the sellers in their respective sets S_i *plus* the valuations of the sellers that are not assigned to any buyer, i.e.,

$$\mathsf{SW}(A) = \mathsf{SW}(S_1, \dots, S_n) = \sum_{b_i \in \mathcal{B}} v_{b_i}(S_i) + \sum_{s \in \mathcal{S}, s \notin \bigcup_i S_i} v_s.$$

We use $\mathsf{OPT}(v_\mathcal{B}, v_\mathcal{S})$ to denote the feasible allocation that maximizes social welfare.

Mechanisms. A (direct revelation) *mechanism* $M = (x, p)$ receives bids $bid_b : 2^\mathcal{S} \to \mathbb{R}_{\geqslant 0}$ from each buyer $b \in \mathcal{B}$ and $bid_s \in \mathbb{R}_{\geqslant 0}$ from each seller $s \in \mathcal{S}$. The bids of the buyers are constrained to be consistent with the class of functions $\mathcal{V}$ of their valuations. Bids represent reported valuations, and need not be truthful. In analogy to our notation for valuations, we use $bid_\mathcal{B}$ and $bid_\mathcal{S}$ for the vector of bids of all buyers or all sellers, respectively.

A mechanism M is defined through an *allocation rule* $x : \mathcal{V}^n \times \mathbb{R}^m_{\geqslant 0} \to \times_{i=1}^n 2^\mathcal{S}$ and a *payment rule* $p : \mathcal{V}^n \times \mathbb{R}^m_{\geqslant 0} \to \mathbb{R}^{n+m}$.

The mapping from bids to feasible allocations can be randomized, in which case $x(bid_\mathcal{B}, bid_\mathcal{S})$ is a random variable. The payments can also be randomized. We interpret the vector of payments as the payments that the buyers need to make to the mechanism, and that the sellers receive from the mechanism. We use the shorthand $p_b(bid_\mathcal{B}, bid_\mathcal{S})$ and $p_s(bid_\mathcal{B}, bid_\mathcal{S})$ to refer to the payment from buyer b to the mechanism and from the mechanism to seller. We require that $p_s = 0$ if seller s keeps her item.

A mechanism is *single-sample* if the only information it is given about the two sets of distributions F_b for $b \in \mathcal{B}$ and F_s for $s \in \mathcal{S}$ is a single sample from each of these distributions.

Utilities. We assume that buyers and sellers have *quasi-linear utilities*, and that they are utility maximizers. The utility $u_b^M(v_b, (bid_\mathcal{B}, bid_\mathcal{S}))$ of buyer b with valuation function v_b in mechanism $M = (x, p)$ under bids $(bid_\mathcal{B}, bid_\mathcal{S})$ is given by her valuation for the items she receives minus payment. That is,

$$u_b^M(v_b, (bid_\mathcal{B}, bid_\mathcal{S})) = \mathbb{E}\left[v_b(x_i(bid_\mathcal{B}, bid_\mathcal{S})) - p_b(bid_\mathcal{B}, bid_\mathcal{S})\right],$$

where the expectation is over the randomness in the mechanism. The utility $u_s^M(v_s, (bid_\mathcal{B}, bid_\mathcal{S}))$ of a seller s in mechanism $M = (x, p)$ under bids $(bid_\mathcal{B}, bid_\mathcal{S})$ is the payment she receives if she sells the item and her valuation for her item otherwise. Formally,

$$u_s^M(v_s, (bid_\mathcal{B}, bid_\mathcal{S})) = \mathbb{E}\left[\mathbb{1}\{s\text{'s item is sold}\} \cdot p_s(bid_\mathcal{B}, bid_\mathcal{S}) + \mathbb{1}\{s\text{'s item is not sold}\} \cdot v_s\right],$$

where the expectation is over the randomness in the mechanism.

Goals. We seek to design mechanisms, and specifically single-sample mechanisms, with the following desirable properties:

(1) **Individual Rationality.** Mechanism $M = (x, p)$ is *individually rational (IR)* if for all $b \in \mathcal{B}$ $u_b^M(v_b, (v_\mathcal{B}, v_\mathcal{S})) \geqslant 0$ and for all $s \in \mathcal{S}$. $u_s^M(v_s, (v_\mathcal{B}, v_\mathcal{S})) \geqslant v_s$.

(2) **Incentive Compatibility.** Mechanism $M = (x, p)$ is *(dominant-strategy) incentive compatible (IC)* or *truthful* if for each buyer b and each seller s, all valuation functions v_b and v_s, all possible bids $bid_{\mathcal{B}}$ by the buyers, and all possible bids by the sellers $bid_{\mathcal{S}}$, it holds that

$$u_b^M(v_b, ((v_b, bid_{\mathcal{B}\setminus\{b\}}), bid_{\mathcal{S}})) \geqslant u_b^M(v_b, (bid_{\mathcal{B}}, bid_{\mathcal{S}})), \text{ and}$$
$$u_s^M(v_s, (bid_{\mathcal{B}}, (v_s, bid_{\mathcal{S}\setminus\{s\}}))) \geqslant u_s^M(v_s, (bid_{\mathcal{B}}, bid_{\mathcal{S}})),$$

where $bid_{\mathcal{B}\setminus\{b\}}$ denotes the set of all the buyer bids but b's and $bid_{\mathcal{S}\setminus\{s\}}$ denotes the set of all the seller bids but s's.

(3) **Budget Balance.** A truthful mechanism $M = (x, p)$ is *weakly budget balanced (WBB)* if

$$\mathbb{E}\left[\sum_{b\in\mathcal{B}} p_b(v_{\mathcal{B}}, v_{\mathcal{S}}) - \sum_{s\in\mathcal{S}} p_s(v_{\mathcal{B}}, v_{\mathcal{S}})\right] \geqslant 0,$$

and it is *strongly budget balanced (SBB)* if the above holds with equality.

(4) **Efficiency.** Finally, a truthful mechanism $M = (x, p)$ provides an *α-approximation* to the optimal social welfare, for some $\alpha \geqslant 1$, if it holds that $\alpha \cdot \mathbb{E}[\mathsf{SW}(x(v_{\mathcal{B}}, v_{\mathcal{S}}))] \geqslant \mathbb{E}[\mathsf{SW}(\mathsf{OPT}(v_{\mathcal{B}}, v_{\mathcal{S}}))]$.

Remark. Our mechanisms will actually satisfy even stronger IR and BB properties in that they will satisfy these conditions "ex post" (i.e., pointwise).

3 Warm-Up: Bilateral Trade

As a warm up we consider the important special case of bilateral trade where there is only a single seller and a single buyer. Note that in this case it's optimal to trade whenever the buyer's value exceeds that of the seller, so $\mathsf{SW}(v_b, v_s) = \max\{v_b, v_s\}$.

We show how to get a 2-approximation with just a single sample! Our mechanism, which we refer to as SamplePrice, (Algorithm 1), couldn't be simpler. It's almost trivial: Draw a sample from the seller distribution F_s and use this sample as a posted price.

Our mechanism differs from the strategies employed in prior work in that it does not choose a posted price that corresponds to the median (or other quantiles) of the seller distribution.

ALGORITHM 1: SamplePrice

```
Sample P ~ F_s
if v_B ≥ P and v_S ≤ P then
    // Let the seller and the buyer trade
    x_B = s, p_B = p_S = P
end
```

Theorem 3. *Algorithm* SamplePrice *is individually rational, truthful, strongly budget balanced and provides, in expectation over the sample, a 2-approximation to the optimal social welfare.*

Proof. It is easy to verify that SamplePrice is IR, IC, and SBB. So all we need to show is that it achieves the claimed approximation guarantee.

To this end let v_s, v_b, and P denote the random variables that correspond to the seller valuation, the buyer valuation, and the price chosen by SamplePrice; and use $\hat{v}_s, \hat{v}_b$, and $\hat{P}$ to denote specific realizations of these random variables. Let $\mathsf{SW}(P, v_s, v_b)$ denote the random variable that corresponds to the social welfare achieved by SamplePrice.

We will use an insight from [8], namely that it suffices to show the approximation ratio for any fixed buyer value $\hat{v}_b$ and truncated seller values $\tilde{v_s} = v_s \wedge \hat{v}_b$ where values of v_s above $\hat{v}_b$ are mapped to $\hat{v}_b$. This simplifies the benchmarking as under this condition it is always optimal to assign the item to the buyer. That is, we want to show that

$$2 \cdot \mathbb{E}\left[\mathsf{SW}(P, v_b, \tilde{v_s}) \mid v_b = \hat{v}_b\right] \geqslant \mathbb{E}\left[\max\{v_b, \tilde{v_s}\} \mid v_b = \hat{v}_b\right] = \hat{v}_b.$$

SamplePrice transfers the item from the seller to the buyer whenever $P \in [\tilde{v_s}, v_b]$, otherwise the seller keeps her item. So,

$$\mathbb{E}\left[\mathsf{SW}(P, v_b, \tilde{v_s}) \mid v_b = \hat{v}_b\right] = \mathbb{E}\left[\hat{v}_b \cdot \mathbb{1}\{P \in [\tilde{v_s}, \hat{v}_b]\}\right] + \mathbb{E}\left[\tilde{v_s} \cdot \mathbb{1}\{P \notin [\tilde{v_s}, \hat{v}_b]\}\right].$$

If we condition with respect to the event $v_s \geqslant \hat{v}_b$ and $v_s < \hat{v}_b$, we have:

$$\mathbb{E}\left[\mathsf{SW}(P, v_b, \tilde{v_s}) \mid v_b = \hat{v}_b\right] \geqslant \hat{v}_b \cdot \mathbb{P}\left(P \in [v_s, \hat{v}_b]\right) + \hat{v}_b \cdot \mathbb{P}\left(v_s > b\right).$$

Let $y = \mathbb{P}\left(v_s > \hat{v}_b\right) \in [0, 1]$. To complete the proof we will show that $\mathbb{P}\left(P \in [v_s, \hat{v}_b]\right) \geqslant (1-y)^2/2$. This will then show the claim as then

$$\begin{aligned}\mathbb{E}\left[\mathsf{SW}(P, v_b, \tilde{v_s}) \mid v_b = \hat{v}_b\right] &\geqslant \hat{v}_b \cdot \mathbb{P}\left(P \in [v_s, \hat{v}_b]\right) + \hat{v}_b \cdot \mathbb{P}\left(v_s > \hat{v}_b\right) \\ &\geqslant \frac{(1-y)^2}{2} \cdot \hat{v}_b + y \cdot \hat{v}_b = (1 + y^2)\frac{\hat{v}_b}{2} \geqslant \frac{\hat{v}_b}{2}.\end{aligned}$$

It remains to prove the non-trivial part of this proof, which is that $\mathbb{P}\left(P \in [v_s, \hat{v}_b]\right) \geqslant (1-y)^2/2$. Recall our notation that $F_s(x) = \mathbb{P}\left(v_s \leqslant x\right)$ for all x, and that we defined $y = \mathbb{P}\left(v_s > \hat{v}_b\right) = 1 - F_s(\hat{v}_b)$. Then with $U \sim U[0, 1]$ and $F_s^{-1}(x) = \inf\{y \in \mathbb{R} \mid F_s(y) \geqslant x\}$ the pseudo-inverse of F_s:

$$\begin{aligned}\mathbb{P}\left(P \in [v_s, \hat{v}_b]\right) &= \mathbb{E}\Big[\mathbb{E}\left[\mathbb{1}\{P \in [v_s, \hat{v}_b]\} \mid P\right]\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\left[\mathbb{1}\{P \geqslant v_s\} \mid P\right] \cdot \mathbb{1}\{P \leqslant \hat{v}_b\}\Big] \\ &= \mathbb{E}\left[F_s(P) \cdot \mathbb{1}\{P \leqslant \hat{v}_b\}\right] \\ &= \mathbb{E}\left[F_s(F_s^{-1}(U)) \cdot \mathbb{1}\left\{F_s^{-1}(U) \leqslant \hat{v}_b\right\}\right] && (1) \\ &\geqslant \mathbb{E}\left[U \cdot \mathbb{1}\left\{F_s^{-1}(U) \leqslant \hat{v}_b\right\}\right] && (2) \\ &= \mathbb{E}\left[U \cdot \mathbb{1}\{U \leqslant F_s(\hat{v}_b)\}\right] = \int_0^{1-y} t\, dt = \frac{(1-y)^2}{2}, && (3)\end{aligned}$$

where in (1) we have used the property that $F_s^{-1}(U)$ is distributed exactly like P (see, for example, [44] for a proof), in (2) that $F_s(F_s^{-1}(x)) \geqslant x$ and in (3) the fact that F_s is non-decreasing. ☐

In Appendix A we show that it's not possible to "derandomize" SamplePrice by simply posting the mean of the seller distribution.

Our next theorem shows that our analysis of SamplePrice is tight, and that's it is best possible among "deterministic" mechanisms that only use a single sample from the buyer or the seller distribution. A proof can be found in Appendix B.

Theorem 4. *Every posted-price mechanism that receives a single sample from the buyer distribution or a single sample from the seller distribution and sets a deterministic price for each sample it may receive has approximation ratio at least* 2*.*

In Appendix C we propose the SampleQuantile mechanism, which receives $\ell \geqslant 1$ samples from F_S and show the following performance guarantee.

Theorem 5. *Mechanism* SampleQuantile *is individually rational, truthful, and budget balanced. For every $\varepsilon \in (0, \frac{4}{e})$, given $\ell = \frac{16e^2}{\varepsilon^2}\log(\frac{4}{\varepsilon})$ samples, it provides in expectation the following approximation guarantee:*

$$\left(\frac{e}{e-1} + \varepsilon\right) \mathbb{E}\left[\mathsf{SW}_\ell(v_s, v_b)\right] \geqslant \mathbb{E}\left[\mathsf{SW}(\mathrm{OPT}(v_s, v_b))\right].$$

4 Double Auctions

We now move on to to more general settings with multiple unit-demand buyers and multiple unit-supply sellers with *identical items*. We present a general technique for turning truthful one-sided mechanisms into truthful two-sided mechanisms.

The one-sided mechanisms we consider are for so-called binary single-parameter problems. In such a problem, an agent i can either win or lose, and has a value v_i for winning. The set of agents that can simultaneously win is given by a set system $\mathcal{I}$. The social welfare of a feasible set $X \in \mathcal{I}$ is simply the sum $\sum_{i \in X} v_i$ of the winning agents' valuations.

Given two set systems $\mathcal{I}$ and $\mathcal{I}'$ on a ground set U, we define its intersection to be the set system that contains all sets $X \subseteq U$ such that $X \in \mathcal{I}$ and $X \in \mathcal{I}'$.

Theorem 6. *Denote by α the approximation guarantee of an offline/online one-sided IC, IR, single-sample mechanism for welfare maximization in a binary single-parameter problem whose feasible solutions correspond to the intersection of a downward-closed set system $\mathcal{I}_\mathcal{B}$ with a m-uniform matroid. Then there is a two-sided single sample IR, IC, SBB mechanism for double auctions with constraints $\mathcal{I}_\mathcal{B}$ on the buyers that yields in expectation a*

$$\left(1 + \frac{1}{2 - \sqrt{3}} \cdot \alpha\right) \approx 1 + 3.73 \cdot \alpha$$

approximation to the expected optimal social welfare. Moreover, the mechanism inherits the same online/offline properties of the one-sided mechanism on the buyer side and it is online random order on the seller side.

We first provide a formal proof of the following special case, and then argue how to generalize it.

Theorem 7. *Let $k = \min\{n, m\}$. There is a IR, IC, SBB, $1 + \frac{1}{2-\sqrt{3}}(1 + O(1/\sqrt{k}))$-approximate single-sample mechanism for unconstrained double auctions that approaches the buyers in online fixed order and the sellers in online random order.*

We give the mechanism for Theorem 7 in Section 4.1, and a proof of its properties in Section 4.2. We explain how to generalize the construction and the proof in Section 4.3.

4.1 Mechanism for Theorem 7

Our mechanism—Two-Sided Rehearsal (Algorithm 2)—runs the one-sided Rehearsal algorithm due to Azar et al. [2] to select the top $k = \min\{n, m\}$ buyers. Azar et al. [2] show that the combined value of the buyers that beat their price provides, in expectation, a $1 + O(\frac{1}{\sqrt{k}})$ approximation to the expected value of the k highest buyers.

Our twist to this mechanism is that we pair buyers b that would be selected by the Rehearsal algorithm with a random seller s, offering them to trade at a price that is the max of the respective buyer's p_{min} and the respective seller's sample v'_s.

This adds the needed component to take into account the valuations on the seller side of the market, and will serve as an insurance that any *good* trade we propose has a good chance of actually happening.

ALGORITHM 2: Two-Sided Rehearsal

```
Let P be the set of the k − 2√k largest buyer samples, together with 2√k copies of the
  (k − 2√k)-largest buyer sample
Let p_min be the smallest element of P
Fix any order on the buyers (or assume buyers arrive online)
for each b_i, in this order do
    if v_{b_i} > p_min then
        Delete from P the highest value p ∈ P such that v_{b_i} > p
        Pick uniformly at random s ∈ S, delete s from S (or assume sellers arrive online)
        Propose a trade to (b_i, s) for the price of max{v'_s, p_min}
        if b_i and s both agree then
            Make the trade at this price
        end
    end
end
```

4.2 Proof of Theorem 7

We begin by showing that our two-sided mechanism inherits IR and IC from its one-sided counterpart, and that it is strongly budget balanced.

Lemma 1. *The* Two-Sided Rehearsal *mechanism is IR, IC, and SBB.*

Proof. In Two-Sided Rehearsal no money is ever received by the mechanism itself. The only exchange of money happens between buyer-seller pairs (b_i, s) that also exchange an item. The mechanism is therefore strongly budget balanced.

It is also clear that the mechanism is individually rational as buyers and sellers would only accept trades at prices that are lower resp. higher than their respective valuations.

Furthermore, the mechanism is truthful for agents on both sides of the market. Each buyer is presented with a trading opportunity once; and the price depends only on the samples and the valuations of previously considered buyers. She can only accept or reject, but never influence it—and will therefore not profit from reporting a lower or higher value. Sellers, on the other hand, are also guaranteed to be considered only once. They, too, have no means of influencing the price they are presented with, and can only accept or reject. ☐

It remains to show the claimed approximation guarantee. Just as in the case of bilateral trade, we will do the bulk of the work for fixed buyer valuations.

Lemma 2. *Fix n and m and let α denote the approximation guarantee of the* Rehearsal *algorithm. The* Two-Sided Rehearsal *mechanism yields in expectation a*

$$\left(1 + \frac{1}{2-\sqrt{3}} \cdot \alpha\right) \approx (1 + 3.73 \cdot \alpha)$$

approximation to the expected optimal social welfare.

In what follows, we use $M = (x, p)$ to refer to the one-sided version of Rehearsal, and $M' = (x', p')$ to refer to the two-sided version. We use B' to denote the set of tentative buyers chosen by M, we use B'_+ to denote the set of buyers that end up with an item in M', and we use S_+ to denote the set of sellers that keep their item in M'. The expected social welfare achieved by M' is:

$$\mathbb{E}[\mathsf{SW}(x'(v_{\mathcal{B}}, v_{\mathcal{S}}))] = \mathbb{E}\Big[\sum_{b \in B'_+} v_b \ + \sum_{s \in S_+} v_s\Big]. \tag{4}$$

The key bit in our proof is the following lemma, which relates the performance of the one-sided mechanism to that of the two-sided mechanism.

Lemma 3. *Let B' denote the set of tentative buyers chosen by the one sided mechanism M, let B'_+ denote the set of buyers that trade in the two-sided mechanism M', and let S_+ denote the set of sellers that keep their item in the two-sided mechanism M'. Then,*

$$\mathbb{E}\Big[\sum_{b \in B'_+} v_b \ + \sum_{s \in S_+} v_s\Big] \geqslant (2 - \sqrt{3}) \cdot \mathbb{E}\Big[\sum_{b \in B'} v_b\Big].$$

Proof. In order to prove the lemma we will show that for any fixed buyer valuations $v_{\mathcal{B}}$, buyer samples $v'_{\mathcal{B}}$, and corresponding set of tentative buyers B', in expectation over the seller valuations $v_{\mathcal{S}}$, the seller samples $v'_{\mathcal{S}}$, and the randomness in the pairing of buyers and sellers,

$$\mathbb{E}\Big[\sum_{b \in B'_+} v_b + \sum_{s \in S_+} v_s \ \mid \ B', v_{\mathcal{B}}, v'_{\mathcal{B}}\Big] \geqslant (2 - \sqrt{3}) \cdot \sum_{b \in B'} v_b.$$

The actual claim then follows by taking expectation over buyer valuations $v_{\mathcal{B}}$, buyer samples $v'_{\mathcal{B}}$, and the corresponding set of tentative buyers B'.

In order to do the analysis let's fix any buyer $b \in B'$ with its value v_b and tentative payment p_b. Let's denote by s the random seller associated to b, and by v_s and v'_s two independent samples from that seller's value distribution. Note that from buyer b's perspective seller s is just a uniform random seller from $\mathcal{S}$.

The contribution of the couple (b, s) to the social welfare is v_b if there is a sale and v_s otherwise, and there is a sale when $v_b \geqslant \max\{p_b, v'_s\} \geqslant v_s$.

To analyze this contribution, we fix some constant $t \geqslant 0$ which will be set later and we define $F(t) = \mathbb{P}(v_s \leqslant t)$ as the probability that a random draw v_s from the distribution F_s of seller s chosen uniformly at random from $\mathcal{S}$ is at most t.

If $v_b \geqslant t$, then the probability to have a sale is at least

$$\begin{aligned}
&\mathbb{P}\left(\{v'_s \leqslant v_b\} \cap \{v_s \leqslant p_b \vee v'_s\} \mid t \leqslant v_b\right) \geqslant \mathbb{P}\left(\{v'_s \leqslant t\} \cap \{v_s \leqslant v'_s\}\right) \\
&\quad \geqslant \mathbb{P}\left(\{v'_s \leqslant t\} \cap \{v_s \leqslant v'_s\} \cap \{v_s \leqslant t\}\right) \\
&\quad \geqslant \mathbb{P}\left(v_s \leqslant v'_s \mid \{v'_s \leqslant t\} \cap \{v_s \leqslant t\}\right) \mathbb{P}\left(\{v_s \leqslant t\} + \cap \{v'_s \leqslant t\}\right) \geqslant \tfrac{1}{2} F(t)^2.
\end{aligned}$$

So the expected contribution is at least $\frac{1}{2}F(t)^2 v_b$.

Else, if $v_b < t$, either there is a sale (and hence the contribution is $v_b \geqslant v_s$) or there is not, and hence the contribution is v_s. So the expected contribution is at least

$$\mathbb{E}\left[v_s \mid t > v_b\right] = \mathbb{E}\left[v_s\right] = \mathbb{E}\left[v'_s\right] \geqslant \mathbb{E}\left[v'_s \mid v'_s \geqslant v_b\right] \cdot \mathbb{P}\left(v'_s \geqslant v_b\right) \geqslant (1 - F(t))v_b.$$

Putting the two cases together, we have that the contribution of buyer b and her random partner s to the expected social welfare is at least

$$v_b \cdot \max_t \min\left\{\tfrac{1}{2} F_{v'_s}(t)^2, (1 - F_{v'_s}(t))\right\} \tag{5}$$

We next show how to lower bound this term. For the sake of simplicity we present here the proof for continuous seller distributions, the argument for general distributions is given in Appendix D. For continuous seller distributions, $\frac{1}{2}F_{v'_s}(t)^2$ is continuous and increasing while $(1 - F_{v'_s}(t))$ is continuous and decreasing, hence there exists a solution $t^\star$ to

$$\tfrac{1}{2} F_{v'_s}(t)^2 = (1 - F_{v'_s}(t)). \tag{6}$$

So a lower bound to (5) is given by

$$v_b \cdot (1 - F_{v'_s}(t^\star)) = v_b \cdot \tfrac{1}{2} F_{v'_s}(t^\star)^2 = v_b \cdot (2 - \sqrt{3}).$$

One concise way to express the progress so far is:

$$\mathbb{E}\Big[\sum_{a \in (B'_+ \,\cup\, S_+) \cap \{b,s\}} v_a \;\Big|\; B', v_{\mathcal{B}}, v'_{\mathcal{B}}\Big] \geqslant (2 - \sqrt{3}) \cdot v_b, \quad \forall b \in B' \tag{7}$$

where the randomness is over the choice of the random seller s and its values v_s and v'_s. The above holds for all buyers b, so we can sum up for all buyers in B', then use linearity of expectation and the fact that $|B'| \leqslant m$ to obtain

$$\begin{aligned}
\sum_{b \in B'} (2 - \sqrt{3}) \cdot v_b &\leqslant \sum_{b \in B'} \mathbb{E}\Big[\sum_{a \in (B'_+ \,\cup\, S_+) \cap \{b,s\}} v_a \;\Big|\; B', v_{\mathcal{B}}, v'_{\mathcal{B}}\Big] \\
&\leqslant \mathbb{E}\Big[\sum_{b \in B'_+} v_b + \sum_{s \in \mathcal{S}_+} v_s \;\Big|\; B', v_{\mathcal{B}}, v'_{\mathcal{B}}\Big],
\end{aligned}$$

as claimed. □

We are now ready to prove the performance guarantee of the two-sided mechanism M'.

Proof of Lemma 2. Recall that we use B'_+ to denote the set of buyers that actually do get an item in our two-sided mechanism M', and that we use S_+ to denote those sellers that do not make any trade and keep their item. With this notation the expected social welfare achieved by our two-sided mechanism M' is

$$\mathbb{E}[\mathsf{SW}(x'(v_{\mathcal{B}}, v_{\mathcal{S}}))] = \mathbb{E}\Big[\sum_{b \in B'_+} v_b + \sum_{s \in S_+} v_s\Big].$$

For a given set of valuations $v_{\mathcal{B}}$ of the buyers, denote by $OPT_k(v_{\mathcal{B}})$ the set of buyers with the k highest values. We can upper bound the expected optimal social welfare by the optimal solution for the buyers plus all seller values

$$\mathbb{E}[\mathsf{SW}(OPT(v_{\mathcal{B}}, v_{\mathcal{S}}))] \leqslant \mathbb{E}\Big[\sum_{b \in OPT_k(v_{\mathcal{B}})} v_b\Big] + \mathbb{E}\Big[\sum_{s \in \mathcal{S}} v_s\Big]. \tag{8}$$

Recall that the one-sided mechanism M' computes a set B' of buyers whose accumulated expected values are at least $\frac{1}{\alpha}$ times the expected one-sided optimum. Hence, for the considered buyers B',

$$\mathbb{E}\Big[\sum_{b \in B'} v_b\Big] \geqslant \frac{1}{\alpha} \cdot \mathbb{E}\Big[\sum_{b \in OPT_k(v_{\mathcal{B}})} v_b\Big]. \tag{9}$$

By combining Inequality (9) with our upper bound on the expected optimal social welfare in Inequality (8), we obtain

$$\mathbb{E}[\mathsf{SW}(OPT(v_{\mathcal{B}}, v_{\mathcal{S}}))] \leqslant \alpha \cdot \mathbb{E}\Big[\sum_{b \in B'} v_b\Big] + \mathbb{E}\Big[\sum_{s \in \mathcal{S}} v_s\Big]. \tag{10}$$

First consider the second term on the right hand side of Inequality (10). In our two-sided mechanism M' sellers trade only if they are matched to a buyer with higher valuation. Therefore, we can replace as follows:

$$\mathbb{E}[\mathsf{SW}(OPT(v_{\mathcal{B}}, v_{\mathcal{S}}))] \leqslant \alpha \cdot \mathbb{E}\Big[\sum_{b \in B'} v_b\Big] + \mathbb{E}\Big[\sum_{s \in S_+} v_s + \sum_{b \in B'_+} v_b\Big].$$

Now consider the first term on the right hand side of Inequality (10). Our two-sided mechanism M' does not make trades for each buyer in B'. Despite the fact that generally $B' \neq B'_+$, as we show in Lemma 3, in expectation, $(2-\sqrt{3}) \cdot \mathbb{E}\left[\sum_{b \in B'} v_b\right]$ is a lower bound on our mechanism's social welfare. Using this we obtain

$$\mathbb{E}[\mathsf{SW}(OPT(v_{\mathcal{B}}, v_{\mathcal{S}}))] \leqslant \frac{\alpha}{2-\sqrt{3}} \cdot \mathbb{E}\Big[\sum_{b \in B'_+} v_b + \sum_{s \in S_+} v_s\Big] + \mathbb{E}\Big[\sum_{s \in S_+} v_s + \sum_{b \in B'_+} v_b\Big].$$

All in all, we get:

$$\mathbb{E}[\mathsf{SW}(OPT(v_{\mathcal{B}}, v_{\mathcal{S}}))] \leqslant \left(1 + \frac{\alpha}{2-\sqrt{3}}\right) \cdot \mathbb{E}\Big[\sum_{b \in B'_+} v_b + \sum_{s \in S_+} v_s\Big],$$

as claimed. □

4.3 Proof of Theorem 6

For the more general result in Theorem 6 we run the given one-sided mechanism on the intersection of the given feasibility constraint $(\mathcal{B}, \mathcal{I}_\mathcal{B})$ with a m-uniform matroid. This gives a set of tentative buyers B' along with tentative buyer prices $p_\mathcal{B}$. We can then randomly match the tentative buyers to sellers, offering buyer-seller pairs (b, s) to trade at price $\max\{p_b, v'_s\}$.

The proof that the resulting mechanism is IR, IC, and SBB is basically identical to that of Lemma 1. They key is that truthfulness of the one-sided mechanism ensures that tentative buyers want an opportunity to trade, and cannot manipulate the price they face for this opportunity.

For the performance analysis we claim that Lemma 2 applies more generally with α being the approximation guarantee of the one-sided mechanism. In fact, the only change to the above proof that is required for this generalization is to redefine OPT_k as the optimal one-sided solution containing at most $k = \min\{n, m\}$ buyers.

5 Combinatorial Double Auctions

Up to this point, our results were for unit-demand buyers and unit-supply sellers with identical items. This means that we have focused on so called *single-parameter* settings. In this section, we give up on this assumption and turn towards *multi-parameter* versions of our techniques, which implies a whole set of complications that our methods need to handle in addition.

Theorem 8. *Denote by α the approximation guarantee of any one-sided IR, IC offline/online single-sample mechanism M_α for maximizing social welfare for XOS valuations. We give a two-sided single-sample mechanism for combinatorial double auctions with XOS buyers and unit-supply sellers that is IR, IC, WBB, and provides in expectation a $(2\alpha + \mathbb{1}\{\alpha < 1.5\})$ approximation. The two-sided mechanism inherits the offline/online properties of the one-sided mechanism on the buyer side and is offline on the seller side.*

We describe the mechanism (resp. reduction) that achieves the properties claimed in Theorem 8 in Section 5.1, and establish that it actually achieves these properties in Section 5.2.

5.1 The Mechanism

The basic idea behind our mechanism 2XOS (Algorithm 3), is to run the given truthful one-sided mechanism M_α on *discounted buyer valuations* and on *a subset of the sellers*. Note that the problem can be viewed as finding a hypermatching in a bipartite hypergraph $G = (\mathcal{B} \cup \mathcal{S}, E, v_\mathcal{B})$ with hyperedge set E defined as all tuples (b, S) s.t. $b \in \mathcal{B}$, $S \subseteq \mathcal{S}$.

First, given valuations $v_\mathcal{S}$ and samples $v'_\mathcal{S}$ for each seller, we determine a subset of the sellers $\hat{S}$ as follows. For each $s \in \mathcal{S}$ we put s in $\hat{S}$ if $v_s \leqslant v'_s$. Otherwise, we will drop s from our considerations. Next we determine discounted valuations. For a given buyer b and a given set of sellers $S \subseteq \mathcal{S}$ let $a_{b,S}$ denote the additive supporting function of buyer b for set S. We define the discounted valuation that buyer b has for the set of sellers $S \subseteq \hat{S}$ as:

$$\hat{v}_b(S) = \sum_{s \in \bar{S}} (a_{b,\bar{S}}(s) - v'_s), \text{ where } \bar{S} = \underset{S^\star \subseteq S \cap \hat{S}}{\operatorname{argmax}} \left\{ v_b(S^\star) - \sum_{s \in S^\star} v'_s \right\}.$$

We note that adjusting the valuations like this retains the XOS property of the original valuations. A proof can be found in Appendix E. The same holds for the gross substitutes class considered in Table 1: after the adjustment, GS valuations remain GS.

Then, we run the one-sided mechanism M_α on the resulting hypergraph $\hat{G} = (\mathcal{B} \cup \hat{\mathcal{S}}, \hat{E}, \hat{v}_\mathcal{B})$ consisting of all buyers, only the sellers in $\hat{S}$, and hyperedge valuations $\hat{v}_\mathcal{B}$. This will lead to an allocation $S_1, \ldots, S_n$ and payments $p_b^{M_\alpha}(S_i)$ for each $b_i \in \mathcal{B}$.

Afterwards, we assign sets $S_1, \ldots, S_n$ to buyer $b_1, \ldots, b_n$ increasing buyer b_i's payment relative to the payment in the one-sided mechanism by the sum of the samples v'_s for $s \in S_i$ and pay each seller $s \in \hat{S}$ whose item has been sold the respective sample v'_s.

The construction given in our mechanism is stated in the value-oracle model, and direct computation of the adjusted valuations would be inefficient. We provide a discussion on how to implement the mechanism efficiently in Section 5.3. For purposes of our analysis, we assume that the one-sided mechanism M_α always assigns each buyer an inclusion-minimal set of items giving the according buyer at least the same utility (this can, e.g., be ensured by employing a simple type of tie-breaking which favors small sets over larger ones).

ALGORITHM 3: 2XOS

```
Set Ŝ = ∅, Ê = ∅
for all s ∈ S do
    Propose to s a price of v'_s
    if s accepts then
        Set Ŝ = Ŝ ∪ {s}
    end
end
for all (b, S) ∈ B × 2^Ŝ do
    v̂_b(S) = Σ_{s∈S̄} (a_{b,S̄}(s) − v'_s),  where S̄ = argmax_{S*⊆S∩Ŝ} {v_b(S*) − Σ_{s∈S*} v'_s}
    Ê = Ê ∪ {(b, S)}
end
Let A be the assignment on Ĝ induced by running M_α on the hypergraph
  Ĝ = (B ∪ Ŝ, Ê, v̂_B), with hyperedge weights v̂_B, presenting buyers to M_α according to its
  input requirements (e.g., offline or in random order)
for all (b, S) ∈ A do
    b pays price p_(b,S) = p_b^{M_α}(S) + Σ_{s∈S} v'_s, where p_b^{M_α}(S) is the price charged to b by M_α
    b gets assigned the items in S
    for each s ∈ S do
        s receives a payment of v'_s
    end
end
```

5.2 Proof of Theorem 8

We start by establishing the individual rationality, truthfulness, and budget balance properties of the two-sided mechanism claimed in Theorem 8.

Lemma 4. *Given that an IR and IC one-sided mechanism M_α is used, the two-sided mechanism* 2XOS *is IC, IR, and WBB.*

Proof. Let's fix any realization of the valuations. We start by showing truthfulness. Fix a seller $s \in \mathcal{S}$: the only interaction s has with the algorithm is by accepting or rejecting the posted price v'_s, which is independent of v_s, in exchange for his item, which is clearly truthful and individually rational.

Fixing a buyer b: the algorithm will ask about his valuation and modify it to

$$\hat{v}_b(S) = \sum_{s \in \bar{S}} (a_{b,\bar{S}}(s) - v'_s), \text{ where } \bar{S} = \operatorname*{argmax}_{S^\star \subseteq S \cap \hat{S}} \left\{ v_b(S^\star) - \sum_{s \in S^\star} v'_s \right\},$$

reflecting that *any* item s will cost at least v'_s. This is done because M_α only involves items, not sellers, so the extra v'_s is charged afterwards.

$$\operatorname*{argmax}_{S' \text{ is available}} \{\hat{v}_b(S') - p_b^{M_\alpha}(S') | \hat{v}_b(S') \geqslant p_b^{M_\alpha}(S')\}$$
$$= \operatorname*{argmax}_{S' \text{ is available}} \left\{ v_b(S') - \sum_{s \in S'} v'_s - p_b^{M_\alpha}(S') \mid v_b(S') \geqslant \sum_{s \in S'} v'_s + p_b^{M_\alpha}(S') \right\},$$

with the left hand side reflecting that M_α is truthful and the right hand side being exactly what buyer b is trying to maximize.

Given that no trade is generated if for all sets $v_b(S') < \sum_{s \in S'} v'_s$, the mechanism is also individually rational. For any trade, the seller's price is v'_s and the buyer's $v'_s + p_b^{M_\alpha}(s) \geqslant v'_s$, so the mechanism is budget balanced. ☐

We now give the proof of the approximation ratio in Theorem 8.

Lemma 5. *Given that an α-approximate one-sided mechanism M_α is used, the two-sided mechanism* 2XOS *is $(2\alpha + \mathbb{1}\{\alpha < 1.5\})$-approximate.*

Proof. Recall that the adjusted buyer valuations in the graph $\hat{G}$ were defined as

$$\hat{v}_b(S) = \sum_{s \in \bar{S}} (a_{b,\bar{S}}(s) - v'_s), \text{ where } \bar{S} = \operatorname*{argmax}_{S^* \subseteq S \cap \hat{S}} \left\{ v_b(S^\star) - \sum_{s \in S^\star} v'_s \right\}$$

Fix a pair (b, S) of buyer and set of items. Also fix a realization $v_{\mathcal{B}}$ of buyers' valuations. Then in expectation over the sellers' valuations and their samples, it holds since $s \in \hat{S}$ if and only if $v_s \leqslant v'_s$:

$$\mathbb{E}[v'_s | s \in \hat{S}] = \mathbb{E}[\max\{v_s, v'_s\}] \leqslant \mathbb{E}[2v_s] \tag{11}$$

Note that for every pair (b, S'), and every realization of buyer and seller valuations and samples implies an according maximizing set in the definition of $\hat{v}_b(S')$, which we will denote as $\bar{S}(S')$. With this, we show

$$\mathbb{E}\left[\hat{v}_b(S)\right] = \mathbb{E}\left[\sum_{s \in \bar{S}(S)} \left(a_{b,\bar{S}(S)}(s) - v'_s \right) \right]$$

$$
\begin{aligned}
&\geqslant \mathbb{E}\left[\sum_{s\in S\cap\hat{S}} \left(a_{b,S}(s) - v'_s\right)_+\right] \\
&= \sum_{s\in S\cap\hat{S}} \mathbb{E}\left[\left(a_{b,S}(s) - v'_s\right)_+\right] \\
&= \sum_{s\in S} \mathbb{E}\left[\left(a_{b,S}(s) - v'_s\right)_+ | s \in \hat{S}\right] \mathbb{P}\left(s \in \hat{S}\right) \\
&\geqslant \frac{1}{2}\sum_{s\in S} \mathbb{E}\left[a_{b,S}(s) - v'_s | s \in \hat{S}\right] \\
&\geqslant \frac{1}{2}\sum_{s\in S} \mathbb{E}\left[a_{b,S}(s) - 2v_s\right] \\
&= \frac{1}{2}\mathbb{E}\left[\sum_{s\in S} a_{b,S}(s)\right] - \mathbb{E}\left[\sum_{s\in S} v_s\right] \\
&= \frac{1}{2}v_b(S) - \mathbb{E}\left[\sum_{s\in S} v_s\right].
\end{aligned}
$$

The first inequality follows from the fact that since $\bar{S}$ maximizes $v_b(S^*) - \sum_{s\in S^*} v'_s$, always choosing those $s \in S \cap \hat{S}$ for which $a_{b,S}(s) - v'_s \geqslant 0$ can only perform worse. Basic transformations and the fact that every $s \in \mathcal{S}$ is included in $\hat{S}$ with probability at least $\frac{1}{2}$ result in the second inequality, at which point we simply plug in Equation 11.

For the social welfare induced by a maximum-welfare assignment OPT in graph G, we have

$$
SW_{OPT} = \sum_{(b,S')\in OPT} v_b(S') + \sum_{s\in S\setminus OPT} v_s \quad .
$$

Note that in our mechanism, each buyer will only be assigned a valuation-maximizing, inclusion-minimal set $\bar{S}$ from the definition of $\hat{v}_b$. Denote by $OPT_{1-sided}$ an optimal assignment (hyper-matching) in the original graph G, i.e., an optimal solution to the one-sided problem with original valuations v_b. Then, for the social welfare induced by our assignment A over $\hat{G}$, it holds

$$
\begin{aligned}
\mathbb{E}\left[SW_A\right] &= \mathbb{E}\left[\sum_{(b,\bar{S})\in A} v_b(\bar{S}) + \sum_{s\in\mathcal{S}\setminus A} v_s\right] \\
&\geqslant \mathbb{E}\left[\sum_{(b,\bar{S})\in A} \hat{v}_b(\bar{S}) + \sum_{s\in\mathcal{S}} v_s\right] \\
&\geqslant \mathbb{E}\left[\sum_{(b,S)\in OPT_{1-sided}} \frac{1}{\alpha}\hat{v}_b(S) + \sum_{s\in\mathcal{S}} v_s\right] \\
&\geqslant \mathbb{E}\left[\sum_{(b,S)\in OPT_{1-sided}} \frac{1}{\alpha}\left(\frac{1}{2}v_b(S) - \sum_{s\in S} v_s\right) + \sum_{s\in\mathcal{S}} v_s\right] \\
&= \mathbb{E}\left[\sum_{(b,S)\in OPT_{1-sided}} \frac{1}{2\alpha}v_b(S) - \sum_{s\in OPT_{1-sided}} \frac{1}{\alpha}v_s + \sum_{s\in\mathcal{S}} v_s\right]
\end{aligned}
$$

$$\geqslant \mathbb{E}\left[\sum_{(b,S)\in OPT_{1-sided}} \frac{1}{2\alpha} v_b(S) + \left(1 - \frac{1}{\alpha}\right) \sum_{s\in\mathcal{S}} v_s\right]$$
$$\geqslant \mathbb{E}\left[\sum_{(b,S)\in OPT} \frac{1}{2\alpha} v_b(S) + \left(1 - \frac{1}{\alpha}\right) \sum_{s\in\mathcal{S}} v_s\right] .$$

Here, the first inequality stems from the fact that the $\hat{v}_b$ are already discounted by $\sum_{s\in\bar{S}} v'_s \geqslant v_s$. Since M_α is an α-approximation to the optimum in $\hat{G}$, also the second inequality also holds true. Finally, we plug in the lower bound to $\hat{v}_b$ proven above. The expectation here is taken over all realizations of buyer valuations, seller valuations, and samples of seller valuations.

We can, for certain, just conclude that $\mathbb{E}\left[SW_{OPT}\right] = \mathbb{E}\left[\sum_{(b,S)\in OPT} v_b(S) + \sum_{s\in\mathcal{S}\setminus OPT} v_s\right] \leqslant 2\alpha\mathbb{E}\left[SW_A\right] + \mathbb{E}\left[SW_A\right]$. However, assuming that $\frac{1}{2\alpha} \leqslant 1 - \frac{1}{\alpha}$, the last of above lower bounds to $\mathbb{E}\left[SW_A\right]$ also yields

$$\mathbb{E}\left[SW_A\right] \geqslant \mathbb{E}\left[\frac{1}{2\alpha}\left(\sum_{(b,S)\in OPT} v_b(S) + \sum_{s\in\mathcal{S}} v_s\right)\right] \geqslant \frac{1}{2\alpha}\mathbb{E}\left[OPT\right] .$$

Since $\frac{1}{2\alpha} \leqslant 1 - \frac{1}{\alpha}$ is equivalent to $\alpha \geqslant \frac{3}{2}$, we get that our mechanism is a $(2\alpha + \mathbb{1}\{\alpha < 1.5\})$-approximation. ☐

5.3 Computational Aspects

We briefly give some remarks regarding the computational complexity of our mechanism. The adjusted valuations $\hat{v}_{\mathcal{B}}$ are stated with an argmax over subsets of S, which—depending on the computational model—may not be an efficient operation.

An alternative, which works for GS, is to for all $S \subseteq \mathcal{S}$ define $\bar{v}_b(S) = \left(\sum_{s\in S\cap\hat{S}} \left(a_{b,S}(s) - v'_s\right)\right)_+$. As before it can be shown that if the original valuations are GS, then the modified valuations are GS as well. A difference between $\bar{v}_{\mathcal{B}}$ and $\hat{v}_{\mathcal{B}}$ is that the former need not be monotone, but monotonicity is not required by poly-time algorithms for GS valuations (see, e.g., [36]). Moreover, the approximation guarantee of the one-sided mechanism run on $\bar{v}_{\mathcal{B}}$ also applies if the resulting assignment is evaluated with the original adjusted valuations $\hat{v}_{\mathcal{B}}$ and the benchmark is the optimal allocation under the original adjusted valuations $\hat{v}_{\mathcal{B}}$, which is the only property of the one-sided mechanism that we used in our proof above (see Appendix F).

In addition, note that while our mechanism is stated for the case of value queries, it can be formulated also for demand queries—as required by poly-time mechanisms for XOS valuations such as [1]—by adjusting the prices proposed instead of the buyer valuations. Here, it is simply necessary to increase the M_α-prices of any $S \subseteq \mathcal{S} \cap \hat{S}$ by $\sum_{s\in S} v'_s$. This does not reflect the *capping* of our adjusted valuations to a minimum contribution of 0 for each seller, but—having the same effect—buyers will never demand the according items.

Finally, observe that if we are allowed to use demand queries (as is the poly-time mechanisms for XOS valuations), then we can also efficiently implement value queries to the adjusted valuations $\hat{v}_{\mathcal{B}}$: First issue a demand query to find the set $S^\star$ in the argmax, and then issue a value query to obtain the value $v_b(S^\star)$ of the corresponding set.

6 Conclusion and Further Directions

We have initiated the study of simple and efficient mechanisms for two-sided markets that use only limited information from the priors, in most cases just a single sample from each distribution. This line of research is of specific relevance for two-sided markets since efficient mechanisms with the desired requirements are only possible in the Bayesian setting and, moreover, the optimal Bayesian mechanism is complicated and known only for restricted cases. Our results are very general, and in several cases even improve on the best known approximation guarantee with perfect knowledge of the distributions. Our results can be extended further by identifying new efficient single-sample mechanisms for the one-sided versions of the problems. Finally, we leave open the problem of devising single sample mechanisms for the challenging problem of optimizing the *gain from trade* in two-sided markets.

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A Counterexample for Posting the Mean of the Seller Distribution

We provide an example that shows that it is not possible to "derandomize" the SamplePrice mechanism, and simply use the mean of the seller distribution as a posted price.

Example 1. *Suppose the buyer deterministically has a very high value* $h \gg 2$*. The seller has a value of* 1 *with probability* ϵ *and a value of* 2 *with probability* $1-\epsilon$*. The mean of the seller distribution is* $2-\epsilon$*, which is just below her high value of* 2*. The social welfare that results from price* $2-\epsilon$ *is* $\epsilon h + (1-\epsilon)2$*, while the optimal social welfare is* h*.*

B Proof of Theorem 4

We give a proof of the lower bound of 2 for "deterministic" sample-based two-sided mechanisms for bilateral trade that are given a single sample from either of the two distributions, and for any given sample that they see post a fixed price.

Proof of Theorem 4. Let v_b and v_s denote the random variables corresponding to the seller and the buyer valuations. We first consider mechanisms that receive a single sample from the seller distribution. Let $p(s)$ be the price posted by the mechanism upon receiving sample s. We distinguish two cases:

Case 1: $\exists\ \hat{s}$ such that $p(\hat{s}) \geqslant 2 \cdot \hat{s}$: In this case consider a seller whose value is exactly $\hat{s}$ and a buyer whose value is exactly $(2-\epsilon)\cdot\hat{s}$ for some $\epsilon > 0$. Clearly, the optimal welfare is $(2-\epsilon)\cdot\hat{s}$. However, in the posted-price mechanism seller and buyer do not trade, which results in social welfare $\hat{s}$ giving an approximation ratio of $2-\epsilon$.

Case 2: $\nexists\ \hat{s}$ such that $p(\hat{s}) \geqslant 2 \cdot \hat{s}$: Consider a seller whose valuation v_s takes a uniformly random value in $\{1, 1/2, 1/2^2, \ldots, 1/2^k\}$ and a buyer with fixed value $v_b = 1$. Clearly, the optimal welfare is 1. Notice that the probability of a trade happening is:

$$\begin{aligned}
\mathbb{P}\left(P(s) \geqslant v_s\right) &= \sum_{i=0}^{k} \mathbb{P}\left(p\left(\frac{1}{2^i}\right) \geqslant v_s\right) \cdot \mathbb{P}\left(s = \frac{1}{2^i}\right) \\
&= \sum_{i=0}^{k} \mathbb{P}\left(p\left(\frac{1}{2^i}\right) \geqslant v_s\right) \cdot \frac{1}{k+1} \\
&\leqslant \sum_{i=0}^{k} \frac{k-i}{k+1} \cdot \frac{1}{k+1} \\
&= \frac{k}{2\cdot(k+1)}
\end{aligned}$$

where we used that $p(1/2^i) < 1/2^{i-1}$ therefore the price posted does not reach the next possible valuation. Clearly, this converges to a 2-approximation as k grows to infinity.

It remains to show the claim for posted-price mechanisms that receive a single sample from the buyer's distribution. As before, let $p(b)$ denote the price posted by the mechanism upon receiving sample b.

Case 1: $\exists\, \hat{b}$ such that $p(\hat{b}) \leqslant \hat{b}/2$: In this case, consider a buyer with value $\hat{b}$ and a seller with value $\hat{b}(1+\epsilon)/2$ for any $\epsilon > 0$. The optimal welfare is clearly $\hat{b}$, but because the posted price is smaller than the seller's value the obtained welfare is a $2/(1+\epsilon)$ approximation.

Case 2: $\exists\, \hat{b}$ such that $p(\hat{b}) > \hat{b}$: in this case if the buyer has value $\hat{b}$ no trade will be made, no matter how small the value of the seller, leading to an unbounded approximation ratio.

Case 3: $\forall\, \hat{b}$ we have that $\hat{b}/2 < p(\hat{b}) \leqslant \hat{b}$: Consider an instance where the buyer's valuation is ϵ or $z \gg 3\epsilon$ with equal probability, while the seller always has value 3ϵ. Clearly, the expected optimal welfare is at least $z/2$. For the mechanism to generate a trade, the sample *and* the real value of the buyer need to be z, by the condition on $p(\hat{b})$. The welfare of the mechanism is $z/4 + 3\epsilon \cdot 3/4$, leading to a 4-approximation as $\epsilon \to 0$. ☐

C Bilateral Trade with Many Samples

We devise a mechanism for simulating the performance of RandomQuantile of Blumrosen and Dobzinski [8] using only access to multiple samples from F_s. As in [8], the analysis works for any distributions F_s and F_b. However, for notational convenience we assume that the probability density function f_s exists as well. We do this so that there is a one to one correspondence between the the value of the seller v_s and its quantile. Formally, we want that the following equation has a solution v_s, for all x:

$$\mathbb{P}\left(F_S(v_s) \leqslant x\right) = x \rightarrow q(x) = v_s.$$

This may not be true for distributions with point masses, but we can handle it using a standard smoothing trick, as used by Rubinstein et al. [38]. In a nutshell, we assume that we draw from a joint probability distribution $F_s, U[0,1]$, where the second coordinate is used for *tie breaking*. This induces a strict ordering on the samples, as the point masses are ordered by the uniform draw and the above equation always has a solution. Notice that sometimes there may be more than one solution, but this is not an issue: the probability that we observe two of them in the same set of samples is 0.

C.1 The SampleQuantile algorithm

The RandomQuantile mechanism draws a quantile z from a carefully chosen distribution, and posts the price p that corresponds to this quantile.

Our mechanism, tries to guess the price p that corresponds to quantile z using the empirical order statistics of the samples drawn. The key point is that it always tries to slightly *underestimate* the quantile of p. This is important: overestimating p will make the deal less desirable to the buyer, while underestimating to the seller. Since RandomQuantile assumes nothing about the buyer, it is safer to only occur extra losses due to uncertainty from the seller's side.

Formally, our mechanism, which we refer to as SampleQuantile has parameters $n \geqslant 0$, $1/e > \delta > 0$, and works as follows:

1. Sample $z \in [1/e, 1]$ with CDF $\ln(e \cdot x)$.
2. Draw ℓ samples from F_s.
3. Sort the samples in increasing order and choose the $(z - \frac{\delta}{2e}) \cdot \ell$-th one. Call that sample p.

4. Post price p and allow the agents to trade.

Notational remark: Above and in the following we will consider the ordinal of the quantile rounded to the nearest smaller integer.

C.2 Analysis of SampleQuantile

We will do a good chunk of our analysis in quantile space, and in particular exploit that the the *quantiles* of the randomly drawn samples from F_s follow the uniform distribution.

We will need the following concentration result:

Lemma 6. *Let $X_{(c\cdot\ell)}$ be the $c\cdot\ell$-th order statistic out of ℓ samples drawn independently from $U[0,1]$.*

$$\mathbb{P}\left(|X_{(c\cdot\ell)} - c| \geqslant \delta\right) \leqslant 2\cdot e^{-2\ell\delta^2}. \tag{12}$$

Proof. We have that:

$$\begin{aligned}\mathbb{P}\left(X_{(c\cdot\ell)} \leqslant c-\delta\right) &= \mathbb{P}\left(\text{more than } c\cdot\ell \text{ samples picked} \leqslant c-\delta\right)\\ &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{\ell} Y_i \geqslant c\right)\\ &= \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{\ell} Y_i - (c-\delta) \geqslant \delta\right)\\ &\leqslant e^{-2\ell\delta^2},\end{aligned}$$

where the inequality follows from the Chernoff bound on the ℓ, i.i.d. Bernoulli random variables Y_i with expectation $c-\delta$, indicating that the corresponding sample was smaller than $c-\delta$. Similarly:

$$\begin{aligned}\mathbb{P}\left(X_{(c\cdot\ell)} \geqslant c+\delta\right) &= \mathbb{P}\left(\text{less than } c\cdot\ell \text{ samples picked} \leqslant c+\delta\right)\\ &\leqslant e^{-2\ell\delta^2}.\end{aligned}$$

Combining the two inequalities gives us the result. ☐

Let $\mathsf{SW}(p)$ denote the social welfare achieved by posting price p. The second main ingredient in our proof will be the following lemma, which asserts that if two prices are close in quantile space, then they also achieve similar expected social welfare.

Lemma 7. *For any $p \leqslant p^\star$ such that*

$$F_s(p) \geqslant F_s(p^\star)(1-\delta), \text{ for some } \delta \in (0,1), \tag{13}$$

then,

$$\mathbb{E}\left[\mathsf{SW}(p)\right] \geqslant \mathbb{E}\left[\mathsf{SW}(p^\star)\right](1-2\delta) \tag{14}$$

Proof. Let us use the shorthand v_b and v_s to denote the random variables that represent the buyer valuation and the seller valuation, respectively.

Since $p \leqslant p^\star$, the only source of inefficiency for price p is that the seller might not accept to trade because the price is too low. This is only $1-\delta$ times less likely to happen than when $p^\star$ is used. Specifically, it happens when $p^\star \geqslant v_s > p$, but this happens rarely, and generates negligible welfare from the buyer's side compared to when $v_s \geqslant p^\star$. Notice that if the buyer accepts $p^\star$, then clearly she would accept p as well. Therefore, assuming the p has been properly selected, the expected social welfare generated by posting price p is:

$$\begin{aligned}
\mathbb{E}\left[\mathsf{SW}(p)\right] &= \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s > p^\star\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{p^\star \geqslant v_s > p\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s \leqslant p \wedge v_b < p\}\right] \\
&\quad + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_s \leqslant p \wedge p^\star > v_b \geqslant p\}\right] + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_s \leqslant p \wedge v_b \geqslant p^\star\}\right] && (15)\\
&= \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s > p^\star\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{p^\star \geqslant v_s > p\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s \leqslant p \wedge v_b < p\}\right] \\
&\quad + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_s \leqslant p \wedge p^\star > v_b \geqslant p\}\right] + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_b \geqslant p^\star\}\right] \mathbb{P}\left(v_s \leqslant p\right) && (16)\\
&\geqslant \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s > p^\star\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{p^\star \geqslant v_s > p\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s \leqslant p \wedge v_b < p\}\right] \\
&\quad + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_s \leqslant p \wedge p^\star > v_b \geqslant p\}\right] + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_b \geqslant p^\star\}\right] \mathbb{P}\left(v_s \leqslant p^\star\right)(1-\delta) && (17)\\
&\geqslant \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s > p^\star\}\right] \\
&\quad + \mathbb{E}\left[v_s \cdot \mathbb{1}\{p^\star \geqslant v_s > p\}\right] + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_b \geqslant p^\star\}\right] \mathbb{P}\left(v_s \leqslant p^\star\right)\delta \\
&\quad + \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s \leqslant p \wedge v_b < p\}\right] + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_s \leqslant p \wedge p^\star > v_b \geqslant p\}\right] \\
&\quad + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_b \geqslant p^\star\}\right] \mathbb{P}\left(v_s \leqslant p^\star\right)(1-2\delta) && (18)\\
&\geqslant \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s > p^\star\}\right] \\
&\quad + \mathbb{E}\left[v_s \cdot \mathbb{1}\{p^\star \geqslant v_s > p\}\right] \mathbb{P}\left(v_b < p^\star\right) + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_b \geqslant p^\star\}\right] \mathbb{P}\left(p^\star \geqslant v_s > p\right) \\
&\quad + \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s \leqslant p \wedge v_b < p\}\right] + \mathbb{E}\left[v_s \cdot \mathbb{1}\{v_s \leqslant p \wedge p^\star > v_b \geqslant p\}\right] \\
&\quad + \mathbb{E}\left[v_b \cdot \mathbb{1}\{v_b \geqslant p^\star\}\right] \mathbb{P}\left(v_s \leqslant p^\star\right)(1-2\delta) && (19)\\
&\geqslant \mathbb{E}\left[\mathsf{SW}(p^\star)\right](1-2\delta), && (20)
\end{aligned}$$

where in Equation (16) we used the independence of v_s and v_b, in Inequality (17) we used Equation (13), in Inequality (18) we reorganized the terms and moved the δ part of the last term around to match the cases in the expected welfare of setting price $p^\star$, and in Inequality (19) we used Equation (13) again and scaled one term by $\mathbb{P}\left(v_b < p^\star\right)$. ∎

We are now ready to prove our main theorem for the SampleQuantile mechanism.

Theorem 9. *Denote by x_{SQ} the allocation rule of* SampleQuantile *and by x_{RQ} the allocation rule of* RandomQuantile*. For any $1/e > \delta > 0$, given ℓ samples we have:*

$$\mathbb{E}\left[SW(x_{SQ}(v_B, v_S))\right] \geqslant \mathbb{E}\left[SW x_{RQ}(v_B, v_S)\right](1-2\delta)(1 - 2 \cdot e^{-2n\left(\frac{\delta}{2e}\right)^2}). \quad (21)$$

Proof. As before we use the shorthands v_b and v_s to denote the random variable corresponding to the buyer's and seller's valuation, respectively.

For any $z \in [1/e, 1]$, we denote with $p^\star(z)$ the quantile price of RandomQuantile for which $F_S(p^\star(z)) = z$. For what concerns our mechanism, we use n samples to calculate an appropriate $P(z)$ such that $F_S(P(z)) \approx z$. We remark that, given z, $p^\star(z)$ is a deterministic value, while $P(z)$ is a random variable, whose randomness is given by the sampling procedure.

Every time we sample from F_s the values themselves are not uniformly distributed, but their quantiles are. This is clear, because

$$\mathbb{P}\left(F_s(v_s) \leqslant x\right) = x.$$

Therefore, the quantiles of the samples drawn behave like order statistics of the uniform distribution; given two samples $s_1 \leqslant s_2$, we might not know their exact quantiles but we do know that $F_s(s_1) \leqslant F_s(s_2)$. From Lemma 6, we have that:

$$\mathbb{P}\left(\left|F_s(P(z)) - \left(F_s(p^\star(z)) - \frac{\delta}{2e}\right)\right| \geqslant \frac{\delta}{2e}\right) \leqslant 2 \cdot e^{-2\ell\left(\frac{\delta}{2e}\right)^2},$$

where the probability is taken over P. However, since $F_s(p^\star(z)) \geqslant 1/e$:

$$\mathbb{P}\left(F_s(p^\star(z)) \geqslant F_s(P(z)) \geqslant F_s(p^\star(z))(1-\delta)\right) \leqslant 2 \cdot e^{-2\ell\left(\frac{\delta}{2e}\right)^2}. \tag{22}$$

Therefore, by applying Lemma 7 for $P(z)$ and $p^\star(z)$ and taking into account the probability given by Inequality (22) that the realization of $P(z)$ does not have the desired property (in which case we assume no welfare was generated) we have:

$$\mathbb{E}\left[\mathsf{SW}(P(z))\right] \geqslant \mathbb{E}\left[\mathsf{SW}(p^\star(z))\right](1-2\delta)(1 - 2 \cdot e^{-2\ell\left(\frac{\delta}{2e}\right)^2}). \tag{23}$$

We obtain the claimed approximation guarantee by taking expectations over the possible values of z, which follows the same distribution as in [8]. ☐

Corollary 1. *For every $\varepsilon \in (0, \frac{4}{e})$, given $\ell = \frac{16e^2}{\varepsilon^2}\log(\frac{4}{\varepsilon})$ samples,* SampleQuantile *provides in expectation the following approximation guarantee of the optimal expected social welfare*

$$\begin{aligned}\mathbb{E}\left[\mathsf{SW}(x_{SQ}(v_B, v_S))\right] &\geqslant (1-\varepsilon)\left(1 - \frac{1}{e}\right)\mathbb{E}\left[\mathsf{SW}(OPT(v_B, v_S))\right] \\ &\geqslant \left(1 - \frac{1}{e} - \varepsilon\right)\mathbb{E}\left[\mathsf{SW}(OPT(v_B, v_S))\right].\end{aligned}$$

Proof. If we use $\delta = \frac{\varepsilon}{4}$ in the SampleQuantile and we set $\ell = \frac{16e^2}{\varepsilon^2}\log(\frac{4}{\varepsilon})$, we get the Corollary directly from Theorem 9 and [8]. ☐

D General Proof of Lemma 3

We show how to generalize the proof of Lemma 3 to the case where distributions are allowed to have atoms.

Proof of Lemma 3. We can proceed exactly as in the proof provided in the body of the paper, except for the derivation of the lower bound on the expected contribution to social welfare of a fixed buyer B_i with fixed valuation v_{B_i} and a random seller s. We claim that this contribution is at least $(2-\sqrt{3}) \cdot v_{B_i}$ as in the continuous case.

If the seller distributions are allowed to have atoms, then equality (6) may not have a solution. In order to overcome we will, as in the body of the paper consider some t and do a case distinction based on whether $v_{B_i} \leqslant t$ or $v_{B_i} > t$. For $v_{B_i} \leqslant t$ we argue as in the continuous case that the expected contribution is at least $\frac{1}{2}F(t)^2 v_{B_i}$. For the case $v_{B_i} < t$ we use a slightly different argument. Namely, using the expected value of the seller as a lower bound we obtain that the expected contribution is at least

$$\mathbb{E}\left[v_s \mid t > v_{B_i}\right] = \mathbb{E}\left[v_s\right] = \mathbb{E}\left[v_s'\right] \geqslant \mathbb{E}\left[v_s' \mid v_s' \geqslant v_{B_i}\right] \cdot \mathbb{P}\left(v_s' \geqslant v_{B_i}\right)$$

$$\geqslant (1 - F(v_{B_i}))v_{B_i} \geqslant \left(1 - F\left(\frac{t+v_{B_i}}{2}\right)\right) v_{B_i}.$$

So the expected contribution of buyer B_i and its random partner s to the social welfare is at least $v_{B_i} \cdot LB(t)$, where

$$LB(t) \geqslant \begin{cases} \frac{1}{2}F(t)^2 & \text{if } t \leqslant v_{B_i} \\ 1 - F\left(\frac{v_{B_i}+t}{2}\right) & \text{if } t > v_{B_i} \end{cases}$$

We now choose $t^\star = \min\{t | F(t) \geqslant \sqrt{3} - 1\}$, i.e., the (jumping) point of function F for which $F(t^\star) \geqslant \sqrt{3} - 1$, and $F(t') < \sqrt{3} - 1$ for all $t' < t^\star$. Using this $t^\star$ we can lower bound the expected contribution of B_i and s as follows:

$$\begin{aligned} &\geqslant v_{B_i} \cdot \min\left\{\tfrac{1}{2}F(t^\star)^2,\, 1 - F\left(\tfrac{v_{B_i}+t^\star}{2}\right)\right\} \\ &\geqslant v_{B_i} \cdot \min\left\{\tfrac{1}{2}2(2-\sqrt{3}),\, 2-\sqrt{3}\right\} \\ &= v_{B_i} \cdot \min\left\{2-\sqrt{3},\, 2-\sqrt{3}\right\} = v_{B_i} \cdot (2-\sqrt{3}), \end{aligned}$$

as claimed. □

E Proof: Adjusted Valuations from Section 5 are XOS

The adjusted valuations $\hat{v}_b$ were defined as

$$\hat{v}_b(S) = \sum_{s \in \bar{S}} (a_{b,\bar{S}}(s) - v'_s), \text{ where } \bar{S} = \operatorname*{argmax}_{S^\star \subseteq S \cap \hat{S}} \left\{ v_b(S^\star) - \sum_{s \in S^\star} v'_s \right\}$$

This captures the fact that only items in $\hat{S}$ are available for trade, and whenever a buyer is assigned some item s, he will be required to pay an additional v'_s later. Therefore, his valuation for S is no more than his valuation for the best subset of it in $\hat{S}$, minus the sum of the according v'_s. This function is XOS because it can be described as the maximum over the following XOS-support: for all a from the support of the original v_b and $s \in \mathcal{S}$, define

$$\hat{a}(s) = (a(s) - v'_s)_+ \text{ if } s \in \hat{S}, \text{ and 0 otherwise.}$$

For the $\bar{v}$, defined as

$$\bar{v}_b(S) = \left(\sum_{s \in S \cap \hat{S}} \left(a_{b,S}(s) - v'_s\right) \right)_+,$$

we define the XOS support

$$\bar{a}(s) = (a(s) - v'_s) \text{ if } s \in \hat{S}, \text{ and 0 otherwise,}$$

and add an additonal function a_0 which is 0 for any $s \in S$. □

F Equivalence of the Adjusted Valuations

Lemma 8. *Any inclusion-minimal assignment ALG made by the algorithm run with adjusted valuations $\bar{v}$ instead of $\hat{v}$ will provide the same approximation to the optimum welfare.*

Proof. Note that the $\bar{v}_b$ are in the class XOS (see Appendix E), so M_α's approximation guarantee holds up. Also, for any pair (b, S) with $b \in B$, $S \subseteq \hat{S}$: $\hat{v}_b(S) \geqslant \bar{v}_b(S)$, since $\hat{v}_b(S)$ results from picking an optimal subset from S.

Let S^{opt} be an inclusion-minimal, utility-preserving set of items assigned to buyer b, which is a subset of b's assigned bundle S^b in an optimal hypermatching $OPT(\hat{G})$ of graph $\hat{G}$ as defined in our algorithm, i.e.

$$\bar{v}_b(S^{opt}) = \sum_{s \in S^{opt}} (a_{b,S^{opt}}(s) - v'_s), \qquad \text{where } S^{opt} = \underset{S^\star \subseteq S^b \cap \hat{S}}{\operatorname{argmax}} \left\{ v_b(S^\star) - \sum_{s \in S^\star} v'_s \right\}.$$

It holds by definition

$$\bar{v}(S^{opt}) = \sum_{s \in Sopt} (a_{b,S^{opt}}(s) - v'_s)$$

and therefore, the weight of an optimal assignment is the same for both $\bar{v}$ and $\hat{v}$. Combining these facts, we get for the weight of the hypermatching $A_{\bar{v}}$ returned by the algorithm when using the valuations $\bar{v}_{\mathcal{B}}$:

$$\sum_{(b,S) \in A_{\bar{v}}} \bar{v}_b(S) \geqslant c \cdot OPT(\bar{G}) = c \cdot OPT(\hat{G}) \implies \sum_{(b,S) \in A_{\bar{v}}} \hat{v}_b(S) \geqslant c \cdot OPT(\hat{G})$$

and this is what we needed in the proof of the approximation ratio. □