PhD. Thesis

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Complex flat manifolds and their moduli spaces

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Abstract.

Although most of the work in this thesis is algebraic, its starting point and examples come from differential topology and geometry. As essential background Chapter I includes sections which describe involuted algebras and Albert's classification of rational positively involuted algebras; representations of finite groups over fields; groups that embed in division algbebras and Amitsur's classification.

The differential topology of the thesis arises in the study of how one can give a flat compact Riemannian manifold a Kählerian/projective structure.

In Chapter II we outline some differential geometry and the theory of Flat Riemannian manifolds, particularly holonomy, we include a description Charlap's classification. Also in this chapter we give a simple proof of a bound for the minimal dimension for a flat compact Riemannian manifold with predescribed holonomy ($m(\Phi) \leq |\Phi|$); the proof requires Amitsur's classification. The notion of complex structures on real manifolds is introduced in Chapter III. Some work on Riemann matrices is required and given. In Chapter IV we parametrise the set of complex structures which give a (real) flat compact Riemannian manifold a Kählerian structure. A parametrisation is also given for complex structures which give a projective structure for certain manifolds with a fixed polarisation. This involves Siegel's generalised upper half plane. In Chapter V we give some examples and give the above parametrisations for certain holonomy groups and representations. Some of the working involves integral representations and cohomology of finite groups.

Finally, the subject of Chapter VI is essentially independent of previous chapters in respect to the work we have done. The chapter concerns subgroups of a product of surface groups, by which we mean the fundamental group of an oriented surface of positive genus. We consider the simultaneous equivalence relations of commensurability and automorphism. In particular, we show that, in a product of

two surface groups in which one factor has genus greater that one, there are infinitely many equivalence classes of normal subdirect products.

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Contents.

Chapter I: Representations of finite groups.

§1 Involuted algebras

Let k be a field of characteristic zero. All algebras will be associative and unitary. Let A be a finite-dimensional algebra over a field k. Let $\mathrm{RTr}_{A/k}$ ($\mathrm{RN}_{A/k}$) : $A \to k$ denote the reduced trace (norm) of A/k.

Definition: Let A be a k-algebra. By an *involution* for A we mean an anti-automorphism $\sigma : A \to A$ (that is σ is k-linear and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$) such that $\sigma^2 = 1$. Given such a $\sigma : A \to A$, $A = (A, \sigma)$ is an *involuted algebra.* An involuted k-algebra homomorphism $\alpha : (A, \sigma) \to (B, \tau)$ is a k-algebra homomorphism $\alpha : A \to B$ satisfying $\alpha \circ \sigma = \tau \circ \alpha$. We then have a category of involuted k-algebras containing products given by

$$\prod_{i=1}^{n}(A_i, \sigma_i) = (\prod_i A_i, \prod_i \sigma_i).$$

Fix $k \subset \mathbb{R}$ a real field and let (A, σ) be an involuted k-algebra. We say that σ is a *positive involution* when $\mathrm{RTr}_{A/k}(x.x^{\sigma}) > 0$ for all non-zero $x \in A$. Let (A, σ) be an involuted division algebra over k. Then, for each n we may define an involuted k-algebra $M_n(A, \sigma) = (M_n(A), \tilde{\sigma})$ where

$$\tilde{\sigma} : M_n(D) \to M_n(D)$$

by

$$(d_{ij})^{\tilde{\sigma}} = ((d_{ji})^{\sigma}).$$

Proposition (1.1) : Let (A, σ) be a positively involuted k-algebra and suppose that A is also finitely generated and k-semisimple, with the following Wedderburn decomposition

$$A = M_{m_1}(D_1) \times \ldots \times M_{m_n}(D_n),$$

where each D_i is a division algebra over k. Then there exists involutions $\sigma_i : D_i \to D_i$ such that

$$(A, \sigma) \cong M_{m_1}(D_1, \sigma_1) \times \ldots \times M_{m_n}(D_n, \sigma_n).$$

Moreover, each (D_i, σ_i) is positive.

Let $a, b \in k$ then denote by $\left(\frac{a, b}{k} \right)$ the quaternion algebra over k which has as its basis 1, i, j, k and multiplication given by

$$i^2 = a.1, \qquad j^2 = b.1, \qquad i.j = -j.i = k.$$

Then $\left(\frac{a, b}{k} \right)$ is a central simple algebra over k when $a, b \neq 0$, and admits essentially only two involutions: *conjugation* $(-)$ and *reversion* $(\sim)$, see [Po]. Let $x = x_0.1 + x_1.i + x_2.j + x_3.k$, then

$$\overline{x} = x_0.1 - x_1.i - x_2.j - x_3.k,$$

$$\widetilde{x} = x_0.1 - x_1.i + x_2.j + x_3.k.$$

A proof for the following may be found in [Pi] or [O'M].

***Proposition* (1.2)** : Let $Q = \left(\frac{a, b}{k} \right)$, then the following are equivalent:

(i) Q is a division algebra;

(ii) $x.\overline{x} \neq 0$ for $0 \neq x \in Q$;

(iii) the quadratic form $\mathcal{Q} : k^3 \to k$, given by $\mathcal{Q}(x_0, x_1, x_2) = x_0^2 - a.x_1^2 - b.x_2^2$, is anisotropic over k.

Over $\mathbb{R}$ there are only three finitely generated division algebras, namely $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H} = \left(\frac{-1, -1}{\mathbb{R}} \right)$, the classical quaternions. Each has a unique positive involution σ given by: $\sigma = 1_{\mathbb{R}}$, complex conjugation and quaternionic conjugation respectively.

We identify $\mathbb{C} \subset \mathbb{H}$ by $\mathbb{C} = \text{span}_{\mathbb{R}}\{1, i\}$. Thus $\mathbb{H}$ can be considered as a $\mathbb{C}$-

space, by $\mathbb{H} = \mathbb{C} + \mathbb{C}.\mathbf{j}$. This induces the standard embedding $\iota : M_e(\mathbb{H}) \to M_{2e}(\mathbb{C})$ of involuted algebras

$$\iota : X = A + B.\mathbf{j} \mapsto \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}.$$

where $A, B \in M_n(\mathbb{C})$.

Over the rationals the positively involuted division algebras are classified by Albert, [Al1]. Let G be a finite group and A an abelian group (written multiplicatively) on which G acts by automorphisms. By a 2-*cocycle* of G with values in A we mean a map $\alpha : G \times G \to A$ satisfying

$$\alpha(x,y).\alpha(xy,z) = \alpha(y,z)^x.\alpha(x,yz) \text{ for all } x,y,z \in G,$$

(the 2-cocycle condition). A 2-cocycle $\alpha : G \times G \to M$ is normalised when

$$\alpha(1,x) = 1 = \alpha(y,1) \text{ for all } x,y \in G.$$

Let $\mathcal{Z}^2(G,A)$ denote the set of normalised 2-cocycles of G with values in A. Note that $\mathcal{Z}^2(G,A)$ is an abelian group with multiplication defined pointwise from A. Let k be a field and $\mathbb{E}/k$ a finite Galois extension with Galois group G. Fix $\alpha \in \mathcal{Z}^2(G,\mathbb{E}^*)$, where $\mathbb{E}^*$ denotes the multiplicative subgroup of $\mathbb{E}$. By the *crossed product of* $\mathbb{E}$ *and* G *with respect to the* 2-*cocycle* α we mean the k-algebra having basis $\{e_x\}_{x \in G}$ over $\mathbb{E}$ and multiplication defined by

$$(\sum_{x \in G} a_x.e_x).(\sum_{y \in G} b_y.e_y) = \sum_{x,y \in G} a_x.b_y.e_x.e_y \qquad (a_x, b_y \in \mathbb{E} \ \forall \ x,y \in G)$$

with

$$e_x.a = a^x.e_x \ (a \in \mathbb{E}) \quad \text{and} \quad e_x.e_y = \alpha(x,y).e_{xy} \qquad \{1.1\}$$

We denote this k-algebra by $(\mathbb{E}/k, G, \alpha)$.

Definition : Let $\mathbb{A}$ be a k-algebra. By a splitting field for $\mathbb{A}$ we mean a k-algebra $\mathbb{E}$ such that $\mathbb{A} \otimes_k \mathbb{E} \cong M_n(\mathbb{E})$, for some n.

Proposition **(1.3)** : The crossed product $(\mathbb{E}/k, G, \alpha)$ is central over k of degree $|G|$, with $\mathbb{E}$ as a self-centralising subfield. $\mathbb{E}$ is also the splitting field for $(\mathbb{E}/k, G, \alpha)$.

We will consider the special case where $\mathbb{E}/k$ is a cyclic extension. Let $\mathbb{E}/k$ be a cyclic extension with cyclic Galois group G. Let $|G| = n$. Let x generate G, then for any $\alpha \in \mathcal{Z}^2(G,A)$ there exists $\xi \in \mathbb{E}^*$ such that

$$\alpha(x^i, x^j) = \begin{cases} 1 & \text{if } 0 \leq i+j < n, \\ \xi & \text{if } n \leq i+j \leq 2n-2 \end{cases}$$

Let $\{e_x\}_{x \in G} = \{e^{(0)}, e^{(1)}, \ldots, e^{(n-1)}\}$ where $e^{(1)} = e$, then the conditions {1.1} reduces to

$$e.a = a^x.e \ (a \in \mathbb{E}) \qquad \text{and} \qquad e^n = \xi.e^{(0)} \qquad \{1.2\}.$$

We denote such a *cyclic k-algebra* by $(\mathbb{E}/k, x, \xi)$. Note that the quaternion algebras are precisely the cyclic algebras defined by an extension of degree two.

Theorem [All] (1.4) : Let (D,σ) be a finitely generated positively involuted division algebra over $\mathbb{Q}$; let $\mathbb{F}$ denote the centre of D and $\mathbb{E}$ denote subfield of $\mathbb{F}$ fixed by σ; and let $d^2 = \dim_{\mathbb{F}} D$ and $g = \dim_{\mathbb{Q}} \mathbb{E}$. Then (D,σ) has one of four types:

(I) $D = \mathbb{E}$ is a totally real algebraic field, σ is the identity,

$$\mathbb{R} \otimes_{\mathbb{Q}} D \cong \mathbb{R} \times \ldots \times \mathbb{R} \qquad \text{(g copies)} \qquad \{1.3\}$$

(II) $D = \left(\frac{a, b}{\mathbb{E}} \right)$, $\mathbb{E}$ is a totally real algebraic field, a is totally negative, b is totally positive and σ is reversion,

$$\mathbb{R} \otimes_{\mathbb{Q}} D \cong M_2(\mathbb{R}) \times \ldots \times M_2(\mathbb{R}) \qquad \text{(g copies)} \qquad \{1.4\}$$

with involution

$$(A_1, \ldots, A_e) \mapsto (A_1^t, \ldots, A_e^t);$$

(III) $D = \left(\frac{a, b}{\mathbb{E}} \right)$, where $\mathbb{E}$ is a totally real algebraic field, a, b $\in \mathbb{E}$ are totally negative and σ is quaternionic conjugation

$$\mathbb{R} \otimes_{\mathbb{Q}} D \cong \mathbb{H} \times \ldots \times \mathbb{H} \qquad \text{(g copies)} \qquad \{1.5\}$$

with conjugation as involution;

(IV) $\mathbb{E}$ is a totally real algebraic field, $\mathbb{F}$ is totally imaginary and

quadratic over $\mathbb{E}$, $D = (\mathbb{K}/\mathbb{Q}, \alpha, \xi)$ where $\mathbb{K}$ is an algebraic number field, and $\mathbb{F}$ is the fixed field of α,

$$\mathbb{R} \otimes_{\mathbb{Q}} D \cong M_d(\mathbb{C}) \times \ldots \times M_d(\mathbb{C}) \qquad \text{(g copies)} \qquad \{1.6\}$$

with involution

$$(A_1, \ldots, A_e) \mapsto (\overline{A}_1^t, \ldots, \overline{A}_e^t).$$

A convenient reference for this result is [Mu].

§2 Representations of finite groups

Let Φ be a finite group and R be a commutative ring. The group ring of Φ over R is denoted by $R[\Phi]$. All representation spaces will be left modules over the group ring.

For V, W $R[\Phi]$-modules, let $\mathrm{Hom}_{R[\Phi]}(V, W)$ denote the group of $R[\Phi]$-homomorphisms of V to W; that is, R-module homomorphisms which commute with the action of Φ. Let $\mathrm{End}_{R[\Phi]}(V)$ denote the group of $R[\Phi]$-endomorphisms.

Let k be a field. By Maschke's Theorem $k[\Phi]$ is semisimple if char $k = 0$ or char $k \not| \, |\Phi|$. If $k[\Phi]$ is semisimple we have the Wedderburn decomposition

$$k[\Phi] \cong M_{m_1}(D_1) \times \ldots \times M_{m_n}(D_n) \qquad \{2.1\}$$

where D_i is a finitely generated division algebras over k. In such a decomposition, each factor $M_{m_i}(D_i)$ there corresponds to a simple $k[\Phi]$-module, V_i. Identify V_i with the first column of $M_{m_i}(D_i)$. In which case,

$$D_i = \mathrm{End}_{k[\Phi]}(V_i).$$

Lemma (2.1) : Let $k \subset \mathbb{E}$ be an extension of fields, and W a $k[\Phi]$-module. Then

$$\mathrm{End}_{\mathbb{E}[\Phi]}(\, \mathbb{E} \otimes_k W \,) = \mathbb{E} \otimes_k \mathrm{End}_{k[\Phi]}(W).$$

Proof: We clearly have that

$$k_2 \otimes_{k_1} \mathrm{End}_{k_1[\Phi]}(W) \subset \mathrm{End}_{k_2[\Phi]}(\mathfrak{S}_k^{\mathbb{E}}(W)).$$

To prove the reverse inclusion, let $\{e_i\}$ be a basis for k_2 over k_1. Since

$$\mathrm{End}_{k_2}(\mathbb{E} \otimes_k W) = k_2 \otimes_{k_1} \mathrm{End}_{k_1}(W).$$

We may write any $g \in \mathrm{End}_{k_2[\Phi]}(\mathbb{E} \otimes_k W)$ as follows

$$g = \sum e_i \otimes f_i$$

where $f_i \in \mathrm{End}_{k_1}(W)$. Also, by comparision of coordinates, it is clear that g commutes with $x \in \Phi$ precisely when each f_i commutes with $x \in \Phi$; that is each $f_i \in \mathrm{End}_{k_1[\Phi]}(W)$. Hence the result follows.

□

Proposition (2.2): Let $k \subset \mathbb{E}$ be an extension of fields of characteristic zero or coprime to $|\Phi|$. Let V and W be $k[\Phi]$-modules having no common isomorphic simple submodule. Then $\mathbb{E} \otimes_k V$ and $\mathbb{E} \otimes_k W$ also have no common isomorphic simple submodule.

Define

$$\sigma : k[\Phi] \to k[\Phi] \qquad \{2.2\}$$

by

$$(\sum a_g.g)^\sigma = \sum a_g.g^{-1}.$$

Then it is clear that σ is an involution of $k[\Phi]$, and furthermore if k is a real field then σ is positive since

$$\mathrm{Tr}_{k[\Phi]/k}((\sum a_g.g).(\sum a_g.g)^\sigma) = \sum a_g^2$$

$$> 0 \text{ if } \sum a_g.g \neq 0.$$

Hence, by (1.2), each of the division algebras D_i admits a positive involution, $\sigma_i : D_i \to D_i$, such that

$$(k[\Phi], \sigma) \cong M_{m_1}(D_1,\sigma_1) \times \ldots \times M_{m_n}(D_n,\sigma_n).$$

If $k = \mathbb{Q}$, then each positively involuted division algebra (D_i, σ_i) in the above decomposition appears in (1.4).

By a *CM-algebra*, we mean a quadruple $A = (A, \mathbb{E}, \tau, \xi)$ where

(i) $\mathbb{E}$ is a totally real algebraic number field of finite degree over $\mathbb{Q}$,

(ii) (A,τ) is a finite dimensional positively involuted $\mathbb{E}$-algebra,

(iii) $\xi \varepsilon A$ such that ξ^2 is a totally negative element of $\mathbb{E}$, and moreover $\mathbb{E}(\xi)$ is totally imaginary,

(iv) $\tau_{/\mathbb{E}} = 1_{\mathbb{E}}$, but $\tau_{/\mathbb{E}(\xi)} \neq 1_{\mathbb{E}(\xi)}$.

Proposition (see (1.2) of [Jo5]) (2.3) : Let (A,τ) be a positively involuted finite dimensional $\mathbb{Q}$-algebra of type (II), (III) or (IV) in Albert's classification. Then there exists a subfield $\mathbb{E} \subset A$ and $\xi \varepsilon A$ such that $A = (A,\mathbb{E},\tau,\xi)$ is a CM-algebra.

Let $\mathbb{E}$ be a totally real field and $\mathbb{E}/\mathbb{Q}$ be a finite extension with $\dim_{\mathbb{Q}}\mathbb{E} = g$. Let $\mathfrak{F}_{\mathbb{E}} = \{ \sigma_\lambda : \mathbb{E} \to \mathbb{R} \}_{1\leq\lambda\leq g}$ denote the set of embeddings of $\mathbb{E}$ into $\mathbb{R}$. For each $\lambda = 1 \to g$, let $\mathbb{R}_\lambda$ denote the field $\mathbb{R}$ considered as an algebra over $\mathbb{E}$, via the embedding σ_λ. Then

$$\mathbb{E}\otimes_{\mathbb{Q}}\mathbb{R} \cong \mathbb{R}_1 \times \ldots \times \mathbb{R}_g \qquad \{2.3\}.$$

Let W be a finite dimensional $\mathbb{E}$-space. Then

$$W\otimes_{\mathbb{Q}}\mathbb{R} = W\otimes_{\mathbb{E}}\mathbb{R}_1 \oplus \ldots \oplus W\otimes_{\mathbb{E}}\mathbb{R}_g.$$

Let $W = V^{(e)}$ be an isotypic $\mathbb{Q}[\Phi]$-module (V simple), and $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. Then, if W is of type (I) let $\mathbb{E} = D$; otherwise we write $D = (D,\mathbb{E},\tau,\xi)$, a CM-algebra, and $\mathbb{F} = \mathbb{E}(\xi)$. Let $g = \dim_{\mathbb{Q}}\mathbb{E}$, and for types (II), (III) and (IV) let $d^2 = \dim_{\mathbb{F}}D$, where $d = 2$ for (II) and (III). Fix an isomorphism as in {2.3}. With

this notation,

Proposition **(2.4)** : The decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{R} = V \otimes_{\mathbb{E}} \mathbb{R}_1 \oplus \ldots \oplus V \otimes_{\mathbb{E}} \mathbb{R}_g$$

is $\mathbb{R}[\Phi]$-isotypic and, for each λ

$$V \otimes_{\mathbb{E}} \mathbb{R}_\lambda \cong (\mathfrak{U}_\lambda)^{(f)}$$

where $\mathfrak{U}_\lambda$ is a simple $\mathbb{R}[\Phi]$-module and

$$f = \begin{cases} 1 & \text{type (I) or (III)} \\ 2 & \text{type (II)} \\ d & \text{type (IV).} \end{cases}$$

Proof: Let $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. Then

$$\mathbb{R} \otimes_{\mathbb{Q}} D \cong \begin{cases} \mathbb{R} \times \ldots \times \mathbb{R} & \text{type (I)} \\ M_2(\mathbb{R}) \times \ldots \times M_2(\mathbb{R}) & \text{type (II)} \\ \mathbb{H} \times \ldots \times \mathbb{H} & \text{type (III)} \\ M_d(\mathbb{C}) \times \ldots \times M_d(\mathbb{C}) & \text{type (IV)} \end{cases}$$

Identifying V with the first column of $M_n(D)$, for some n, corresponding to the factor $M_n(D)$ in the Wedderburn decomposition of $\mathbb{Q}[\Phi]$. Hence $V \otimes_{\mathbb{E}} \mathbb{R}_\lambda$ may be identified with the first f columns of

$$\begin{cases} M_n(\mathbb{R}) & \text{type (I)} \\ M_{2n}(\mathbb{R}) & \text{type (II)} \\ M_n(\mathbb{H}) & \text{type (III)} \\ M_{d.n}(\mathbb{C}) & \text{type (IV)} \end{cases}$$

where f is given above.

□

§3 Groups that embed in division algebras

Let Φ be a finite group. We say that Φ embeds in a division ring R when it appears as a subgroup of the multiplicative group of R. Let $\mathfrak{H}$ denote the set of all finite groups which embed in a rational division algebra.

We say that Φ has periodic cohomology when there exists $n \varepsilon \mathbb{N}$ such that $H^{i+n}(\Phi,\mathbb{Z}) \cong H^i(\Phi,\mathbb{Z})$ for all $i \geq 1$. A generalised quaternion group, $Q(2^\alpha)$ of order 2^α ($\alpha \geq 3$), is a group with the following presentation

$$Q(2^\alpha) = < X, Y : X^{2^{\alpha-2}} = Y^2, Y^4 = 1, YXY^{-1} = X^{-1} > \qquad \{3.1\}.$$

Theorem [Ca-Ei] (3.1) : For a finite group G the following statements are equivalent:

(i) G has periodic cohomology;

(ii) every abelian subgroup of G is cyclic:

(iii) every p-subgroup of G is cyclic or a generalised quaternion group;

(iv) every Sylow subgroup of G is cyclic or a generalised quaternion group.

Proposition (see [Sm]) (3.2) : If $G \varepsilon \mathfrak{H}$, then G has periodic cohomology.

Proof: We shall show that G satisfies (ii) of (3.1). Let H be an abelian subgroup of G. Let $G \subset R^*$. Then the division subring of R generated by H is commutative, and hence a field. Since H is finite and embeds into a field it must be cyclic, see [He].

□

Not all finite groups with periodic cohomology embed in a division ring. For example, D_{2m} has cohomology of period 4 if m is odd, but does not appear in Amitsur's classification, (3.7). We note that this group also provides an example of a finite group with periodic cohomology which does not act freely on a (3-)sphere. This was first indicated by Milnor and is a consequence of the following.

Proposition [Mi] (3.3) : Suppose that a finite group Φ acts without fixed points on a manifold M^n having the mod 2 homology of the sphere. Then any element of order 2 in Φ belongs to the centre.

Define the polyhedral groups, D_{2m} ($m \geq 1$), T_{12}, O_{24} and I_{60} as follows:

$$D_{2m} = \left\langle A, B : A^m = B^2 = 1,\ BAB^{-1} = A^{-1} \right\rangle$$

$$T_{12} = \left\langle A, B, C : A^3 = B^2 = C^2 = 1,\ ABA^{-1} = C,\ ACA^{-1} = BC \right\rangle$$

$$O_{24} = \left\langle A, B, C, D : A^3 = B^2 = C^2 = D^2 = 1,\ BC = CB,\ DAD^{-1} = A^{-1} \right.$$
$$\left. ABA^{-1} = C,\ ACA^{-1} = BC,\ DBD^{-1} = CB,\ DCD^{-1} = C^{-1} \right\rangle$$

$$I_{60} = \left\langle A, B, C : A^3 = B^2 = C^5 = ABC = 1 \right\rangle$$

The dihedral group D_m acts on the two sided polygon with m sides; the tetrahedral group T_{12} acts on the regular tetrahedron (4 vertices, 6 edges, 4 faces); the octahedral group O_{24} acts on the regular octahedron (6 verices, 12 edges, 8 faces); and the icosahedral group I_{60} acts on the regular icosahedron (12 vertices, 30 edges, 20 faces). For details of the actions see [Wo].

Proposition (see [Wo]) (3.4) : Every finite subgroup of SO(3) is a cyclic, dihedral, tetrahedral, octahedral or icosahedral group.

Let $- : \mathbb{H} \to \mathbb{H}$ denote conjugation of the classical quaternions. Consider a norm on $\mathbb{H}$ given by $|x|^2 = x.\overline{x}$. Identify $S^3 = \{x \in \mathbb{H} : |x| = 1\}$ and $\mathbb{R}^3$ with the space of pure imaginary quaternions. The Euclidean norm of $\mathbb{R}^3$ then corresponds to the restriction fo the norm of $\mathbb{H}$. Let $\pi : S^3 \to SO(3)$ be the map defined by $\pi(x)(y) = x.y.x^{-1}$. The kernel of π is then $\{+1, -1\}$.

Define the binary polyhedral groups as follows:

(i) the binary dihedral group, of order 4m, by

$$\mathfrak{D}^*_{4m} = \pi^{-1}(\mathrm{D}_{2m})$$

(ii) the binary tetrahedral group, of order 24, by

$$\mathfrak{T}^* = \pi^{-1}(\mathrm{T}_{12})$$

(iii) the binary octahedral group, of order 48, by

$$\mathfrak{O}^* = \pi^{-1}(\mathrm{O}_{24})$$

(iv) the binary icosahedral group, of order 60, by

$$\mathfrak{I}^* = \pi^{-1}(\mathrm{I}_{60}).$$

Apart from cyclic groups, the binary polyhedral groups are the only finite subgroups of the classical quaternions: see Theorem 11 of [Am].

Clearly the binary polyhedral groups also act without fixed points on S^3 by the multiplication in $\mathbb{H}$.

We now describe the classification of finite groups which embed into division rings, given by Amitsur.

Let m and r be positive integers such that (m,r) = 1, and put n = the order of r (mod m) and s = (r−1,m). Denote by $G_{m,r}$ a group having the following presentation

$$G_{m,r} = < A, B : A^m = 1,\ B^n = A^t, BAB^{-1} = A^r > \qquad \{3.2\},$$

where t = m/s. This group has order m.n and is clearly metacyclic.

A synthesis of a result of Zassenhaus, [Za] and Lemmas 1-2 of [Am] give the following:

Lemma (3.5) : Let G be a finite group. If all the Sylow subgroups of G are cyclic, then G has a presentation {3.2} with (n, t)=1. If G has a presentation {3.2} then all the odd Sylow subgroups of G are cyclic and the even Sylow subgroups of G are generalised quaternion if and only if $n = 2.n'$, $m = 2^{\alpha}.m'$, $s = 2.s'$ with $\alpha \geq 2$, m', s', n' odd numbers and such that $(n,t) = (s,t) = 2$, and $r \equiv -1\ (2^{\alpha})$.

Let $G_{m,r}$ have presentation {3.2}. For a prime p|m we introduce the following notation

(i) let α_p be the integer such that p^{α_p} is the highest power of p|m;

(ii) let $n = n_p$ be the minimal integer satisfying $r^{n_p} \equiv 1\ (mp^{-\alpha_p})$;

(iii) let δ_p be the minimal integer satisfying $p^{\delta_p} \equiv 1\ (mp^{-\alpha_p})$.

With s defined as above, let $\mathfrak{U}_{m,r}$ denote the the following cyclic algebra

$$\mathfrak{U}_{m,r} = (\mathbb{Q}(\zeta_m), \sigma_r, \zeta_s)$$

where ζ_m is a primitive m^{th} root of unity, $\sigma = \sigma_r$ is the automorphism of $\mathbb{Q}(\zeta_m)$ given by $\zeta_m \mapsto \zeta_m^r$, and ζ_s is a primitive s^{th} root of unity contained in the fixed field of σ in $\mathbb{Q}(\zeta_m)$. $G_{m,r}$ embeds in a division ring if and only if $\mathfrak{U}_{m,r}$ is a division algebra, and then the linear span of the elements of $G_{m,r}$ is isomorphic to $\mathfrak{U}_{m,r}$, see Theorem 3 of [Am]. In all cases $\mathfrak{U}_{m,r}$ is central simple of degree n over the fixed field of σ in $\mathbb{Q}(\zeta_m)$.

Theorem [Am] (3.6) : Let $G_{m,r}$ be a finite group having presentation {3.2}; $G_{m,r} \varepsilon$ $\mathfrak{H}$ if and only if $G_{m,r}$ has periodic cohomology and one of the following holds

(i) $n=s=2$ and $r \equiv -1$ (m)

(ii) For every prime q|n there exists a prime p|m such that

$q \nmid n_p$ and that either

(a) $p \neq 2$ and $(q, (p^{\delta_p}-1)/s) = 1$

or (b) $p = q = 2$ $G_{m,r}$ contains a non-cyclic Sylow subgroup and $m/4 \equiv \delta_p \equiv 1 \pmod 2$.

We give Amitsur's classification as follows.

Theorem [Am] (3.7) : Let G be a finite group. G ε $\mathfrak{H}$ if and only if

(i) $G \cong G_{m,r}$ satisfying one of the conditions of (3.4);

(ii) $G \cong \mathfrak{T}^* \times G_{m,r}$ where $G_{m,r}$ as in (i) and 2 has odd order mod m;

(iii) $G \cong \mathfrak{O}^*$ or $\mathfrak{I}^*$.

Chapter II: Flat manifolds.

Let M be a real manifold and let $\mathfrak{I}(M)$ denote the set of smooth functions on M. The field structure of $\mathbb{R}$ induces an $\mathbb{R}$-algebra structure on $\mathfrak{I}(M)$. $\mathfrak{I}(M)$ is known as the algebra of *smooth functions* on M. Let $p \in M$, then a *tangent vector* to M at p is a derivative of $\mathfrak{I}(M)$ at p. Let T_pM denote the *tangent space* of all tangent vectors to M at the point p. If M has real dimension n then T_pM is a real vector space of dimension n.

Let $\mathfrak{X}^1(M)$ denote the set of vector fields of M. Then $\mathfrak{X}^1(M)$ is a module over $\mathfrak{I}(M)$ defined pointwise. Recall that by a *Riemann metric*, g, on a smooth manifold M, we mean a tensor field of type $(0,2)$, $g : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathfrak{I}(M)$, such that the associated $\mathbb{R}$-bilinear form $g : T_pM \times T_pM \to \mathbb{R}$ is symmetric and positive definite.

By a *connection* on M we mean an $\mathfrak{I}(M)$-linear map from $\mathfrak{X}^1(M)$ into the derivations of $\mathfrak{X}^1(M)$; that is, if we denote the connection by $X \mapsto \nabla_X$, then

$$\nabla_X(f.Y) = f.\nabla_X(Y) + X(f).Y \qquad \text{for all } f \in \mathfrak{I}(M),\ Y \in \mathfrak{X}^1(M).$$

∇_X is known as the *covariant derivative* at X.

Let M be a smooth manifold and ∇ a connection on M. Let T denote the *torsion tensor field* and R denote the *curvature tensor field* R of ∇. These are tensors of type $(1,2)$ and $(1,3)$ respectively. On vector fields and one-forms they are given by

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y]$$

$$R(X,Y,Z) = \nabla_X\nabla_Y(Z) - \nabla_Y\nabla_X(Z) - \nabla_{[X,Y]}(Z).$$

A connection is said to be *torsion-free* if $T(X,Y) = 0$ for all $X,Y \in \mathfrak{X}^1(M)$ and *flat* if $R(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{X}^1(M)$.

On a riemannian manifold there exists a torsion-free connection. This connection is uniquely determined if we want parallel translations to preserve the Riemann metric. We shall not go into this at this point, but direct the reader to

[Wo]. This unique connection is known as the *Levi-Cività* (or *riemannian*) *connection.* A *flat* riemannian manifold is then a riemannian manifold with flat Levi-Cività connection.

§1 Flat manifolds

We shall give a brief account of some of the theory of flat riemannian manifolds. For a more detailed account we again direct the reader to [Wo].

For $n > 0$ let E(n) denote the group of rigid motions of n-dimensional Euclidean space. We may topologies E(n) in the obvious way, we get $E(n) \cong O_n \times \mathbb{R}^n$, where $\cong$ denotes homeomorphism. Rotations correspond to O_n and translations to $\mathbb{R}^n$. A subgroup G of E(n) is said to be *uniform* when $\mathbb{R}^n/G$ is compact. A subgroup G of E(n) is said to be a *Bieberbach* group when G is torsion-free, uniform and discrete in E(n).

Let M be a compact riemannian manifold of dimension n and let $G = \pi_1(M)$, the fundamental group of M. Consider G as a subgroup of E(n) through the action of G on the universal covering of M. We have $M \cong \mathbb{R}^n/G$, but M is a compact riemannian manifold, hence G must be torsion-free, uniform and discrete. Hence G is a Bieberbach group.

Let N denote the subgroup of G of translations in E(n). Then, [Bi], N is a free abelian group generated by n linearly independent translations. Let $\Phi = \Phi(M)$ denote the holonomy group of M. It is well known that Φ is finite, since M is flat, and that $G/N \cong \Phi$, see [Au-Ma].

Theorem (see [Au-Ku]) (1.1) : Let G be a finitely generated group. Then G is the fundamental group of a compact flat riemannian manifold of dimension n if and only if G is torsion free and fits into a short exact sequence

$$0 \rightarrow N \rightarrow G \rightarrow \Phi \rightarrow 1 \qquad \{1.1\}$$

where

(i) N is a maximal free abelian subgroup of G,

and (ii) Φ is finite.

If M is a compact flat Reimann manifold such that $G = \pi_1(M)$, the {1.1} is called the holonomy short exact sequence. Let $\rho : \Phi \rightarrow GL_{\mathbb{Z}}(N)$ denote the operator homomorphism of {1.1}. Then, N is maximal abelian in G if and only if ρ is faithful. It is well known that the realisation of ρ corresponds to the holonomy representation $\rho' : \Phi \rightarrow GL_{\mathbb{R}}(\widetilde{M})$, where $\widetilde{M}$ denote the universal covering of M.

Theorem [Au-Ku] (1.2) : Any finite group can occur as the holonomy group of a compact flat riemannian manifold.

Let Φ be a finite group, write $\Phi = F/R$ where F is a finitely generated free group, R is the group of relations of Φ and both F and R are non-abelian. Let $G = F/[R,R]$ and $N = R/[R,R]$. Then we have a short exact seqence,

$$0 \rightarrow N \rightarrow G \rightarrow \Phi \rightarrow 1.$$

The result follows by verifying that G is torsion free and the conditions (i) and (ii) of (1.1) are satisfied. See [Au-Ku] or [Wo] for details. A proof of this theorem is also contained in [Ch].

Let Φ be a finite group. Denote by $m(\Phi)$ the minimal dimension of a compact flat riemannian manifold have holonomy group isomorphic to Φ. By (1.1) $m(\Phi)$ equals the minimal rank of a $\mathbb{Z}[\Phi]$-module N which has a torsion-free extension

{1.1} satisfying conditions (i) and (ii).

The minimal dimension is finite for all finite groups, and we may give a simple bound for $m(\Phi)$ using (1.1): let d denote the minimal number of generators for F, then by the schreier's formula, see [Ma-Ka-So], R is free on $|F_{/R}|.(d-1)+1$, hence $m(\Phi) \leq |\Phi|.(d-1)+1$.

Fix Φ a finite group, and let Δ be a subgroup $\iota : \Delta \to \Phi$ the injection map. Denote by ι^* $(= \mathrm{Res}^{\Phi}_{\Delta}) : H^2(\Phi, N) \to H^2(\Delta, N)$ the restriction map induced from ι.

Definition [Ch] : Let N be a finitely generated $\mathbb{Z}[\Phi]$-module. We say that α is *special* when $\iota^*(\alpha) \neq 0$ for all cyclic subgroups Δ of Φ which have prime order. Such an α is called a special point for N.

Let $\alpha \in H^2(\Phi, {}_{\rho}N)$ denote the cohomology class corresponding to {1.1}. G is torsion-free precisely when α is special, [Ch]. By a Φ-manifold we mean a compact riemannian manifold which has holonomy group Φ. The manifold must be flat since Φ is finite, [Ch].

Let $\mathcal{S}(\Phi)$ denote the category, whose objects are pairs, (M, α), with M a finitely generated $\mathbb{Z}[\Phi]$-module and $\alpha \in H^2(\Phi, M)$ a special point. Let $(M, \alpha), (N, \beta) \in \mathcal{S}(\Phi)$, then by a morphism in $\mathcal{S}(\Phi)$, $(f, A) : (M, \alpha) \to (N, \beta)$, we mean a pair of group homomorphisms $f : M \to N$ and $A : \Phi \to \Phi$ such that

$$f(x.m) = A(x).f(m) \qquad \text{for all } x \in \Phi, m \in M$$

$$f_*(\alpha) = A_*(\beta).$$

Where $A_* : H^2(\Phi, N) \to H^2(\Phi, N_A)$ and N_A is the $\mathbb{Z}[\Phi]$-module with Φ-action factored through A.

Theorem [Ch] (2.3) : There is a bijection between the isomorphism classes of the category $\mathcal{E}(\Phi)$ and the isometry classes of Φ-manifolds. A manifold in the isometry class corresponding to (N, α) has dimension equal to $\mathrm{rk}_{\mathbb{Z}}(N)$.

Let p be a prime and let $\mathbb{Z}_p$ denote the cyclic group of order p generated by x. We describe the indecomposable $\mathbb{Z}[\mathbb{Z}_p]$-modules are as follows, [Cu-Re] and [Re]. Let ζ be a primitive p^{th} root of unity and $\mathbb{Z}[\zeta]$ denote the ring of integers in $\mathbb{Q}(\zeta)$, the cyclotomic field of order p. Let $C = C(\mathbb{Q}(\zeta))$ denote the ideal class group of order $h = h_p$. Let $A_1, \ldots, A_h \in C$ be a set of representatives for the ideal classes. We consider each ideal as a $\mathbb{Z}[\mathbb{Z}_p]$-module by defining

$$x.a = \zeta.a$$

for all $a \in A_i$, for all i. Also, for each i choose an $a_i \in A_i$ and let (A_i, a_i) denote the $\mathbb{Z}[\mathbb{Z}_p]$-module such that $(A_i, a_i) \cong_{\mathbb{Z}} A_i \oplus \mathbb{Z}$ and

$$x.(a,0) = (x.a, 0)$$

$$x.(0,1) = (a_i, 1)$$

for all $a \in A_i$ and $n \in \mathbb{Z}$. This definition makes sense because the action of $\mathbb{Z}_p$ on A_i satisfies

$$\sum_{r=0}^{p-1} x^r = 0.$$

Then a full set of indecomposable $\mathbb{Z}[\mathbb{Z}_p]$-modules is

$$\mathbb{Z}\ ,\ A_1\ ,\ \ldots\ ,\ A_h\ ,\ (A_1, a_1)\ ,\ \ldots\ ,\ (A_h, a_h)$$

for $a_i \in A_i$, see [Cu-Re]. Where $\mathbb{Z}$ here to denote trivial module of rank one.

So let

$$M = \mathbb{Z}^{\alpha} \oplus \Big(\bigoplus_{i=1}^{h} A_i^{\beta_i} \Big) \oplus \Big(\bigoplus_{i=1}^{h} (A_i, a_i)^{\gamma_i} \Big) \qquad \{3.1\}$$

be an arbitrary $\mathbb{Z}[\mathbb{Z}_p]$-module. Put $\beta = \sum_{i=1}^{h} \beta_i$ and $\gamma = \sum_{i=1}^{h} \gamma_i$. Then, [Ch],

$$H^2(\mathbb{Z}_p, M) \cong \mathbb{Z}_p^{\alpha}.$$

Hence, there exists torsion-free extensions of M by Φ if and only if $\alpha > 0$. Also, it is clear that M is faithful if and only if $\beta + \gamma > 0$.

§3 The minimal dimension problem

The minimal dimension problem for flat riemannian manifolds is as follows: for each finite group Φ, what is the minimal dimension, denoted $m(\Phi)$, in which a Φ-manifold can appear? It is clear from the end of the last section that we may compute the minimal dimension, for cyclic groups of prime order, as $m(\mathbb{Z}_p) = p$; which corresponds to $\alpha = \beta = 1$. To improve the bound on $m(\Phi)$ given in the last section for solvable groups we have the following result proved by Symonds, [Sy].

Theorem [Sy] (3.1) : Let Φ be a finite group. If Φ is solvable then $m(G) \leq |\Phi|$, with equality if and only if G is cyclic of prime order.

Proof: We first consider some special cases. If $|\Phi|$ is prime then the result follows by [Ch]. If $|\Phi| = p.q$, p and q primes, then Φ is one of the following three types $\Phi = C_p \times C_q$, C_{p2} or D_{pq}. Where D_{pq} is the non-abelian metacyclic group of order pq. If $\Phi = C_p \times C_q$ consider the $\mathbb{Z}[\Phi]$-module $M = \mathbb{Z} \oplus \mathbb{Z}[\zeta_p] \oplus \mathbb{Z}[\zeta_q]$, where ζ_p (resp. ζ_q) is a primitive p^{th} (resp. q^{th}) root of unity. Then, by the results in Table 1 of Chapter V, $H^2(\Phi, M)$ contains a special point; also M is clearly faithful, hence $m(\Phi) \leq p-1 + q - 1 + 1 = p + q - 1 \leq p.q$. If $\Phi = C_{p2}$ consider the $\mathbb{Z}[\Phi]$-module $M = \mathbb{Z} \oplus \mathbb{Z}[\zeta_{p2}]$, where ζ_{p2} is a $p^{2\,th}$ root of unity. Φ has only one subgroup of prime order, namely C_p, hence again by Table 1, $H^2(\Phi, M)$ contains a special point. M is clearly faithful and so $m(\Phi) \leq p(p - 1) + 1 \leq p^2$. In fact, it can be shown that $m(\Phi) = p(p - 1) + 1$, [Sy]. Finally suppose $\Phi = D_{pq}$, then let $M = \mathrm{ind}_{C_p}^{D_{pq}}(\mathbb{Z}) \oplus \mathrm{ind}_{C_q}^{D_{pq}}(\mathbb{Z})$. It is easily verified that in this case $H^2(\Phi, M)$ contains a special point, M is faithful and hence $m(\Phi) \leq p + q \leq p.q$.

We complete the prove by induction on $|\Phi|$. Let H be a normal subgroup of

Φ such that Φ/H is cyclic, such a subgroup exists since Φ is solvable. We may assume that $m(H) \leq |H| - 1$, otherwise the result follows by the special cases. Let M be the $\mathbb{Z}[H]$-module realising the minimal dimension for H. Let $N = \mathrm{ind}_H^\Phi(M) \oplus \mathbb{Z}$. N is a faithful $\mathbb{Z}[\Phi]$-module, since $\mathrm{ind}_H^\Phi(M)$ is faithful and $H^2(\Phi, N)$ admits a special. But $\mathrm{rk}_{\mathbb{Z}}(N) = |\Phi/H|.(|H| - 1) \leq |\Phi| - 1$. Hence $m(\Phi) \leq |\Phi| - 1$.

□

We shall extend this result to all finite groups.

For a finite group Φ, we say that two $\mathbb{Z}[\Phi]$-modules, M and N, are $\mathbb{Q}$-isomorphic if $\mathbb{Q}\otimes_{\mathbb{Z}}M \cong_{\mathbb{Q}[\Phi]} \mathbb{Q}\otimes_{\mathbb{Z}}N$. We say that a $\mathbb{Q}$-isomorphism class, $\mathcal{C}$, of $\mathbb{Z}[\Phi]$-modules is *special* when there exists an $M \in \mathcal{C}$ such that $H^2(\Phi, M)$ admits a special point. Let $\mathfrak{D}(\Phi)$ denote the class of coverings of Φ that have finite abelian kernel; to be more precise, let

$$\mathfrak{D}(\Phi) = \Big\{ (\Psi, \theta) : \ 0 \to T \to \Psi \xrightarrow{\theta} \Phi \to 1 \text{ is exact and T is finite abelian} \Big\}.$$

If M is a $\mathbb{Z}[\Phi]$-module and $(\Psi, \theta) \in \mathfrak{D}(\Phi)$ denote by ${}^{\theta}M$ the $\mathbb{Z}[\Psi]$-module induced from M by θ.

Proposition (3.2) : Let Φ be a finite group and let $\mathcal{C}$ be a $\mathbb{Q}$-isomorphism class of finitely generated torsion-free $\mathbb{Z}[\Phi]$-modules. Then,

$\mathcal{C}$ is special $\Leftrightarrow$ $(\forall\, N \in \mathcal{C})\ (\exists\, (\Psi, \theta) \in \mathfrak{D}(\Phi))\ H^2(\Psi; {}^{\theta}N)$ contains a special point.

Proof (Compare Chpt. 4 §3 of [Sy]) : ($\Rightarrow$) Let $M, N \in \mathcal{C}$, and let $H^2(\Phi, M)$ contain a special point. Let $\sigma : M \to N$ and $\tau : N \to M$ be $\mathbb{Z}[\Phi]$-injections, these clearly exist

since $\mathbb{Q}\otimes_{\mathbb{Z}}M \cong_{\mathbb{Q}[\Phi]} \mathbb{Q}\otimes_{\mathbb{Z}}N$. Since $H^2(\Phi, M)$ contains a special point there exists a torsion free group G which fits into

$$0 \rightarrow M \rightarrow G \rightarrow \Phi \rightarrow 1.$$

Hence,

$$0 \rightarrow M_{/\tau(N)} \rightarrow G_{/\tau(N)} \rightarrow \Phi \rightarrow 1$$

is exact. Let $T = M_{/\tau(N)}$ and $\Psi = G_{/\tau(N)}$. Note that T is finite abelian. Thus we have the following short exact sequence

$$0 \rightarrow T \rightarrow \Psi \xrightarrow{\theta} \Phi \rightarrow 1$$

and $(\Psi, \theta) \varepsilon \mathfrak{D}(\Phi)$. The result then follows, since $H^2(\Psi, {}^{\theta}N)$ has a special point corresponding to

$$0 \rightarrow N \rightarrow G \rightarrow \Psi \rightarrow 1.$$

($\Leftarrow$) Suppose we have $(\Psi, \theta) \varepsilon \mathfrak{D}(\Phi)$ and a $\mathbb{Z}[\Phi]$-module $N \varepsilon \mathfrak{C}$ such that $H^2(\Psi; {}^{\theta}N)$ contains special points. Let this point correspond to the sequence

$$0 \rightarrow N \rightarrow G \xrightarrow{\pi} \Psi \rightarrow 1.$$

Let $M = \ker(\theta \circ \pi)$, then we have a short exact sequence

$$0 \rightarrow M \rightarrow G \rightarrow \Phi \rightarrow 1.$$

That M is abelian and torsion-free is clear. So M is a well defined $\mathbb{Z}[\Phi]$-module. Also, $M\otimes_{\mathbb{Z}}\mathbb{Q} \cong N\otimes_{\mathbb{Z}}\mathbb{Q}$, hence $M \varepsilon \mathfrak{C}$. Since G is torsion-free $H^2(\Phi, M)$ contains a special point.

□

Fix Φ a finite group. Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be a complete set of irreducible $\mathbb{Q}[\Phi]$-modules, hence,

$$\mathbb{Q}[\Phi] \cong \mathcal{M}_1^{e_1} \oplus \ldots \oplus \mathcal{M}_n^{e_n} \qquad \text{for some } e_1, \ldots, e_n$$

and

$$\mathcal{M}_i \cong_{\mathbb{Q}\Phi} \mathcal{M}_j \quad \Leftrightarrow \quad i = j.$$

Proposition (see (73.6) of [Cu-Re]) (3.3) : Let R be a principal ideal domain with quotient field k. Then every $k[\Phi]$-module is the extension of scalars of an $R[\Phi]$-module.

Thus the problem of finding $\mathbb{Z}[\Phi]$-modules with special points does not require a classification of $\mathbb{Z}[\Phi]$-modules, but only a classification of the cohomology group $H^2(T, \Phi)$ where T is finite abelian. This is by no means an easy problem either, but it is hoped that this will prove useful, for instance in finding bounds for the minimal dimension for all finite groups. The author hopes to explore this in later publications.

Chapter III: Complex and projective manifolds.

§1 Complex manifolds

Let M be a complex manifold. By a *smooth complex (resp. holomorphic) vector bundle* $\mathcal{E}$ we mean a quintuple (E, M, π, F) where

(i) E is a complex manifold and can be written

$$E = \bigcup_{p \in M} E_p$$

with E_p a complex vector space for each $p \in M$;

(ii) The projection map $\pi : E \to M$ given by $\pi_{/E_p} : E_p \to \{p\}$ is smooth (resp. holomorphic);

(iii) F is a complex vector space;

(iv) there exists a locally finite covering $\{U_i\}$ of M such that for each $p \in M$ there exists a U_i containing p and a smooth (resp. holomorphic) equivalence $h_i : \pi^{-1}(U_i) \to U_i \times F$ for which $h_{i/E_p} : E_p \to \{p\} \times F$ is a vector space isomorphism.

For all $p \in M$ we may give sets $\pi^{-1}(p)$ the structure of a complex vector space; this follows from (iii) of the definition. The vector spaces $\pi^{-1}(p)$ are the *fibres* of the vector bundle.

Let M be a complex manifold. By a vector bundle map $\alpha : \mathcal{E} \to \mathcal{E}'$, $\mathcal{E} = (E, M, \pi, \mathbb{C}^m)$ and $\mathcal{E}' = (E', M, \rho, \mathbb{C}^n)$ we mean a holomorphic map $\alpha : E \to E'$ such that

(i) $\pi = \rho \circ \alpha$

(ii) $\alpha_{/\pi^{-1}(p)}$ is $\mathbb{C}$-linear for each $p \in M$.

Note that (ii) is a correct definition because (i) implies that α maps fibres in $\mathcal{E}$ to fibres in $\mathcal{E}'$. Let $\mathfrak{B}_{\mathbb{C}}(M)$ denote the category of (holomorphic) vector bundles over M.

The *transition maps* for a holomorphic vector bundle $\mathcal{E} = (E, M, \pi, \mathbb{C}^m)$ are,

$$h_{ij} : U_i \cap U_j \to GL_m(\mathbb{C})$$

defined by, $h_{ij} = h_i \circ (h_j)^{-1}$. We have the cocycle condition,

$$h_{ik} = h_{ij} \cdot h_{jk} \qquad U_i \cap U_j \cap U_k \neq \emptyset.$$

In $\mathfrak{B}_{\mathbb{C}}(M)$ we have the following constructions:

(i) the Whitney sum $\oplus$;

(ii) the tensor product $\otimes_{\mathbb{C}}$;

(iii) the exterior product $\wedge$;

(iv) the complex dual $*$;

(v) conjugation $-$;

(vi) the mixed dual $\bullet$.

Each is well defined by corresponding complex vector space constructions, [At]. If V is a complex vector space then the complex dual is given by

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}),$$

and the mixed dual by

$$V^\bullet = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$$

Corresponding to the vector space isomorphism

$$\wedge^n(V \oplus V') \cong \underset{p+q=n}{\oplus} \wedge^p(V) \otimes \wedge^q(V')$$

(see chapter 4 §5 of [Bo]) we have the following isomorphism holomorphic vector bundles

$$\wedge^n(\mathcal{E} \oplus \mathcal{E}') \cong \underset{p+q=n}{\oplus} \wedge^p(\mathcal{E}) \otimes \wedge^q(\mathcal{E}').$$

Similarly, for vector spaces we have,

$$V^\bullet = V^* \oplus \overline{V^*},$$

hence,

$$\mathcal{E}^\bullet = \mathcal{E}^* \oplus \overline{\mathcal{E}^*}.$$

Let M be a smooth real manifold. By an *almost complex structure* for M we mean a tensor field $J : \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$, of type (1,1), such that $J^2 = -1$. Clearly

M must have even real dimension for there to exist an almost complex structure. However, this is not a sufficient condition: S^{4n} ($n \geq 1$) does not admit an almost complex structure, see **(41.20)** of [St]; in this reference a quasi complex manifold is smooth real manifold admitting an almost complex structure. Note that a smooth real manifold admits an almost complex structure if and only if the smooth tangent bundle TM admits the structure of a smooth complex vector bundle.

By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ as real vector spaces we may consider an underlying real manifold of M. We have a well defined almost complex structure on this real manifold. By a *complex structure* for M we mean an almost complex structure which induces the local complex coordinate system of M.

Let M have complex structure J. By a *Hermitian structure* on M we mean a Riemann metric $g : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \rightarrow \mathcal{F}(M)$ on the underlying real manifold such that $g \circ (J \times J) = g$. A *Hermitian* manifold is a complex manifold admitting a Hermitian structure. By a *Kählerian structure* we mean a Hermitian structure such that $\nabla_X(J) = 0$ for all $X \in \mathfrak{X}^1(M)$, where ∇_X is the (Levi-Cività) connection of g. Equivalently, we could say that J is uniquely determined by J_p, at some point $p \in M$, by parallel translation. Again a *Kählerian* manifold is a complex manifold admitting a kählerian structure. We note that the kählerian condition is defined only locally, hence any complex submanifold of a kählerian manifold is kählerian.

We say that a complex manifold is *flat* if its underlying real manifold is flat. Let M be a compact flat kählerian manifold of complex dimension n, let $G = \pi_1(M)$ be the fundamental group of M and $\Phi = \Phi(M)$ be the holonomy group of M. We have the following short exact sequence of groups

$$0 \rightarrow N \rightarrow G \rightarrow \Phi \rightarrow 1 \qquad \{1.1\}$$

where:

(i) N ($\cong \mathbb{Z}^{2n}$) is a faithful $\mathbb{Z}[\Phi]$-module {1.2};

(ii) G is torsion free {1.3};

(iii) Φ is finite {1.4}.

We shall describe the nature of such short exact sequences associated to compact flat kählerian manifolds and later to compact flat complex projective manifolds.

Let Φ be finite group and N a $\mathbb{Z}[\Phi]$-module (torsion free, as usual). Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We say that N admits a *complex structure* when there exists a map $t \in \mathrm{End}_{\mathbb{R}[\Phi]}(N_{\mathbb{R}})$ such that $t^2 = -1$.

Theorem [Jo-Re] (1.1) : Consider the short exact sequence of groups {1.1} satisfying conditions {1.2} – {1.4}. G is the fundamental group of a (compact flat) kählerian manifold with holonomy group Φ if and only if N admits a complex structure.

Let Φ be a finite group and N a $\mathbb{Z}[\Phi]$-module. We clearly have a complex structure on the $\mathbb{Z}[\Phi]$-module $N \oplus N$; $t : N_{\mathbb{R}} \oplus N_{\mathbb{R}} \to N_{\mathbb{R}} \oplus N_{\mathbb{R}}$ defined by $t(x,y) = (y,-x)$. Thus, if N occurs in a short exact sequence {3.1}, of a compact flat riemannian manifold the, then we can construct a torsion free group G' to complete the following diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & G & \to & \Phi & \to & 1 \\ & & \downarrow \Delta & & \downarrow & & \downarrow \mathrm{Id}_{\Phi} & & \\ 0 & \to & N \oplus N & \to & G' & \to & \Phi & \to & 1 \end{array} \qquad \{1.5\}$$

where Δ is the diagonal map $\Delta(x) = (x,x)$ for $x \in N$. Hence, the short exact

sequence {3.5} corresponds to a compact flat kählerian manifold. We have prove the following

Theorem [Jo-Re] (1.1) : Any finite group is the holonomy group of a compact flat kählerian manifold.

Recall the definition of n-dimensional complex projective space $\mathbb{P}_n(\mathbb{C})$. Define $\sim$ on $\mathbb{C}^{n+1} - \{0\}$ by $(z_1, \ldots, z_{n+1}) \sim (\zeta_1, \ldots, \zeta_{n+1})$ if and only if $z_i = \lambda.\zeta_i$ for some $\lambda \in \mathbb{C}$ for all i. Then

$$\mathbb{P}_n(\mathbb{C}) = \mathbb{C}^{n+1} - \{0\}_{/\sim}.$$

The Fubini-Study metric gives $\mathbb{P}_n(\mathbb{C})$ the structure of a complex manifold of dimension n. Let $U_j = \{ (z_i) \in \mathbb{P}_n(\mathbb{C}) : z_j \neq 0\}$. Then $\{U_j\}_{1 \leq j \leq n}$ is an open covering of $\mathbb{P}_n(\mathbb{C})$. Define $\phi_j : U_j \to \mathbb{C}^n$ by

$$\phi_j(z_1, \ldots, z_{n+1}) = (z_1/z_j, \ldots, z_{j-1}/z_j, z_{j+1}/z_j, \ldots, z_{n+1}/z_j).$$

It is easily checked that these are local coordinates for a complex manifold.

By a *projective manifold* we mean a complex manifold which can me holomorphically embedded into $\mathbb{P}_n(\mathbb{C})$ for some n. It is well-known that the Fubini-Study metric gives $\mathbb{P}_n(\mathbb{C})$ the structure of a kählerian manifold, see for example [Mo-Ko]. Hence any projective manifold is also kählerian.

Consider again the short exact sequences {1.1}, satisfying {1.2} . . {1.4} with G the fundemental group of some compact complex manifold with holonomy group Φ, $N \cong \mathbb{Z}^{2n}$, where n is the complex dimension of M. Let $N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$.

Suppose N does admit a complex structure, t. Let $(N_{\mathbb{R}})^t$ denote the complex vector space which has underlying real space $N_{\mathbb{R}}$ and complex scalar multiplication given by $(x + yi).v = x.v + y.t(v)$ for all $v \in N_{\mathbb{R}}$. We say that N admits a projective structure if, in addition to its complex structure, the complex torus

$(N_{\mathbb{R}})^t/N$ is projective as a complex manifold. We say that a complex torus is *algebraic* when it is projective as a complex manifold.

Theorem [Jo4] (1.3) : Consider the short exact sequence of groups {1.1} satisfying conditions {1.2} – {1.4}. G is the fundamental group of a compact flat projective manifold with holonomy group Φ if and only if N admits a projective structure.

For use in the next section we state the Kodaira embedding theorem.

Let M be a complex manifold. In $\mathfrak{B}_C = \mathfrak{B}_C(M)$ we have the obvious notion of isomorphisms, denoted $\alpha : \mathcal{E} \xrightarrow{\cong} \mathcal{E}'$ where $\mathcal{E} = (E, M, \pi, \mathbb{C}^m)$ and $\mathcal{E}' = (E', M, \rho, \mathbb{C}^n)$. We must have $m = n$. Let the vector bundles $\mathcal{E}$ and $\mathcal{E}'$ have transition maps h_{ij} and h'_{ij}, respectively, with respect to one locally finite covering of M, (such a covering exists by refinement). Then, $\mathcal{E} \cong \mathcal{E}'$ if and only if $n = m$ and there exists a holomorphic maps $\alpha_i : U_i \to GL_n(\mathbb{C})$ such that

$$h_{ij}(p) = \alpha_i(p).h_{ij}(p).(\alpha_j(p))^{-1} \qquad \text{for all } p \in U_i \cap U_j.$$

Let M be a complex manifold. By a (*holomorphic*) *line bundle* over M we mean a vector bundle $\mathcal{E} = (E, M, \pi, \mathbb{C})$. Let

$$\text{Pic}(M) = \{ \text{ holomorphic line bundles over M} \}/\cong.$$

This has a natural group structure: let $\mathcal{E}$ and $\mathcal{E}'$ be represntatives for classes in Pic(M) with transition functions h_{ij} and h'_{ij} respectively with respect to one locally finite covering, on transition functions the group structure is given by,

(i) $\{h_{ij}\}.\{h'_{ij}\} = \{h_{ij}.h'_{ij}\}$,

(ii) $\{h_{ij}\}^{-1} = \{h_{ij}{}^{-1}\}$

(iii) and with identity the trivial line bundle $(\mathbb{C} \times M, M, \mathbb{C}, \pi_M)$.

Let M be a smooth manifold and k a commutative ring. A *sheaf* $\mathcal{S}$ (of k-modules) on M is a triple $\mathcal{S} = (S, M, \pi)$ where

(i) S is a topological space;

(ii) π is a local homeomorphism of S onto M;

(iii) $\pi^{-1}(p)$ is a k-module for all $p \in M$;

(iii) the map $(s,t) \mapsto \alpha.s + \beta.t$ is continuous on $\pi^{-1}(p)$, where $\alpha, \beta \in k$, for all $p \in M$.

The k-module $S_p = \pi^{-1}(p)$ is called the *stalk* in $\mathcal{S}$ over p. A *sheaf homomorphism* $\alpha : \mathcal{S} \to \mathcal{S}'$ is a continuous map of $\mathcal{S}$ into $\mathcal{S}'$ such that

(i) $\pi' \circ \alpha = \pi$;

(ii) $\alpha_{/S_x}$ is a k-module homomorphism.

Note that (ii) is a correct definition because (i) implies that α maps stalks in $\mathcal{S}$ to stalks in $\mathcal{S}'$. Thus we have a category of k-sheaves over the manifold M.

With the above notation let $H^i(M, \mathcal{S})$ denote the i^{th} cohomology group of M with coefficients in $\mathcal{S}$. For details we direct the reader to [Mo-Ko].

Let $\mathcal{O}_p$ ($\mathcal{O}_p^*$) denote the set of germs at p of all (non-vanishing) holomorphic functions. Let

$$\mathcal{O} = \bigcup_{p \in M} \mathcal{O}_p \qquad (\mathcal{O}^* = \bigcup_{p \in M} \mathcal{O}_p^*).$$

Then $\mathcal{O}$ ($\mathcal{O}^*$) is a sheaf, the $\mathbb{R}$-sheaf of germs of (non-vanishing) holomorphic functions on M, with topology generated by the following open sets: let $\phi \in \mathcal{O}_p$ ($\mathcal{O}_p^*$); for each holomorphic function f, with $f_p = \phi$, and neighbourhood U of p define an open set by

$$\mathcal{U}(\phi, f, U) = \{ f_q : q \in U \}.$$

Complex line bundles and sheaves are connected by the isomorphism

$$\mathrm{Pic}(M) \cong H^1(M, \mathcal{O}^*)$$

which is immediate from the definition of sheaf cohomology.

The notion of complexes and exact sequences exists in the category of sheaves. We have:

Theorem (Long exact sequence) (1.4) : Let M be a smooth manifold and

$$0 \to \mathcal{R} \xrightarrow{\alpha} \mathcal{S} \xrightarrow{\beta} \mathcal{T} \to 0$$

a short exact sequence of sheaves over M. Then there exists a map δ such that

$$\to \quad H^0(M,\mathcal{R}) \xrightarrow{\alpha_*} H^0(M,\mathcal{S}) \xrightarrow{\beta_*} H^0(M,\mathcal{T}) \xrightarrow{\delta} H^1(M,\mathcal{R})$$

$$\to \quad H^1(M,\mathcal{R}) \xrightarrow{\alpha_*} H^1(M,\mathcal{S}) \xrightarrow{\beta_*} \quad . \ . \ . \ .$$

is an exact sequence of groups.

Let $\mathbb{Z}$ denote the sheaf of germs constant integer valued functions on M. The sheaf cohomology of M with coefficients in $\mathbb{Z}$ is the the same as integer cohomology of M, [Sp]. We the following exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

where the map $\mathcal{O} \to \mathcal{O}^*$ is given by $f \mapsto e^{2\pi i f}$. Hence by applying the Long Exact Sequence Theorem we have an induced map $\delta : H^1(M,\mathcal{O}^*) \to H^2(M,\mathbb{Z})$. Let $\mathcal{E}$ be a complex line bundle on M. We call its image $\delta(\mathcal{E}) \in H^2(M,\mathbb{Z})$ the Chern class of $\mathcal{E}$.

Define

$$A^n(M) = \{ \text{ smooth sections of } \wedge^n(TM^{\bullet}) \}$$

$$A^{r,s}(M) = \{ \text{ smooth sections of } \wedge^p(TM^*) \otimes \wedge^q(\overline{TM^*}\},$$

the sets of n-forms and (r,s)-forms respectively.

We have the following identity,

$$A^n(M) = \bigoplus_{r+s=n} A^{r,s}(M).$$

To define exterior differentiation $d : A^n(M) \to A^{n+1}(M)$ (locally) we introducing bases

$$\frac{\partial}{\partial z_1}, \ . \ . \ . \ , \frac{\partial}{\partial z_n} \qquad \text{for } T_pM,$$

$$\frac{\partial}{\partial \overline{z}_1}, \ . \ . \ . \ , \frac{\partial}{\partial \overline{z}_n} \qquad \text{for } \overline{T_pM}$$

and dual bases

$$dz_1, \ . \ . \ . \ , dz_n \qquad \text{for } T_pM^*$$

$$d\overline{z}_1, \ . \ . \ . \ , d\overline{z}_n \qquad \text{for } \overline{T_pM}^*.$$

Then, for $r + s = n$, define

$$d : A^{r,s}(M) \to A^{r+1,s}(M) \oplus A^{r,s+1}(M) \subset A^{n+1}(M)$$

on local basis elements of $A^{r,s}(M)$ by,

$$d(f_{\underline{i}\,\underline{j}}.dz_{i_1} \wedge \ . \ . \ \wedge dz_{i_r} \otimes d\overline{z}_{j_1} \wedge \ . \ . \ \wedge d\overline{z}_{j_s})$$

$$= \sum_{\alpha=1}^{n} \frac{\partial}{\partial z_1}(f_{\underline{i}\,\underline{j}}).dz_\alpha \wedge dz_{i_1} \wedge \ . \ . \ \wedge dz_{i_r} \otimes d\overline{z}_{j_1} \wedge \ . \ . \ \wedge d\overline{z}_{j_s}$$

$$+ (-1)^r \sum_{\alpha=1}^{n} \frac{\partial}{\partial \overline{z}_1}(f_{\underline{i}\,\underline{j}}).dz_{i_1} \wedge \ . \ . \ \wedge dz_{i_r} \otimes d\overline{z}_\alpha \wedge d\overline{z}_{j_1} \wedge \ . \ . \ \wedge d\overline{z}_{j_s}$$

It is easily verified that,

$$d^2 = 0.$$

We have now a complex $\{A^n(M), d\}$, with which we define the de Rham cohomology of M as

$$H^n_{dR}(M) = \frac{\ker(d : A^n(M) \to A^{n+1}(M))}{\mathrm{im}(d : A^{n-1}(M) \to A^n(M))}.$$

Theorem (de Rham, see [Mo-Ko]) (1.5) :

$$H^n_{dR}(M) = H^n(M, \mathbb{C}).$$

Let ω be a Hermitian metric of M: then ω uniquely determines a $(1,1)$ form of M. We denote this again by $\omega \ \varepsilon \ A^{11}(M)$. Then it can be shown that the earlier conditions for such a metric to be kählerian are equivalent to $d(\omega) = 0$. Suppose that M is kählerian with metric ω. Then ω induces a class $[\omega] \ \varepsilon \ H^2(M, \mathbb{C})$, by (1.6).

Let $\mathcal{S}$ be a holomorphic line bundle on M. We say that the line bundle $\mathcal{S}$ is

positive when there exists kählerian metric ω on M such that

$$\delta(\mathfrak{S})\otimes 1 = \tfrac{1}{2\pi i}[\omega],$$

where $\delta(\omega) \varepsilon H^2(M,\mathbb{Z})$ is the chern class of $\mathfrak{S}$, and $\delta(\mathfrak{S})\otimes 1 \varepsilon H^2(M,\mathbb{C})$.

Theorem (Kodaira's Embedding Theorem, [Ko] (1.6) : Let M be a compact complex manifold. M is projective if and only if M admits a positive line bundle.

§2 Riemann matrices and projective embeddings

We shall work with Riemann matrices: the notions of complex tori and Riemann matrices are equivalent for our purposes. For convenience in later sections, we shall work over a commutuative ring k contained in the real number field.

Let V a k-module of finite rank. By a lattice in V we mean a free abelian group, Λ, of maximal rank contained in V, so $\mathrm{rk}_{\mathbb{Z}}(\Lambda) = m$. For a k-module V let $V_{\mathbb{R}}$ denote the $\mathbb{R}$-space $\mathbb{R}\otimes_k V$. By a *Riemann matrix* over k we mean a triple (V,Λ,t) where

(i) V is a k-module;

(ii) Λ is a lattice in V;

(iii) $t : V_{\mathbb{R}} \to V_{\mathbb{R}}$ such that $t^2 = -1$.

For a Riemann matrix (V,Λ,t), let $(V_{\mathbb{R}})^t$ denote the complex vector space with underlying real space isomorphic to $V_{\mathbb{R}}$ and with complex multiplication given by

$$(x + yi).v = x.v + y.t(v) \text{ for all } v \varepsilon V_{\mathbb{R}}.$$

We naturally identify V with $k\otimes_{\mathbb{Z}}\Lambda$.

By a map of Riemann matrices $\phi : (V, \Lambda, t) \to (U, \Omega, s)$ we mean a k-linear map $\phi : V \to U$ such that

(i) $\phi(\Lambda) \subset \Omega$;

(ii) $(1 \otimes \phi) \circ t = s \circ (1 \otimes \phi)$

where $1 \otimes \phi : \mathbb{R} \otimes_k V \to \mathbb{R} \otimes_k V$ is the induced map.

We have a category of Riemann matrices over k, which we denote by $\mathfrak{R}_k$. Over $\mathbb{Z}$ we consider Riemann matrices as pairs (Λ, t). To any $(V, \Lambda, t) \in \mathfrak{R}_k$ we associate the complex torus $(V_{\mathbb{R}})^t/_{\Lambda}$. We say that $(V, \Lambda, t) \in \mathfrak{R}_k$ is *algebraic* when the complex torus $(V_{\mathbb{R}})^t/_{\Lambda}$ is algebraic.

Let $\mathcal{V}$ be a complex vector space. By a meromorphic function, $F : \mathcal{V} \to \mathbb{C}$, we mean a quotient of holomorphic functions $f, g : \mathcal{V} \to \mathbb{C}$,

$$F = f/g$$

where g is not identically zero.

Let $(V, \Lambda, t) \in \mathfrak{R}_k$, let $L : (V_{\mathbb{R}})^t \times V_{\mathbb{R}} \to \mathbb{C}$ and $J : \Lambda \to \mathbb{C}$ satisfy:

(i) L is $\mathbb{C}$-linear in the first variable and $\mathbb{Z}$-linear when the second variable is restricted to $\Lambda \subset V_{\mathbb{R}}$;

(ii) $L(\lambda, \mu) \equiv L(\mu, \lambda) \pmod{\mathbb{Z}} \; \forall \; \lambda, \mu \in \Lambda$;

(iii) $J(\lambda + \mu) - J(\lambda) - J(\mu) \equiv L(\lambda, \mu) \pmod{\mathbb{Z}} \; \forall \; \lambda, \mu \in \Lambda$.

Then, by a *theta function* of (V, Λ, t) of *type* (L, J) we mean a non-zero meromorphic function Θ on $V_{\mathbb{R}}$ such that

$$\Theta(x + \lambda) = \Theta(x).e^{2\pi i[L(x, \lambda) + J(\lambda)]} \qquad x \in V_{\mathbb{R}}, \; \lambda \in \Lambda.$$

The theta functions of a fixed type (L, J) form a real vector space, [La].

Let $(V, \Lambda, t) \in \mathfrak{R}_k$, then by a *Riemann form* for (V, Λ, t) we mean a non-degenerate alternating bilinear form $B : V \times V \to k$ such that, if $\mathfrak{B} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ denotes the realisation of B, then

(i) $\mathfrak{B}(\Lambda,\Lambda) \subset \mathbb{Z} \subset k$.

(ii) $\widetilde{\mathfrak{B}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\widetilde{\mathfrak{B}}(x,y) = \mathfrak{B}(t(x),y) \quad \forall x,y$$

is symmetric and positive definite.

Suppose $(V,\Lambda,t) \in \mathfrak{R}_k$ admits a Riemann form, $B : V \times V \to k$. Then we may define a type for (V,Λ,t). Define a Hermitian form on $(V_{\mathbb{R}})^t$ by

$$H(x,y) = \mathfrak{B}(ix,y) + i\mathfrak{B}(x,y)$$

where, again $\mathfrak{B}$ is the realisation of B. Define $L : (V_{\mathbb{R}})^t \times \Lambda_{\mathbb{R}} \to \mathbb{C}$ by

$$L(x,y) = \tfrac{1}{2i}.H(x,y),$$

and $J : \Lambda \to \mathbb{C}$ by

$$J(\lambda+\mu) - J(\lambda) - J(\mu) \equiv L(\lambda,\mu) \pmod{\mathbb{Z}} \text{ for all } \lambda,\mu \in \Lambda.$$

Then (L,J) is a type.

If Θ is an entire theta function of type (L,J) on $(V,\Lambda,t) \in \mathfrak{R}_k$, then for $x,y \in (V_{\mathbb{R}})^t$ define

$$\mathfrak{B}(x,y) = L(x,y) - L(y,x).$$

Then $\mathfrak{B}$ is $\mathbb{R}$-bilinear, alternating and real valued and the form $\widetilde{\mathfrak{B}} = \mathfrak{B} \circ (t \times 1)$ is symmetric. Also, $\mathfrak{B}$ takes integer values on Λ. Let $\mathfrak{B} = 1 \otimes (\mathfrak{B}_{/\Lambda \times \Lambda}) : V \times V \to k$. We say the theta function is *non-degenerate* when the associated form $\mathfrak{B}$ is non-degenerate. In this case the form $\mathfrak{B} : V \times V \to k$ is a Riemann form for V.

***Theorem* (2.1)** : Let $(V,t,\Lambda) \in \mathfrak{R}_k$. The following are equivalent:

(i) (V,t,Λ) is algebraic;

(ii) (V,t,Λ) admits a Riemann form;

(iii) (V,t,Λ) admits a positive entire theta function.

We have already shown the equivalence of (ii) and (iii). We give a brief outline a proof of (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) using the Kodaira embedding theorem.

Let $\mathcal{E}$ be a positive line bundle on $X = V/\Lambda$. Then the chern class of $\mathcal{E}$ gives a 2-cocycle $\zeta \in Z^2(\Lambda, \mathbb{Z})$. We are using the natural isomorphism $H^2(\Lambda, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$. Define $B : \Lambda \times \Lambda \to \mathbb{Z}$ by $B(x,y) = \zeta(x,y) - \zeta(y,x)$ and extend to k. Then B is non-degenerate and alternating, and $\mathfrak{B} \circ (1 \times t)$ is symmetric and positive definite, where $\mathfrak{B}$ is the realisation of B.

Now let Θ be a positive entire theta function on (V, t, Λ). Define an action of Λ on $\mathbb{C} \times V$, $\alpha : \Lambda \times (\mathbb{C} \times V) \to \Lambda \times (\mathbb{C} \times V)$, by $(\lambda, (z,x)) \mapsto (\Theta(x).z, z + \lambda)$. Let $E = \mathbb{C} \times V/\alpha$. Then, $\mathcal{E} = (E, X, \pi_2, \mathbb{C})$ is a positive line bundle on X.

The implication (iii) $\Rightarrow$ (i) is essentially a theorem of Lefscetz. For a positive entire theta function Θ on a Riemann matrix $(V, t, \Lambda) \in \mathfrak{R}_k$, let $\mathcal{L}(\Theta)$ denote the space of all entire theta functions which have the same type as Θ. Let $\{\Theta_1, \ldots, \Theta_n\}$ be a $\mathbb{C}$-basis for $\mathcal{L}(\Theta)$, then,

Theorem ([Lef1], see [La]) (2.2) : The map

$$x \mapsto (\Theta_1(x), \ldots, \Theta_n(x))$$

is an embedding of V/Λ into $\mathbb{P}_n(\mathbb{C})$.

Finally, we briefly describe the notion of the ring of endomorphisms of a Riemann matrix, this ring is classically known as the ring of complex multiplications of the Riemann matrix.

We say that a Riemann matrix (V, Λ, t) over k is *simple* when V contains no k-submodule U such that $t(U_{\mathbb{R}}) \subset U_{\mathbb{R}}$. Let X and Y be complex tori, by an *isogeny* $\phi : X \to Y$ we mean a surjective holomorphic map which has finite kernel. We say

that Riemann matrices (V,Λ,t), (W,Ω,s) are *isogenous*, written $(V,\Lambda,t) \sim (W,\Omega,s)$, when there exists an isogeny $\phi : (V_{\mathbb{R}})^t/\Lambda \to (W_{\mathbb{R}})^s/\Omega$. ϕ induces, and is induced form an $\mathbb{R}$-isomorphism $\phi' : V_{\mathbb{R}} \to W_{\mathbb{R}}$ such that $\phi'(\Lambda) \subset \Omega$, see [La].

Proposition (Poincaré's complete reducibility) (2.3) : Let (V,Λ,t) be an algebraic Riemann matrix over k. Then,

$$(V,\Lambda,t) \sim (V_1,\Lambda_1,t_1)^{e_1} \oplus \ldots . \oplus (V_n,\Lambda_n,t_n)^{e_n}$$

where (V_i,Λ_i,t_i), for $1 \leq i \leq n$, are pairwise non-isomorphic simple algebraic Riemann matrices over k. Moreover, the isomorphism types (V_i,Λ_i,t_i) and multiplicities e_i are unique up to order.

For $(V,\Lambda,t) \varepsilon \mathfrak{R}_k$ let $\mathrm{End}_{\mathfrak{R}_k}(V,\Lambda,t)$ denote the ring of $\mathfrak{R}_k$-endomorphisms of (V,Λ,t).

Corollary (2.4) : If (V,Λ,t) is algebraic then $\mathrm{End}_{\mathfrak{R}_k}(V,\Lambda,t)$ is semisimple, let $D_i = \mathrm{End}_{\mathfrak{R}_k}(X_i,t_i)$, then D_i is a division algebra over k, and

$$\mathrm{End}_{\mathfrak{R}_k}(V,\Lambda,t) \cong M_{e_1}(D_1) \times \ldots \times M_{e_n}(D_n).$$

Let $(V,\Lambda,t) \varepsilon \mathfrak{R}_k$ be algebraic. It is well known, see [Mu], that $\mathrm{End}_{\mathfrak{R}_k}(V,\Lambda,t)$ admits a positive involution σ. This involution is the Rosati involution and can be constructed as follows. To $(V,\Lambda,t) \varepsilon \mathfrak{R}_k$ define the dual Riemann matrix $(V,\Lambda,t)^* = (V^*,\Lambda^*,t^*) \varepsilon \mathfrak{R}_k$, where V^* is the k-dual of V, Λ^* is the $\mathbb{Z}$-dual of Λ and t^* is the $\mathbb{R}$-dual of t. Let $B : V \times V \to k$ be a Riemann form for (V,Λ,t). We then have a map $\hat{B} : V \to V^*$ given by $\hat{B}(x)(y) = B(x,y)$. It is easily verified that $\hat{B}$ is a Riemann matrix isomorphism, $\hat{B} : (V,t) \xrightarrow{\sim} (V,t)^*$. Define the Rossati involution by

$$\alpha^{\sigma} = \hat{B}^{-1} \circ \alpha^{*} \circ \hat{B},$$

where α^{*} is the k-dual of α. Let $A = \mathrm{End}_{\mathfrak{R}_k}(V, \Lambda, t)$. Using results on positively involuted algebras, it can be shown that we have the following decomposition of involuted algebras

$$(A,\sigma) \cong M_{e_1}(D_1,\widetilde{\sigma}_1) \times \ldots \times M_{e_n}(D_n,\widetilde{\sigma}_n)$$

where σ_i is the Rosati involution of D_i $(= \mathrm{End}_{\mathfrak{R}_k}(V_i, \Lambda_i, t_i))$.

§3 The existence of complex and projective structures

We fix Φ a finite group and a commutative ring $k \subset \mathbb{R}$. Let W a finitely generated $k[\Phi]$-module, and put $W_{\mathbb{R}} = W \otimes_k \mathbb{R}$ which we consider as a $\mathbb{R}[\Phi]$-module. We say that W admits a *complex structure* when there exists a map $t \in \mathrm{End}_{\mathbb{R}[\Phi]}(W_{\mathbb{R}})$ such that $t^2 = -1$. Denote by $\mathfrak{I}_k(W,\Phi)$ the set of all such complex structures on W. Note that, $\mathfrak{I}_k(W,\Phi) = \mathfrak{I}_{\mathbb{R}}(W_{\mathbb{R}},\Phi) \subset \mathrm{End}_{\mathbb{R}[\Phi]}(\mathcal{W})$.

Let $k \subset \mathbb{E} \subset \mathbb{R}$ be a tower of rings. Put $U = W \otimes_k \mathbb{E}$ and let $U_1, \ldots, U_f$ be $\mathbb{E}[\Phi]$-submodules of U such that $U = U_1 \oplus \ldots \oplus U_f$. Then we have an injection

$$\prod_{i=1}^{f} \mathfrak{I}_{\mathbb{E}}(U_i,\Phi) \quad \rightarrow \quad \mathfrak{I}_k(W,\Phi). \qquad \{3.1\}$$

given by

$$(t_1, \ldots, t_f) \mapsto t_1 \oplus \ldots \oplus t_f.$$

Proposition (3.1) : Suppose $k \subset \mathbb{E} \subset \mathbb{R}$ is a tower of fields. Let V be a $k[\Phi]$-module and $U = W \otimes_k \mathbb{E}$. If $U_1, \ldots, U_f$ are the isotypic $\mathbb{E}[\Phi]$-components of U then

$$\mathfrak{I}_k(W,\Phi) = \prod_{i=1}^{f} \mathfrak{I}_{\mathbb{E}}(U_i,\Phi).$$

Proof : Let $t : W_{\mathbb{R}} \rightarrow W_{\mathbb{R}}$, then by (I.2.2) $\mathbb{R} \otimes_{\mathbb{E}} U_i$ and $\mathbb{R} \otimes_{\mathbb{E}} U_j$ have no common simple submodules. Hence, $t(\mathbb{R} \otimes_{\mathbb{E}} U_i) \subset \mathbb{R} \otimes_{\mathbb{E}} U_i$. Thus $t = t_1 \oplus \ldots \oplus t_f$ where $t_i : \mathbb{R} \otimes_{\mathbb{E}} U_i \rightarrow \mathbb{R} \otimes_{\mathbb{E}} U_i$. Also it is clear that

$$t \in \mathfrak{I}_k(W,\Phi) \Leftrightarrow t_i \in \mathfrak{I}_{\mathbb{E}}(U_i,\Phi) \text{ for all i.}$$

Hence the result follows.

□

We have the following special cases,

***Proposition* (3.2)** : If $W = W_1 \oplus \ldots \oplus W_n$ is the isotypic $k[\Phi]$-decomposition of W, then

$$\mathfrak{I}_k(W,\Phi) = \prod_{i=1}^{n} \mathfrak{I}_k(W_i,\Phi).$$

***Proposition* (3.3)** : If $W_{\mathbb{R}} = \mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_n$ the isotypic $\mathbb{R}[\Phi]$-decomposition of $W_{\mathbb{R}}$, then

$$\mathfrak{I}_k(W,\Phi) = \prod_{i=1}^{n} \mathfrak{I}_{\mathbb{R}}(\mathcal{W}_i,\Phi).$$

We say that a simple $\mathbb{R}[\Phi]$-module $\mathcal{V}$ is of $\mathbb{R}$-type if $\mathrm{End}_{\mathbb{R}[\Phi]}(\mathcal{V}) = \mathbb{R}$. Let T = I, II, III or IV and let V be a simple $\mathbb{Q}[\Phi]$-module. We say that V is of *type* T if $\mathrm{End}_{\mathbb{Q}[\Phi]}(V)$ is of type T in Albert's classification of positively involuted rational division algebras. We extend these definitions to isotypic modules in the obvious way.

Let $W = V^{(e)}$ be an isotypic $\mathbb{Q}[\Phi]$-module (V simple), and $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. If W is of type (I) let $\mathbb{E} = D$, otherwise we write $D = (D,\mathbb{E},\tau,\xi)$, a CM-algebra, and $\mathbb{F} = \mathbb{E}(\xi)$. Let $g = \dim_{\mathbb{Q}}\mathbb{E}$, and for types (II), (III) and (IV) let $d^2 = \dim_{\mathbb{F}} D$, where $d = 2$ for (II) and (III).

Theorem (see (3.1) of [Jo4]) (3.4) : Let N be a $\mathbb{Z}[\Phi]$-module, $W = \mathbb{Q}\otimes_{\mathbb{Z}}N$ and $W_{\mathbb{R}} = \mathbb{R}\otimes_{\mathbb{Z}}W$. The following are equivalent:

(i) N admits a complex structure;

(ii) each $\mathbb{Q}[\Phi]$-isotypic component of W admits a complex structure;

(iii) each $\mathbb{R}[\Phi]$-isotypic component of $W_{\mathbb{R}}$ admits a complex structure;

(iv) each $\mathbb{Q}[\Phi]$-simple summand of type I in W occurs with even multiplicity;

(v) each $\mathbb{R}[\Phi]$-simple summand of $\mathbb{R}$-type in $W_{\mathbb{R}}$ occurs with even multiplicity.

Proof: By (3.2) and (3.3) we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). We will show that (iv) $\Leftrightarrow$ (v), (iii) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (iii). It is clear that at each stage in the proof we may assume the module to be isotypic.

(iv) $\Leftrightarrow$ (v) : Assume that $W = V^{(e)}$ is $\mathbb{Q}[\Phi]$-isotypic (V $\mathbb{Q}[\Phi]$-simple). If W is of type (III) or (IV) there is nothing to prove, by Albert's classification. If W is of type (II) then the multiplicity of the $\mathbb{R}[\Phi]$-simple module in $W_{\mathbb{R}}$ is 2e, by (I.2.4). Hence we are reduced to consider the case of W being type (I), but then the result follows by hypothesis.

(iii) $\Rightarrow$ (v) : Assume that $W_{\mathbb{R}} = \mathcal{V}^{(f)}$ is $\mathbb{R}[\Phi]$-isotypic ($\mathcal{V}$ $\mathbb{R}[\Phi]$-simple) and of $\mathbb{R}$-type. So $\mathrm{End}_{\mathbb{R}[\Phi]}(\mathcal{V}) \cong \mathbb{R}$ and $\mathrm{End}_{\mathbb{R}[\Phi]}(W_{\mathbb{R}}) \cong M_f(\mathbb{R})$. Suppose their exists $T \in M_f(\mathbb{R})$ such that $T^2 = -1$. Then we could give $\mathbb{R}^f$ the structure of a complex vector space. Hence f must be even.

(iv) $\Rightarrow$ (ii) : In chapter IV we shall give a complete parametrisation of $\mathfrak{I}_{\mathbb{Q}}(W,\Phi)$ for any W, and this implcation will follow form that. For now, we shall indicate the existence of a single complex structure. Again we may assume that $W = V^{(e)}$ is $\mathbb{Q}[\Phi]$-isotypic (V $\mathbb{Q}[\Phi]$-simple).

If W is of type (I) then by hypothesis e is even. Write $W = (V \oplus V)^{(e/2)}$, hence $W_{\mathbb{R}} = (V_{\mathbb{R}} \oplus V_{\mathbb{R}})^{(e/2)}$. Define $t : (V_{\mathbb{R}} \oplus V_{\mathbb{R}})^{(e/2)} \to (V_{\mathbb{R}} \oplus V_{\mathbb{R}})^{(e/2)}$ by $t : ((x_1,y_1), \ldots, (x_{e/2},y_{e/2})) \to ((y_1,-x_1), \ldots, (y_{e/2},-x_{e/2}))$. This clearly is a

complex structure for W.

Now suppose W is of type (II), (III) or (IV). Corresponding to the simple $\mathbb{Q}[\Phi]$-module V there is a unique factor in the Wedderburn decomposition of $\mathbb{Q}[\Phi]$. Let this factor be A ($= M_m(D)$, for some m). Identify V with the first column of A, with coefficients in D. Fix isomorphisms $V \cong D^{(m)}$ and $W \cong D^{(em)}$ and consider V and W as a right D-spaces. Let $\mathfrak{D} = \mathbb{R} \otimes_{\mathbb{Q}} D$, then $W_{\mathbb{R}} \cong \mathfrak{D}^{(em)}$. We define $t : W_{\mathbb{R}} \to W_{\mathbb{R}}$ by $t(1 \otimes x_1, \ldots, 1 \otimes x_{em}) = (\frac{1}{\sqrt{-\xi^2}} \otimes \xi.x_1, \ldots, \frac{1}{\sqrt{-\xi^2}} \otimes \xi.x_{em})$. Then $t^2 = -1$, since $\xi^2 = -1$ and t is a complex structure since $\xi \in D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. This completes the proof.

□

Let W be a $k[\Phi]$-module and $t \in \mathrm{End}_{\mathbb{R}[\Phi]}(W_{\mathbb{R}})$ a complex structure. By a *Riemann form* for the pair (W, t) we mean a non-degenerate alternating k-bilinear form

$$B : W \times W \to k,$$

with realisation $\mathfrak{B} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$, such that

$$\widetilde{\mathfrak{B}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R} \text{ defined by } \quad \widetilde{\mathfrak{B}}(x, y) = \mathfrak{B}(t(x), y) \text{ for all } x, y$$

is symmetric and positive definite.

This definition is justified by,

***Proposition* (3.5) :** Suppose k is an algebraic number field with R as a ring of algebraic integers. Let W be a $k[\Phi]$ module and $t \in \mathrm{End}_{\mathbb{R}[\Phi]}(W_{\mathbb{R}})$ a complex structure. The pair (W, t) admits a Riemann form if and only if there exists a $R[\Phi]$-lattice $\Lambda \subset V$ such that the Riemann matrix (W, t, Λ) admits a Riemann form.

The reverse implication is trivial. The foward implication will follow from a Lemma.

Lemma (3.6) : Let k be an algebraic number field. Let Λ_1 and Λ_2 be $\mathbb{Z}$-lattices in a finite dimensional k-space V, and let $t \in \mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$ such that $t^2 = -1$. The Riemann matrix (V, t, Λ_1) admits a Riemann form if and only if (V, t, Λ_2) admits a Riemann form.

Proof: Since k is finite dimensional over $\mathbb{Q}$, there exists integers n, m such that $m.\Lambda_2 \subset \Lambda_1$ and $n.\Lambda_1 \subset \Lambda_2$. Suppose that (V, t, Λ_1) admits a Riemann form $\beta_1 : V \times V \to k$. Let $\beta = m^2.\beta_1 : V \times V \to k$, we claim that β is a Riemann form for (V, t, Λ_2). We need only verify that β takes integer values on $\Lambda_2 \times \Lambda_2$. Let $x, y \in \Lambda_2$, then $\beta(x, y) = m^2.\beta_1(x, y) = \beta_1(m.x, m.y) \in \mathbb{Z}$, since $m.x, m.y \in \Lambda_1$. Similarly, if (V, t, Λ_2) admits a Riemann form β_2, then (V, t, Λ_1) admits a Riemann form $n^2.\beta_2$. The result follows.

□

Proof of (3.5) : R is a P.I.D with k as the field of fractions, hence, by (3.3), there exists an $R[\Phi]$-lattice Λ in V. Since there exists a Reimann form on (W, t) their clearly exists some $\mathbb{Z}$-lattice, Λ', such that (W, t, Λ') admits a Riemann form. The result now follows from the Lemma, since we may consider Λ as a $\mathbb{Z}$-lattice.

□

Let W be a $k[\Phi]$-module. By a *projective structure* for W we mean a complex structure t such that the pair (W, t) admits a Riemann form. Let $\mathfrak{S}_k(W, \Phi, B)$ denote the set of projective structures on W such that (W, t) admits a Riemann form B.

As an immediate corollary to (3.5) we have,

***Proposition* (3.7)** : Let N be a $\mathbb{Z}[\Phi]$-module. Then, N admits a projective structure if and only if $\mathbb{Q}\otimes_{\mathbb{Z}}N$ admits a projective structure.

For any $\mathbb{Q}[\Phi]$-module we shall define a "canonical form" determined by its rational endomorphisms. Let $W = V^{(e)}$ be an isotypic $\mathbb{Q}[\Phi]$-module (V simple), and $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. Recall our previous notation. If W is of type (I) let $\mathbb{E} = D$, otherwise we write $D = (D,\mathbb{E},\tau,\xi)$, a CM-algebra, and $\mathbb{F} = \mathbb{E}(\xi)$, as in **(1.3)**. Let $g = \dim_{\mathbb{Q}}\mathbb{E}$, and for types (II), (III) and (IV) let $d^2 = \dim_{\mathbb{F}}D$.

We define a *canonical form* $B : W \times W \to \mathbb{Q}$ for W as follows.

If W is of type (I) and e is even then let

$$S : V \times V \to \mathbb{E}$$

be any positive definite symmetric $\mathbb{E}$-bilinear form and define

$$S^D : (V \oplus V) \times (V \oplus V) \to \mathbb{E}$$

by

$$S^D((x_1,x_2),(y_1,y_2)) = S(x_1)S(y_2) - S(x_2).S(y_1)$$

(S^D is then a non-degenerate skew symmetric $\mathbb{E}$-bilinear form).

Define β by

$$(W,\beta) = (\,(V \oplus V), S^D)^{(e/2)}.$$

If W is of type (II), (III) or (IV) define

$$\eta : D \times D \to \mathbb{E}$$

by

$$\eta(x, y) = \mathrm{Tr}_{\mathbb{E}}(x^{\tau}y\xi).$$

Corresponding to the simple $\mathbb{Q}[\Phi]$-module V there is a unique factor in the Wedderburn decomposition of $\mathbb{Q}[\Phi]$. Let this factor be A ($= M_m(D)$, for some m). We may identify V with the first column of A, with coefficients in D. We fix an isomorphisms $V \cong D^{(m)}$ and $W \cong D^{(em)}$ and consider V and W as a right D-spaces.

Define β by

$$(W,\beta) = (D,\eta)^{(em)}$$

For isotypic $\mathbb{Q}[\Phi]$-modules of define the *canonical form* to be $B = \mathfrak{R}_{\mathbb{E}/\mathbb{Q}}(\beta) : W \times W \to \mathbb{Q}$. For an arbitrary $\mathbb{Q}[\Phi]$-module we define a *canonical form* by defining forms on its isotypic components. The canonical form is an non-degenerate alternating $\mathbb{Q}$-bilinear form.

Lemma (3.8) : Let W be a $\mathbb{Q}[\Phi]$-module with canonical form $B : W \times W \to \mathbb{Q}$. If summands of type (I) in W occur with even multiplicity then $\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B) \neq \emptyset$.

Proof : Since the canonical form is defined on the isotypic components of W and the set of complex structures on W splits on the isotypic components of W we may assume W is isotypic. In chapter IV we shall also give a parametrisation of $\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B)$ for the canonical form. For now we indicate the existences of a single projective structure on W. Let $W = V^{(e)}$, V simple. Consider the complex structures, t, constructed in (3.4). Let $\mathfrak{B} : W_{\mathbb{R}} \times W_{\mathbb{R}} \to \mathbb{R}$ denote the realisation of B. We need only verify that $\mathfrak{B} \circ (t \times 1)$ is symmetric and positive definite for each type.

Suppose that W is of type (I), then e is even by hypothesis. Let $\mathfrak{s} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ denote the realisation of $S : V \times V \to \mathbb{Q}$. Then it is easily verified that

$$\mathfrak{B} \circ (t \times 1) = \mathfrak{s} \perp \ldots \perp \mathfrak{s} \qquad \text{(e copies)}.$$

Hence $\mathfrak{B} \circ (t \times 1)$ is symmetric and positive definite, by the definition of S.

Now suppose that W has type (II), (III) or (IV). By the constructions of B and t we may restrict our verification to $\mathfrak{D} = \mathbb{R} \otimes_{\mathbb{Q}} D$. Let $\mathfrak{E} : \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}$ denote the realisation of $\mathfrak{R}_{\mathbb{E}/\mathbb{Q}}(\eta) : D \times D \to \mathbb{Q}$. Then

$$\begin{aligned}
\mathfrak{E} \circ (t \times 1) &= \mathrm{Tr}_{\mathbb{R}}(t(1 \otimes x)^{\tau}.(1 \otimes y).(1 \otimes \xi)) \\
&= \mathrm{Tr}_{\mathbb{R}}((\tfrac{1}{\sqrt{-\xi^2}} \otimes \xi.x)^{\tau}.(1 \otimes y).(1 \otimes \xi)) \\
&= \mathrm{Tr}_{\mathbb{R}}((1 \otimes x)^{\tau}.(1 \otimes y))
\end{aligned}$$

Since,

$$\begin{aligned}(1\otimes\xi.x)^{\tau}.(1\otimes\xi) &= (1\otimes\xi).(1\otimes\xi.x)^{\tau}\\ &= (-1\otimes\xi)^{\tau}.(1\otimes\xi.x)^{\tau}\\ &= (\ (1\otimes\xi.x).(1\otimes\xi)\ )^{\tau}\\ &= (1\otimes x)^{\tau}.\end{aligned}$$

Hence, $\mathfrak{B}\circ(t \times 1)$ is symmetric, and positive definite since τ is a positive involution.

□

By **(3.4)**, we have,

***Corollary* (3.9)** : Let W be a $\mathbb{Q}[\Phi]$-module. Then W admits a complex structure if and only if W admits a projective structure.

Denote by $\mathfrak{K}_{flat}$ ($\mathfrak{P}_{flat}$) the class of fundamental groups of compact flat kählerian (complex projective) manifolds. Let $G \in \mathfrak{K}_{flat}$, then G fits into a short exact sequence

$$0 \rightarrow N \rightarrow G \rightarrow \Phi \rightarrow 1$$

such that the $\mathbb{Z}[\Phi]$-module N admits a complex structure. Hence $\mathbb{Q}\otimes_{\mathbb{Z}}N$ admits a complex structure. By **(3.8)**, $\mathbb{Q}\otimes_{\mathbb{Z}}N$ admits a projective structure. N also then admits a projective structure by **(3.6)**. Finally $G \in \mathfrak{P}_{flat}$ by **(1.4)**. Hence, we have,

***Theorem* [Jo5] (3.10)** : We have the following equality,

$$\mathfrak{K}_{flat} = \mathfrak{P}_{flat}.$$

If we were to consider fundamental groups which have a type of certain finite holonomy group then the result follows by much simpler considerations. We shall consider the special case of nilpotent holonomy groups. We shall only need to consider the notion of a field of CM-type rather than the more general notion of a CM-algebra.

By a field k of *CM-type* we mean a totally imaginary algebraic number field quadratic over a totally real field, see [Sh-Ta]. Let $k \subset \mathbb{R}$ be a field and A a finite dimensional k-algebra. Let $t \in \mathrm{End}_{\mathbb{R}}(A_{\mathbb{R}})$ be a complex structure, $t^2 = -1$. We say that t is a *complex algebra structure* when we can consider $(A_{\mathbb{R}})^t$ as a complex algebra with imaginary scalar multiplication given by t.

Proposition **(3.11)** : Let $\mathbb{K}$ be a field of CM-type and quadratic over the totally real field $\mathbb{E}$. Let $\dim_{\mathbb{Q}}\mathbb{E} = g$. Let R denote the ring of algebraic integers in $\mathbb{K}$. If t is a complex algebra structure for $\mathbb{K}$ then the Riemann matrix $(\mathbb{K}, t, R)$ is algebraic.

Proof: Let $\dim_{\mathbb{Q}}(\mathbb{E}) = g$. Then

$$\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{E} = \mathbb{R}_1 \times \ldots \times \mathbb{R}_g$$

where $g = \dim_{\mathbb{Q}}\mathbb{E}$, and

$$\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{K} = \mathbb{R}_1\otimes_{\mathbb{E}}\mathbb{K} \times \ldots \times \mathbb{R}_g\otimes_{\mathbb{E}}\mathbb{K}.$$

Each $\mathbb{R}_\lambda\otimes_{\mathbb{E}}\mathbb{K} \cong \mathbb{C}$, $\lambda = 1, \ldots, g$, hence $\mathbb{K}$ has exactly 2^g complex structures, corresponding to the g choices of $\pm i \in \mathbb{C}$. By fixing a choice of complex structure we fix isomorphisms $\alpha_\lambda : \mathbb{R}_\lambda\otimes_{\mathbb{E}}\mathbb{K} \xrightarrow{\sim} \mathbb{C}$. Write $k = \mathbb{Q}(\xi)$, where $\xi^2 \in \mathbb{E}$ and ζ is choosen so that $\alpha_\lambda(\xi) = \eta_\lambda.i$ where $\eta \in \mathbb{R}$ and $\eta > 0$. Define $B : \times\mathbb{K} \to \mathbb{Q}$ by

$$B(x,y) = \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\overline{x}.y.\xi).$$

B is skew symmetric and it may easily be verified to be a Riemann form for our fixed complex structure.

□

Lemma (3.12) : If C_m denotes the cyclic group of order m, then

$$\mathbb{Q}[C_m] \cong \prod_{r|m} \mathbb{Q}(\xi_r),$$

where, for each $r \mid m$, ξ_r is a primitive r^{th} root of unity. Moreover, for each $r \geq 3$, $\mathbb{Q}(\xi_r)$ is totally imaginary and quadratic over $\mathbb{Q}(\xi_r + \overline{\xi}_r)$.

Corollary (3.13) : Let C be a cyclic group and W be a $\mathbb{Q}[C]$-module. Then any complex structure for W is a projective structure.

Proof: It is clear that we need only show that any irreducible $\mathbb{Q}[C]$-module of dimension greater than 1 (if one exists) admits a projective structure. We may identify such a module with $\mathbb{Q}(\xi)$, where ξ is a primitive root of unity. We claim that any complex structure for $\mathbb{Q}(\xi)$ is also a complex algebra structure. We need only verify that $x.t = t.x$ for all $x \in \mathbb{Q}(\xi)_{\mathbb{R}}$. But t is $\mathbb{R}$-linear so this is equivalent to

$$(1 \otimes \xi).t = t.(1 \otimes \xi).$$

However, multiplication by $1 \otimes \xi$ is just the C-action on $\mathbb{Q}(\xi)_{\mathbb{R}}$, and t commutes with this C-action, so any complex structure gives $\mathbb{Q}(\xi)$ the structure of a complex algebra. The result now follows from (3.11).

□

Proposition (3.14) : Let Φ, Θ be finite groups and $\pi : \Phi \to \Theta$ a group epimorphism. Let W be a $\mathbb{Q}[\Phi]$-module. Suppose that the action of Φ on W factors through π. Let V be the corresponding $\mathbb{Q}[\Theta]$-module. Then

(i) W is irreducible $\Leftrightarrow$ V is irreducible

(ii) W admits a complex structure $\Leftrightarrow$ V admits a complex structure

(iii) W admits a projective structure $\Leftrightarrow$ V admits a projective structure.

Corollary (3.15) : Let Φ be a finite abelian group and W a $\mathbb{Q}[\Phi]$-module. Then any complex structure for W is a projective structure. Any irreducible $\mathbb{Q}[\Phi]$-module of dimension greater than 1 admits a projective structure.

Proof: We may assume $W = V^{(e)}$ is isotypic (V $\mathbb{Q}[\Phi]$-simple). Let $\rho : \Phi \to GL_{\mathbb{Q}}(V)$ denote the action of Φ on V. Let $C = \Phi/\ker(\rho)$. Then V factors through a faithful irreducible action $\rho' : C \to GL_{\mathbb{Q}}(V)$. C is finite abelian and ρ' embeds C into a division ring, hence C is cyclic. The result follows from (3.13) and (3.14).

□

Proposition (3.16) : Let Θ be a subgroup of a finite group Φ. Let V be a $\mathbb{Q}[\Theta]$-module. Let $W = \mathrm{Ind}_{\Theta}^{\Phi}(V)$. Then

(i) V is faithful $\Rightarrow$ W is faithful

(ii) V is irreducible $\Rightarrow$ W is irreducible

(iii) a complex structure on V induces a complex structure on W

(iv) a projective structure on V induces a projective structure on W.

Proof: (i) and (ii) are well known, see [Se].

(iii) We make the following identification $W_{\mathbb{R}} = \mathbb{R}[\Phi] \otimes_{\mathbb{R}[\Theta]} V_{\mathbb{R}}$. Let $t \in \mathfrak{I}(V,\Theta)$ and define $T : \overline{W} \to \overline{W}$ by

$$T(x \otimes \overline{v}) = x \otimes t(\overline{v}).$$

Clearly, $T \in \mathfrak{I}(W, \Phi)$.

(iv) Let $t \in \mathfrak{S}(V, \Theta, B)$ with $B : V \times V \to \mathbb{Q}$ a Riemann form. Let $C = \{c_1, \ldots, c_r\}$ be a set of coset representatives for Φ/Θ and $\{e_1, \ldots, e_m\}$ an $\mathbb{Q}$-basis for V. Then $\{ c_i \otimes e_j : 1 \leq i \leq r;\ 1 \leq j \leq m \}$ is a $\mathbb{Q}$-basis for W. Define $F : W \times W \to \mathbb{Q}$ by

$$F(c \otimes e_i,\, c' \otimes e_j) = \begin{cases} 0 & \text{if } c \neq c' \\ E(e_i, e_j) & \text{if } c = c' \end{cases}$$

where c, $c' \in C$. Then F is a Riemann form for $(W, \iota(t))$. If Λ is a $\mathbb{Z}[\Theta]$-lattice such that (V, t, Λ) is an algebraic Riemann matrix, then $\Omega = \mathrm{Ind}_{\Theta}^{\Phi}(\Lambda) = \mathbb{Z}[\Phi] \otimes_{\mathbb{Z}[\Theta]} \Lambda$ is a lattice in W such that $(W, \iota(t), \Omega)$ is algebraic.

□

We introduce some finite groups. For $\alpha \geq 4$ let D_{2^α} denote the dihedral group with presentation

$$D_{2^\alpha} = \langle\, X, Y : X^r = Y^2 = 1\,,\ Y^{-1}XY = X^{-1} \,\rangle; \qquad r = 2^{\alpha-1}.$$

For $\alpha \geq 4$ let SD_{2^α} denote the special dihedral group with presentation

$$SD_{2^\alpha} = \langle\, X, Y : X^r = Y^2 = 1\,,\ Y^{-1}XY = X^{-1+r/2} \,\rangle; \qquad r = 2^{\alpha-1}.$$

For $\alpha \geq 3$ let $Q(2^\alpha)$ denote the quaternion group and define

$$Q_{2^\alpha} = \langle\, X, Y : X^s = Y^2\ ,\ Y^4 = 1\,,\ Y^{-1}XY = X^{-1} \,\rangle; \qquad s = 2^{\alpha-2}.$$

Proposition [Ro, Ec-Mu] (3.17) : Let Φ be a finite nilpotent group and let V be a faithful irreducible $\mathbb{Q}[\Phi]$-module. Then V is induced from a representation of a subgroup H of Φ, which can be written $H = \Theta \times \Psi$, where Ψ is a cyclic group of odd order and Θ is a cyclic dihedral special dihedral or quaternion group. If Φ is a p-group (even with $p = 2$) then V is induced from a representation of a cyclic subgroup.

Theorem (3.18) : Let Φ be a finite nilpotent group and W a $\mathbb{Q}[\Phi]$-module. Then every irreducible $\mathbb{Q}[\Phi]$ module of $\mathbb{Q}$-dimension greater than 1 admits a projective structure. Thus, W admits a complex structure if and only if W also admits a projective structure.

Proof: Let $\rho : \Phi \to GL_{\mathbb{Q}}(W)$ denote the action of Φ on W. Let $\Theta = \Phi/\ker(\rho)$. Let W' denote the $\mathbb{Q}[\Theta]$-module through which W factors. Θ is also nilpotent. We first consider two special cases:

Suppose Φ has odd order and W is irreducible. Then W' is a faithful and irreducible and so is induced by a faithful irreducible representation of a cyclic subgroup of Θ, by **(3.17)**. This representation of a cyclic group admits a projective structure by **(3.13)**. Hence we have the result by **(3.16)**.

Now suppose Φ is a 2-group. We show that every irreducible $\mathbb{Q}[\Phi]$-module of $\mathbb{Q}$-dimension greater than 1 (if one exists) admits a projective structure. Suppose W is such a module, then $|\Theta| > 2$ by the definition of W. By **(3.17)** $W' = \mathrm{Ind}_C^{\Theta}(V)$ where C is a cyclic subgroup of Θ and V factors through a representation of a cyclic group of order 2. But V must be faithful, hence C has order 2. Let D be a subgroup of Θ of order 4 containing C. One must exist. We claim that D is cyclic. Otherwise $D \cong C(2) \times C(2)$. Put $U = \mathrm{Ind}_C^D(V)$, then U is not irreducible. This is a contradiction since $W' (= \mathrm{Ind}_D^{\Theta}(U))$ is irreducible. Thus W' is induced from the faithful irreducible representation of a cyclic subgroup of order 4. Hence we again have the result by **(3.16)**.

We return to the general case in the theorem: let Φ be a finite nilpotent group. Write $\Phi = \Theta \times \Psi$; where Θ is a 2-group and 2 does not divide the order of Ψ. Any irreducible $\mathbb{Q}[\Phi]$ module, U, then has the form $V \otimes_{\mathbb{Q}} W$; where V (resp. W) is an irreducible Θ (resp. Ψ) module.

If $\dim_{\mathbb{Q}}(V) = 1$ then U admits a projective structure with respect to the action of Φ if and only if W admits a projective strucure with respect to the action of Ψ. Hence we consider the case where both V and W have $\mathbb{Q}$-dimension greater than 1. Let ρ denote the action of Θ on V; put $\Theta_o = \Theta/\ker(\rho)$ and let ρ' denote the faithful action of Θ_o on V. Let $\Gamma = \langle X \rangle \subset \Theta_o$ denote the cyclic subgroup of

order 4 from which ρ' is induced. Let $\pi : \Theta \to \Theta_o$ denote the canonical projection. Choose $x \in \Theta$ such that $\pi(x) = X$. Let $s : V_{\mathbb{R}} \to V_{\mathbb{R}}$ by $s = \rho(x) \otimes 1$. Then s is a projective structure for V. Let $E : V \times V \to \mathbb{Q}$ be a Riemann form for (V, s), let $\mathcal{E}$ denote its realisation. Let $t : W_{\mathbb{R}} \to W_{\mathbb{R}}$ be any projective structure for W. Let $F : W \times W \to \mathbb{Q}$ be a Riemann form for (W, t), let $\mathcal{F}$ denote its realisation. Identify $U_{\mathbb{R}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} W_{\mathbb{R}}$, and let $T : U_{\mathbb{R}} \to U_{\mathbb{R}}$ by $T(v \otimes w) = v \otimes t(w)$. This is a complex structure for U. Let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be $\mathbb{Q}$-bases for V and W respectively. Identify these with the $\mathbb{R}$-bases for $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. Consider the form $G : U \times U \to \mathbb{Q}$ by $G(e_i \otimes f_p, e_j \otimes f_q) = \tilde{E}(e_i, e_j).F(f_p, f_q)$, where $\tilde{E}(v_1, v_2) = E(s(v_1), v_2)$ for all $v_1, v_1 \in V$. G is skew symmetric since E is symmetric while F is skew symmetric. Also, the associated form $\tilde{G}$ is positive definite; this follows from

$$\begin{aligned}\tilde{G}(e_i \otimes f_p, e_j \otimes f_q) &= G(T(e_i \otimes f_p), e_j \otimes f_q) \\ &= G(e_i \otimes t(f_p), e_j \otimes f_q) \\ &= \tilde{E}(e_i, e_j).\tilde{F}(f_p, f_q).\end{aligned}$$

Thus Ω is a Riemann form for (U, T), and hence U admits a projective structure.

Let Λ be a $\mathbb{Z}[\Theta]$-lattice of V such that the Riemann matrix (V, s, Λ) has Riemann form B and let Ω be a $\mathbb{Z}[\Psi]$-lattice in W such that (W, t, Ω) has Riemann form F. Let $\mathfrak{L} = \Lambda \otimes_{\mathbb{Z}} \Omega$, then $\mathfrak{L}$ is a $\mathbb{Z}[\Phi]$-lattice for U. Also, $(U, T, \mathfrak{L})$ is an algebraic Riemann matrix with Riemann form G: it is easily checked that G takes integral values on $\mathfrak{L} \times \mathfrak{L}$.

□

Chapter IV : Complex structures.

Let M be a flat compact riemannian manifold of dimension n. Then M is covered by a flat n-torus $T \to M$, and the group of deck transformations of this covering is isomorphic to the holonomy group, Φ, of M. Let

$$0 \to N \to G \to \Phi \to 1,$$

denote the holonomy exact sequence of M, and let $\widetilde{M}$ denote the universal covering of M. Then $T = \widetilde{M}/N$. A complex structure for M induces a complex structure for T, and any complex for T which commutes with the action of Φ induces a complex structure for M. Moreover, if M is a complex manifold then $T \to M$ is a holomorphic covering.

The following is well-known.

***Proposition* (0.1) :** Let M' be a holomorphic covering of a connected complex manifold M. Then,

(i) M is kählerian if and only if M' is kählerian,

and, if M' is a finite covering, then

(ii) M is complex projective if and only if M' is complex projective.

Hence we have bijections,

$$\{\text{kählerian (projective) structures for X}\}$$
$$\leftrightarrow \{\text{kählerian (projective) structures for } \hat{X} \text{ which are } \Phi\text{-invariant}\}.$$

Now let M be a flat compact complex projective manifold. T is a flat algebraic torus. By a *polarisation* for M we mean a polarisation for the complex torus T. Write $T = V^t/N$, where $V \cong \widetilde{M}$ and $t : V \to V$ is a complex structure on V induced form the complex structure on M. Then we shall think of a polarisation of M

as just a Riemann form for the Riemann matrix (V, N, t).

In this chapter we shall parametrise the kählerian structures of a flat riemannian manifold and the projective structures for certain manifolds with respect to a canonical polarisation.

Let G be a Lie group (real or complex), X a topological space and $\alpha : G \times X \to X$ a smooth group action. Fix $x_0 \in X$ and let H denote the stabiliser of x_0 under the action of G on X. If α is transitive then we may identify X with the homogeneous space G/H. We say that X has parameter space G/H, and write $X = G/H$.

§1 Complex structures

Let Φ be a finite group and N a finitely generated torsion free $\mathbb{Z}[\Phi]$-module. Put $W = \mathbb{Q} \otimes_{\mathbb{Z}} N$. We shall first parametrise $\mathfrak{I}_{\mathbb{Q}}(W, \Phi)$. We introduce some notation.

For a ring R let

$$\mathfrak{I}(R) = \{ x \in R : x^2 = -1 \}.$$

Proposition **(1.1)** : Let $e > 0$, then we have the following parametrisations,

(i) $$\mathfrak{I}(M_e(\mathbb{R})) = \begin{cases} \emptyset & \text{if e is odd} \\ SL_e(\mathbb{R})/SL_{e/2}(\mathbb{C}) & \text{if e is even} \end{cases}$$

(ii) $$\mathfrak{I}(M_e(\mathbb{C})) = \bigcup_{r+s=e}^{\circ} \left(SL_e(\mathbb{C})/SL_r(\mathbb{C}) \times SL_s(\mathbb{C}) \right)$$

(iii) $$\mathfrak{I}(M_e(\mathbb{H})) = \left\{ A \in M_n(\mathbb{H}) : \iota(A) \in \mathfrak{I}(M_e(\mathbb{C})) \right\}$$

where $\iota : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ is the standard embedding.

Proof: (i) Firstly suppose e is odd then if $\mathfrak{J}(\, M_e(\mathbb{R})\,) \neq \emptyset$ we may give $\mathbb{R}^e$ with the structure of a complex vector space, but this is clearly impossible. Let e be even, and write $e = 2n$. Then $\mathfrak{J}(\, M_{2n}(\mathbb{R})\,)$ is clearly non-empty, since $J_n \in \mathfrak{J}(\, M_{2n}(\mathbb{R})\,)$, where,

$$\mathbf{J_n} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix} \qquad \{2.1\}.$$

Moreover, any $J \in \mathfrak{J}(\, M_{2n}(\mathbb{R})\,)$ has minimal polynomial $m_J(t) = t^2 + 1$. Thus, all such J have Normal Form J_n. This follows from the Jordan normal form over C since if

$$A = \tfrac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}$$

then

$$A.\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.A^{-1} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Thus we have a transitive action

$$SL_{2n}(\mathbb{R}) \times \mathfrak{J}(\, M_{2n}(\mathbb{R})\,) \;\rightarrow\; \mathfrak{J}(\, M_{2n}(\mathbb{R})\,)$$

given by

$$(\, A\,,\, J\,) \;\rightarrow\; A.J.A^{-1}.$$

Also, the Cauchy-Riemann embedding of $M_n(\mathbb{C}) \subset M_{2n}(\mathbb{R})$ corresponds to identifying

$$M_n(\mathbb{C}) = \{A \in M_{2n}(\mathbb{R}) : A.J_n = J_n.A\,\}.$$

Thus $\mathrm{Stab}(J_n) = M_n(\mathbb{C})$, and we have the result. Note that by a simple rearrangement of coordinates we could take the following matrix as a Jordan normal form

$$\mathbf{J}'_n = \begin{bmatrix} 0 & iI_n \\ -iI_n & 0 \end{bmatrix}$$

instead of J_n.

(ii) Let $J \in \mathfrak{I}(M_e(\mathbb{C}))$ then it is easily seen that the Jordan normal form for J is

$$\begin{bmatrix} iI_p & 0 \\ 0 & -iI_q \end{bmatrix}$$

for some p and q such that $p + q = e$. Hence the action

$$SL_e(\mathbb{C}) \times \mathfrak{I}(M_e(\mathbb{C})) \rightarrow \mathfrak{I}(M_e(\mathbb{C}))$$

given by

$$(A , J) \rightarrow A.J.A^{-1}$$

has a unique orbit space corresponding to each pair (p,q) with $p + q = e$. It is easy to shown that for $A \in M_e(\mathbb{C})$ we have

$$A.\begin{bmatrix} iI_p & 0 \\ 0 & -iI_q \end{bmatrix} = \begin{bmatrix} iI_p & 0 \\ 0 & -iI_q \end{bmatrix}.A$$

if and only if

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

for $A_{11} \in M_p(\mathbb{C})$ and $A_{22} \in M_q(\mathbb{C})$. That is, $A \in M_p(\mathbb{C}) \times M_q(\mathbb{C})$. Again the result follows.

(iii) This case follows trivially from (ii).

□

Let $k \subset \mathbb{R}$ be a commutative ring. If V a k-module then as before define

$$\mathfrak{I}_k(V) = \{ t \in \mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}}) : t^2 = -1 \}$$

where $V_{\mathbb{R}} = \mathbb{R} \otimes_k V$.

As an immeadiate corollary to (1.1) we have,

***Proposition* (1.2)** : Let V be a finitely generated free k-module of rank e. Then,

$$\mathfrak{I}_k(\,V\,) = \begin{cases} \emptyset & \text{if e is odd} \\ \mathrm{SL}_e(\mathbb{R})/_{\mathrm{SL}_{e/2}(\mathbb{C})} & \text{if e is even.} \end{cases}$$

***Proposition* (1.3)** : Let W be an isotypic $\mathbb{Q}[\Phi]$, $W = V^{(e)}$ where V is irreducible. Let $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. Then,

(i) if W has type (I)

$$\mathfrak{I}_{\mathbb{Q}}(W,\Phi) = \begin{cases} \emptyset & \text{if e is odd} \\ \left(\mathfrak{I}(\,M_{e/2}(\mathbb{R})\,)\right)^g & \text{if e is even,} \end{cases}$$

(ii) if W has type (II)

$$\mathfrak{I}_{\mathbb{Q}}(W,\Phi) = \left(\mathfrak{I}(\,M_e(\mathbb{R})\,)\right)^g,$$

(iii) if W has type (III) then

$$\mathfrak{I}_{\mathbb{Q}}(W,\Phi) = \left(\mathfrak{I}(\,M_e(\mathbb{H})\,)\right)^g,$$

(iv) if W has type (IV) then

$$\mathfrak{I}_{\mathbb{Q}}(W,\Phi) = \left(\mathfrak{I}(\,M_{ed}(\mathbb{C})\,)\right)^g$$

where d is the degree of D over its centre.

***Proof*:** Let $\mathcal{W} = W\otimes_{\mathbb{Q}}\mathbb{R}$ and $\mathcal{V} = V\otimes_{\mathbb{Q}}\mathbb{R}$, so $\mathcal{W} = \mathcal{V}^{(e)}$. For types (II), (III) and (IV) fix $D = (D,\mathbb{E},\tau,a)$ and for type (I) let $\mathbb{F} = \mathbb{E} = D$. Then

$$\mathcal{V} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_e,$$

where $\mathcal{V}_k = V\otimes_{\mathbb{E}}\mathbb{R}_k$, is an isotypic $\mathbb{R}[\Phi]$-decomposition of $\mathcal{V}$. Thus we may write

$$\mathfrak{I}_{\mathbb{Q}}(W,\Phi) = \mathfrak{I}_{\mathbb{R}}(\mathcal{V}_1^{(e)},\Phi)\ \mathrm{x}\ \ldots\ \mathrm{x}\ \mathfrak{I}_{\mathbb{Q}}(\mathcal{V}_g^{(e)},\Phi)$$

Moreover,

$$\mathrm{End}_{\mathbb{R}[\Phi]}(\mathcal{V}_k) \cong \begin{cases} \mathbb{R} & \text{(I)} \\ M_2(\mathbb{R}) & \text{(II)} \\ \mathbb{H} & \text{(III)} \\ M_d(\mathbb{C}) & \text{(IV)} \end{cases}$$

Hence,

$$\mathfrak{I}_{\mathbb{R}}(\mathcal{V}_k^{(e)},\Phi) \quad = \quad \begin{cases} \mathfrak{I}(\ M_e(\mathbb{R})\ ) & \text{(I)} \\ \mathfrak{I}(\ M_{2e}(\mathbb{R})\ ) & \text{(II)} \\ \mathfrak{I}(\ M_e(\mathbb{H})\ ) & \text{(III)} \\ \mathfrak{I}(\ M_{ed}(\mathbb{C})\ ) & \text{(IV)} \end{cases}$$

Hence, the result follows from **(1.1)**.

□

Recall our original notation: let N be a finitely generated $\mathbb{Z}[\Phi]$-module, $W = \mathbb{Q}\otimes_{\mathbb{Z}}N$. Let T denote the torus $W_{\mathbb{R}}/N$. We are now in a position to parametrise the set of complex structures on T which commute with the Φ action, denote such a set by $\mathfrak{I}(T,\Phi)$.

For $t \in \mathfrak{I}_{\mathbb{Q}}(W,\Phi)$, let $\mathbb{T}_t$ denote the complex torus

$$\mathbb{T}_t = W^t/N.$$

Fix $t_0 \in \mathfrak{I}_{\mathbb{Q}}(W,\Phi)$, then for any $t \in \mathfrak{I}_{\mathbb{Q}}(W,\Phi)$ there exists an $\alpha \in SL_{\mathbb{R}}(W_{\mathbb{R}})$ such that $t = \alpha.t_0.\alpha^{-1}$, by **(III.3.3)**. We clearly have,

$$\mathbb{T}_t = W^{t_0}/\alpha(N),$$

and,

Theorem **(1.4)** : Let N be a finitely generated $\mathbb{Z}[\Phi]$-module, and $W = \mathbb{Q}\otimes_{\mathbb{Z}}N$. Let T $= V_{\mathbb{R}}/N$, then

$$\mathfrak{I}(T,\Phi) = \mathrm{Aut}_{\mathbb{Z}[\Phi]}(N)\backslash\mathfrak{I}_{\mathbb{Q}}(W,\Phi).$$

§2 Projective structures I.

Let Φ be a finite group and N a finitely generated $\mathbb{Z}[\Phi]$-module. Put $W = \mathbb{Q}\otimes_{\mathbb{Z}}N$. Recall the definition of the canonical form for a $\mathbb{Q}[\Phi]$-module, see chapter III. We introduce a canonical form (again see chapter III) on W as follows. Split W into its $\mathbb{Q}[\Phi]$-isotypic components

$$W = W_1 \oplus \ldots \oplus W_f,$$

we split each W_i by its simple factor

$$W_i = V_i^{(e_i)}.$$

Let $D_i = \mathrm{End}_{\mathbb{Q}[\Phi]}(V_i)$. Write $D_i = (D_i, \mathbb{E}_i, \tau_i, \xi_i)$, a CM-algebra, if D_i (and therefore V_i) is of type (II), (III) or (IV) otherwise let $\mathbb{E}_i = D_i$. We define a canonical form $\beta_i : W_i \times W_i \to \mathbb{E}_i$ as in chapter III. Let $B_i = \mathfrak{R}_{\mathbb{E}_i/\mathbb{Q}}(\beta_i) : W_i \times W_i \to \mathbb{Q}$, and put

$$B = B_1 \perp \ldots \perp B_f : W \times W \to \mathbb{Q}.$$

Then as a corollary to (III.3.2) we have,

Proposition (2.1) :

$$\mathfrak{S}_{\mathbb{Q}}(W, \Phi, B) = \prod_{i=1}^{f} \mathfrak{S}_{\mathbb{Q}}(W_i, \Phi, \mathfrak{B}_i).$$

Hence, to parametrise $\mathfrak{S}_{\mathbb{Q}}(W, \Phi, B)$ for B canonical, we may assume that W is isotypic.

Let $\mathbb{E}$ be a totally real field and $\mathbb{E}/\mathbb{Q}$ be a finite extension with $\dim_{\mathbb{Q}}\mathbb{E} = g$. Let $\mathfrak{F}_{\mathbb{E}} = \{ \sigma_\lambda : \mathbb{E} \to \mathbb{R} \}_{1\leq\lambda\leq g}$ denote the set of embeddings of $\mathbb{E}$ into $\mathbb{R}$. Write

$$\mathbb{E}\otimes_{\mathbb{Q}}\mathbb{R} \cong \mathbb{R}_1 \times \ldots \times \mathbb{R}_g \qquad \{2.1\}.$$

where $\mathbb{R}_\lambda$ denotes the field $\mathbb{R}$ considered as an algebra over $\mathbb{E}$, via the embedding σ_λ for each $\lambda = 1, \ldots, g$.

Proposition (2.2) : Let W be a finite dimensional $\mathbb{E}$-space. Write $W_{\mathbb{R}} = \mathbb{R}\otimes_{\mathbb{Q}}W$ and $W_{\mathbb{R}_\lambda} = \mathbb{R}_\lambda\otimes_{\mathbb{E}}W$. Let $\beta : W \times W \to \mathbb{E}$ be an $\mathbb{E}$-bilinear form and put

$$\mathfrak{B}_\lambda = \mathfrak{S}_{\mathbb{R}_\lambda/\mathbb{E}}(\beta) : W_{\mathbb{R}_\lambda} \times W_{\mathbb{R}_\lambda} \to \mathbb{R}$$

and

$$\mathfrak{B} = \mathfrak{S}_{\mathbb{R}/\mathbb{Q}}(\mathfrak{R}_{\mathbb{E}/\mathbb{Q}}(\beta)) : W_{\mathbb{R}} \times W_{\mathbb{R}} \to \mathbb{R}.$$

Then we have the following orthogonal decomposition

$$(W_{\mathbb{R}},\mathfrak{B}) = (W_{\mathbb{R}_1},\mathfrak{B}_1) \perp \ldots \perp (W_{\mathbb{R}_g},\mathfrak{B}_g).$$

Proof: Let

$$1 = e_1 + \ldots + e_g$$

be the decomposition of $1 \varepsilon \mathbb{R}\otimes_{\mathbb{Q}}\mathbb{E}$ by central idempotents corosponding to {3.1}. Since $\{e_1, \ldots, e_g\}$ is a $\mathbb{R}$-linearly independent set and $\dim_{\mathbb{R}}(\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{E}) = g$, $\{e_1, \ldots, e_g\}$ is an $\mathbb{R}$-basis for $\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{E}$. Let $\{\omega_1, \ldots, \omega_n\}$ be a basis for W / $\mathbb{E}$. Then $\{e_\lambda.(1\otimes\omega_1), \ldots, e_\lambda.(1\otimes\omega_n)\}$ is an $\mathbb{R}$-basis for $\mathcal{W}_\lambda$. Also $\mathfrak{B} = 1\otimes \mathrm{Tr}_{\mathbb{E}/\mathbb{Q}} \circ \mathfrak{S}_{\mathbb{R}/\mathbb{E}}(\beta)$ and $\mathfrak{S}_{\mathbb{R}/\mathbb{E}}(\beta)$ are $\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{E}$-linear. The result follows since $\{e_1, \ldots, e_g\}$ are mutually orthogonal.

□

We fix the following notation. Let $W = V^{(e)}$ be an isotypic $\mathbb{Q}[\Phi]$-module (V simple), and $D = \mathrm{End}_{\mathbb{Q}[\Phi]}(V)$. Then, if W is of type (I) let $\mathbb{E} = D$, otherwise we write $D = (D,\mathbb{E},\tau,\xi)$, a CM-algebra, and $\mathbb{F} = \mathbb{E}(\xi)$, as in (1.3). Let $g = \dim_{\mathbb{Q}}\mathbb{E}$, and for types (II), (III) and (IV), let $d^2 = \dim_{\mathbb{F}}D$.

Corollary (2.3) : Let W be an isotypic $\mathbb{Q}[\Phi]$-module and $B : W \times W \to \mathbb{Q}$ a canonical form for W. By the definition of B we may write $B = \mathfrak{R}_{\mathbb{E}/\mathbb{Q}}(\beta)$ where $\beta : W \times W \to \mathbb{E}$ be an $\mathbb{E}$-bilinear form. Define

$$\mathfrak{B}_\lambda = \mathfrak{S}_{\mathbb{R}_\lambda/\mathbb{E}}(\beta) : W_{\mathbb{R}_\lambda} \times W_{\mathbb{R}_\lambda} \to \mathbb{R}_\lambda$$

Then,

$$\mathfrak{S}_{\mathbb{Q}}(W, \Phi, B) = \prod_{\lambda=1}^{g} \mathfrak{S}_{\mathbb{R}}(W_{\mathbb{R}_\lambda}, \Phi, \mathfrak{B}_\lambda).$$

§3 An elementary device: tensor products.

Let $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $- : \mathbb{A} \to \mathbb{A}$ denote the standard positive involution. We denote by $(\mathfrak{U}, \beta)$ a finite dimensional $\mathbb{A}$-space $\mathfrak{U}$ with an $\mathbb{A}$-form β, (that is β is either symmetric, skew symmetric, Hermitian or skew Hermitian). Note that by Hermitian we mean (i) β $\mathbb{A}$-linear in the second variable and (ii) $\beta(x,y) = \overline{\beta(y,x)}$ $\forall x, y \in \mathfrak{U}$. A morphism between two such objects $(\mathfrak{U}, \beta)$, $(\mathfrak{U}', \beta')$ is an $\mathbb{A}$-linear map $\alpha : \mathfrak{U} \to \mathfrak{U}'$ such that the forms are preserved. If $\mathcal{A} : \mathbb{A}^p \times \mathbb{A}^p \to \mathbb{A}$ is an $\mathbb{A}$-form then we identify $\mathcal{A} = (a_{i,j})_{1 \le i,j \le p}$, the matrix for $\mathcal{A}$ with respect to the standard basis for $\mathbb{A}^p$. We have a coproduct for the above as follows. Define

$$\mathcal{A}\nabla\beta : \mathfrak{U}^p \times \mathfrak{U}^p \to \mathbb{A}$$

by $$\mathcal{A}\nabla\beta(\,((x_1, \ldots, x_p), (y_1, \ldots, y_p)\,) = \sum_{i,j=1}^{p} a_{i,j}.\beta(x_i, y_j).$$

(Note that in terms of matrices this product is just the tensor product.)

And let

$$(\mathfrak{U}, \beta)\nabla\mathcal{A} = (\mathfrak{U}^p, \mathcal{A}\nabla\beta).$$

Proposition (3.1) : The form $\mathcal{A}\nabla\beta$ is non-degenerate precisely when $\mathcal{A}$ and β are non-degenerate.

Proof: ($\Leftarrow$) is easy.

($\Rightarrow$) Firstly it is clear that β is non-degenerate. Suppose $\mathcal{A}$ is degenerate, then we may choose $\underline{\lambda} = (\lambda_1, \ldots, \lambda_p) \neq \underline{0}$ such that

$$\underline{\lambda}.\mathcal{A} = \underline{0}.$$

Let $y = (y_1, \ldots, y_p) \in \mathcal{U}$. Then, for $0 \neq x \in \mathcal{U}$,

$$\begin{aligned}\mathcal{A}\nabla\beta(\ (\lambda_1.x, \ldots, \lambda_p.x)\ ,\ (y_1, \ldots, y_p)\ ) &= \sum_{i,j=1}^{p} a_{i,j}.\beta(\lambda_i.x, y_j) \\ &= \sum_j (\ \sum_i \lambda_i a_{i,j}).\beta'(x, y_j) \\ &= 0.\end{aligned}$$

But this contradicts $\mathcal{A}\nabla\beta$ being non-degenerate. Hence no such $\underline{\lambda}$ exists, and hence $\mathcal{A}$ is non-degenerate.

□

Let $\mathcal{V}$ be an $\mathbb{A}$-space of dimension m, and β: $\mathcal{V} \times \mathcal{V} \to \mathbb{A}$ an $\mathbb{A}$-form. By a *frame* $\mathcal{F} = (\mathcal{U}, \beta', \mathcal{A})$ we mean a triple with $(\mathcal{U}, \beta')$ an $\mathbb{A}$-space with an $\mathbb{A}$-form β' and $\mathcal{A} \in M_p(\mathbb{A})$. We say that $(\mathcal{V}, \beta)$ is *framed* by $(\mathcal{U}, \beta', \mathcal{A})$ when there exists an isomorphism

$$(\mathcal{V}, \beta) \cong (\mathcal{U}, \beta')\nabla\mathcal{A} \qquad \{3.1\}.$$

Thus if $\mathcal{U}$ is of dimension n, n|m, and if we let m = p.n then $\mathcal{A} \in M_p(\mathbb{A})$. If β is Hermitian (skew Hermitian) then we take β' to be Hermitian and then $\mathcal{A}$ is Hermitian (skew Hermitian).

Fix a frame $\mathcal{F} = (\mathcal{U}, \beta', \mathcal{A})$ for a pair $(\mathcal{V}, \beta)$. Let $\theta \in \mathrm{End}_{\mathbb{A}}(\mathcal{V})$ and suppose θ has the following form

$$\theta(x_1, \ldots, x_p) = (\ (\sum_\alpha \theta_{1,\alpha}.x_\alpha, \ldots, \sum_\alpha \theta_{p,\alpha}.x_\alpha)\ )$$

for $(x_1, \ldots, x_p) \in \mathcal{U}^{(p)} = \mathcal{V}$, where each $\theta_{\alpha,i}$ is scalar, then we say that θ is an $\mathcal{F}$-morphism. Let $\mathrm{Scal}_p(\mathbb{A})$ denote the ring of p x p scalar matrices. Then $\mathrm{Scal}_m(\mathbb{A}) \cong \mathbb{A}$, hence if θ is an $\mathcal{F}$-morphism then we may identify $\theta = (\theta_{i,j}) \in M_p(\mathbb{A})$.

***Proposition* (3.2) :** We continue with the above notation, suppose β is skew Hermitian and $\theta \in \mathrm{End}_{\mathbb{A}}(\mathcal{V})$ is an $\mathcal{F}$-morphism. Then

$$\beta\circ(\theta \text{ x } 1) \text{ is Hermitian} \Leftrightarrow \begin{cases} \mathcal{A}\circ(\theta \text{ x } 1) \text{ is Hermitian if } \mathcal{A} \text{ is skew Hermitian} \\ \qquad \text{or} \\ \mathcal{A}\circ(\theta \text{ x } 1) \text{ is skew Hermitian if } \mathcal{A} \text{ is Hermitian.} \end{cases}$$

***Proof*:** Let $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_p) \in \mathcal{U}^{(p)} = \mathcal{V}$. Then

$$\begin{aligned} \beta(\theta(x), y) &= \beta\left(\left(\sum_\alpha \theta_{1,\alpha}.x_\alpha, \ldots, \sum_\alpha \theta_{p,\alpha} x_\alpha\right), (y_1, \ldots, y_p) \right) \\ &= \sum_{i,j} a_{i,j}.\beta'\left(\sum_\alpha \theta_{i,\alpha} x_\alpha, y_j\right) \\ &= \sum_{i,j} a_{i,j} \cdot \sum_\alpha \theta_{i,\alpha} \cdot \beta'(x_\alpha, y_j) \\ &= \sum_{\alpha,j} \left(\sum_i \theta_{i,\alpha}.a_{i,j}\right) \cdot \beta'(x_\alpha, y_j) \end{aligned}$$

Hence the result follows, since $A\circ(\theta \text{ x } 1) = \left(\sum_i \theta_{i,\alpha} a_{i,j}\right)_{1\le\alpha,j\le p}$.

□

***Proposition* (3.3) :** Suppose there exist an $\mathcal{F}$-morphism $\theta \in \mathrm{End}_{\mathbb{A}}(\mathcal{V})$. If $\beta\circ(\theta \text{ x } 1) \gg 0$ then β' is definite and by choosing $\beta' \gg 0$ we must have $A\circ(\theta \text{ x } 1) \gg 0$. Conversely, if β' is Hermitian and $\mathcal{A}\circ(\theta \text{ x } 1), \beta' \gg 0$ then $\beta\circ(\theta \text{ x } 1) \gg 0$.

***Proof*:** If $\beta\circ(\Theta \text{ x } 1) \gg 0$ then it can be seen that β' is definite by the proof of **(3.2)**. So suppose $\beta' \gg 0$, and take $x = {}^{\mathrm{T}}(\lambda_1.x, \ldots, \lambda_p.x)$. Then

$$\beta(\theta(x), x) = \left(\sum_{\alpha,j} \lambda_\alpha \cdot \left(\sum_i \theta_{i,\alpha} a_{i,j}\right) \cdot \lambda_j\right) \cdot \beta'(x,x).$$

So $\beta'(x,x) > 0$, thus $\beta(\theta(x), x) > 0 \Rightarrow \left(\sum_{\alpha,j} \lambda_\alpha \cdot \left(\sum_i \theta_{i,\alpha} a_{i,j}\right) \cdot \lambda_j\right) > 0$.

Hence $A\circ(\theta \text{ x } 1) \gg 0$. The remainder of the proposition follows simply by choosing an orthonormal basis for $\mathcal{U}$.

□

For $\mathcal{A} \in M_p(\mathbb{A})$ let $\mathcal{A}^{(k)}$ denote the following matrix

$$\mathcal{A}^{(k)} = \begin{bmatrix} \mathcal{A} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathcal{A} \end{bmatrix} \in M_{kp}(\mathbb{A}).$$

Then, we easily have the following.

***Proposition* (3.4)** : Suppose $(\mathcal{V}, \beta) \cong (\mathcal{U}, \beta')\nabla\mathcal{A}$ where $\mathcal{A} \in M_p(\mathbb{A})$.

(i) If $(\mathcal{U}, \beta') = (\mathcal{U}_1, \beta'_1) \perp \ldots \perp (\mathcal{U}_n, \beta'_n)$,

then

$$(\mathcal{V}, \beta) \cong (\mathcal{U}_1, \beta'_1)\nabla\mathcal{A} \perp \ldots \perp (\mathcal{U}_n, \beta'_n)\nabla\mathcal{A}.$$

In particular,

(ii) if $(\mathcal{U}, \beta') \cong (\mathcal{W}, \beta'') \perp \ldots \perp (\mathcal{W}, \beta'')$ (k copies),

then

$$(\mathcal{V}, \beta) \cong (\mathcal{W}, \beta'')\nabla\mathcal{A}^{(k)}$$

(iii) If $(\mathcal{W}, \mathcal{B}) \cong (\mathcal{V}, \beta) \perp \ldots \perp (\mathcal{V}, \beta)$ (e copies),

then

$$(\mathcal{W}, \mathcal{B}) \cong (\mathcal{V}, \beta)^{(e)}\nabla\mathcal{A}$$
$$\cong (\mathcal{V}, \beta)\nabla\mathcal{A}^{(e)}.$$

§4 Complex structures II

Recall the notation of §§1-2: let Φ be a finite group and let N be a finitely generated $\mathbb{Z}[\Phi]$-module. As before put $W = \mathbb{Q} \otimes_{\mathbb{Z}} N$ and let $T = W_{\mathbb{R}/N}$, a real torus. Let $B : W \times W \to \mathbb{Q}$ denote a canonical form. In this § we shall parametrise the projective structures of the T which have Riemann form B, we denote this set by $\mathfrak{S}(T, \Phi, B)$. We first give a parametrisation of $\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B)$ for W isotypic of type (I), (II) and (IV), for W of type (III) a parametrisation of $\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B)$ seems slightly beyond our grasp.

Write $W = (V)^e$ (V $\mathbf{Q}[\Phi]$-simple). Let $D = \mathrm{End}_{\mathbf{Q}[\Phi]}(V)$ and write $D = (D, \mathbb{E}, \tau, \xi)$ if D is of type (II), (III) or (IV) otherwise let $\mathbb{E} = D$. By definition, we may write $B = \mathfrak{R}_{\mathbb{E}/\mathbf{Q}}(\beta)$ where $\beta : W \times W \to \mathbb{E}$.

Let $\dim_{\mathbb{E}} V = n$, then we have the following natural embedding given by tensor product,

$$\begin{array}{ccc} GL_e(\mathbb{E}) \subset GL_{en}(\mathbb{E}) = GL_{\mathbb{E}}(W) \\ A \quad \mapsto \quad A \otimes I_n \end{array}$$

where $\otimes$ denotes the tensor product of matrices. Fix an identification

$$\mathbb{E} \otimes_{\mathbf{Q}} \mathbb{R} = \mathbb{R}_1 \times \ldots \times \mathbb{R}_g$$

and for ease of notation let $V_\lambda = \mathbb{R}_\lambda \otimes_{\mathbb{E}} V$ and $W_\lambda = \mathbb{R}_\lambda \otimes_{\mathbb{E}} W$ for $\lambda = 1, \ldots, g$.

If W is of type (I) or (III) then each V_λ is a simple $\mathbb{R}[\Phi]$-module; otherwise we have the following $\mathbb{R}[\Phi]$-simple decomposition

$$V_\lambda = \mathfrak{U}_\lambda^{(f)},$$

where,

$$f = \begin{cases} 2 & \text{type (II)} \\ d & \text{type (IV)} \end{cases}$$

For $\mathbb{E} \subset k \subset \mathbb{R}$, define

$$G_{(k)} = M_e(D) \cap SU(k \otimes_{\mathbb{E}} W, 1 \otimes \beta)$$

G is clearly an algebraic group defined over $\mathbb{E}$. Let $\mathbb{H} = \mathfrak{R}_{\mathbb{E}/\mathbf{Q}}(G)$. It is clear that $\mathbb{H}_{\mathbb{R}}$ act on $\mathfrak{S}_{\mathbf{Q}}(W,\Phi,B)$ by

$$\begin{array}{ccccc} \mathbb{H}_{\mathbb{R}} \times \mathfrak{S}_{\mathbf{Q}}(W,\Phi,B) & \to & \mathfrak{S}_{\mathbf{Q}}(W,\Phi,B) \\ (\alpha \quad , \quad t) & \to & \alpha^{-1}.t.\alpha. \end{array}$$

We have

$$\mathbb{H}_{\mathbb{R}} = M_e(\mathbb{R} \otimes_{\mathbf{Q}} D) \cap SU(W_{\mathbb{R}}, 1 \otimes B)$$

But

$$SU(W_{\mathbb{R}}, 1 \otimes B) = SU(V_1^{(e)} \oplus \ldots \oplus V_g^{(e)}, \mathfrak{B}_1 \perp \ldots \perp \mathfrak{B}_g)$$

and

$$M_e(\mathbb{R}\otimes_{\mathbb{Q}}D) = \begin{cases} M_e(\mathbb{R}) \text{ x } \ldots \text{ x } M_e(\mathbb{R}) & \text{type (I)} \\ M_{2e}(\mathbb{R}) \text{ x } \ldots \text{ x } M_{2e}(\mathbb{R}) & \text{type (II)} \\ M_e(\mathbb{H}) \text{ x } \ldots \text{ x } M_e(\mathbb{H}) & \text{type (III)} \\ M_{ed}(\mathbb{C}) \text{ x } \ldots \text{ x } M_{ed}(\mathbb{C}) & \text{type (IV)} \end{cases}$$

So let

$$(\mathbb{H}_{\mathbb{R}})_\lambda = \begin{cases} M_e(\mathbb{R}) \cap SU(V_\lambda^{(e)}, \mathfrak{B}_\lambda) & \text{type (I)} \\ M_{2e}(\mathbb{R}) \cap SU(\mathfrak{U}_\lambda^{(2e)}, \mathfrak{B}_\lambda) & \text{type (II)} \\ M_e(\mathbb{H}) \cap SU(V_\lambda^{(e)}, \mathfrak{B}_\lambda) & \text{type (III)} \\ M_{ed}(\mathbb{C}) \cap SU(\mathfrak{U}_\lambda^{(ed)}, \mathfrak{B}_\lambda) & \text{type (IV)} \end{cases}$$

then

$$\mathbb{H}_{\mathbb{R}} = (\mathbb{H}_{\mathbb{R}})_1 \text{ x } \ldots \text{ x } (\mathbb{H}_{\mathbb{R}})_g.$$

We consider types (I), (II) and (IV), leaving a discussion of type (III) to the end.

Type (I) : Suppose W is of type (I). By (III.3.4) and (III.3.9) e must be even for there to exist a complex structure, write e = 2n. Let $\mathcal{A} = J_n' \in \mathbb{R}^{2n}$ and $\mathfrak{B}_\lambda' = \mathfrak{S}_{\mathbb{R}_\lambda/\mathbb{E}}(S)$: $V_\lambda \text{ x } V_\lambda \to \mathbb{R}_\lambda$. Then by interchanging coordinates it may easily be verified that

$$(W_\lambda, \mathfrak{B}_\lambda) \cong (V_\lambda, \mathfrak{B}_\lambda')\nabla\mathcal{A}.$$

Fix the frame $\mathfrak{F} = (V_\lambda, \mathfrak{B}_\lambda', \mathcal{A})$ for $(W_\lambda, \mathfrak{B}_\lambda)$. Identify $\mathbb{R}^{2n}$ with rows of 2n elements of $\mathbb{R}$. Let $\{x_1, \ldots, x_{2n}\}$ be a symplectic basis for $\mathbb{R}^{2n}$ with respect to the form $\mathcal{A}$, that is,

$$\mathcal{A}(x_i, x_j) = \begin{cases} 1 & i = j + n; \\ -1 & i = j - n; \\ 0 & \text{otherwise.} \end{cases}$$

Any $\mathbb{R}[\Phi]$-Endomorphism of W_λ is clearly $\mathfrak{F}$-framed. Let $t \in \mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$, t is $\mathfrak{F}$-

framed, so we may give $\mathbb{R}^{2n}$ the structure of a complex vector space by defining

$$(a + b.i).x = a.x + b.t(x) \qquad \text{for } x \in \mathbb{R}^{2n}.$$

Hence we consider each $x_i \in \mathbb{C}^n$. Let $X_1 = {}^T(x_1 \mid \ldots \mid x_n)$ and $X_2 = {}^T(x_{n+1} \mid \ldots \mid x_{2n})$ then, for $i = 1$ or 2, $X_i \in M_n(\mathbb{C})$ and is invertible since the rows of each X_i form a $\mathbb{C}$-basis for $\mathbb{C}^n$. Let $X = \begin{bmatrix} X_1 \\ \overline{X_2} \end{bmatrix}$ and define $\Theta \in M_{2n}(\mathbb{R})$ by

$$i.X = \Theta.X.$$

Then by (3.2) and (3.3), $t \in \mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$ if and only if ${}^T\Theta.\mathcal{A} \in M_{2n}(\mathbb{R})$ is symmetric and positive definite. If ${}^T\Theta.\mathcal{A}$ is symmetric then ${}^T\Theta.\mathcal{A} = {}^T({}^T\Theta.\mathcal{A}) = -\mathcal{A}.\Theta$ and $(\mathcal{A}.\Theta)^{-1} = \Theta.\mathcal{A}$. Hence, $t \in \mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$ if and only if $(-\Theta).\mathcal{A}$ is symmetric and positive definite. In the notation of Weyl, see [Wey1], [Wey2] or [Al1], $-\Theta$ is a generalised Riemann matrix with principal matrix $\mathcal{A}$. Let $H = -i.{}^TX.\mathcal{A}.\overline{X}$. Then H is hermitian,

$$\begin{aligned} {}^TH &= {}^T(-i.{}^TX.\mathcal{A}.\overline{X}) \\ &= -i.{}^T\overline{X}.{}^T\mathcal{A}.X \\ &= i.{}^T\overline{X}.\mathcal{A}.X \\ &= \overline{(-i.{}^TX.\mathcal{A}.\overline{X})}. \end{aligned}$$

By the classical theory of Riemann matices $(-\Theta).\mathcal{A}$ is symmetric and positive definite if and only if

$${}^TX.\mathcal{A}.X = 0 \qquad \{4.1\}$$

and

$$H \text{ is positive definite} \qquad \{4.2\}.$$

With $X = \begin{bmatrix} X_1 \\ \overline{X_2} \end{bmatrix}$, {4.1} may be written

$$X_1.{}^TX_2 - X_2.{}^TX_1 = 0 \qquad \{4.3\}$$

and {4.2} may be written

$$i.(X_2.{}^T\overline{X}_1 - X_1.{}^T\overline{X}_2) \gg 0 \qquad \{4.4\}.$$

Let $\mathcal{Z} = X_1.X_2^{-1}$, then {4.3} is equivalent to

$${}^T\mathcal{Z} = \mathcal{Z} \qquad \{4.5\}$$

and {4.4} is equivalent to

$$\text{Im } \mathcal{Z} \gg 0 \qquad \{4.6\}.$$

Conversely, let $\mathcal{Z} \varepsilon M_n(\mathbb{C})$ and put $X = \left[\frac{\mathcal{Z}}{I_n}\right]$, Then $\mathbb{R}^{2n}$ is given complex coordinates and so we have a uniquely defined complex structure on $\mathbb{R}^{2n}$. If we again define $\Theta \varepsilon M_{2n}(\mathbb{R})$ by

$$i.X = \Theta.X$$

then conditions {4.5} and {4.6} imply that ${}^T\Theta.\mathcal{A} \varepsilon M_{2n}(\mathbb{R})$ is symmetric and positive definite, by the reverse of the above argument.

Let

$$(III_p) = \{ \mathcal{Z} \varepsilon M_p(\mathbb{C}) : {}^T\mathcal{Z} = \mathcal{Z} \text{ and } \text{Im}(\mathcal{Z}) \gg 0 \}.$$

(III_p) is sometimes written in the form $\mathfrak{H}_p$ and is known as the generalised Siegel upper half plane or the Siegel space of type (III). We have the following identification,

$$\mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda) = (III_{e/2})$$

where e is defined as above.

The classical theory of Riemann matrices and the importance of Riemann matrices to algebraic geometry, as here, may be found in a paper by Lef schetz [Lef1]. For a description of the connection between Riemann matrices and generalised Riemann matrices the reader is directed to [Wey1−2] and [Al1], as decribed in [Al2].

With respect to the basis given by the rows of X it is now clear that

$$(\mathfrak{H}_{\mathbb{R}})_\lambda = SU(V_\lambda^{(e)}, \mathfrak{B}_\lambda) \cap Gl_e(\mathbb{R}) = Sp_n(\mathbb{R}).$$

We describe the action $(\mathfrak{H}_{\mathbb{R}})_\lambda$ on $\mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$ as an action on (III_n). Let $t \varepsilon \mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$ and define X and Θ as above. X satisfies {4.5} and {4.6}. Let $\alpha \varepsilon (\mathfrak{H}_{\mathbb{R}})_\lambda = Sp_n(\mathbb{R})$, then $\alpha(t) = \alpha.t.\alpha^{-1}$. Let $t' = \alpha.t.\alpha^{-1}$, then it is easily verified that the matrix of complex coordinates is given by $X' = \alpha X$. Write

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in M_n(\mathbb{R})$. Then

$$\alpha X = (aX_1 + bX_2, cX_1 + dX_2),$$

hence the action of α on (III_n) maps $\mathcal{Z}$ to $\mathcal{Z}'$, where

$$\begin{aligned}\mathcal{Z}' &= (aX_1 + bX_2)/(cX_1 + dX_2)^{-1} \\ &= (a\mathcal{Z} + b)/(c\mathcal{Z} + d)^{-1}.\end{aligned}$$

Hence $(\mathbb{H}_{\mathbb{R}})_\lambda$ acts on (III_n) as the group of fractional transformations; in particular, the action of $(\mathbb{H}_{\mathbb{R}})_\lambda$is transitive, [Si]. This corresponds to the usual identification of

$$(III_n) = Sp_n(\mathbb{R})_{/U(n)}.$$

Fix $\mathbb{A} = \mathbb{R}, \mathbb{H}$ or $\mathbb{C}$ according to W being of type (II), (III) or (IV) respectively. Let $\mathcal{D}_\lambda = End_{\mathbb{R}[\Phi]}(V_\lambda)$, $\mathcal{D}_\lambda$ has a positive involution induced from the group algebra $\mathbb{R}[\Phi]$, and by proposition 4 of [We] we may fix an isomorphism

$$(\mathcal{D}_\lambda, \tau) \cong (M_f(\mathbb{A}), \sigma) \qquad \{4.7\}$$

where σ denotes the standard involution of $M_f(\mathbb{A})$.

Corresponding to the identification of V as the first column of A, we identify V_λ with the first f columns of $A \otimes_{\mathbb{E}} \mathbb{R}_\lambda = M_{fm}(\mathbb{A})$. Let,

$$\begin{aligned} X &= [\, X_1 | \ldots | X_f | \, 0 \ldots 0] \\ Y &= [\, Y_1 | \ldots | Y_f | \, 0 \ldots 0] \end{aligned} \quad \in M_{fm}(\mathbb{A}) \qquad \{4.8\}$$

and $\mathcal{B}'_\lambda$ denote the restriction of $\mathcal{B}_\lambda$ to V_λ; we clearly have,

$$\mathcal{B}'_\lambda(X, Y) = Tr_{\mathbb{R}}(X.\Xi.{}^T\bar{Y}) \qquad \{4.9\}$$

where $\Xi \in M_{fm}(\mathbb{A})$. In fact Ξ is the matrix corresponding to the action of $\xi \in D$ on V. Thus

$$\Xi = \begin{bmatrix} \Xi' & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \Xi' \end{bmatrix} \qquad \{4.10\}$$

where $\Xi' \in M_f(\mathbb{A})$ is the image of $\xi \in D$ under the isomorphism $D \otimes_{\mathbb{E}} \mathbb{R} \cong M_f(\mathbb{A})$.

Type (II) : Let W be of type (II). Then we have $\mathfrak{U}_\lambda$ identified with the first column of $M_{2m}(\mathbb{R})$. Let

$$\mathfrak{F} : \mathfrak{U}_\lambda \times \mathfrak{U}_\lambda \to \mathbb{R}$$

by $$\mathfrak{F}(X,Y) = x_1.y_1 + \ldots + x_{2m}.y_{2m}$$

where $X = {}^T(x_1, \ldots, x_{2m})$ and $Y = {}^T(y_1, \ldots, y_{2m})$.

Let $\Xi' = (\xi'_{i,j})_{1\le i,j\le 2} \varepsilon M_2(\mathbb{R})$, then $\xi'_{1,1} = \xi'_{2,2} = 0$, $\xi'_{1,2} = -\xi'_{2,1}$ and

$$\mathfrak{B}'_\lambda((X_1,X_2),(Y_1,Y_2)) = \mathfrak{F}(X_1,Y_2).\xi'_{2,1} + \mathfrak{F}(X_2,Y_1).\xi'_{1,2} \quad .$$

Let $\mathcal{A}' = {}^T\Xi'$ and $\mathcal{A} = {}^T\Xi$, then,

$$(V_\lambda, \mathfrak{B}'_\lambda) \cong (\mathfrak{U}_\lambda, \mathfrak{F}_\lambda)\nabla\mathcal{A}'.$$

And, by (3.4),

$$(W_\lambda, \mathfrak{B}_\lambda) \cong (\mathfrak{U}_\lambda, \mathfrak{F})\nabla\mathcal{A}.$$

Let $\mathfrak{T} = (\mathfrak{U}_\lambda, \mathfrak{F}, \mathcal{A})$ be the framing. Any $\mathbb{R}[\Phi]$-endomorphism of $\mathcal{W}$ is $\mathfrak{T}$-framed. Let $t \varepsilon \mathfrak{S}_{\mathbb{R}}(W_\lambda,\Phi,\mathfrak{B}_\lambda)$, t is $\mathfrak{T}$-framed, so we may give $\mathbb{R}^{2e}$ the structure of a complex vector space. Let $\{x_1, \ldots, x_{2e}\}$ be a symplectic basis for $\mathbb{R}^{2e}$ with respect to $\mathcal{A}$. Hence we consider each $x_i \varepsilon \mathbb{C}^e$. Let $X_1 = {}^T(x_1 \mid \ldots \mid x_e)$ and $X_2 = {}^T(x_{e+1} \mid \ldots \mid x_{2e})$ then, for i = 1 or 2, $X_i \varepsilon M_e(\mathbb{C})$ and is invertible since the rows of each X_i form a $\mathbb{C}$-basis for $\mathbb{C}^e$. Let $X = \left[\frac{X_1}{X_2}\right]$ and define $\Theta \varepsilon M_{2e}(\mathbb{R})$ by

$$i.X = \Theta.X.$$

Then by (3.2) and (3.3), $t \varepsilon \mathfrak{S}_{\mathbb{R}}(W_\lambda,\Phi,\mathfrak{B}_\lambda)$ if and only if ${}^T\Theta.\mathcal{A} \varepsilon M_{2n}(\mathbb{R})$ is symmetric and positive definite. Thus to similar working to type (I) we may identify

$$\mathfrak{S}_{\mathbb{R}}(W_\lambda,\Phi,\mathfrak{B}_\lambda) = (III_e).$$

Also, it is clear that $(\mathbb{H}_{\mathbb{R}})_\lambda = Sp_e(\mathbb{R})$ and acts transitively on (III_e) as the group of fractional transformations as in type (I).

Type (IV) : Let W be of type (IV) and define

$$\mathfrak{H}_\lambda : W_\lambda \times W_\lambda \to \mathbb{C}$$

by $$\mathfrak{H}_\lambda(x,y) = \mathfrak{B}_\lambda(x,y) + i\mathfrak{B}_\lambda(ix,y),$$

and

$$\mathcal{H}'_\lambda : V_\lambda \times V_\lambda \to \mathbb{C}$$

by $$\mathcal{H}'_\lambda(x,y) = \mathcal{B}_\lambda(x,y) + i\mathcal{B}_\lambda(ix,y).$$

It is clear that

$$(W_\lambda, \mathcal{H}_\lambda) \cong (V_\lambda, \mathcal{H}'_\lambda)^{(e)}$$

and is easily verified that $\mathcal{H}'_\lambda$ is skew-Hermitian.

Proposition (4.1) : Suppose t is complex linear, then

(i) $\mathcal{B}_\lambda \circ (t \times 1)$ is symmetric $\Leftrightarrow$ $\mathcal{H}_\lambda \circ (t \times 1)$ is Hermitian

(ii) $\mathcal{B}_\lambda \circ (t \times 1) \gg 0 \Leftrightarrow \mathcal{H}_\lambda \circ (t \times 1) \gg 0$.

***Proof*:** Since $\mathcal{B}_\lambda = \mathrm{Re}\mathcal{H}_\lambda$, (ii) and "$\Leftarrow$" of (i) are clear. Suppose that $\mathcal{B}_\lambda \circ (t \times 1)$ is symmetric, then

$$\begin{aligned}
\mathcal{H}_\lambda(ty,x) &= \mathcal{B}_\lambda(ty,x) + i\mathcal{B}_\lambda(ity,x) \\
&= \mathcal{B}_\lambda(tx,y) + i\mathcal{B}_\lambda(ix,ty) \\
&= \mathcal{B}_\lambda(tx,y) + i\mathcal{B}_\lambda(y,tix) \\
&= \mathcal{B}_\lambda(tx,y) - i\mathcal{B}_\lambda(itx,y) \\
&= \overline{\mathcal{H}_\lambda(tx,y)}.
\end{aligned}$$

The result follows. □

Relative to the identifications given in {4.8} and {4.9} it is easily verified that

$$\mathcal{H}'_\lambda(X,Y) = \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{X}.Y.\Xi\ ]$$

We may carry out the working as follows:

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{Y}.X.\Xi\ ] &= \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\Xi.{}^{T}X.\overline{Y}\ ] \\
&= \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}X.\overline{Y}.{}^{T}\Xi\ ] \\
&= \mathrm{Tr}_{\mathbb{C}}[\ \overline{{}^{T}\overline{X}.Y.{}^{T}\overline{\Xi}}\ ]
\end{aligned}$$

$$= \overline{\mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{X}.Y.\Xi\ ]}.$$

Hence,

$$\mathrm{Tr}_{\mathbb{R}}[\ {}^{T}\overline{Y}.X.\Xi\ ] = \mathrm{Re}\ \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{Y}.X.\Xi\ ]$$
$$= \mathrm{Re}\ \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{X}.Y.\Xi\ ].$$

Thus,

$$\mathfrak{B}'_{\lambda}(X,Y) = \mathrm{Re}\ \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{X}Y\Xi\ ],$$

and
$$\mathfrak{B}'_{\lambda}(i.X,Y) = \mathrm{Re}\ \mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{iX}.Y.\Xi\ ]$$
$$= \mathrm{Re}\ -i.\mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{X}Y\Xi\ ]$$
$$= \mathrm{Im}\ .\mathrm{Tr}_{\mathbb{C}}[\ {}^{T}\overline{X}Y\Xi\ ].$$

Now let $\mathcal{S}$ denote the standard Hermitian form on $\mathcal{U}_{\lambda}$, that is $\mathcal{S}(X,Y) = {}^{T}\overline{X}Y$, also let $\mathcal{A} = {}^{T}\Xi$, then,

$$(W_{\lambda},\mathcal{H}_{\lambda}) \cong (\mathcal{U}_{\lambda},\mathcal{S})\nabla\mathcal{A}.$$

Let $\mathfrak{F} = (\mathcal{U}_{\lambda},\mathcal{S},\mathcal{A})$ be the framing. But Ξ is central in $M_{ed}(\mathbb{C})$, and so scalar. Thus, the signature of $i.\mathcal{A}$ is $(0,ed)$ or $(ed,0)$. Choose a $\mathbb{C}$-basis $\{x_1, \ldots x_{ed}\}$ so that $\mathcal{A} = \pm i.I_{ed}$. Again any $\mathbb{R}[\Phi]$-endomorphism of W_{λ} is $\mathfrak{F}$-framed. Write $X = {}^{T}(x_1 \mid \ldots \mid x_{ed})$. Let $t \in \mathfrak{S}_{\mathbb{R}}(W_{\lambda},\Phi,\mathfrak{B}_{\lambda})$ and define $\Theta \in M_{ed}(\mathbb{C})$ by

$$t(X) = \Theta.X.$$

Hence, ${}^{T}\Theta.\mathcal{A}$ is Hermitian and positive definite.

As described by Satake in an appendix of 'Algebraic structures of symmetric domains', this is a bounded domain of Siegel type I, which is identified with a single point since A is diagonal.

Hence we have proved the following.

Theorem (4.2) : Let $W = V^{(e)}$ be an isotypic $\mathbb{Q}[\Phi]$-module and $B : V \times V \to \mathbb{Q}$ a canonical form. Then,

(i) if W has type (I)

$$\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B) = \begin{cases} \left((III_{e/2})\right)^g & \text{if e is even} \\ \emptyset & \text{if e is odd} \end{cases} \qquad \{4.11\}$$

(ii) if W has type (II)

$$\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B) = \left((III_e)\right)^g \qquad \{4.12\}$$

(iii) if W has type (IV)

$\mathfrak{S}_{\mathbb{Q}}(W,\Phi,B)$ has one element.

Moreover, $\mathbb{H}_{\mathbb{R}}$ act trasnsitively on the parameter space for each type.

Finally, we remark on the remaining case of type (III). Here each V_λ is simple and we may identify V_λ with the first column of $M_m(\mathbb{H})$. As before let $\mathfrak{B}'_\lambda$ denote the restriction of $\mathfrak{B}_\lambda$ to V_λ, then for $X, Y \in V_\lambda$, we have,

$$\mathfrak{B}'_\lambda(X,Y) = \mathrm{Tr}_{\mathbb{R}}(X.\Xi.{}^{T}\overline{Y}) \qquad \{4.13\}$$

where $\Xi \in M_m(\mathbb{H})$. Also

$$(\mathbb{H}_{\mathbb{R}})_\lambda = M_e(\mathbb{H}) \cap SU(V_\lambda^{(e)}, \mathfrak{B}_\lambda)$$

Consider $\mathbb{H}^e$ as rows of e elements. Let $\mathfrak{H} : \mathbb{H}^e \times \mathbb{H}^e \to \mathbb{H}$ denote the skew Hermitian form

$$\mathfrak{H}(x,y) = x.\Xi.{}^{T}\overline{y}.$$

It is clear that $SU(\mathbb{H}^e,\mathfrak{H}) \subset (\mathbb{H}_{\mathbb{R}})_\lambda$. By the usual injections $\mathbb{H}^e \subset \mathbb{C}^{2e}$, and $\iota : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$ we may idenify $\mathfrak{H}$ with a skew Hermitian form $\mathfrak{H} : \mathbb{C}^{2e} \times \mathbb{C}^{2e} \to \mathbb{C}$. We have, $SU(\mathbb{H}^e,\mathfrak{H}) \subset SU(\mathbb{C}^{2e},\mathfrak{H}) = SU(n,n)$. We may identify $SU(\mathbb{H}^e,\mathfrak{H}) = SU(\mathbb{C}^{2e},\mathfrak{H}) \cap O(\mathbb{C}^e,\mathfrak{F})$ where $\mathfrak{F} : \mathbb{C}^{2e} \times \mathbb{C}^{2e} \to \mathbb{C}$ by $\mathfrak{F}(x,y) = \mathfrak{H}(x.j,y)$ is Hermitian, [Po].

Let

$(\mathrm{II}_n) = \{\, Z \,\varepsilon\, M_n(\mathbb{C}) : {}^{T}Z = -Z\,,\ I - Z^{T}\overline{Z} \gg 0 \text{ and is Hermitian} \,\}.$

(II_n) is the Siegel space of type (II), it is a bounded symmetric domain. Then modulo its centre $SU(\mathbb{H}^e, \mathfrak{K})$ acts transitively on (II_e) as the group of fractional transformations, see [Si]. As a subgroup of $(\mathbb{H}_{\mathbb{R}})_\lambda$, $SU(\mathbb{H}^e, \mathfrak{K})$ acts without fixed points on $\mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$. Hence we may write $\mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$ as a disjoint union

$$\mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda) = \bigcup_{\alpha \epsilon \Delta} (\mathrm{II}_e)$$

of copies of (II_e) over a, possibly infinite, index set Δ. The elements of Δ correspond to orbits under the action of $SU(\mathbb{H}^e, \mathfrak{K})$ on $\mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$.

To apply the work of Shimura, [Sh], we considered only those projective structures t on $W = V^{(e)}$ (V $\mathbb{Q}[\Phi]$-simple) where there is an embedding of $\mathrm{End}_{\mathbb{Q}[\Phi]}(V)$ into the endomorphism algrebra $\mathcal{A}_0(W, N, t)$ which is compatible with the involutions and the projective structure. Let $t \,\varepsilon\, \mathfrak{S}_{\mathbb{Q}}(W, \Phi, B)$ and write

$$t = t_1 \oplus \ldots \oplus t_e,$$

where $t_\lambda \,\varepsilon\, \mathfrak{S}_{\mathbb{R}}(W_\lambda, \Phi, \mathfrak{B}_\lambda)$ for each λ. The involuted algebra $\mathrm{End}_{\mathbb{Q}[\Phi]}(V)$ embeds in $\mathcal{A}_0(W, N, t)$, as above, if and only if each t_λ commutes with the right action of $\mathbb{H}$ on W_λ; which is equivalent to $t_\lambda \,\varepsilon\, M_e(\mathbb{R})$ for each λ. Hence we at least must have e even for any such projective structures to exists. However, a simple application of (3.2) to a framing $\mathcal{F} = (V_\lambda, \mathfrak{B}'_\lambda, \mathcal{A})$, with $\mathcal{A}$ Hermitian, shows that no projective structures with such an embedding of endomorphism algebras exists.

Recall our original notation: let N be a finitely generated $\mathbb{Z}[\Phi]$-module, $W = \mathbb{Q} \otimes_{\mathbb{Z}} N$. Let T denote the torus $W_{\mathbb{R}}/N$. Because of the problems just described we suppose that W has no summands of type (III). We are now in a position to parametrise the set of projective structures on T which commute with the Φ-action and have Riemann form B, the canonical form. We denote such a set by $\mathfrak{S}(T, \Phi, B)$.

Essentially the work has been done in §1: let $\Gamma \subset \mathbb{H}_{\mathbb{R}}$ denote the maximal

arithmetic subgroup such that $\Gamma(N) \subset N$. As in §1, it follows easily that,

***Theorem* (4.3) :** We have the following parametrisation,

$$\mathfrak{S}(T,\Phi,B) = {}_{\Gamma\backslash}\mathfrak{S}_Q(W,\Phi,B).$$

Moreover, $\mathfrak{S}(T,\Phi,B)$ is then an irreducible bounded symmetric domain.

***Proof* :** We need only verify irreducibility, but this follows from (3.4) of [Jo2].

□

Chapter V: Holonomy groups and representations.

§1 Integral representations and cohomology of finite groups

Let G be a group and R a commutative ring. Let N be an R[G]-module. We say that N is decomposable when we can write

$$N = N_1 \oplus N_2$$

for non-zero R[G]-modules N_1 and N_2. Otherwise we say that N is indecomposable. The following is well known.

Theorem [Jon] (1.1) : Let G be a finite group. Then, the number of isomorphism classes of indecomposable $\mathbb{Z}$[G]-modules is finite if and only if every Sylow p-subgroup of G is cyclic of order p or p^2.

Let p be a prime and let $\mathbb{Z}_p$ denote the cyclic group of order p. Let ζ be a primitive p^{th} root of unity and $\mathbb{Z}[\zeta]$ denote the ring of integers in $\mathbb{Q}(\zeta)$, the cyclotonic field of order p. Let $C = C(\mathbb{Q}(\zeta))$ denote the ideal class group of order $h = h_p$. Let $A_1, \ldots, A_h \in C$ be a set of representatives for the ideal classes. Then a full set of indecomposible $\mathbb{Z}[\mathbb{Z}_p]$-modules is

$$\mathbb{Z}\ ,\ A_1\ ,\ \ldots\ ,\ A_h\ ,\ (A_1, a_1)\ ,\ \ldots\ ,\ (A_h, a_h)$$

for $a_i \in A_i$, see §2 of Chapter II.

So let

$$N = \mathbb{Z}^{\alpha} \oplus \Big(\bigoplus_{i=1}^{h} A_i^{\beta_i} \Big) \oplus \Big(\bigoplus_{i=1}^{h} (A_i, a_i)^{\gamma_i} \Big) \qquad \{1.1\}$$

be an arbitrary $\mathbb{Z}[\mathbb{Z}_p]$-module, put $\beta = \sum_{i=1}^{h} \beta_i$ and $\gamma = \sum_{i=1}^{h} \gamma_i$. Then, [Ch],

$$H^2(\mathbb{Z}_p, N) \cong \mathbb{Z}_p^{\alpha}.$$

Hence, there exists torsion-free extensions of N by Φ if and only if $\alpha > 0$. Also, it is clear that N is faithful if and only if $\beta + \gamma > 0$.

Again let p be a prime, denote by D_{2p} the dihedral group of order 2p,

$$D_{2p} = \left\langle a, b : a^2 = b^p = 1 \ , \ bab = a \right\rangle \qquad \{1.2\}.$$

Then,

$$0 \ \rightarrow \ \mathbb{Z}_p \ \rightarrow \ D_{2p} \ \rightarrow \ \mathbb{Z}_2 \ \rightarrow \ 1 \qquad \{1.3\}$$

is a split short exact seqence. Although the first example of a dihedral group is the Klein 4-group (p = 2) we do not discuss this group in this section because of (1.1).

Let $p > 2$. We may describe the indecomposable $\mathbb{Z}[D_{2p}]$-modules as follows, [Le] and [Ch-Va3]. Again, let ζ be a primitive p^{th} root of unity. Put $R = \mathbb{Z}[\zeta]$ and $R_0 = \mathbb{Z}[\zeta + \overline{\zeta}]$, let $C_0 = C(\mathbb{Q}(\zeta + \overline{\zeta}))$ the ideal class group of $\mathbb{Q}(\zeta + \overline{\zeta})$ and let $V_1, \ldots, V_r$ be a full set of representatives for these ideal classes, $r = r_p \ (= |C_0| \ )$ the ideal class number. Also, for each $i = 1 \rightarrow r$, choose $v_i \ \varepsilon \ V_iR$ not in $(1 - \zeta).V_iR$. We describe the indecomposable modules in ten types $N_1 - N_{10}$, see [Ch-Va].

$N_1 \cong_{\mathbb{Z}} \mathbb{Z}$ the trivial module of rank one;

$N_2 \cong_{\mathbb{Z}} \mathbb{Z}$ $a.n = -n$, $b.n = n$;

$N_3 \cong_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z}$ $a.(n_1, n_2) = (n_2, n_1)$,

$b.(n_1, n_2) = (n_1, n_2)$;

for $i = 1, \ldots, r$,

$N_4^i \cong_{\mathbb{Z}} V_iR$ $a.x = \overline{x}$, $b.x = \zeta.x$;

$N_5^i \cong_{\mathbb{Z}} V_iR$ $a.x = -\overline{x}$, $b.x = \zeta.x$;

$N_6^i \cong_{\mathbb{Z}} V_iR \oplus \mathbb{Z}$ $a.(x, n) = (-\overline{x} + n.v_i, n)$,

$b.(x, n) = (\zeta.x + n.v_i, n)$;

$N_7^i \cong_{\mathbb{Z}} V_iR \oplus N_2$ $a.(x, n) = (\overline{x} - n.v_i, -n)$,

$b.(x, n) = (\zeta.x + n.v_i, n)$;

$N_8^i \cong_{\mathbb{Z}} V_iR \oplus N_3$ $a.(x, n_1, n_2) = (\overline{x} + (n_1 - n_2).v_i, n_2, n_1)$,

$b.(x, n_1, n_2) = (\zeta.x + [n_1 + n_2.(1 - 2\zeta)].v_i, n_1, n_2)$;

$$N_9^i \cong_{\mathbb{Z}} V_iR \oplus N_3 \qquad a.(x,n_1,n_2) = (-\overline{x} + (n_1 + n_2).v_i, n_2, n_1),$$

$$b.(x,n_1,n_2) = (\zeta.x + (n_1 + n_2).v_i, n_1, n_2);$$

$$N_{10}^i \cong_{\mathbb{Z}} V_iR \oplus V_iR \oplus N_3$$

$$a.(x_1,x_2,n_1,n_2) = (\overline{x} + (n_1 - n_2).v_i, \overline{x} + (n_1 + n_2).v_i, n_2, n_1),$$

$$b.(x_1,x_2,n_1,n_2) = (\zeta.x_1 + [n_1 + n_2(1 - 2\zeta)].v_i, \zeta.x_2 + (n_1 + n_2).v_i, n_1, n_2).$$

Let N be an arbitrary $\mathbb{Q}[D_{2p}]$-module. Then the $\mathbb{Z}[D_{2p}]$-isomorphism class of N is uniquely determined by the following invariants [Ch-Va],

$$rk_{\mathbb{Z}}N\ ,\ H^i(D_{2p},N)\ i = 1 \to 4,\quad E(N) \text{ and } I(N),$$

where

$E(N)$ = total number of modules of type N_6, N_8 and N_{10} that occur in the indecomposable decomposition of N.

$I(N)$ = the product, in C_0, of all the ideals occuring in the indecomposable decomposition of N.

Proposition **(1.2)** : Let G be a finite group, and K a normal subgroup such that ($|K|, |G/K|$) = 1. Let N be a $\mathbb{Z}[G]$-module and $[N]_K$ denote the restriction of N to K. Then

$$H^i(G,N) = H^i(K,[N]_K)^{G/K} \oplus H^i(G/K, N^K).$$

This is well known and follows from the Lyndon-Hochschild-Serre spectral sequence for the short exact sequence

$$0 \to K \to G \to G/K \to 1 \qquad \{1.4\}.$$

We continue with the notation of (1.2).

Proposition **(1.3)** : Let $\iota : K \to G$ be injection and $s : G/K \to G$ be a right splitting for {1.4}. Let $s^* : H^i(G,N) \to H^i(G/K,[N]_{G/K})$ denote the induced map in cohomology. Then s^* is surjective.

Proof: That s^* is surjective follows immediate since s is a right splitting for {3.5}. The rest is a corollary of **(1.2)**.

□

There is a unique cyclic subgroup of D_{2p} of order p, denote this by $\mathbb{Z}_p$, and precisely p subgroups of order 2; the order-2 subgroups are all conjugate, since they Sylow 2-subgroups, in this way $\mathbb{Z}_p$ acts transitive on the set of Sylow 2-subgroups.

For a $\mathbb{Z}[D_{2p}]$-module N let $[N]_p$ ($[N]_A$) denote the retriction of N to $\mathbb{Z}_p$ (A $= \left\langle ab^i : (ab^i)^2 = 1 \right\rangle$ a 2-subgroup of D_{2p}).

Lemma **(1.4)** : Let N be a $\mathbb{Z}[D_{2p}]$-module. Let $[N]_A$ and $[N]_{A'}$ denote the restriction of N to any two 2-subgroups of D_{2p}. Then $[N]_A \cong [N]_{A'}$ as $\mathbb{Z}[\mathbb{Z}_2]$-modules.

Proof: The proof is by a simple case-by-case analysis using the classification of $\mathbb{Z}[D_{2p}]$-modules.

□

Let $\iota_p : \mathbb{Z}_p \to D_{2p}$ ($\iota_A : A \to D_{2p}$, $A \cong \mathbb{Z}_2$) be injection and let $\iota_p^* : H^2(D_{2p},N) \to H^2(\mathbb{Z}_p,[N]_p)$ ($\iota_A^* : H^2(D_{2p},[N]_A) \to H^2(A,[N]_A)$) denote the induced map in cohomology, for any $\mathbb{Z}[D_{2p}]$-module N. Let A and A' be 2-subroups of D_{2p}, then by **(1.4)**,

$$H^2(A,[N]_A) \cong H^2(A',[N]_{A'}).$$

Also, by (1.3),

$$\iota_A^* : H^2(D_{2p}, [N]_A) \to H^2(A, [N]_A)$$

is surjective for any A. Let $\mathbb{Z}_2 = \left\langle a : a^2 = 1 \right\rangle$ and $\iota_2 = \iota_{\mathbb{Z}_2} : \mathbb{Z}_2 \to D_{2p}$. Thus, to find special points for a $\mathbb{Z}[D_{2p}]$-module N we need only consider the cyclic subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_p$ of D_{2p}.

Recall the classification of indecomposable $\mathbb{Z}[\mathbb{Z}_p]$-modules as given above. We shall say that an indecomposable $\mathbb{Z}[\mathbb{Z}_p]$-module N has type $\mathbb{Z}$ if it is trivial, has type α if $N \cong A_i$ for some i, and type β if $N \cong (A_i, a_i)$ again for some i. Table 1 gives the types in the restrictions of indecomposable $\mathbb{Z}[D_{2p}]$-module to $\mathbb{Z}_2$ and $\mathbb{Z}_p$. For working see [Ch-Va] and [Le].

Table 1

	N_j^i	$[N_j^i]_2$	$[N_j^i]_p$
i =	1	$\mathbb{Z}$	$\mathbb{Z}$
	2	α	$\mathbb{Z}$
	3	β	$2.\mathbb{Z}$
	4	$\frac{1}{2}(p-1).\beta$	α
	5	$\frac{1}{2}(p-1).\beta$	α
	6	$\frac{1}{2}(p-1).\beta \oplus \mathbb{Z}$	β
	7	$\frac{1}{2}(p-1).\beta \oplus \alpha$	β
	8	$\frac{1}{2}(p+1).\beta$	$\beta \oplus \mathbb{Z}$
	9	$\frac{1}{2}(p-1).\beta$	$\beta \oplus \mathbb{Z}$
	10	$p.\beta$	$2.\beta$

These fully determine the 2^{nd} cohomology of the restrictions, see [Ch].

In Table 2 we give the 2^{nd} cohomology groups of the indecomposable $\mathbb{Z}[D_{2p}]$-module and their restrictions to $\mathbb{Z}_2$ and $\mathbb{Z}_p$. The 1^{st} column is from [Ch-Va].

Table 2

N_j^i	$H^2(D_{2p}, N_j^i)$	$H^2(\mathbb{Z}_2, N_j^i)$	$H^2(\mathbb{Z}_p, N_j^i)$
i = 1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_p$
2	$\mathbb{Z}_p$	0	$\mathbb{Z}_p$
3	$\mathbb{Z}_p$	0	$\mathbb{Z}_p \oplus \mathbb{Z}_p$
4	0	0	0
5	0	0	0
6	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
7	0	0	0
8	0	0	$\mathbb{Z}_p$
9	$\mathbb{Z}_p$	0	$\mathbb{Z}_p$
10	0	0	0

Hence, from table 2 and the remarks proceeding table 1, we may deduce the following.

Proposition (1.5) : Let N be a $\mathbb{Z}[D_{2p}]$-module then $H^2(D_{2p}, N)$ has special points if and only if the 2-torsion and p-torsion of $H^2(D_{2p}, N)$ is non-zero.

§2 Examples

We first consider trivial holonomy: up to a connection preserving diffeomorphism the flat torus is the unique flat compact riemannian manifold in each dimension, [Ch]. It follows that the set of Kählerian structures on a 2n-dimensional torus T is parametrised by $\mathfrak{I}(T,1)$ as follows

$$\mathfrak{I}(T,1) = {}_{SL_{2n}(\mathbb{Z})}\backslash^{SL_{2n}(\mathbb{R})}/_{SL_n(\mathbb{C})}.$$

Let $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a non-degenerate skew symmetric bilinear form. Since the holonomy representation is trivial we may consider B as a canonical form for the torus T. Hence, the set of projective structures on T is parametrised by $\mathfrak{S}(T,1,B)$,

$$\mathfrak{S}(T,1,B) = {}_{Sp_n(\mathbb{Z})}\backslash^{Sp_n(\mathbb{R})}/_{U(n)}.$$

In dimension 2 we have

$$\mathfrak{I}(T,1) = \mathfrak{S}(T,1,B).$$

This is the well known result that any Kählerian structure in real dimension 2 is a projective structure. The torus is the only flat riemannian manifold which admits a admit a complex structure in dimension 2.

Let p be a prime, we now consider cyclic groups of prime order as holonomy groups. Let ζ be a primitive p^{th} root of unity and let C denote the ideal class group of $\mathbb{Q}(\zeta)$, as in §1. Let C_p denote the set of orbits under the action of the Galois group $Gal(\mathbb{Q}(\zeta):\mathbb{Q})$ on C, also if p is odd let C_2' denote the orbit set of the action of $\mathbb{Z}_2$ ($\subset Gal(\mathbb{Q}(\zeta):\mathbb{Q})$) on C, if $p = 2$ then C is trivial, and we take $C_2' = \{1\}$ also. We have the following classification.

Theorem [Ch] (2.1) : There is a bijection between the isometry classes of $\mathbb{Z}_p$-manifolds and the set of 4-tuples $(\alpha, \beta, \gamma, A) \varepsilon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times C$ such that

$$\alpha\ ,\ \gamma + \beta > 0,$$

also $A \varepsilon C_2'$ if $(\alpha, \gamma) = (1,0)$, and $A \varepsilon C_p$ otherwise.

For such a 4-tuple, $(\alpha, \beta, \gamma, A)$, satisfying the condition of the theorem let $X_p(\alpha, \beta, \gamma, A)$ denote a representative for isometry class corresponding to $(\alpha, \beta, \gamma, A)$.

Again let N be an arbitrary $\mathbb{Z}[\mathbb{Z}_p]$-module as defined in {1.1}. Then it is easily seen that

$$\mathbb{Q} \otimes_{\mathbb{Z}} N \cong \mathbb{Q}^{\alpha+\gamma} \oplus \mathbb{Q}(\zeta)^{\gamma+\beta} \qquad \{2.1\}$$

where $\mathbb{Q}$ denotes the 1-dimensional trivial $\mathbb{Q}[\mathbb{Z}_p]$-module. If p is odd then the simple summand $\mathbb{Q}(\zeta)$ (which we identify with its endomorphism ring) is totally imaginary and quadratic over the totally real field $\mathbb{Q}(\zeta + \overline{\zeta})$, and hence is of type (IV).

Hence, from **(III.3.4)** and **(III.3.8)**, we have,

Corollary **(2.2)** : Let $(\alpha, \beta, \gamma, A)$ be a quadruple satisfying the conditions of **(2.1)**, then we may choose the representative $X_p(\alpha, \beta, \gamma, A)$ to be complex projective if and only if $\alpha + \gamma$ is even and also $\gamma + \beta$ if $p = 2$.

Let $(\alpha, \beta, \gamma, A)$ be a quadruple as above. Let $N = N(\alpha, \beta, \gamma, A)$ denote the $\mathbb{Z}[\mathbb{Z}_p]$-module corresponding to the holonomy representationand put $W = \mathbb{Q} \otimes_{\mathbb{Z}} N$. Suppose $\alpha + \gamma$ is even and if $p = 2$ suppose $\gamma + \beta$ is even. Let $m = (\alpha + \beta)/2$ and $n = (\gamma + \beta)/2$ if $p = 2$ or $g = (p-1)/2$ if $p \neq 2$. Then, by (IV.1.3) and (IV.1.1), the set of complex structures of W is

$$\mathfrak{T}_{\mathbb{Q}}(W,\mathbb{Z}_p) = \begin{cases} SL_{2m}(\mathbb{R})/SL_m(\mathbb{C}) \times SL_{2n}(\mathbb{R})/SL_n(\mathbb{C}) & \text{if } p = 2 \\ SL_{2m}(\mathbb{R})/SL_m(\mathbb{C}) \times \Big(\overset{\circ}{\bigcup}_{r+s=\gamma+\beta} SL_{\gamma+\beta}(\mathbb{C})/SL_r(\mathbb{C})xSL_s(\mathbb{C}) \Big)^g & \text{otherwise.} \end{cases}$$

Let $B : W \times W \to \mathbb{Q}$ be a canonical form. Then, by (IV.4.2), the set of projective structures of W having Riemann form B is

$$\mathfrak{S}_{\mathbb{Q}}(W,\mathbb{Z}_p,B) = \begin{cases} Sp_m(\mathbb{R})/U(m) \times Sp_n(\mathbb{R})/U(n) & \text{if } p = 2 \\ Sp_m(\mathbb{R})/U(m) & \text{otherwise.} \end{cases}$$

We note that the minimal dimension for a flat complex projective $\mathbb{Z}_p$-manifold is $p + 1$ if p is odd and is 4 if $p = 2$.

Let p be an odd prime. It is clear from the classification that, a $\mathbb{Z}[D_{2p}]$-module is faithful if and only if there is an indecomposable of type $N_4 - N_{10}$ in the decomposition of N. We do not find it convenient at this point to give a criterion for this in terms of the invariants for N listed above. However, for any prime p, the minimal dimension for a flat riemannian D_{2p}-manifold is $p + 1$, [Ch-Va3]. For $p \neq 2$ this is easily seen from above.

We consider kählerian and projective structures as before, we determine the $\mathbb{Q}$-points of the indecomposable $\mathbb{Z}[D_{2p}]$-modules for $p \neq 2$ as follows.

Table 3

	N_j^i	$\mathbb{Q}\otimes_{\mathbb{Z}} N_j^i$
i =	1	$\mathbb{Q}$
	2	$\underset{\sim}{\mathbb{Q}}$
	3	$\mathbb{Q} \oplus \underset{\sim}{\mathbb{Q}}$
	4	$\mathbb{Q}(\zeta)$
	5	$\mathbb{Q}(\zeta)$
	6	$\mathbb{Q} \oplus \mathbb{Q}(\zeta)$
	7	$\underset{\sim}{\mathbb{Q}} \oplus \mathbb{Q}(\zeta)$
	8	$\mathbb{Q} \oplus \underset{\sim}{\mathbb{Q}} \oplus \mathbb{Q}(\zeta)$
	9	$\mathbb{Q} \oplus \underset{\sim}{\mathbb{Q}} \oplus \mathbb{Q}(\zeta)$
	10	$\mathbb{Q}[D_{2p}]$

where $\mathbb{Q}$ ($\underset{\sim}{\mathbb{Q}}$) is the trivial (non-trivial) 1-dimensional $\mathbb{Q}[D_{2p}]$-module.

If $p \neq 2$ then we have,

$$\mathbb{Q}[D_{2p}] \cong \mathbb{Q} \oplus \underset{\sim}{\mathbb{Q}} \oplus \mathbb{Q}(\zeta) \oplus \mathbb{Q}(\zeta)$$

otherwise $\mathbb{Q}[D_4]$ has four non-isomorphic irreducible modules of rank one, each is obviously totally real.

If $p \neq 2$, then

$$\mathrm{End}_{\mathbb{Q}[D_{2p}]}(\underset{\sim}{\mathbb{Q}}) = \mathbb{Q}$$

and

$$\mathrm{End}_{\mathbb{Q}[D_{2p}]}(\ \mathbb{Q}(\zeta)\ ) = \ \mathbb{Q}(\zeta + \overline{\zeta}).$$

These are of type (I) in Albert's classification. Thus, from **(III.3.4)** and **(III.3.8)** we see that the minimal dimension for a flat complex projective D_{2p}-manifold is 2.(p + 1). Let N be the $\mathbb{Z}[D_{2p}]$-module corresponding to the holonomy represenation for a manifold of this minimal dimension. Put $W = \mathbb{Q}\otimes_{\mathbb{Z}}N$. Then

$$W = \begin{cases} \mathbb{Q}^{(2)} \oplus \underset{\sim}{\mathbb{Q}}^{(2)} \oplus \underset{\approx}{\mathbb{Q}}^{(2)} & \text{if } p = 2 \\ \mathbb{Q}^{(2)} \oplus \underset{\sim}{\mathbb{Q}}^{(2)} \oplus \mathbb{Q}(\zeta)^{(2)} & \text{otherwise.} \end{cases}$$

where $\underset{\sim}{\mathbb{Q}}$ and $\underset{\approx}{\mathbb{Q}}$ are distinct non-trivial modules of dimension 1.

If $p \neq 2$ then let $g = (p-1)/2$. Then, by (IV.1.3) and (IV.1.1), the set of complex structures of W is

$$\mathfrak{I}_{\mathbb{Q}}(W, D_{2p}) = \begin{cases} \mathcal{G}L_2(\mathbb{R})/\mathbb{C}^* \times \mathcal{G}L_2(\mathbb{R})/\mathbb{C}^* \times \mathcal{G}L_2(\mathbb{R})/\mathbb{C}^* & \text{if } p = 2, \\ \mathcal{G}L_2(\mathbb{R})/\mathbb{C}^* \times \mathcal{G}L_2(\mathbb{R})/\mathbb{C}^* \times \left(\mathcal{G}L_2(\mathbb{R})/\mathbb{C}^*\right)^g. \end{cases}$$

Let $B : W \times W \to \mathbb{Q}$ be a canonical form, then, by (IV.4.2), the set of projective structures of W having Riemann form B is

$$\mathfrak{S}_{\mathbb{Q}}(W, \mathbb{Z}_p, B) = Sp_1(\mathbb{R}) \quad \times Sp_1(\mathbb{R}) \quad \times \left(Sp_1(\mathbb{R}) \quad\right)^{p-1}.$$

Note that any minimal dimension flat compact Kählerian manifold with holonomy group D_{2p} (p any prime) is projective.

For our last example in this section we consider the groups having the following presentation

$$G_{m,r} = < A, B : A^m = 1, B^a = A^b, BAB^{-1} = A^r > \qquad \{2.2\},$$

where m = a.b, a and b $\neq 2$ distinct primes. These are a special case of the groups introduced in §3 of chapter I. With a = 3, b = 7 and r = 4 we have the group of order 63, which is the smallest non-cyclic odd order group which embeds into a division ring, [Am]. Groups which embed into division algebras give convenient

examples were there exists irreducible rational representations of type (IV). The group $G_{m,r}$ embeds into a division ring if and only if either

(i) $a = 2$ and $r \equiv -1$ (m),

or

(ii) $r \equiv 1$ (a) and $a^2 \not| b - 1$.

These conditions follow from (I.3.6), infact from (I.3.7) it can be deduced that any group of order $a^2.b$ which embeds into a division ring has a presentation {2.2} satisfying (i) and (ii). All Sylow subgroups of such a $G_{m,r}$ are cyclic.

Lemma (2.5) : If $G_{m,r}$ is not cyclic then there are precisely 5 conjugacy classes of cyclic subgroups of $G_{m,r}$. If $G_{m,r}$ is cyclic then there are 6.

Proof: If $G_{m,r}$ is cyclic then the result follows easily. So suppose $G_{m,r}$ is not cyclic. By the Sylow theorems there is one conjugacy class of subgroups of orders a^2 and b. Also, since any subgroup of a cyclic group is characteristic there is just one conjugacy class of subgroups of order a. It is easily verified that the subgroup $<A> \cong C_{ab}$ is invarient under conjugation, hence so are its subroups of order a and b. Since there is one conjugacy class of subgroups with these orders they must be the only subgroups in $G_{m,r}$ of orders a and b. Hence there can be only one subgroup of order a.b, which is clearly $<A>$. Thus there there are 5 conjugacy classes of cyclic subgroups.

□

Corresponding to these conjugacy class there are five irreducible $\mathbb{Q}[G_{m,r}]$-modules. We have

$$\mathbb{Q}[G_{m,r}] \cong \mathbb{Q} \times \mathbb{Q}(\zeta_a) \times \mathbb{Q}(\zeta_{a^2}) \times M_a(\mathbb{K}) \times \mathfrak{A}_{m,r} \qquad \{2.3\}$$

where:

(i) $\mathbb{Q}$ is the trivial module which is clear of type (I);

(ii) $\mathbb{Q}(\zeta_a)$ and $\mathbb{Q}(\zeta_{a^2})$ are modules factored from the following exact sequence

$$1 \rightarrow C_b \rightarrow G_{m,r} \rightarrow C_{a^2} \rightarrow 1 \qquad \{2.4\}$$

$C_b = \langle A^{\alpha} \rangle$ they both are of type (IV);

(iii) $\mathbb{K} \subset \mathbb{Q}(\zeta_b)$ with $[\mathbb{Q}(\zeta_b) : \mathbb{K}] = a$, $\mathbb{K}$ is of type (I) if $a = 2$, and of type (IV) otherwise;

(iv) $\mathfrak{A}_{m,r}$ is a division algebra over $\mathbb{Q}$ of type (III) if $a = 2$, and of type (IV) otherwise.

Although there are only finitely many isomorphism classes of indecomposable integral representations of $G_{m,r}$ (this follows from (I.3.1) and (V.1.1), since $|G_{m,r}| = a^2.b$), a classification of such modules is not known. However, we need only consider the rational representations of $G_{m,r}$ to describe complex and projective structures.

Let W_3 denote the $\mathbb{Q}[G]$-module corresonding to the factor $M_a(\mathbb{K})$ in {2.3}. We may write an arbitrary $\mathbb{Q}[G]$-module as

$$W = \mathbb{Q}^{(e_0)} \oplus \mathbb{Q}(\zeta_a)^{(e_1)} \oplus \mathbb{Q}(\zeta_{a^2})^{(e_2)} \oplus W_3^{(e_3)} \oplus \mathfrak{A}_{m,r}{}^{(e_4)}$$

Then W is faithful if and only if $e_2 + e_4, e_3 + e_4 > 0$ and W admits a complex stucture if and only if e_0 is even and e_1, e_3 are even if $a = 2$. Let these last conditions be satisfied and put $e_0 = 2.m$ and $e_1 = 2.n$, $e_3 = 2.q$ if $a = 2$. Let $g = a.(a-1)/2$, which is the dimension of the totally real subfield contained in the centre of $\mathfrak{A}_{m,r}$ and let $d = a^2$ which is the degree of $\mathfrak{A}_{m,r}$ over its centre. Also let

$$h = \begin{cases} (b-1)/4a & \text{if } a = 2 \\ (b-1)/2a & \text{otherwise.} \end{cases}$$

Then, by (IV.1.3) and (IV.1.1), the set of complex structures of W is

$$\mathfrak{T}_{\mathbb{Q}}(W,G) = \mathfrak{T}_{\mathbb{Q}}(\mathbb{Q}^{(e_0)},G) \oplus \mathfrak{T}_{\mathbb{Q}}(\mathbb{Q}(\zeta_a)^{(e_1)},G) \oplus \mathfrak{T}_{\mathbb{Q}}(\mathbb{Q}(\zeta_{a^2})^{(e_2)},G) \oplus \mathfrak{T}_{\mathbb{Q}}(W_3^{(e_3)},G) \oplus \mathfrak{T}_{\mathbb{Q}}(\mathfrak{A}_{m,r}{}^{(e_4)},G)$$

where

$$\mathfrak{T}_{\mathbb{Q}}(\mathbb{Q}^{(e_0)},G) = SL_{2m}(\mathbb{R})/SL_m(\mathbb{C})$$

$$\mathfrak{I}_{\mathbf{Q}}(\mathbf{Q}(\zeta_a)^{(e_1)},\mathrm{G}) = \begin{cases} \mathrm{SL}_{2n}(\mathbb{R})/\mathrm{SL}_n(\mathbb{C}) & \text{if a = 2} \\ \overset{\circ}{\bigcup}_{r+s=e_1} \mathrm{SL}_{e_1}(\mathbb{C})/\mathrm{SL}_r(\mathbb{C}) \times \mathrm{SL}_s(\mathbb{C}) & \text{otherwise} \end{cases}$$

$$\mathfrak{I}_{\mathbf{Q}}(\mathbf{Q}(\zeta_{a^2})^{(e_2)},\mathrm{G}) = \Big(\overset{\circ}{\bigcup}_{r+s=e_2} \mathrm{SL}_{e_2}(\mathbb{C})/\mathrm{SL}_r(\mathbb{C}) \times \mathrm{SL}_s(\mathbb{C}) \Big)^{\mathrm{g}}$$

$$\mathfrak{I}_{\mathbf{Q}}(\mathrm{N}_3^{(e_3)},\mathrm{G}) = \begin{cases} \mathrm{SL}_{2q}(\mathbb{R})/\mathrm{SL}_q(\mathbb{C}) & \text{if a = 2} \\ \overset{\circ}{\bigcup}_{r+s=e_3} \mathrm{SL}_{e_3}(\mathbb{C})/\mathrm{SL}_r(\mathbb{C}) \times \mathrm{SL}_s(\mathbb{C}) & \text{otherwise} \end{cases}$$

$$\mathfrak{I}_{\mathbf{Q}}(\mathrm{A}_{m,r}{}^{(e_4)},\mathrm{G}) = \begin{cases} \mathfrak{I}(\mathrm{M}_{e_1}(\mathbb{H}))^{\mathrm{h}} & \text{if a = 2} \\ \Big(\overset{\circ}{\bigcup}_{r+s=d.e_4} \mathrm{SL}_{d.e_4}(\mathbb{C})/\mathrm{SL}_r(\mathbb{C}) \times \mathrm{SL}_s(\mathbb{C}) \Big)^{\mathrm{h}} & \text{otherwise} \end{cases}$$

Let $\mathrm{B} = \mathrm{B}_0 \perp \ldots \perp \mathrm{B}_4 : \mathrm{W} \times \mathrm{W} \to \mathbf{Q}$ be a canonical form for

$$\mathrm{W} = \mathbf{Q}^{(e_0)} \oplus \mathbf{Q}(\zeta_a)^{(e_1)} \oplus \mathbf{Q}(\zeta_{a^2})^{(e_2)} \oplus \mathrm{W}_3^{(e_3)} \oplus \mathfrak{A}_{m,r}{}^{(e_4)}$$

Then, by (IV.4.2), the set of projective structures of W having Riemann form B is

$$\begin{aligned} \mathfrak{S}_{\mathbf{Q}}(\mathrm{W},\mathrm{G},\mathrm{B}) = {} & \mathfrak{S}_{\mathbf{Q}}(\mathbf{Q}^{(e_0)},\mathrm{G},\mathrm{B}_0) \oplus \mathfrak{S}_{\mathbf{Q}}(\mathbf{Q}(\zeta_a)^{(e_1)},\mathrm{G},\mathrm{B}_1) \\ & \oplus \mathfrak{S}_{\mathbf{Q}}(\mathbf{Q}(\zeta_{a^2})^{(e_2)},\mathrm{G},\mathrm{B}_2) \oplus \mathfrak{S}_{\mathbf{Q}}(\mathrm{W}_3^{(e_3)},\mathrm{G},\mathrm{B}_3) \\ & \oplus \mathfrak{S}_{\mathbf{Q}}(\mathfrak{A}_{m,r}{}^{(e_4)},\mathrm{G},\mathrm{B}_4) \end{aligned}$$

where

$$\mathfrak{S}_{\mathbf{Q}}(\mathbf{Q}^{(e_0)},\mathrm{G},\mathrm{B}_0) = \mathrm{Sp}_m(\mathbb{R})/\mathrm{U(m)}$$

$$\mathfrak{S}_{\mathbf{Q}}(\mathbf{Q}(\zeta_a)^{(e_1)},\mathrm{G},\mathrm{B}_1) = \begin{cases} \mathrm{Sp}_n(\mathbb{R})/\mathrm{U(n)} & \text{if a = 2} \\ 1 & \text{otherwise} \end{cases}$$

$$\mathfrak{S}_{\mathbf{Q}}(\mathbf{Q}(\zeta_{a^2})^{(e_2)},\mathrm{G},\mathrm{B}_2) = \begin{cases} \mathrm{Sp}_q(\mathbb{R})/\mathrm{U(q)} & \text{if a = 2} \\ 1 & \text{otherwise} \end{cases}$$

$$\mathfrak{S}_{\mathbf{Q}}(\mathrm{W}_3^{(e_3)},\mathrm{G},\mathrm{B}_3) = 1$$

$$\mathfrak{S}_{\mathbf{Q}}(\mathfrak{A}_{m,r}{}^{(e_4)},\mathrm{G},\mathrm{B}_4) = 1 \qquad \text{if a} \neq 2.$$

Here 1 denotes the set with one element.

Chapter VI :Subgroups of a product of surface groups.

By a surface group of genus g, denoted Σ_g^+, we mean the fundamental group of the oriented surface of genus $g \geq 1$. Σ_g^+ has the following presentation

$$\Sigma_g^+ = \langle X_1, \ldots, X_{2g} : \prod_{k=1}^{g} [X_k , X_{g_i+k}] \rangle.$$

where $[X,Y]$ is the commutator $[X,Y] = X.Y.X^{-1}.Y^{-1}$. We consider normal subgroups of a product of two suface groups $\Theta = \Sigma_{g_1}^+ \times \Sigma_{g_2}^+$. Although the work in this chapter will be independent of other chapters, there is a well known connection between surface groups and discrete groups. The fundamental group of an oriented surface of positive genus g, act as the group of deck transformations on the universal covering which we identified with the hyperbolic plane, $\mathfrak{H}$. The fundamental group acts discontinously and so is Fuchsian, [Be]. If we take the upper half plane model for $\mathfrak{H}$,

$$\mathfrak{H} = \{z \in \mathbb{C} : \text{Im } z > 0\},$$

then Σ_g^+ acts as a subgroup of the group of fractional transformations of $\mathfrak{H}$. Σ_g^+ is a discrete subgroup of $Sp_1(\mathbb{R})/\{\pm 1\} = PSL_2(\mathbb{R})$.

Definition : Let $G = G_1 \times \ldots \times G_n$ be a product of groups and let $\pi_i : G \to G_i$ denote the natural projections. Then by a *subdirect product* of $G_1 \times \ldots \times G_n$ we mean a subgroup H such that $\pi_i(H) = G_i$.

Fix $\Theta = \Sigma_{g_1}^+ \times \Sigma_{g_2}^+$ a product of surface groups, let $\pi_i : \Theta \to \Sigma_{g_i}^+$ denote projection. Let H be a normal subgroup of Θ. If $\pi_i(H)$ has finite index d_i in $\Sigma_{g_i}^+$ then we may identify this image with a surface group with genus $h_i = 1 + d_i(g_i - 1)$. If $\pi_i(H)$ does not have finite index, then we may identify $\pi_i(H) = F_\infty$, the free group of infinite rank. We shall only consider the finite index case, and we are then reduced to consider H as a subdirect product of a product of two surface groups.

Definition : Let H, H' be subgroups of a arbitrary group G. We say that H and H' are *commensurable* when $H \cap H'$ has finite index in both H and H'. If H and H' are both normal in G then H and H' are commensurable if and only if both H and H' having finite index in $H.H'$. Let $\mathcal{M}$ be a subgroup of the group of automorphisms of Θ. We say that H is $\mathcal{M}$-*commensurable* to H' when there exists an $\alpha \in \mathcal{M}$ such that $\alpha(H)$ is commensurable to H'.

Note that, if H is maximal in its commensurability class then so is $\alpha(H)$ for any group automorphism α of Θ. Hence two maximal normal subgroups are $\mathcal{M}$-commensurable if and only if they are $\mathcal{M}$-equivalent.

We will show that there are infinitely many $\mathrm{Aut}(\Theta)$-commensurability class of normal subdirect products in a product of two suface groups. In a product of two free groups it can be shown that there are only finitely many such classes, see [Jo3].

§1 Symplectic modules.

By a *symplectic module* $(A\ , \langle\ ,\ \rangle)$ over $\mathbb{Z}$ we mean a finitely generated free abelian group A with a skew symmetric bilinear form $\langle\ ,\ \rangle : A \times A \to \mathbb{Z}$.

Let $\mathcal{A}$ denote the category of symplectic modules, with as its morphisms all group isomorphisms that either preserve or negate the forms. That is, for

$(A_1\ , \langle\ ,\ \rangle_1), (A_2\ , \langle\ ,\ \rangle_2) \in \mathcal{A}$ and $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A_1\ , A_2)$

$$\langle\ \alpha(x)\ ,\ \alpha(y)\ \rangle = \epsilon(\alpha).\langle\ x,y\ \rangle \qquad \text{for all } x,y \in A_1,$$

where $\epsilon(\alpha) = \pm 1$. We will often write A for $(A\ , \langle\ ,\ \rangle) \in \mathcal{A}$ when the form is fixed. As a matter of notation, if $A \in \mathcal{A}$ and $A = B \oplus C$ with B orthogonal to C, we write

$$A = B \perp C.$$

For $A \in \mathcal{A}$, let $\mathrm{rad}(A) = \{\ a \in A : \langle\ a,x\ \rangle = 0\ \forall\ x \in A\ \}$. We say that $A \in \mathcal{A}$ is *non-degenerate* when $\mathrm{rad}(A) = 0$. If B is a submodule of $A \in \mathcal{A}$ then let

$$B^{\perp} = \{x \in A : \langle x, b \rangle = 0 \ \forall \ b \in B\}.$$

The following is standard.

***Proposition* (1.1) :** Let $A \in \mathcal{A}$, then $A = \widetilde{A} \perp \mathrm{rad}(A)$, where $\widetilde{A}$ is non-degenerate and unique up to an $\mathcal{A}$-morphism of A.

We say that $H = (H, \langle , \rangle_H) \in \mathcal{A}$ is a *hyperbolic plane of length a* when H has a basis e_1, e_2 such that $\langle e_1, e_2 \rangle_H = a$. The following result is due to Frobenius.

***Proposition* [Fr] (1.2) :** For $A \in \mathcal{A}$ there exists a sequence of positive integers $(a_1, \ldots, a_p)$ such that

(I) $A \cong_{\mathcal{A}} H(a_1) \perp \ . \ . \ \perp H(a_p) \perp \mathrm{rad}(A)$

where each $H(a_i)$ is a hyperbolic plane of length a_i and these lengths satisfy

(II) $a_i \mid a_{i+1}$ for $1 \leq i \leq p - 1$.

Moreover, A uniquely determines a sequence with these properties.

***Proof*:** To prove existence it suffices to consider $A = (A, \langle , \rangle)$ non-degenerate. Choose $e, e' \in A$ such that

$$\langle e, e' \rangle = \min\{\langle x, y \rangle > 0 : x, y \in A\}.$$

Let $a_1 = \langle e, e' \rangle$, and the a_1 is the length of the hyperbolic plane

$$H(a_1) = \mathrm{span}_{\mathbb{Z}}(e, e').$$

We claim that $A = H(a_1) \perp H(a_1)^{\perp}$. Since $H(a_1)$ is non-degenerate we need only show that $A = H(a_1) + H(a_1)^{\perp}$. Let $x \in A$, then

$$\langle x, e \rangle = k(x).a_1 + r(x),$$

$$\langle x, e' \rangle = k'(x).a_1 + r'(x)$$

where $k(x), r(x), k'(x), r'(x) \in \mathbb{Z}$ and $|r(x)|, |r'(x)| < a_1$. But this contradicts the definition of a_1, unless $r(x) = r'(x) = 0$ for all x. Define

$$y = x + k(x).e' - k'(x).e,$$

then $y \varepsilon H(a_1)^{\perp}$. Hence $x \varepsilon H(a_1) + H(a_1)^{\perp}$ and the claim follows. Hence we may inductively define a sequence satisfying (I); define a basis $\{e_1, \ldots, e_{2p}\}$ for A_1 by $H(a_j) = \text{span}_{\mathbb{Z}}(e_j, e_{p+j})$. For $1 \leq m \leq p$ let $g_m = \text{g.c.d.}(a_m, \ldots a_p)$. So $g_m \leq a_m$ for all m. Write $g_m = \lambda_m.a_m + \ldots + \lambda_p.a_p$ and let

$$x = \sum_{i=m}^{p} \lambda_i.e_i \qquad , \qquad y = \sum_{i=m}^{p} e_{p+i}.$$

Then $a_m \leq \langle x, y \rangle = \lambda_m.a_m + \ldots + \lambda_p.a_p = g_m$. Hence, $a_m = \text{g.c.d.}(a_m, \ldots a_p)$ and so condition (II) follows.

We prove uniqueness, again it suffices to consider $A = (A, \langle , \rangle)$ non-degenerate, since all complements to rad(A) in A are $\mathcal{A}$-isomorphic. Let $(a_1, \ldots, a_p)$ and $(a'_1, \ldots, a'_p)$ be two sequences saisfying (I) and (II). Let $m \varepsilon \{1, \ldots, p\}$ be the least integer such that $a_m \neq a'_m$. Asume that $a_m > a'_m$. Let $X = A \otimes_{\mathbb{Z}} \mathbb{Z}_{/a_m}$ and $\prec , \succ : X \times X \to \mathbb{Z}_{/a_m}$ be the skew symmetric $\mathbb{Z}_{/a_m}$-bilinear form induced form $\langle , \rangle$. Note that X is a finite set. Define

$$R = \{x \varepsilon X : \prec x, y \succ = 0 \ \forall \ y \varepsilon X\}.$$

The cardinality of R is determined by the sequences $(a_1, \ldots, a_p)$ and $(a'_1, \ldots, a'_p)$. But the condition $a_m > a'_m$ gives two different values for this cardinality. Hence we must have $a_m = a'_m$ and $(a_1, \ldots, a_p) = (a'_1, \ldots, a'_p)$.

□

For each $A \varepsilon \mathcal{A}$, let $\mathfrak{S}(A)$ denote the sequence of integers for A as given in (1.2). We have the following corollary.

***Proposition* (1.3)** : Let $A_1, A_2 \varepsilon \mathcal{A}$, then

$$A_1 \cong_{\mathcal{A}} A_2 \quad \Leftrightarrow \quad \left\{ \begin{array}{c} \text{rk}_{\mathbb{Z}}(A_1) = \text{rk}_{\mathbb{Z}}(A_2) \\ \mathfrak{S}(A_1) = \mathfrak{S}(A_2) \end{array} \right\} \ .$$

For $A = (A\ ,\langle\ ,\ \rangle) \varepsilon\ \mathcal{A}$ let $A^* = \mathrm{Hom}_{\mathbb{Z}}(A\ ,\ \mathbb{Z})$ and $\lambda_A\colon A \to A^*$ the map given by $\lambda_A(x)(y) = \langle x\ ,\ y\rangle$ for all $x,y\ \varepsilon\ A$. Hence $\mathrm{rad}(A) = \ker(\lambda_A)$. We say that A is *non-singular* when λ_A is an isomorphism. Note that,

***Proposition* (1.4) :** $A\ \varepsilon\ \mathcal{A}$ is non-singular $\Leftrightarrow\ \mathfrak{S}(A) = (1, \ldots ,1)$ (p copies) where

$$\mathrm{rk}_{\mathbb{Z}}(A) = 2.p\ .$$

The category $\mathcal{A}$ is of particular interest because of the following example. Let Σ_+^g denote the surface of genus g. Consider the following ho mology and cohomology groups, $H_n(\Sigma_+^g, \mathbb{Z})$, $H^n(\Sigma_+^g, \mathbb{Z})$. It is standard that, $H_1(\Sigma_+^g, \mathbb{Z}) = (\Sigma_g^+)^{ab}$ and $H^2(\Sigma_+^g, \mathbb{Z}) \cong H_2(\Sigma_+^g, \mathbb{Z}) \cong \mathbb{Z}$ for all g. Fix a generator μ_g for $H_2(\Sigma_+^g, \mathbb{Z})$. Let $\cap$ (resp. $\cup$) denote the cap (resp. cup) product, and $p = (-)\cap\mu_g : H^1(\Sigma_+^g, \mathbb{Z}) \to H_1(\Sigma_+^g, \mathbb{Z})$ the Poincare' Duality map. Consider the form

$$\langle\ ,\ \rangle : H_1(\Sigma_+^g, \mathbb{Z})\ \mathrm{x}\ H_1(\Sigma_+^g, \mathbb{Z}) \to \mathbb{Z}$$

given by

$$\langle x\ ,\ y\rangle_g = (\ p^{-1}(x) \cup p^{-1}(y)\ ) \cap \mu_g.$$

This form is skew symmetric (and non-singular) so we consider $H_1(\Sigma_+^g, \mathbb{Z})\ \varepsilon\ \mathcal{A}$.

Let $\alpha\ \varepsilon\ \mathrm{Aut}(H_1(\Sigma_+^g, \mathbb{Z}))$. We say that α has a *lifting* to $\mathrm{Aut}(\Sigma_g^+)$ when there exists an $\widetilde{\alpha}\ \varepsilon\ \mathrm{Aut}(\Sigma_g^+)$ such that the following diagram commutes

$$\begin{array}{ccc} \Sigma_g^+ & \xrightarrow{\ \widetilde{\alpha}\ } & \Sigma_g^+ \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ H_1(\Sigma_+^g, \mathbb{Z}) & \xrightarrow{\ \alpha\ } & H_1(\Sigma_+^g, \mathbb{Z}) \end{array}$$

where φ denotes the abelianisation map.

Let $\hat{\alpha} : \Sigma_g^+ \to \Sigma_g^+$ be an automorphism. Since $\Sigma_+^g = K(\Sigma_g^+, 1)$ it follows that α is realisable by a homeomorphism $\hat{\alpha} : \Sigma_+^g \to \Sigma_+^g$, see [Ep], and the induced map of α on $H_1(\Sigma_+^g, \mathbb{Z})$ is a morphism in $\mathcal{A}$. Conversely, any automorphism of $H_1(\Sigma_+^g, \mathbb{Z})$, which is also a morphism of $\mathcal{A}$, can be expressed, using a result of Hua-Reiner [Hu-Re], as a product of automorphisms which are realisable geometrically as homeomorphisms of Σ_+^g. Thus we have the following well known result.

***Proposition* (1.5) :** $\alpha \ \varepsilon \ \mathrm{Aut}(H_1(g))$ has a lifting to $\mathrm{Aut}(\Sigma_g^+) \Leftrightarrow \alpha \ \varepsilon \ \mathrm{Aut}_{\mathcal{A}}(H_1(\Sigma_+^g, \mathbb{Z}))$.

An alternative proof, which is algebraic, may be found in [Ma-Ka-So].

Fix $A = (A \ , \langle \ , \ \rangle) \ \varepsilon \ \mathcal{A}$, non-singular of rank 2n. Let $\mathcal{C} = \mathcal{C}(A)$ denote the category of pairs $(N \ , C) \ \varepsilon \ \mathcal{A} \times \mathcal{A}$ such that $A = N \oplus C$. If $(N \ , C)$, $(N', C') \ \varepsilon \ \mathcal{C}$ then a morphism $\alpha : (N \ , C) \to (N', C')$ of $\mathcal{C}$ is a map $\alpha \ \varepsilon \ \mathrm{Hom}_{\mathcal{A}}(A \ , A)$ such that $\alpha(N) = N'$ and $\alpha(C) = C'$. For arbitrary submodules N of A there is no restriction on the sequence $\mathfrak{S}(N) \ \varepsilon N^p$, $p \leq n$. For direct summands of A the situation is slightly different.

***Proposition* (1.6) :** Let N be a direct summand of A, and let L denote the number of non-unity entries in $\mathfrak{S}(N)$. Then,

$$0 \leq L \leq \frac{\mathrm{rk}_{\mathbb{Z}}(A) - \mathrm{rk}_{\mathbb{Z}}(N) - \mathrm{rk}_{\mathbb{Z}}\mathrm{rad}(N)}{2} \qquad \{1.1\}.$$

***Proof*:** If $L = 0$ the result is trivial, so assume that $L > 0$. Let $(b_1, \ldots, b_L)$ denote the sequence of non-unity entries in $\mathfrak{S}(N)$. Choose a basis $\{e_1, \ldots, e_{2p+\delta}\}$ for N such that

$$N = H_1(b_1) \perp \ldots \perp H_L(b) \perp H_{L+1}(1) \perp \ldots \perp H_{2p}(1) \perp \mathrm{rad}(N),$$

where,

(i) $H_i(b_i) = \text{span}_{\mathbb{Z}}(e_i, e_{p+i}) \qquad 1 \leq i \leq L$

(ii) $H_i(1) = \text{span}_{\mathbb{Z}}(e_j, e_{p+j}) \qquad L+1 \leq j \leq p$

(iii) $\text{rad}(N) = \text{span}_{\mathbb{Z}}(e_{2p+1}, \ldots, e_{2p+\delta})$

Fix $k \in \{1, \ldots, 2p+\delta\}$, and let $\phi_k \in A^*$ be any map which extends the map $\phi_k \in N^*$ given by

$$\phi_k(e_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Since A is non-singular, we may choose $f_k \in A$ such that

$$\phi_k(x) = \langle x, f_k \rangle, \qquad \text{for all } x \in A.$$

That is, the f_k satisfy,

$$\langle e_i, f_k \rangle = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{O}_0 = \{e_1, \ldots, e_{2p+\delta}\}$$

$$\mathcal{O}_1 = \{f_{2p+1}, \ldots, f_{2p+\delta}\}$$

$$\mathcal{O}_2 = \{f_1, \ldots, f_L\}$$

$$\mathcal{O}_3 = \{f_{p+1}, \ldots, f_{p+L}\}.$$

We claim that

$$\mathcal{O} = \bigcup_{i=1}^{3} \mathcal{O}_i$$

is a linearly independent set of elements in A. For suppose

$$\sum_{i=1}^{2p+\delta} \mu_i.e_i + \sum_{j=1}^{\delta} \lambda_{2p+j}.f_{2p+j} + \sum_{k=1}^{L} \lambda_k.f_k + \sum_{l=1}^{L} \lambda_{p+l}.f_{p+l} = 0 \qquad \{1.2\}.$$

By simple arguments we will show that all coefficients are zero.

(1) For $1 \leq n \leq \delta$ apply $\langle e_{2p+n}, - \rangle$ to {1.2}. Then,

$$\lambda_n = 0$$

since in each case $e_n \in \text{rad}(N)$. We have reduced {1.2} to

$$\sum_{i=1}^{2p+\delta} \mu_i.e_i + \sum_{k=1}^{L} \lambda_k.f_k + \sum_{l=1}^{L} \lambda_{p+l}.f_{p+l} = 0 \qquad \{1.3\}.$$

(2) Suppose there exists $\lambda_1, \ldots, \lambda_L, \lambda_{p+1}, \ldots, \lambda_{p+L}$ satisfying {1.3} for

some μ_i not equal to zero. Choose $\lambda_1, \ldots, \lambda_L, \lambda_{p+1}, \ldots, \lambda_{p+L}$ such that

$$\sum_{k=1}^{L} |\lambda_k| + \sum_{l=1}^{L} |\lambda_{p+l}| \qquad \text{is minimal.}$$

Note that this sum is non-zero: not all the λ_k, λ_{p+l} can be zero since the e_i are linearly independent. For $1 \leq n \leq L$ apply $\langle e_n, - \rangle$ and $\langle e_{p+n}, - \rangle$ to {1.3}. Then we have

$$\left.\begin{aligned} \mu_{p+n}.b_n + \lambda_n = 0 \\ \mu_n.b_n - \lambda_{p+n} = 0 \end{aligned}\right\} \qquad 1 \leq n \leq L.$$

But $b_1 \neq 1$ and divides b_n for all n, hence b_1 divides λ_n and λ_{p+n} for all n. Let

$$\lambda'_k = \lambda_k / b_1 \qquad 1 \leq k \leq L,$$

$$\lambda'_{p+l} = \lambda_{p+l} / b_1 \qquad 1 \leq l \leq L.$$

Hence by the choice of $\lambda_1, \ldots, \lambda_L, \lambda_{p+1}, \ldots, \lambda_{p+L}$ the coefficients $\lambda'_1, \ldots, \lambda'_L$, $\lambda'_{p+1}, \ldots, \lambda'_{p+L}$ do not satisfy {1.3}. Hence,

$$x = \sum_{k=1}^{L} \lambda'_k.f_k + \sum_{l=1}^{L} \lambda'_{p+l}.f_{p+l}$$

is not an element of N. However $b_1.x$ is an element of N, and this contradicts th fact that N is a direct summand of A. Hence $\mu_i = 0$ for all i.

(3) Finally, we see that the rest of the coeficients of {1.2} are zero by applying $\langle e_n, - \rangle$ and $\langle e_{p+n}, - \rangle$ for $1 \leq n \leq L$.

Since $\mathcal{O}$ is an independent set containing $2.(p + \delta + L)$ elements we have

$$2.(p + \delta + L) \leq 2.n$$

Hence,

$$\begin{aligned} 2.L &\leq 2.n - 2.p - 2.\delta \\ &= rk_{\mathbb{Z}}(A) - rk_{\mathbb{Z}}(N) - rk_{\mathbb{Z}}rad(N) \end{aligned}$$

□

For $S = (a_1, \ldots, a_p) \in \mathbb{N}^p$ let $\mathfrak{L}(S)$ denote the number of non-unity entries in S, and $\mathfrak{L}(\emptyset) = 0$. Let $p, q, d \in \mathbb{N}$ such that $p + q \leq n$ and $2p + d \leq 2n$. Then for any sequences $S = (a_1, \ldots, a_p) \in \mathbb{N}^p$ and $S' = (a'_1, \ldots, a'_p) \in \mathbb{N}^q$ with $\mathfrak{L}(S) \leq$

$\min(d + p - n, p)$ and $\mathfrak{L}(S') \leq \min(n + q - 2d, p)$ then we may construct a $(N, C) \in \mathcal{C}$ such that $rk_{\mathbb{Z}}(C) = d$, $\mathfrak{S}(N) = S$ and $\mathfrak{S}(C) = S'$. We indicate a verification for N, the rest follows similarly. Fix $\epsilon_1, \ldots, \epsilon_{2n}$ as a symplectic basis for A, that is

$$\langle \epsilon_i, \epsilon_j \rangle = \begin{cases} 1 & \text{if } j = n + i \\ -1 & \text{if } i = n + j \\ 0 & \text{otherwise} \end{cases} .$$

Let $(b_1, \ldots, b_L)$ be the sequence of non-unity elements of S, $L = \mathfrak{L}(S)$; if every entry is equal to one let $L = 0$. Define

$$e_i = \epsilon_i + \epsilon_{p+\delta+1} \qquad \text{for } 1 \leq i \leq L$$

$$e_{n+j} = \epsilon_j + (b_j - 1).\epsilon_{n+p+\delta+j} \qquad \text{for } 1 \leq j \leq L$$

Let

$$N = \text{span}_{\mathbb{Z}}(e_1, \ldots, e_L, \epsilon_{L+1}, \ldots, \epsilon_p, e_{n+1}, \ldots, e_{n+L}, \epsilon_{n+L+1}, \ldots, \epsilon_{n+p}, \epsilon_{p+1}, \ldots, \epsilon_{p+\delta})$$

Then N is a direct summand of A and it is easily verified that $\mathfrak{S}(N) = (a_1, \ldots, a_p)$.

§2 Symplectic product structures.

By a (2-*fold*) *product structure* $\mathcal{P}$ of a group G we mean a 3-tuple $\mathcal{P} = (G ; G_1 , G_2)$ with G_1 and G_2 subgroups of G such that $G = G_1 \circ G_2$. For such a product structure let $\pi_i : G \to G_i$ denote the obvious projections. Then by a *subdirect product* of $\mathcal{P}$ we mean a subgroup H of G such that $\pi_i(H) = G_i$. Let $\mathcal{N}(\mathcal{P})$ denote the class of subdirect products of $\mathcal{P}$ that are normal in G. Also for $\mathcal{P}$ let $\mathcal{P}^{ab}$ denote the product structure (G_1^{ab}, G_2^{ab}) for the abelianised group G^{ab}. Abelianisation induces a bijection $\varphi : \mathcal{N}(\mathcal{P}) \to \mathcal{N}(\mathcal{P}^{ab})$, by $\varphi(N) = N^{ab}$, see (1.2) of [Jo3]. Define

$$\text{Aut}(\mathcal{P}) = \{\alpha : G \to G \text{ group automorphism } : \alpha(G_i) \subset G_i \text{ for } i = 1 , 2\}.$$

***Definition*:** For $i = 1, 2$ let $(A_i , \langle , \rangle_i) \varepsilon \mathcal{A}$ be non-singular. By a *symplectic product structure* $\mathcal{Q} = (A , \langle , \rangle ; A_1 , A_2)$ we mean a product structure $(A ; A_1 , A_2)$, with $(A , \langle , \rangle) \varepsilon \mathcal{A}$ where $\langle , \rangle = \langle , \rangle_1 \perp \langle , \rangle_2$. Define

$$\mathrm{Aut}(\mathcal{Q}) = \{ \alpha \varepsilon \mathrm{Aut}_{\mathcal{A}}(A) : \alpha(A_i) \subset A_i \}.$$

Let $\mathcal{N}(\mathcal{Q})$ denote the class of (normal) subdirect products of a symplectic product structure $\mathcal{Q}$. For the remainder of this section fix a symplectic product structure $\mathcal{Q} = (A , \langle , \rangle ; A_1 , A_2)$ with $\mathrm{rk}_{\mathbb{Z}}(A_i) = 2.n_i > 0$.

We say that a subgroup H of A is *framed* in $\mathcal{Q}$ when there exist

(i) subgroups N_i, C_i with $A_i = N_i \oplus C_i$

(ii) a group isomorphism $\phi : C_1 \rightarrow C_2$ such that

$$N = N_1 \oplus N_2 \oplus \Delta(\phi)$$

where $\Delta(\phi) = \{ c + \phi(c) : c \varepsilon C_1 \}$.

We call the 5-tuple $(N_1, N_2, C_1, C_2, \phi)$ a *framing* for N. Note that $N_i = N \cap A_i$ for $i = 1, 2$. Clearly any such framed subgroup is a subdirect product of $\mathcal{Q}$.

Let $\mathcal{F} = \mathcal{F}(\mathcal{Q})$ denote the category whose objects are the framed subgroups of $\mathcal{Q}$. If $N, M \varepsilon \mathcal{F}$ then a morphism $\alpha : N \rightarrow M$ in $\mathcal{F}$ is a map $\alpha \varepsilon \mathrm{Aut}(\mathcal{Q})$ such that there exist framings $\mathcal{F}_N = (N_1, N_2, C_1, C_2, \phi)$, $\mathcal{F}_M = (M_1, M_2, D_1, D_2, \psi)$ for N and M with

(i) $\alpha(N_i) = M_i$, $\alpha(C_i) = D_i$

(ii) and

$$\begin{array}{ccc} C_1 & \xrightarrow{\ \phi\ } & C_2 \\ {\scriptstyle \beta_1}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ D_1 & \xrightarrow{\ \psi\ } & D_2 \end{array}$$

a commuting diagram for $\beta_i = \alpha_{/C_i}$.

***Proposition* (2.1) :** Let N, M ε $\mathcal{F}$ with $(N_1, N_2, C_1, C_2, \phi)$ a framing for N. Then N is Aut($\mathcal{Q}$)-equivalent to M if and only if there exists a frame $(M_1, M_2, D_1, D_2, \psi)$ for M and a morphism α in $\mathcal{F}$ satisfying (i) and (ii) above.

***Definition* :** For H ε $\mathcal{N}(\mathcal{Q})$ define the *diagonal rank* to be

$$\mathrm{drk}(H) = \mathrm{rk}_{\mathbb{Z}}\left(H_{/(H \cap A_1 + H \cap A_2)} \right).$$

***Proposition* (2.2) :** For any H ε $\mathcal{N}(\mathcal{Q})$ there exists an N ε $\mathcal{F}$ containing H with finite index. Moreover, drk(H) = drk(N).

***Proof*:** Let $\pi_i : A \to A_i$ denote projection. Let $H_i = H \cap A_i$. Since A_i is free abelian there exists a unique direct summand N_i of A_i containing H_i with finite index. Let

$$N = H \oplus N_1 \oplus N_2,$$

then $N \cap A_i = N_i$. Since $N_1 \oplus N_2$ is a direct summand of A and $N_1 \oplus N_2 \subset N$ we may write

$$N = N_1 \oplus N_2 \oplus \Delta$$

for some Δ. Note that,

$$\Delta \cap A_i = \Delta \cap N \cap A_i = \Delta \cap N_i = \{0\},$$

hence the maps $\pi_{i/\Delta}$ are injective. Let $C_i = \pi_i(\Delta)$ and $\phi = \pi_2 \circ (\pi_{1/\Delta})^{-1}$. Clearly $\varphi : C_1 \to C_2$ is an isomorphism and

$$N = N_1 \oplus N_2 \oplus \Delta(\phi).$$

Since H is a subdirect product of $\mathcal{Q}$,

$$A_i = N_i + C_i.$$

Also, if $x \varepsilon N_i \cap C_i$ then $\pi_i^{-1}(x) \varepsilon N_i \cap \Delta = \{0\}$. So $N_i \cap C_i = \{0\}$ and

$$A_i = N_i \oplus C_i.$$

It follows that $(N_1, N_2, C_1, C_2, \phi)$ is a framing for N. Hence N ε $\mathcal{F}(\mathcal{Q})$ contains H with finite index. The result now follows since $\mathrm{drk}(N) = \mathrm{drk}(H) = \mathrm{rk}_{\mathbb{Z}}(\Delta) =$

$rk_{\mathbb{Z}}(C_i)$, i = 1, 2.

□

If H ε $\mathcal{N}(\mathbb{Q})$ is maximal in its commensurability class, then by (2.2) H is framed. Conversely, suppose H is framed by $(H_1 , H_2 , C_1 , C_2 , \phi)$, then H is a direct summand of A since C_1 is a complement. Hence H must be maximal in its commensurability class. Thus two framed subgroups of A are Aut($\mathbb{Q}$)-commensurable if and only if they are Aut($\mathbb{Q}$)-equivalent.

For $1 \leq d \leq \min(n_1, n_2)$ let $\mathcal{Z}(\mathbb{Q},d)$ denote the set of all Aut($\mathbb{Q}$)-commensurability classes of subdirect products of $\mathbb{Q}$ that have diagonal rank equal to d.

***Proposition* (2.3)** : If d = 0, 1 then $|\mathcal{Z}(\mathbb{Q},d)| = 1$.

***Proof*:** Clearly $\mathcal{Z}(\mathbb{Q},0) = \{ A \}$. Suppose $H' \varepsilon \mathcal{N}(\mathbb{Q})$ with $drk(H') = 1$. Let H ε $\mathcal{D}$ contain H' with finite index. Put $H_i = H \cap A_i$ then we may choose $\widetilde{H}_i \subset H_i$ such that

(i) $H_i = \widetilde{H}_i \perp rad(H_i)$, and,

(ii) there exists a framing $\mathfrak{F}_H = (H_1 , H_2 , C_1 , C_2 , \phi)$ for H with C_i orthogonal to $\widetilde{H}_i$.

This implies that $C_i \oplus rad(H_i)$ is a hyperbolic planes of length 1. The result follows since there is clearly only one automorphism class of framed subgroup of diagonal rank equal to one.

□

***Proposition* (2.4)** : If $2 \leq d \leq 2.(\min(n_1 , n_2) - 1)$ then $|\mathcal{Z}(\mathbb{Q},d)| = \infty$.

Proof: Without loss we may assume that $2 \leq n_2 \leq n_1$, hence $d \leq 2.(n_2 - 1)$. Let $k \in \mathbb{N}^+$. Let $(N_i^{(k)}, C_i^{(k)}) \in \mathcal{C}(A_i)$ for $i = 1,2$ such that,

(i) $rk_{\mathbb{Z}}(C_i^{(k)}) = d$

(ii) $\mathcal{G}(N_1^{(k)}) = (1, \ldots, 1) \in \mathbb{N}^{2n_1-d}$

and $\mathcal{G}(N_2^{(k)}) = (1, \ldots, 1, k) \in \mathbb{N}^{2n_2-d-1}$.

Let $N(k) \in \mathcal{F}(\mathcal{Q})$ with framing $(N_1^{(k)}, N_2^{(k)}, C_1^{(k)}, C_2^{(k)}, \phi^{(k)})$ for some group isomorphism $\phi^{(k)} : C_1^{(k)} \rightarrow C_2^{(k)}$. Suppose N(k) is $Aut(\mathcal{Q})$-commensurable to $N(k')$. By the remarks preceeding **(2.3)**, N(k) is then $Aut(\mathcal{Q})$-equivalent to $N(k')$, since each is framed. Then since $N_1^{(k)}$, $N_2^{(k)}$ and $N_1^{(k')}$, $N_2^{(k')}$ are uniquely determined by N(k) and $N(k')$ we require that $\mathcal{G}(N_2^{(k)}) = \mathcal{G}(N_2^{(k')})$ by **(1.3)** and the definition of $Aut(\mathcal{Q})$. This is a contradiction unless $k = k'$. Hence $\{N(k)\}_{k \in \mathbb{N}}$ represents an infinite set of $Aut(\mathcal{Q})$-commensurable classes in $\mathcal{Z}(\mathcal{Q}, d)$.

□

For $r \in \mathbb{N}^+$, let I_r denote the r x r identity matrix and let O denote the zero matrix whose size will be determined by its position. Fix

$$\Lambda = \begin{bmatrix} O & I_r \\ -I_r & O \end{bmatrix}$$

Let $Sp_{2r}^*(\mathbb{Z})$ denote the <u>full</u> symplectic group of 2r x 2r matrices, that is $Z \in Sp_{2r}^*(\mathbb{Z})$ satisfies

$$Z.\Lambda.Z^t = \pm \Lambda.$$

Let K_r denote the following set of double cosets,

$$K_r = {}_{Sp_{2r}^*(\mathbb{Z})}\backslash GL_{2r}(\mathbb{Z}) / {}_{Sp_{2r}^*(\mathbb{Z})} .$$

Lemma **(2.5)** : For all $r \geq 2$, $|K_r| = \infty$.

Proof: Let $E_{i,j}$ denote the (i , j) elementary 2r x 2r matrix. Define

$$Y(\lambda) = I_{2r} + \lambda.(E_{1,n} + E_{2n,n+1})$$

and $$X(\lambda) = \Lambda + \lambda.(E_{1,2n} + E_{2n,1})$$

for any $\lambda \varepsilon \mathbb{Z}$. Then,

$$Y(\lambda).\Lambda.Y(\lambda) = X(2\lambda).$$

Let $Z \varepsilon Sp^*_{2r}(\mathbb{Z})$ and λ, μ be distinct primes. Then, by a comparison of the (1 , 2n) entries, it can be seen that

$$Z.X(2\lambda).Z^t \neq \pm X(2\mu).$$

Now suppose that there exist S, T $\varepsilon Sp^*_{2r}(\mathbb{Z})$ such that $S.Y(\lambda).T = Y(\mu)$. Then

$$S.Y(\lambda).T.\Lambda.T^t.Y(\lambda)^t.S^t = \pm Y(\mu).\Lambda.Y(\mu)^t$$

$$\Rightarrow \quad S.Y(\lambda).\Lambda.Y(\lambda)^t.S^t = \pm Y(\mu).\Lambda.Y(\mu)^t$$

$$\Rightarrow \quad S.X(2\lambda).S^t = \pm X(2\mu)$$

But this is a contradiction, thus no such S and T exist, and $\{Y(\lambda)\}_{\lambda \text{ prime}}$ is an infinite set of coset representatives in K.

□

Proposition (2.6): Suppose $2n_1 = 2n_2$ (= 2n). Then

(a) $|\mathfrak{Z}(\mathbb{Q}, 2n-1)| = \infty$ and (b) $|\mathfrak{Z}(\mathbb{Q}, 2n)| = \infty$.

Proof: Let $\{\Phi_\lambda\}_{\lambda \epsilon \Lambda}$ be a complete set of coset representatives in K_n. So $|\Lambda| = \infty$ by the Lemma. Fix a symplectic basis $\{e^i_1, \ldots, e^i_{2n}\}$ for A_i, i = 1, 2. We first prove:

(a) For each $\lambda \varepsilon \Lambda$ we construct a $M^{(\lambda)} \varepsilon \mathcal{N}(\mathbb{Q})$ with $drk(M^{(\lambda)}) = 2n - 1$. Let $M^{(\lambda)}_i = span_Z(e^i_1)$ and $D^{(\lambda)}_i = span_Z(e^i_2, \ldots, e^i_{2n})$ and define $\psi^{(\lambda)} : D^{(\lambda)}_1 \to D^{(\lambda)}_2$ such that $\psi^{(\lambda)}(e^1_2) = e^2_2$ and $\psi^{(\lambda)}$ restricted to $span_Z(e^i_3, \ldots, e^i_{2n})$ has matrix Φ_λ with respect to these bases. Let $M^{(\lambda)} = M^{(\lambda)}_1 \oplus M^{(\lambda)}_2 \oplus \Delta(\psi^{(\lambda)})$ which has framing ($M^{(\lambda)}_1$, $M^{(\lambda)}_2$, $D^{(\lambda)}_1$, $D^{(\lambda)}_2$, $\psi^{(\lambda)}$). If $M^{(\lambda)}$ is Aut(Q)-commensurable to $M^{(\mu)}$ then $\lambda = \mu$, since condition (ii) in the definition of a framed automorphism implies that Φ_λ

and Φ_μ must be in the coset in K_n. Hence $\left\{M^{(\lambda)}\right\}_\Lambda$ is an infinite set of representatives for classes in $\mathcal{Z}(2n-1)$.

(b) For $\lambda \in \Lambda$ let $\phi^{(\lambda)}$: $A_1 \to A_2$ be the isomorphism which has matrix Φ_λ with respect to the fixed bases. Let $N^{(\lambda)} = \Delta(\phi^{(\lambda)})$, which has unique framing $(0\ ,\ 0\ ,\ A_1\ ,\ A_2\ ,\ \phi_\lambda)$, where 0 is the trivial group. Again, it is seen that $\left\{N_\lambda\right\}_\Lambda$ is an infinite set of representatives for classes in $\mathcal{Z}(2n)$.

□

We summarise our results as follows.

Corollary (2.7) : For $0 \leq d \leq \min(n_1, n_2)$

$$|\mathcal{Z}(\mathcal{Q}, d)| = \begin{cases} 1 & \text{if } d = 0, 1 \\ \infty & \text{otherwise} \end{cases} .$$

§3 Normal subdirect products of a product of two surface groups.

Fix $g_1, g_2 \in \mathbf{N}$ and then let $\mathcal{P}$ denote the product structure $(\Theta\ ;\ \Sigma^+_{g_1}, \Sigma^+_{g_2}))$, $g_i \geq 1$. We suppose that $\min(g_1, g_2) \geq 2$. Let $\mathcal{Q} = \mathcal{P}^{ab}$, which we consider as a symplectic product structure with form $\langle\ ,\ \rangle : \Theta^{ab} \times \Theta^{ab} \to \mathbb{Z}$ given by $\langle\ ,\ \rangle = \langle\ ,\ \rangle_1 \perp \langle\ ,\ \rangle_2$ where $\langle\ ,\ \rangle_i = \langle\ ,\ \rangle_{g_i} : H_1(\Sigma^{g_i}_+, \mathbb{Z}) \times H_1(\Sigma^{g_i}_+, \mathbb{Z}) \to \mathbb{Z}$, see §1.

We clearly have,

$$\mathrm{Aut}(\mathcal{P}) = \mathrm{Aut}(\Sigma^+_{g_1}) \times \mathrm{Aut}(\Sigma^+_{g_2})$$

So, by (1.1) and (1.6) of [Jo1] we have,

(i) if $g_1 \neq g_2$ then

$$\mathrm{Aut}(\mathcal{P}) = \mathrm{Aut}(\Theta)$$

otherwise,

(ii) if $g_1 = g_2$ $(=g)$ then $\mathrm{Aut}(\mathcal{P})$ is contained with index 2 in

$$\mathrm{Aut}(\Theta) \cong \mathrm{Aut}(\Sigma_g^+) \int \mathbb{Z}_2$$

In (ii) the non-trivial coset is represented by the swap map $(X , Y) \mapsto (Y , X)$, which we denote by τ. It is clear that the image of a (normal) subdirect product of $\mathcal{P}$, by τ, is also a (normal) subdirect product. Hence, from above, it is also clear that the class of normal subdirect products of $\mathcal{P}$ is invarient under the action of $\mathrm{Aut}(\Theta)$. The following is now clear.

***Proposition* (3.1) :** Let $N, N' \varepsilon \mathcal{N}(\mathcal{P})$. If N is $\mathrm{Aut}(\Theta)$-commensurable to N' then one of the following holds

(i) N is $\mathrm{Aut}(\mathcal{P})$-commensurable to N'

(ii) N is $\mathrm{Aut}(\mathcal{P})$-commensurable to $\tau(N')$;

(ii) can occur only when $g_1 = g_2$.

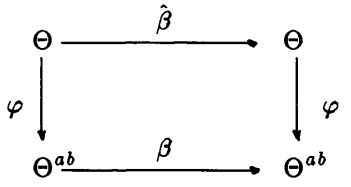
If $g_1 = g_2$ then let τ also denote the swap map in $\mathcal{Q} = \mathcal{P}^{ab}$. Then for $N \varepsilon \mathcal{N}(\mathcal{Q})$ we have $\tau(N)^{ab} = \tau(N^{ab})$. The following is clear.

***Lemma* (3.2) :** If $g_1 = g_2$ and $N \varepsilon \mathcal{F}(\mathcal{Q})$ is framed by $(N_1 , N_2 , C_1 , C_2 , \phi)$ then $\tau(N) \varepsilon \mathcal{F}(\mathcal{Q})$ is framed by $(N_2 , N_1 , C_2 , C_1 , \phi^{-1})$.

***Proposition* (3.3) :** Let $H, N \varepsilon \mathcal{N}(\mathcal{P})$. Then

H is $\mathrm{Aut}(\mathcal{P})$-commensurable to $N \Leftrightarrow H^{ab}$ is $\mathrm{Aut}(\mathcal{Q})$-commensurable to N^{ab} .

***Proof*:** Let $\beta : \Theta^{ab} \to \Theta^{ab}$ be a group isomorphism; the result will follow if we show that the following diagram



(φ the abelianisation map)

can be completed with $\hat{\beta} \in \mathrm{Aut}(\mathcal{P})$ only when $\beta \in \mathrm{Aut}(\mathcal{Q})$. But this follows from **(1.5)**.

□

By **(3.1)** we have the following.

***Corollary* (3.4)** : Let H, N $\in \mathcal{N}(\mathcal{P})$. H is $\mathrm{Aut}(\Theta)$-commensurable to N if and only if either

(i) N^{ab} is $\mathrm{Aut}(\mathcal{Q})$-commensurable to M^{ab}

or (ii) N^{ab} is $\mathrm{Aut}(\mathcal{Q})$-commensurable to $\tau(M)^{ab}$.

Again, (ii) can occur only when $g_1 = g_2$.

If $N \in \mathcal{N}(\mathcal{P})$ then, by the remark at the beginning of §2, the abelianisation $N^{ab} \in \mathcal{N}(\mathcal{Q})$. We define the *diagonal rank* in $\mathcal{N}(\mathcal{P})$ by $\mathrm{drk}(N) = \mathrm{drk}(N^{ab})$. Let $\mathfrak{B}(\Theta, d)$ denote the set of $\mathrm{Aut}(\Theta)$-commensurability classes of the normal subdirect products of Θ that have diagonal rank equal to d. By combining **(2.7)** and **(3.4)** we have,

***Theorem* (3.5)** : For $0 \leq d \leq \min(g_1, g_2)$,

$$|\mathfrak{B}(\Theta, d)| = \begin{cases} 1 & \text{if } d = 0, 1 \\ \infty & \text{otherwise} \end{cases} .$$

At this point it seems quite difficult to construct representatives for all the classes in $\mathfrak{B}(d)$ for $d \geq 1$ because a solution to this problem must include, at the very least, a full set of coset representatives for K_r. However, since any class in $\mathfrak{B}(\Theta, d)$ is represented by the pre-image (in Θ) of a framed subdirect product of Θ^{ab}, we may construct an infinite number of representatives.

To illustrate we give a representative for the single class of $\mathfrak{B}(\Theta, 1)$.

Fix presentations

$$\Sigma_{g_i}^{+} = \langle\, X_1^i, \ldots, X_{2g_i}^i : \prod_{k=1}^{g_i} [X_k^i\ ,\ X_{g_i+k}^i] \,\rangle$$

for $i = 1, 2$. Let φ denote abelianisation and let $\epsilon_k^i = \varphi(X_k^i)$ for all k. Then $\{\ \epsilon_k^i\ \}_{1 \leq k \leq 2g_i}$ is a symplectic basis for $H_1(\Sigma_+^{g_i}, \mathbb{Z})$. Let $H_i = \mathrm{span}_{\mathbb{Z}}(\ \epsilon_1^i, \ldots, \epsilon_{2g_i-1}^i\ )$ and $\Delta = \mathrm{span}_{\mathbb{Z}}(\ \epsilon_{2g_1}^1 + \epsilon_{2g_2}^2\ )$. By **(2.3)** it is clear that $H = H_1 \oplus H_2 \oplus \Delta$ is a representative for the class in $\mathfrak{B}(\mathbb{Q}, d)$. Let N denote the subgroup of Θ generated by

$$(X_k^1\ ,\ 1) \qquad 1 \leq k \leq 2g_1 - 1\ ,$$

$$(1\ ,\ X_k^2) \qquad 1 \leq k \leq 2g_2 - 1\ ,$$

$$(X_{2g_1}^1\ ,\ X_{2g_2}^2)\ ,$$

and $\quad (\ [X_k^1\ ,\ X_{2g_1}^1]\ ,\ 1) \qquad 1 \leq k \leq 2g_1\ .$

Then N is a representative for the class in $\mathfrak{B}(1)$, since clearly $N = \varphi^{-1}(H)$.

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