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A Numerical Model of the Seasonal Thawing of Permafrost in the Swamp-lake Landscapes

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The theoretical description of the temperature field in the soils during freezing or thawing is carried out using solutions of Stefan's problem. A mathematical model based on the equations of thermal conductivity for frozen and thawed layers. We consider the areas in which there are lakes or bogs. We distinguished the following layers in the vertical structure of the zone of permafrost: thawed soil, frozen soil, water, ice, snow. We offer a simplified numerical algorithm for solving of one-dimensional (in the vertical direction) heat conduction problems with moving boundaries of phase transition with the formation of new and cancellation of existing layers. A simplified numerical algorithm for solving one-dimensional (in the vertical direction) heat conduction problems with moving boundaries of phase transition with the formation of new and cancellation of existing layers is offering.

Keywords: permafrost, Stefan's problem, thawed and frozen soil, small dimensional numerical model.
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Introduction

In connection with the change in global temperature is of interest assess the response of permafrost to climate change. We consider the areas in which there is a lake or swamp. Since the vertical temperature gradients are usually larger than the horizontal one, so all physical process assumes one-dimensional in the vertical direction in the description of the heat transfer. We distinguish the following layers in the vertical direction: the snow, ice, water, thawed soil , frozen soil. The theoretical description of the temperature field in the water and soils during their freezing and thawing is carried out using solutions of Stefan problem [1]. A mathematical model based on the equations of thermal conductivity for frozen and thawed areas. At the borders of phase transition (freezing-thawing) the conditions of equality of temperatures to the phase

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transition temperature and the Stefan condition are posed. There is an extensive literature on the mathematical modeling of permafrost (see, Eg, [2–8]).

In this paper the numerical model of small dimension is proposed for discription vertical temperature distribution in thawed and frozen layers taking into account the formation of new and revocation of the existing layers ([9]). One can allocate two types of water bodies: a) reservoirs, which freeze to the bottom in the winter, and the ice melt throughout the depth and the top layer of the bottom defrost in the summer ("shallow" water bodies); b) reservoirs that do not freeze to the bottom in the winter ("deep" water). There are various options for the location of frozen and thawed layers (Tab. 1). For "deep" waters there are three variants.
Variant 1: water layer, layer of thawed soil, layer of frozen soil, layer of thawed soil.
Variant 2: snow–ice layer, water layer, layer of thawed soil, layer of frozen soil, layer of thawed soil.
Variant 3: water layer, ice layer, water layer, layer of thawed soil, layer of frozen soil, layer of thawed soil.

For "shallow" water bodies seven variants are considered. When switching from one variant to another the layers are added or deleted.

Table 1. Variants of the location for frozen and thawed layers in the swamp-lake landscapes

variant N	water	ice (snow)	water	thawed soil	frozen soil	thawed soil	frozen soil	thawed soil
1	+					+	+	+
2		+	+			+	+	+
3		+			+	+	+	+
4		+					+	+
5	+	+					+	+
6	+	+			+	+	+	+
7	+			+	+	+	+	+
8	+	+	+			+	+	+

Fig. 1 illustrates a vertical structure for variants 6 and 7, Fig. 2 shows a scenario of switching from one version to another for the "shallow" reservoirs.

1. Mathematical model of dynamics of freezing and thawing of permafrost

The vertical temperature distribution in every layer are determined by solving the heat equation, satisfying the appropriate boundary conditions:

$$\frac{\partial T_i}{\partial t} = K_i \frac{\partial^2 T_i}{\partial z^2}. \tag{1}$$

Here $T_i(t, z)$ is a temperature of i-th layer ($h_{i-1} \leqslant z \leqslant h_i$), K_i is a thermal diffusivity, t is time, z is a vertical coordinate (downward).

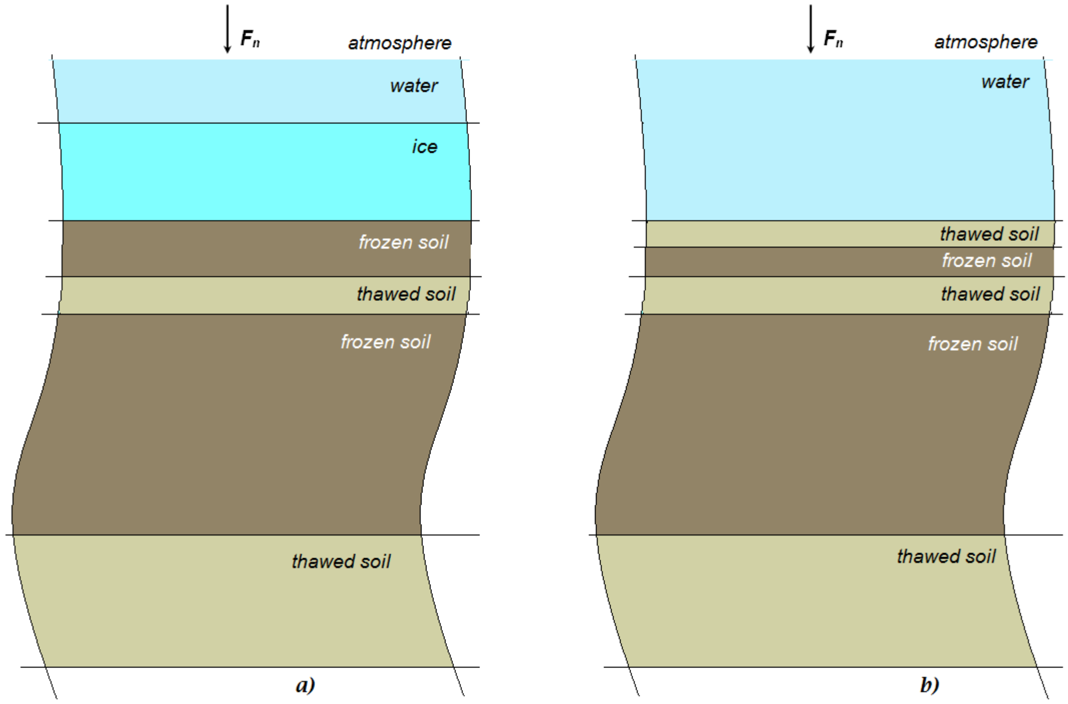

Fig. 1. Variants 6 (a) and 7 (b)

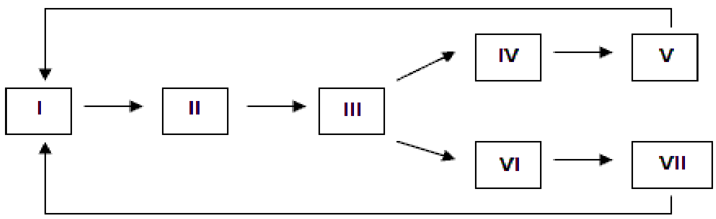

Fig. 2. The scheme of switching from one variant to another

Boundary conditions. The condition on the boundary of the atmosphere – water ($z = 0$)

$$\left(K\frac{\partial T}{\partial z}\right) = -\frac{F_n}{c_w \rho_w}, \tag{2}$$

the condition on the boundary of the atmosphere – snow+ice ($z = 0$)

$$T = T_{ice}. \tag{3}$$

The condition at the bottom of the pond (boundary of i-th $h_{i-1} \leqslant z \leqslant h_i$ and $(i+1)$-th $h_i \leqslant z \leqslant h_{i+1}$ layers, $z = h_i = h_{bt}$)

$$T_i = T_{i+1}, \qquad \left(\lambda\frac{\partial T}{\partial z}\right)_i = \left(\lambda\frac{\partial T}{\partial z}\right)_{i+1}. \tag{4}$$

On moving boundaries of phase transition ($z = h_j$)

$$T_j = T_{j+1} = T_{ph}, \qquad \rho_j L_j \frac{dh_j}{dt} = \left(\lambda \frac{\partial T}{\partial z}\right)_j - \left(\lambda \frac{\partial T}{\partial z}\right)_{j+1}. \tag{5}$$

Here ρ_w is water density, c_w is the specific heat capacity of water, F_n is the total heat flow on the boundary of the atmosphere–water, T_{ice} is temperature on ice surface, $z = h_i$ is coordinate of the boundary between i-th and $(i+1)$-th layers, λ is coefficient of heat conductivity, T_{ph} is phase transition temperature, ρ_j is density of j-th layer, $L_j = L_w \cdot W_j$ is volumetric latent heat of melting environment in j-th layer, $W_j = \dfrac{\Omega_{jw}}{\Omega_j}$ is soil humidity in j-th layer, Ω_{jw} is the volume of water in the soil, Ω_j is the volume of the soil. The total heat flow (F_n) and temperature on the boundary snow–ice (T_{ice}) is defining according to the formulas, described in [10].

On the lower boundary of permafrost layer the conditions (5) are supplemented with setting of a geothermal gradient in the layer on thawed soil $G = \dfrac{\partial T}{\partial z}$, ($G = 0.02 - 0.05^0$ C/m [11]).

The initial conditions:

$$T_i(0, z) = T_i^0, \qquad \delta_i = \delta_i^0. \tag{6}$$

Consider an arbitrary i-th layer: $h_{i-1} \leqslant z_i \leqslant h_i$, $\delta_i = h_i - h_{i-1}$, where δ_i is the thickness of i-th layer. We introduce new independent variables t, ξ:

$$\hat{t} = t, \qquad \xi_i = \frac{z_i - h_{i-1}}{\delta_i}, \qquad 0 \leqslant \xi_i \leqslant 1.$$

Since

$$\frac{\partial}{\partial z_i} = \frac{1}{\delta_i}\frac{\partial}{\partial \xi_i}, \qquad \frac{\partial^2}{\partial z_i^2} = \frac{1}{\delta_i^2}\frac{\partial^2}{\partial \xi_i^2}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + w_i \frac{\partial}{\partial \xi_i},$$

$w_i = \dfrac{\partial \xi_i}{\partial t} = \dfrac{1}{\delta_i}\left[(\xi_i - 1)\dfrac{dh_{i-1}}{dt} - \xi_i \dfrac{dh_i}{dt}\right]$, then equation (1) and boundary conditions (2)–(5) written as

$$\frac{\partial T_i}{\partial t} + w_i \frac{\partial T_i}{\partial \xi_i} = \frac{K_i}{\delta_i^2}\frac{\partial^2 T_i}{\partial \xi_i^2}, \tag{7}$$

$$\left(\frac{K_i}{\delta_i}\frac{\partial T_i}{\partial \xi_i}\right)_{\xi_i=0} = -\frac{F_n}{c_w \rho_w}, \qquad T_i|_{\xi_i=0} = T_{ice}, \qquad T_i|_{\xi_i=1} = T_{i+1}|_{\xi_{i+1}=0}, \tag{8}$$

$$\left(\frac{\lambda_i}{\delta_i}\frac{\partial T_i}{\partial \xi_i}\right)_{\xi_i=1} = \left(\frac{\lambda_{i+1}}{\delta_{i+1}}\frac{\partial T_{i+1}}{\partial \xi_{i+1}}\right)_{\xi_{i+1}=0}, \qquad T_j|_{\xi_j=1} = T_{j+1}|_{\xi_{j+1}=0} = T_{ph}, \tag{9}$$

$$\rho_j L_j \frac{dh_j}{dt} = \left(\frac{\lambda_j}{\delta_j}\frac{\partial T_j}{\partial \xi_j}\right)_{\xi_j=1} - \left(\frac{\lambda_{j+1}}{\delta_{j+1}}\frac{\partial T_{j+1}}{\partial \xi_{j+1}}\right)_{\xi_{j+1}=0}. \tag{10}$$

2. Numerical algorithm

Let us consider the solution of the formulated problems on coarse (three-point) grid for each of layers: $\xi_{i1} = 0$, $\xi_{i2} = 0.5$, $\xi_{i3} = 1$.

We approximate the equation (7) using the implicit difference scheme and directional differences for the convective terms [12]. Grid equations corresponding to the differential equation (7) for i-th layer are of the form:

$$\frac{T_{i,2} - T_{i,2}^n}{\Delta t} + \frac{w^- T_{i,3} - 2w^0 T_{i,2} + w^+ T_{i,1}}{2} = \frac{K_i(T_{i,1} - 2T_{i,2} + T_{i,3})}{(0.5\delta_i^n)^2}, \tag{11}$$

$w_{i,2} = -\left(\dfrac{dh_i}{dt} + \dfrac{dh_{i-1}}{dt}\right)/(2\delta_i^n)$, $w^0 = |w_{i,2}|$, $w^- = w_{i,2} - |w_{i,2}|$, $w^+ = w_{i,2} + |w_{i,2}|$, here Δt is time step, $t_{n+1} = t_n + \Delta t$, $T_{i,k}^n = T_i(t_n, \xi_{i,k})$, $T_{i,k} = T_i(t_{n+1}, \xi_{i,k})$.

Boundary conditions (8)–(10) for the difference grid are represented as:
on the border of the atmosphere–water

$$\left[1 + \frac{8K_i\Delta t}{(\delta_i^n)^2}\right] T_{i,1} - \frac{8K_i\Delta t}{(\delta_i^n)^2} T_{i,2} = T_{i,1}^n + \frac{4\Delta t F_n}{c_w \rho_w} \quad (12)$$

on the border of the atmosphere– snow+ice

$$T_{i,1} = T_{ice}, \quad (13)$$

on the water bottom $z = h_i$

$$T_{i,3} = T_{i+1,1}, \quad (14)$$

$$\frac{\lambda_i}{\delta_i}(T_{i,3} - T_{i,2}) = \frac{\lambda_{i+1}}{\delta_{i+1}}(T_{i+1,2} - T_{i+1,1}). \quad (15)$$

At the borders of phase transition $z = h_j$

$$T_{j,3} = T_{j+1,1} = T_{ph}, \quad (16)$$

$$\rho_j L_j \frac{h_j^{n+1} - h_j^n}{\Delta t} = \lambda_j \frac{T_{j,3} - T_{j,2}}{0.5\delta_j^n} - \lambda_{j+1} \frac{T_{j+1,2} - T_{j+1,1}}{0.5\delta_{j+1}^n}. \quad (17)$$

If δ_j becomes less than a predetermined small value ε ($\delta_j < \varepsilon$), the corresponding layer will be cancelled. If as a result of melting or freezing a new k-th layer is formed, the initial thickness $\delta_k = \varepsilon$ and temperatures is put for this layer.

The following algorithm is used for the obtained tasks. Let is known all parameters (temperature distributions in layers under study and the positions of phase transition) on n-th time step. Then a finding the unknown parameters at time t_{n+1} is performed in two stages. The first step is to define the temperature distributions in the selected layers (taking into account the relations (11)–(16)) by solution of systems of linear algebraic equations of small dimension. In the second phase it is clarified the position of the phase boundary by the numerical solution of equation (17).

3. Calculations samples

Model calculations were performed for a swamp depth of 25 cm, 50 cm, 75 cm and 100 cm by weather data 2010–2011 years for the weather station Dudinka. Year 2010 can be considered as "cold" (about 240 days with mean daily negative temperature, the average temperature for the period from 1 October to 30 April was -20.72^oC); year 2011 is "warm" (190 days with mean daily negative temperature and the average temperature for October to April was -15.22^oC). There were determined the depth of seasonal thawing of permafrost soils, the temperature of the melt layer, the time intervals of the existence for layer of thawed soil. In Tabs. 2, 3 are given the results of calculations shown the impact of the swamp depths and weather data on the main characteristics of the permafrost.

From the freezing-thawing of soil calculation results it follows that with increasing depth of "shallow" water it is increased the period of existence of the thawed soil layer. For sufficiently

Table 2. The thickness of the layer of thawing frozen soil (cm)

swamp depth (cm)	25	50	75	100
2010	122.9	118.5	114.2	108.5
2011	143.2	139.38	135.04	144.6

Table 3. The count of days existence of the thawed soil layer

swamp depth (cm)	25	50	75	100
2010	212	228	243	264
2011	242	292	311	365

great depths of the lake a layer of thawed soil does not freeze completely during the year. With further increase of the depth of the reservoir it does not freeze to the bottom ("deep" water) and thawed layer exists throughout the year.

In Fig. 3 the calculated vertical temperature distributions are given. In Fig. 3a is given temperature distribution for variant 7, in Fig. 3b is given for variant 2.

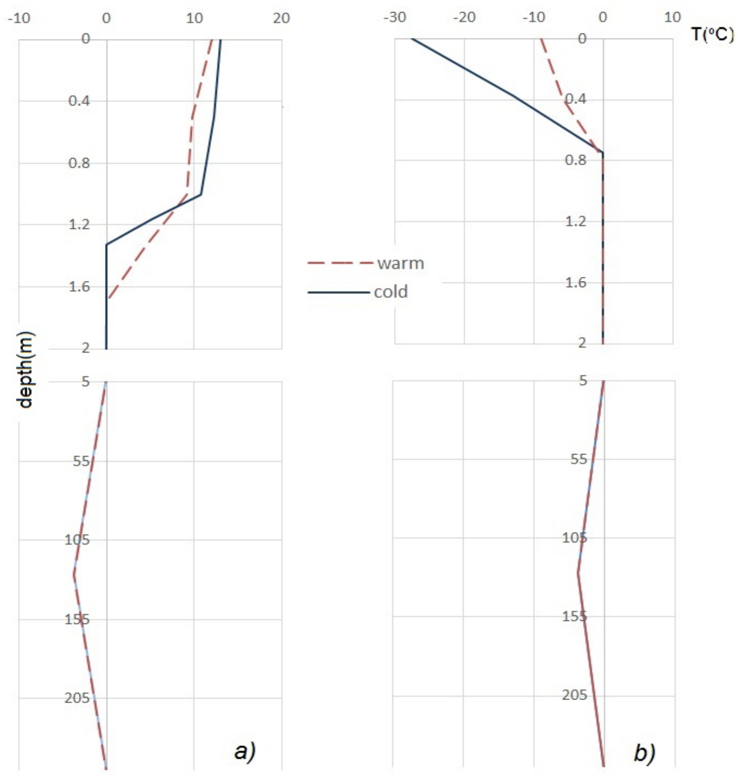

Fig. 3. The calculated temperature in layers for different weather data: a) for summer (variant 7), b) for winter (variant 2). Water depth is 1 m

The proposed numerical algorithm allows to describe the dynamics of annual freezing-thawing permafrost in the swamp–lake district landscapes and to assess the impact of climate change.

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Численная модель сезонного оттаивания вечной мерзлоты в болотно-озерных ландшафтах

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Теоретическое описание температурного поля в почвах при их промерзании или оттаивании осуществляется с помощью решений задач Стефана. Математическая модель основывается на уравнениях теплопроводности для мерзлых и талых слоев. Рассматриваются территории, на которых имеются озера или болота. Выделяются следующие слои в вертикальной структуре зоны вечной мерзлоты: талый грунт, мерзлый грунт, вода, лед, снег. Предлагается упрощенный численный алгоритм решения одномерных (в вертикальном направлении) задач теплопроводности с подвижными границами фазового перехода с образованием новых и аннулированием существующих слоев.

Ключевые слова: вечная мерзлота, задачи Стефана, мерзлые и талые слои, малоразмерная численная модель.