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Boundary Version of the Morera Theorem for a Matrix Ball of the Second Type

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In this article we prove a boundary Morera theorem for a matrix ball of the second type.

Keywords: matrix ball, automorphism, Poisson kernel, Morera theorem.

In this article we consider a boundary version of the Morera theorem for a matrix ball of the second type. Our starting point is Nagel and Rudin's result (see [1]), which says that if f is a continuous function on the boundary of a ball in $\mathbb{C}^n$ and the integral

$$\int_0^{2\pi} f\left(\psi(e^{i\varphi},\ 0,\ \ldots,\ 0)\right) e^{i\varphi} d\varphi = 0,$$

for all (holomorphic) automorphisms ψ of a ball, then the function f is holomorphically extends into the ball. For classical domains an analog of a boundary Morera theorem was obtained in [7].

Let $\mathbb{C}\,[m\times m]$ be the space of $[m\times m]$-matrices with complex elements. We denote by $\mathbb{C}^n\,[m\times m]$ the Cartesian product of n copies of $\mathbb{C}\,[m\times m]$:

$$\mathbb{C}^n\,[m\times m] = \mathbb{C}\,[m\times m]\times .....\times \mathbb{C}\,[m\times m]\,.$$

Set (see, for example [2])

$$B_I = \left\{Z\in\mathbb{C}^n\,[m\times m]:\ \ I-\langle Z,Z\rangle > 0\right\},$$

where $\langle Z,Z\rangle = Z_1Z_1^* + Z_2Z_2^* + ... + Z_nZ_n^*$ is a 'scalar' product, I is the identity matrix $[m\times m]$, $Z_\nu^* = \overline{Z'}_v$ is the adjoint and transposed matrix to Z_ν, $\nu = 1,2,...,n$. B_I is called a *matrix ball* (of the first type). Here $I-\langle Z,Z\rangle > 0$ means that the Hermite matrix $I-\langle Z,Z\rangle$ is positively defined, i.e. all eigen values are positive.

The skeleton of B_I is the set

$$X_I = \left\{Z\in\mathbb{C}^n\,[m\times m]:\ \ \langle Z,Z\rangle = I\right\}.$$

The domain B_{II} in spaces $\mathbb{C}^n\,[m\times m]$:

$$B_{II} = \left\{Z\in\mathbb{C}^n\,[m\times m]: I-\langle Z,Z\rangle > 0,\quad Z'_v = Z_\nu,\ \nu = 1,2,...,n\right\}, \tag{1}$$

where I is, as usual, the identity matrix of order m, is called a *matrix ball of the second type* (see [3]).

The skeleton of this domain is the following manifold:

$$X_{II} = \left\{Z\in\mathbb{C}^n\,[m\times m]: \langle Z,Z\rangle = I,\quad Z'_v = Z_\nu,\ \nu = 1,2,...,n\right\}.$$

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Lemma 1. *The domain B_{II} has the following properties:*
1) B_{II} is bounded;
2) B_{II} is a complete circular domain;
3) B_{II} and its skeleton X_{II} are invariant under unitary transformations.

Proof. **1.** The definition of the domain implies that each diagonal element of the matrix $\langle Z, Z\rangle$ is positive and less than 1, and the sum of the squares of the modules of all elements in Z_ν, $\nu = 1, ..., n$, does not exceed m. This implies that the matrix ball of the second type is bounded.

2. If $Z \in B_{II}$ and $\alpha \in \mathbb{C}, |\alpha| \le 1$, then

$$I - \langle \alpha Z, \alpha Z\rangle = I - |\alpha|^2 \langle Z, Z\rangle = I(1 - |\alpha|^2) + |\alpha|^2 (I - \langle Z, Z\rangle) > 0.$$

3. Invariance under unitary transformations means that if U is a unitary matrix of order m, then for $Z \in B_{II}$ we have $UZ \in B_{II}$ and $ZU \in B_{II}$. Indeed,

$$I - \langle UZ, UZ\rangle = I - UZ_1\overline{Z_1}U^* - UZ_2\overline{Z_2}U^* - ..... - UZ_n\overline{Z_n}U^* =$$
$$= I - U\left(Z_1\overline{Z_1} + Z_2\overline{Z_2} + ... + Z_n\overline{Z_n}\right)U^* = I - U\langle Z, Z\rangle U^* = U(I - \langle Z, Z\rangle)U^* > 0,$$

and

$$\langle ZU, ZU\rangle = \langle Z, Z\rangle .$$

The invariance of the skeleton is proved similarly. □

We consider normalized Lebesgue measures μ in B_{II} and σ on the skeleton X_{II}, i.e.

$$\int_{B_{II}} d\mu(Z) = 1 \quad \text{and} \quad \int_{X_{II}} d\sigma(Z) = 1.$$

We define the space $H^1(B_{II})$ as follows: a function f belongs to $H^1(B_{II})$ if it is holomorphic in B_{II} and

$$\sup_{0<r<1} \int_{X_{II}} |f(rZ)|\, d\sigma(Z) < \infty.$$

We fix a point $\Lambda^0 \in X_{II}$ ($\Lambda^0 = (\Lambda^0_1, ..., \Lambda^0_n)$) and consider the following embedding of a unit disk Δ in the domain B_{II}

$$\left\{W \in \mathbb{C}^n [m \times m] : \ W_\nu = \xi\Lambda^0_\nu, \ |\xi| < 1, \ \nu = 1, ..., n, \right\}. \tag{2}$$

By this embedding the boundary T of the disk Δ transforms into the disk on X_{II}. If ψ is an automorphism of the domain B_{II}, then the set (2) under the action of this automorphism becomes some analytic disk with the boundary on X_{II}.

Theorem 1. *Let f be a continuous function on X_{II}. If f satisfies*

$$\int_T f(\psi(\xi\Lambda^0))d\xi = 0 \tag{3}$$

for all automorphisms ψ of the domain B_{II}, then the function f has a holomorphic extension F in B_{II} of the class $C(\overline{B}_{II})$.

Proof. On X_{II} the subgroup of the automorphisms leaving 0 fixed acts transitively (see [3]). Since X_{II} is invariant with respect to unitary transformations, the condition (3) is satisfied for any point $\Lambda \in X_{II}$.

First of all, we parametrize manifold X_{II} as follows: for $Z \in X_{II}$ we put $Z = e^{i\theta}U$, where $0 \leqslant \theta \leqslant 2\pi$, and in the matrix U_1 the element $u^{(1)}_{11}$ in the left top corner is positive. We denote the manifold of such matrices by X^+. This way we parametrize not the whole set X_{II}, but some smaller set, which differs from X_{II} by a set of zero measure.

The normalized Lebesgue measure $d\sigma$ can be written as (Lemma 8.4 in [2])

$$d\sigma = \frac{d\theta}{2\pi} d\sigma_1(U) = \frac{1}{2\pi i} \frac{d\xi}{\xi} d\sigma_1(U),$$

where $\xi = e^{i\theta}$, and the measure σ_1 is positive on X^+.

Multiplying equality (3) by $d\sigma_1$ and integrating over X^+, from (3) we obtain

$$\int_{X_{II}} f(\psi(Z)) z_{pq}^{\nu} d\sigma(Z) = 0, \tag{4}$$

where z_{pq}^{ν} are components of vector $Z = (Z_1, Z_2, \ . \ . \ . \ , Z_n)$, $p, q = 1, ..., m$, $\nu = 1, ..., n$.

We consider the automorphism ψ_A translating the point $A = (A_1, ..., A_n)$ from B_{II} into 0 (see [3]). It is defined up to a generalized unitary transformation.

Then we substitute the automorphism ψ_A^{-1} in (4) instead of ψ and change variables $W = \psi_A^{-1}(Z)$. We get

$$\int_{X_{II}} f(W) \psi_{pq}^{A,\nu}(W) d\sigma(\psi_A(W)) = 0, \tag{5}$$

where $\psi_{pq}^{A,\nu}$ are components of the automorphism ψ_A.

By Corollary 7.7 from [2] we have

$$d\sigma(\psi_A(W)) = P(A, W) d\sigma(W),$$

where $P\,(A, W)$ is an invariant Poisson kernel for the matrix ball B_{II} of the second type.

Then, from the condition (5) we have that

$$\int_{X_{II}} f(W) \psi_{pq}^{A,\nu}(W) P(A, W) d\sigma(W) = 0 \tag{6}$$

for all points $A = (A_1, ..., A_n)$ from B_{II} and all $p, q = 1, ..., m$, $\nu = 1, ..., n$.

Thus, taking into account the properties of the Poisson integral of continuous functions, Theorem 1 follows from the next assertion.

Theorem 2. *If for $f \in L^1(X_{II})$ the equality (6) holds for all automorphisms ψ_A of domain B_{II}, then f is a radial boundary value of some function $F \in H^1(B_{II})$.*

Proof. The invariant Poisson kernel for a matrix ball of the second type has the form

$$P(A, W) = \frac{\left(\det(I - A_1\bar{A}_1 - ..... - A_n\bar{A}_n)\right)^{\frac{(m+1)n}{2}}}{\left|\det(I - A_1\bar{W}_1 - ..... - A_n\bar{W}_n)\right|^{(m+1)n}} =$$

$$= \frac{\left(\det(I - A_1\bar{A}_1 - ..... - A_n\bar{A}_n)\right)^{\frac{(m+1)n}{2}}}{\left(\det(I - A_1\bar{W}_1 - ..... - A_n\bar{W}_n)\right)^{\frac{(m+1)n}{2}} \left(\det(I - W_1\bar{A}_1 - ..... - W_n\bar{A}_n)\right)^{\frac{(m+1)n}{2}}}.$$

We write the elements of matrices A and W in the vector form:

$$A = (A_1, ..., A_n) = \left(a_{11}^1, ..., a_{1m}^1; ....; a_{m1}^1, ..., a_{mm}^1; ....; \ a_{11}^n, ..., a_{1m}^n; ...\right.$$

$$\left. ...; a_{m1}^n, ..., a_{mm}^n\right) = \left(\left\|a_{pq}^1\right\|, ..., \left\|a_{pq}^n\right\|\right),$$

$$W = (W_1, ..., W_n) = \left(w_{11}^1, ..., w_{1m}^1; ....; w_{m1}^1, ..., w_{mm}^1; ....; \ w_{11}^n, ..., w_{1m}^n; ...\right.$$

$$...; w^n_{m1}, ..., w^n_{mm}) = \left(\left\|w^1_{pq}\right\|, ..., \left\|w^n_{pq}\right\|\right),$$

where $\left\|a^\nu_{pq}\right\| = \left\|a^\nu_{qp}\right\|$, $\left\|w^\nu_{pq}\right\| = \left\|w^\nu_{qp}\right\|$, $p, q = 1, ..., m$, $\nu = 1, ..., n$.

We shall compute

$$\sum_{p,q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial P(A, W)}{\partial \bar{a}^\nu_{pq}}. \tag{7}$$

Denote

$$I - W_1\bar{A}_1 - ... - W_n\bar{A}_n = \|\alpha_{sj}\| \qquad (s, j = 1, ..., m),$$

where

$$\alpha_{sj} = \delta_{sj} - \sum_{k=1}^{m} \sum_{\nu=1}^{n} w^\nu_{sk}\bar{a}^\nu_{jk}, \quad a^\nu_{jk} = a^\nu_{kj}, \quad w^\nu_{sk} = w^\nu_{ks}, \quad s, j = 1, ..., m,$$

and δ_{sj} is the Kronecker symbol.
Calculations show that

$$\sum_{q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)}{\partial \bar{a}^\nu_{pq}} =$$

$$= \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n) - \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)[p, p],$$

where $\det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)[p, p]$ denotes the cofactor of the element α_{pp} in the matrix $I - W_1\bar{A}_1 - ... - W_n\bar{A}_n$.

Then

$$\sum_{p,q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)}{\partial \bar{a}^\nu_{pq}} =$$

$$= m \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n) - \sum_{p=1}^{m} \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)\,[p, p].$$

Similarly

$$\sum_{p,q=1}^{m} \sum_{\nu=1}^{n} \bar{a}^\nu_{pq} \frac{\partial \det(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n)}{\partial \bar{a}^\nu_{pq}} =$$

$$= m \det(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n) - \sum_{p=1}^{m} \det(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n)\,[p, p].$$

Therefore, the expression (7) is equal to

$$\frac{m\,(m+1)}{2} nP(A, W) \left[\frac{\sum\limits_{p=1}^{m} \det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)[p, p]}{\det(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)} - \right.$$

$$\left. - \frac{\sum\limits_{p=1}^{m} \det(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n)[p, p]}{\det(I^{(m)} - A_1\bar{A}_1 - ... - A_n\bar{A}_n)} \right] =$$

$$= \frac{m\,(m+1)}{2} nP(A, W)[\mathrm{Sp}\,(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n)^{-1} - \mathrm{Sp}\,(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n)^{-1}]. \tag{8}$$

Here Sp, as usual, is the matrix trace.

An automorphism of the domain B_{II} has the form (see [3])

$$\psi_A(W) = \bar{R}^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\sum_{\nu=1}^{n}\left(W_\nu - A_\nu\right)R_{\nu k}, \quad k = 1, ..., n,$$

where R is a block matrix satisfying the condition

$$R'\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)\bar{R} = I.$$

If the condition (6) is satisfied for the components of the map $\psi_A(W)$, the same condition is satisfied for the components of the map

$$\varphi_A(W) = \left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\sum_{\nu=1}^{n}\left(W_\nu - A_\nu\right),$$

since matrices R, $\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)$ are nonsingular and depend only on A. Then from (6) we get

$$\int_{X_{II}} f(W)\varphi_{pq}^{A,\nu}(W)P(A,W)d\sigma(W) = 0, \tag{9}$$

where $\varphi_{pq}^{A,\nu}(W)$ are the components of the map $\varphi_A(W)$, $(p, q = 1, ..., m, \ \nu = 1, ..., n)$.

Now we compute the sum

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\varphi_{p,q}^{A,\nu}.$$

It is obvious that this expression is equal to Sp $\langle\varphi_A(W), A\rangle$, since

$$\begin{aligned}\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\varphi_{p,q}^{A,\nu} = \text{Sp}\, \Big[&\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\times \\ &\times\left(W_1\bar{A}_1 + ... + W_n\bar{A}_n - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)\Big] = \\ = \text{Sp}\, \Big[&\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1}\times \\ &\times\left(\left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right) - \left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)\right)\Big] = \\ = \text{Sp}\, \Big[&\left(I - W_1\bar{A}_1 - ... - W_n\bar{A}_n\right)^{-1} - \left(I - A_1\bar{A}_1 - ... - A_n\bar{A}_n\right)^{-1}\Big],\end{aligned} \tag{10}$$

Using this, we get from (9)

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\frac{\partial F(A)}{\partial\bar{a}_{pq}^{\nu}} = 0, \tag{11}$$

where

$$F(A) = \int_{X_{II}} f(W)P(A,W)d\sigma(W) \tag{12}$$

is the Poisson integral of the function f.

The function $F(A)$ is real analytic in the domain B_{II}. We expand $F(A)$ in a Taylor series in a neighborhood of 0,

$$F(A) = \sum_{|\alpha|,|\beta|\geq 0} C_{\alpha,\beta}a^{\alpha}\bar{a}^{\beta},$$

where $\alpha = \left(\|\alpha_{pq1}\|, ..., \|\alpha_{pqn}\|\right)$ and $\beta = \left(\|\beta_{pq1}\|, ..., \|\beta_{pqn}\|\right)$, $(p, q = 1, ..., m)$ are matrices with nonnegative integer elements and

$$|\alpha| = \sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\alpha_{pq\nu}, \qquad a^{\alpha} = \prod_{p,q=1}^{m}\prod_{\nu=1}^{n}a_{pq\nu}^{\alpha_{pq\nu}}.$$

Then (11) implies

$$\sum_{p,q=1}^{m}\sum_{\nu=1}^{n}\bar{a}_{pq}^{\nu}\frac{\partial F(A)}{\partial\bar{a}_{pq}^{\nu}}=\sum_{|\alpha|,|\beta|}|\beta|\,C_{\alpha,\beta}a^{\alpha}\bar{a}^{\beta}=0.$$

It follows that for $|\beta|>0$ all coefficients $C_{\alpha,\beta}$ are equal to zero. So, the function $F(A)$ is holomorphic in B_{II} and belongs to the class $H^1(B_{II})$.

If f is continuous on X_{II}, then the function F belongs to $C(\bar{B}_{II})$ and its boundary values on X_{II} concide with f. □

The proof of this theorem shows that it remains true if the conditions (3) and (6) are satisfied only for those automorphisms ψ_A, for which the point $A=(A_1,...,A_n)$ lies in some open set $V\subset B_{II}$. Therefore the following statement is true.

Theorem 3. *If a function $f\in L^1(X_{II})$ satisfies the condition* (6) *for all points lying in some open set $V\subset B_{II}$ and for all components of the automorphism ψ_A, then f is a radial boundary value for some function $F\in H^1(B_{II})$ on X_{II}.*

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Граничный вариант теоремы Морера для матричного шара второго типа

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В этой статье доказывается граничная теорема Морера для матричного шара второго типа.

Ключевые слова: матричный шар, автоморфизм, ядро Пуассона, теорема Морера.