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On the Asymptotic of Homological Solutions to Linear Multidimensional Difference Equations

Natalia A. Bushueva*
Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny, 79, Krasnoyarsk, 660041
Russia

Konstantin V. Kuzvesov†
Multifunctional Center
9 May, 12, Krasnoyarsk, 660125
Russia

Avgust K. Tsikh‡
Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny, 79, Krasnoyarsk, 660041
Russia

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Given a linear homogeneous multidimensional difference equation with constant coefficients, we choose a pair (γ, ω), where γ is a homological k-dimensional cycle on the characteristic set of the equation and ω is a holomorphic form of degree k. This pair defines a so called homological solution by the integral over γ of the form ω multiplied by an exponential kernel. A multidimensional variant of Perron's theorem in the class of homological solutions is illustrated by an example of the first order equation.

Keywords: difference equation, asymptotic, amoebas of algebraic sets, logarithmic Gauss map

Introduction

In this paper we consider linear homogeneous difference equations. In one-dimensional case they can be written as

$$f(x+k) + a_{k+1}(x)f(x+k-1) + \cdots + a_0(x)f(x) = 0, \tag{1}$$

where $f(x)$ is an unknown function of a discrete argument $x \in \mathbb{Z}$ (or $\mathbb{Z}_+$) with values in $\mathbb{C}$. Equations (1) were studied in detail in [1–3]. In the case of constant coefficients (when all a_j do not depend on x) one associates with the equation (1) its *characteristic polynomial*

$$P(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0. \tag{2}$$

The roots $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ of this polynomial generate the space of solutions to (2) as exponential solutions; for example, if all roots are different, then $\lambda_1^x, \ldots, \lambda_k^x$ are a base of the solution space.

In the case of variable coefficients an important role is played by the limit characteristic polynomial, which coefficients a_k are equal to the limits of functions $a_k(x)$ as $x \to +\infty$. But in this case we can speak only of the effect of the roots of the limit characteristic polynomial on the asymptotic of the solutions to the equation (1), as follows from Poincaré's theorem.

*nbushueva@sfu-kras.ru
†kuzvesov@list.ru
‡atsikh@sfu-kras.ru

Theorem (Poincaré [1], see also [3]). *Assume that the coefficients $a_j(x)$ of equation (1) have finite limits*

$$\lim_{x\to+\infty} a_j(x) =: a_j, \quad j = 0, \ldots, k-1,$$

and that the roots $\lambda_1, \ldots, \lambda_k$ of the limit characteristic polynomial all have different absolute values.

Then for any nonvanishing solution $f(x)$ to the equation (1) the limit

$$\lim_{x\to+\infty} \frac{f(x+1)}{f(x)}$$

exists and is equal to one of the characteristic roots λ_j.

The question whether the limits of the ratios $\dfrac{f(x+1)}{f(x)}$ attain all the values of roots of the limit characteristic polynomial (when all base solutions $f(x)$ of the equation (1) are run over) is answered by Perron's theorem.

Theorem (Perron [2], see also [3]). *Assume that all conditions of the Poincaré theorem hold for the equation (1), and moreover $a_0(x) \neq 0$ for all $x \in \mathbb{Z}$. Then there are k solutions $f_1(x), \ldots, f_k(x)$ of this equation such that*

$$\lim_{x\to+\infty} \frac{f_j(x+1)}{f_j(x)} = \lambda_j, \quad j = 1, \ldots, k.$$

Now consider the multidimensional case. Let $f(x) = f(x_1, \ldots, x_n)$ be a complex-valued function of a discrete argument $x \in \mathbb{Z}^n$. We consider the linear shift operators on the vector space of such functions:

$$\delta_j f(x) = f(x + e_j) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n), \quad j = 1, \ldots, n.$$

Using notation $\delta = (\delta_1, \ldots, \delta_n)$ we can associate to every polynomial

$$P(x, z) = \sum_{\alpha\in A} a_\alpha(x) z^\alpha$$

a difference equation

$$P(x, \delta) f(x) = \sum_{\alpha\in A} a_\alpha(x) f(x + \alpha) = 0;$$

here $A \subset \mathbb{Z}^n_+$ is a finite set of indices $\alpha = (\alpha_1, \ldots, \alpha_n)$.

In the case of constant coefficients $a_\alpha(x) \equiv a_\alpha$ the polynomial

$$P(z) = \sum_{\alpha\in A} a_\alpha z^\alpha$$

is said to be *characteristic* and each its solution $z = \lambda = (\lambda_1, \ldots, \lambda_n)$ defines an elementary exponential solution $f(x) = \lambda^x = \lambda_1^{x_1} \ldots \lambda_n^{x_n}$. But now the characteristic set

$$V = \{z \in \mathbb{C}^n : \ P(z) = 0\}$$

is not finite, so there exist many ways to compose solutions from the elementary exponents. For example, if the characteristic polynomial $P(z)$ has no multiple factors, then all exponential solutions can be written as the integral [4]

$$f(x) = \int_V z^x d\mu(z), \tag{3}$$

where $d\mu$ is a measure with the support on the characteristic set.

In the present paper we introduce a subclass of exponential solutions for which the measure $d\mu$ in (3) is given by the pair (γ, ω), where $\gamma \in Z_k(V)$ is a k-dimensional homological cycle on V and $\omega \in \Omega^k(V)$ is a closed holomorphic differential form on V of degree k. That is, we consider solutions given by the integral

$$f(x) = \int\limits_{\gamma} z^x \omega(z), \quad \gamma \in Z_k(V), \quad \omega \in \Omega^k(V), \quad k = 1, \ldots, n-1. \tag{4}$$

We call them *admissible* or *homological solutions*, since they depend only on the homology class of γ. The restriction of the integral (4) on the ray $L_q = \{x = q \cdot l;\ l \in \mathbb{N}\}$ with the directing vector $q = (q_1, \ldots, q_n) \in \mathbb{Z}^n \setminus \{0\}$ turns into the Laplace integral

$$f(x)|_{L_q} = \int\limits_{\gamma} \omega(z) e^{l \cdot \langle q, \ln z \rangle},$$

with the parameter l and the phase

$$\varphi(z) = \langle q, \ln z \rangle = q_1 \ln z_1 + \cdots + q_n \ln z_n.$$

Consequently, the behaviour of a homological solution $f(x)$ along radial directions can be studied by the method of stationary phase (see [5]). The stationary (critical) points of the phase φ are exactly the values of the inversion $z = \gamma^{-1}(q)$ of the logarithmic Gauss map γ for the characteristic set V (see formula (8) in section 2).

In the paper [6] solutions (4) were considered only for $k = n-1$, i.e. for half-dimensional cycles γ. For such solutions a multidimensional analog of Poincaré's theorem was proved in [6], where instead of the ratio $f(x+1)/f(x)$ the authors considered the vector (see section 2)

$$\left(\frac{f(x+e_1)}{f(x)}, \ldots, \frac{f(x+e_n)}{f(x)} \right)$$

restricted on the ray L_q. This vector we call a *Horn vector*.

The main purpose here is to show by an example of one-order equation that the study of all dimensions $k = 1, \ldots, n$ in (4) allows to obtain a multidimensional Perron theorem (Theorem 3).

1. Basic definitions and some known facts around the concept of amoeba

Let us recall same notions and definitions we shall use. Denote by $\mathbb{T}^n = (\mathbb{C} \setminus \{0\})^n$ the complex algebraic torus.

Definition 1 ([7]). The *amoeba* $\mathcal{A}_V$ of an algebraic set $V \subset \mathbb{T}^n$ is the image of V under the logarithmic map $Log : \mathbb{T}^n \to \mathbb{R}^n$ defined by the formula

$$Log : (z_1, \ldots, z_n) \to (log|z_1|, \ldots, log|z_n|).$$

An important notion in the study of amoebas is the following one.

Definition 2 ([8]). The *contour* $\mathcal{C}_V$ *of the amoeba* $\mathcal{A}_V$ is defined to be the set of critical values of the logarithmic map Log restricted to V.

The structure of the contour is described with the help of *the logarithmic Gauss map*

$$\gamma_V : V \to \mathbb{CP}_{n-1},$$

which to any nonsingular point $z \in V$ associates the complex normal $\gamma_V(z)$ to the hypersurface $\log V$ at the point $\log z$ (here $\log z_j = \log|z_j| + i \arg z_j$ is the complete (complex) logarithm).

In the case of a hypersurface

$$V = \{z \in \mathbb{T}^n : P(z) = 0\},$$

when V is the zero set of a single polynomial $P(z)$, the logarithmic Gauss map admits the following analytic expression

$$(z_1, \dots, z_n) \to (z_1 \frac{\partial P}{\partial z_1} : \dots : z_n \frac{\partial P}{\partial z_n}).$$

For the surfaces of codimension greater than 1 the corresponding expression for the logarithmic Gauss map see in [9].

Theorem ([10])**.** *A point of a hypersurface V is critical for the map $Log|_V$ if and only if its image under the logarithmic Gauss map belongs to the real projective subspace $\mathbb{RP}_{n-1} \subset \mathbb{CP}_{n-1}$.*

According to this statement the contour $\mathcal{C}_V$ of the amoeba $\mathcal{A}_V$ is the set $\mathrm{Log}(\gamma^{-1}(\mathbb{RP}_{n-1}))$. The boundary $\partial\mathcal{A}_V$ of the amoeba belongs to the contour $\mathcal{C}_V$ but in general $\mathcal{C}_V$ is larger. We say that the boundary $\partial\mathcal{A}_V$ comprises the *external* part of the contour $\mathcal{C}_V$, while the rests of the contour we call its *internal* part.

Sometimes it is more useful to study the contour of the amoeba looking at the compactified amoeba.

By the *compactified amoeba* $\overline{\mathcal{A}_V}$ of a projective algebraic set $\overline{V} \subset \mathbb{CP}_n$ defined in the homogeneous coordinates $(Z_0 : \dots : Z_n)$ we call the image of this variety under *the moment map* $\mu : \mathbb{CP}_n \to \Sigma_n$

$$(Z_0 : \dots : Z_n) \to \frac{(|Z_0|, \dots, |Z_n|)}{|Z_0| + \dots + |Z_n|}$$

into the standard simplex $\Sigma_n = \{t \in \mathbb{R}^{n+1} : t_j \geqslant 0, t_0 + \dots + t_n = 1\}$ [11].

Remark. The projective space $\mathbb{CP}_n$ is the union of the complex torus $\mathbb{T}^n$ and $n+1$ hypersurfaces $\{Z_j = 0\}$, $j = 0, \dots, n$. The amoeba $\mathcal{A}_V$ corresponds to the points of V in the complex torus $\mathbb{T}^n$, the compactified amoeba $\overline{\mathcal{A}_V}$ corresponds to $\mathcal{A}_V$ with the $(n+1)$ compactified amoebas of hypersurfaces $\overline{V}_j = \overline{V} \bigcap \{Z_j = 0\}$ of one dimension less.

Definition 3. The *contour of a compactified amoeba* is the image of the set of critical values of the projection $\mathrm{Log}|_V$ under the moment map μ.

Example 1. The amoeba of the complex line $z_1 + z_2 + 1 = 0$ in $\mathbb{T}^2$ is shown on Fig. 1 (left). The contour of this amoeba consists only of the boundary $\partial\mathcal{A}_V$. The compactified amoeba of this line is shown on Fig. 1 (right) as the shaded triangle.

Theorem ([11,12])**.** *Let $n \geqslant 3$. The compactified amoeba $\overline{\mathcal{A}_V}$ of the hyperplane*

$$V = \{z \in \mathbb{T}^n : P = b_0 + b_1 z_1 + \dots + b_n z_n = 0\}, \quad b_j \neq 0,$$

is an n-dimensional polyhedron with $2(n+1)$ hyperfaces in the simplex Σ_n defined by the inequalities

$$t_j \geqslant 0, \quad \sum_{l=0}^{n} t_l = 1, \quad \beta_j t_j \leqslant \sum_{k \neq j} \beta_j t_k, \quad j = 0, \dots, n,$$

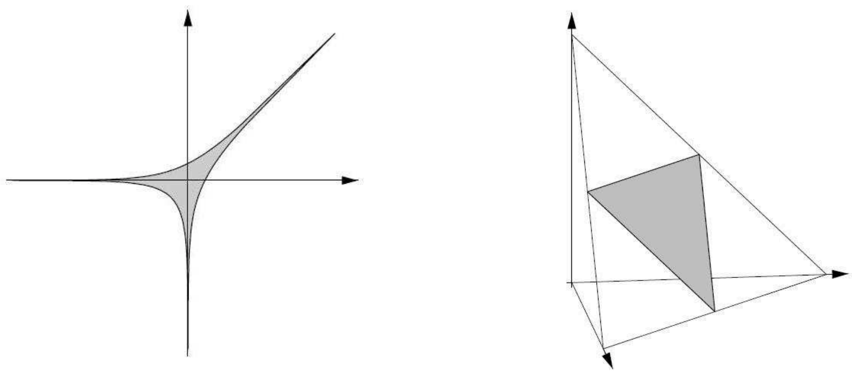

Fig. 1. Amoeba for a complex line in $\mathbb{C}^2$ and its compactified variant

where $\beta_j = |b_j|$. *The external part of its contour (the boundary* $\partial\overline{\mathcal{A}_V}$*) consists of* $(n+1)$ *simplicial faces of* $\overline{\mathcal{A}_V}$

$$\left\{ t \in \Sigma_n : \beta_j t_j = \sum_{k \neq j} \beta_k t_k \right\}, \quad j = 0, \ldots, n,$$

and the internal part consists of $(2^n - n - 2)$ *polyhedrons of the form*

$$\left\{ t \in \Sigma_n : \sum_{k \in I} \beta_k t_k = \sum_{l \notin I} \beta_l t_l \right\}, \quad I \subset \{0, \ldots, n\}, \quad 2 \leqslant \#I \leqslant n-1.$$

We see that in the case $n = 2$ the internal part of the contour of the amoeba is empty (we saw this also in Example 1). For $n = 3$ the compactified amoeba of the hyperplane $z_1 + z_2 + z_3 + 1 = 0$ in $\mathbb{C}^3$ is an octahedron (Fig. 2). On the left of Fig. 2 the external part of the contour is coloured, it consists of $n+1 = 4$ faces of the octahedron, which correspond to the boundaries of connected components of $\mathbb{R}^3 \setminus \mathcal{A}_V$. The remaining $2^n - (n+1) - 1 = 3$ internal pieces of the contour are parallelograms, each dividing the octahedron in two quadrangular pyramids (on the right of Fig. 2). In accordance with the remark to the definition of the compactified amoeba, the four non coloured faces of the octahedron correspond to amoebas of smaller dimension, namely, to amoebas of lines $\overline{V}_j$.

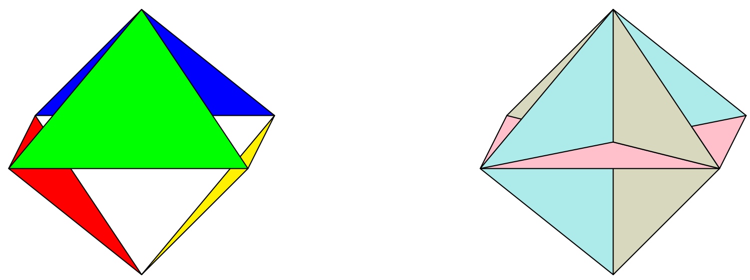

Fig. 2. The external and internal parts of the contour of the compactified amoeba for the complex hyperplane $z_1 + z_2 + z_3 + 1 = 0$ in $\mathbb{C}^3$

Further, we shall need some more general facts about amoebas.

1. The complement $\mathbb{R}^n \setminus \mathcal{A}_V$ consists of a finite number of connected components $\{E\}$, each is open and convex, and each preimage $\mathrm{Log}^{-1}(E)$ is the domain of convergence of the corresponding Laurent series for the rational function $1/P$ centred at the origin, see [7].

2. There exists an injective mapping

$$\nu : \ \{E\} \to \mathbb{Z}^n \cap N_P$$

such that the normal cone of the Newton polyhedron N_P at the point $\nu(E)$ coincides with the recession cone of the component E. The integer vector $\nu(E)$ is called the *order* of the component E, and we shall denote by E_ν the component of the order ν, see [11].

3. The number of connected components is at least equal to the number of vertices of the polytope N_P and is at most equal to the total number of integer points of N_P:

$$\#\mathrm{vert} N_P \leqslant \#\{E\} \leqslant \#\{\mathbb{Z}^n \cap N_P\}.$$

2. Fundamental solutions to equations with constant coefficients

In [6] a class of *fundamental* solutions to the scalar difference equations with constant coefficients

$$P(\delta)f(x) = 0 \tag{5}$$

was defined. Like homological solutions (4) these fundamental solutions are defined by integrals, but the integration cycles here lie outside the characteristic set. Namely, in [6] to each connected component E_ν of the amoeba complement $\mathbb{R}^n \setminus \mathcal{A}_V$ a fundamental solution is associated by means of the integral

$$\mathcal{P}_\nu(x) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\nu} \frac{z^x}{P(z)} \frac{dz}{z}, \tag{6}$$

where $\Gamma_\nu = \mathrm{Log}^{-1} u$ is an n-dimensional real torus defined by an arbitrary point $u \in E_\nu$ (Fig. 3), and $\dfrac{dz}{z}$ is the differential form $\dfrac{dz_1}{z_1} \wedge \cdots \wedge \dfrac{dz_n}{z_n}$. The integral (6) satisfies the relation

$$\sum_{\alpha \in A} a_\alpha \mathcal{P}_\nu(x+\alpha) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\nu} z^x \frac{dz}{z} = \delta_{x,0},$$

where $\delta_{x,0}$ is the function equal to zero for all $x \in \mathbb{Z}^n \setminus \{0\}$, and at the point 0 its value is equal to 1. Thus, $\mathcal{P}(x)$ is a fundamental solution.

Now, a certain class of solutions to equation (5) can be obtained as linear combinations of fundamental solutions

$$f(x) = \sum_\nu m_\nu \mathcal{P}_\nu(x), \quad \sum_\nu m_\nu = 0. \tag{7}$$

In fact, besides (5), the class of solutions (7) satisfies the extended system of difference equations

$$\begin{cases} P(\delta)f(x) := \sum\limits_{\alpha \in A} a_\alpha f(x+\alpha) = 0, \\ \sum\limits_{\alpha \in A} (x_n \alpha_i - x_i \alpha_n) a_\alpha f(x+\alpha) = 0, \quad i = 1, \dots, n-1, \end{cases}$$

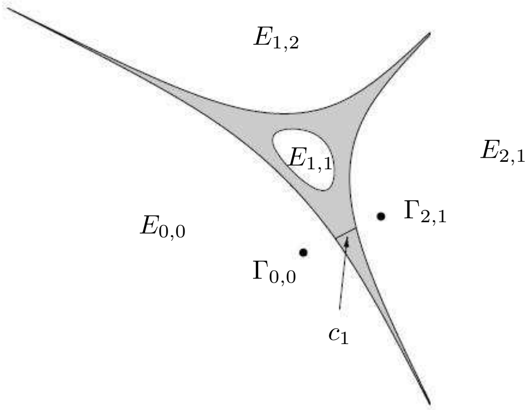

Fig. 3. Components of the amoeba complement for the polynomial $z^2w - 4zw + zw^2 + 1$ and some integration cycles Γ_ν

which is called the *associated system* for the equation (5). This system is holonomic, i.e. the dimension of the space of its solutions is finite. As $x \to \infty$ along the ray $x = a + lq$ with the directing vector $q = (q_1, \dots, q_n)$ its limit characteristic system is

$$\begin{cases} P(z) = 0, \\ \dfrac{z_i P'_{z_i}}{z_n P'_{z_n}} = \dfrac{q_i}{q_n}, \quad i = 1, \dots, n-1. \end{cases} \tag{8}$$

The roots $z = \lambda(q)$ of the algebraic system of equations (8) are exactly the preimages $\gamma^{-1}(q)$ of the logarithmic Gauss map $\gamma : V \to \mathbb{CP}_{n-1}$ (see the analytic definition of γ in section 1). The asymptotic behaviour of solutions (7) is described by the following theorem.

Theorem (Leinartas, Passare, Tsikh [6]). *If for the direction $q \in \mathbb{QP}_{n-1}$ the roots $\lambda_{(j)}(q)$ of the limit characteristic system (8) are such that the absolute values of all monomials $[\lambda_{(j)}(q)]^q$ are different, then for any solution $f(x)$ of the form (7) non-vanishing on the sequence $\{a + lq\}$, $a \in \mathbb{Z}^n$, the limit of the Horn vector*

$$\lim_{l \to \infty} \left(\frac{f(x + e_1)}{f(x)}, \dots, \frac{f(x + e_n)}{f(x)} \right) \Bigg|_{x = a + lq}$$

is equal to one of the characteristic roots $\lambda_{(p)}(q)$.

In [6] we investigated the connection between combinations (7) of fundamental solutions (6) and homological solutions (4) in the case $k = n - 1$. In section 3 we shall complete the list of fundamental solutions (see formula (17)) and describe their connection with homological solutions (4) for $k \leqslant n - 1$ (see Proposition 1).

3. Multidimensional Version of the Perron Theorem for the First Order Difference Equation

Consider a scalar difference equation of the first order

$$b_0 f(z) + b_1 f(x + e_1) + \dots + b_n f_n(x + e_n) = 0, \tag{9}$$

with the linear function as the characteristic polynomial

$$P(z) = b_0 + b_1 z_1 + \cdots + b_n z_n. \tag{10}$$

In this case the characteristic set $V = \{z \in \mathbb{T}^n : \ P(z) = 0\}$ is a hypersurface. We assume that all coefficients $b_j \neq 0$.

We introduce the following notion in order to formulate results in terms of the contour of the amoeba.

Definition 4. *The logarithmic Horn vector of the function $f(x)$ along the direction $q \in \mathbb{RP}_{n-1}$* is defined to be the vector

$$\left(\log \left| \frac{f(x+e_1)}{f(x)} \right|, \ldots, \log \left| \frac{f(x+e_n)}{f(x)} \right| \right) \Big|_{x=ql}.$$

Theorem 1. *The limit positions of the logarithmic Horn vector for fundamental solutions (6) to the scalar first order difference equation (9) fill the external contour of the amoeba $\mathcal{A}_V$.*

Proof. In the case of the first order scalar difference equation the theorem by Leinartas-Passare-Tsikh [6] describes the asymptotic behaviour of solutions $f(x)$ only for directions q, corresponding to the external part of the contour of the amoeba $\mathcal{A}_V$ of the characteristic set, because according to the definition, the fundamental solution $\mathcal{P}_\nu(a + lq)$ is equal to zero for directions q, corresponding to the internal part of the contour of $\mathcal{A}_V$. □

Recall that an admissible solutions of the equation (9) has the form

$$f(x) = \int\limits_\gamma z^x \ \omega(z), \quad \gamma \in Z_k(V), \ \omega \in \Omega_k(V). \tag{11}$$

Fundamental solutions (6), described in the previous section, are defined by integrals over cycles $\gamma \in H_{n-1}(V)$ of maximal dimension. We use them to obtain solutions to the equation (10) with asymptotic behaviour only along the directions q that correspond to the internal part of the contour of the amoeba $\mathcal{A}_V$ of the characteristic set $V = \{z \in \mathbb{T}^n : P(z) = 0\}$.

The remaining solutions with asymptotic along the directions q, corresponding to the *internal* part of the contour of $\mathcal{A}_V$, are given by the cycles $\tau \in H_k(V)$ of smaller dimension $k < n-1$.

Let us find a section of V by a plane S such that a point of the internal part of the contour of the amoeba $\mathcal{A}_V$ lies in the external part of the contour of the section $V \cap S$.

Lemma 1. *Let the point a belong to the internal part of the contour of the amoeba $\mathcal{A}_V$ of the complex hypersurface $V = \{z \in \mathbb{T}^n : \ \ b_0 + b_1 z_1 + \cdots + b_n z_n = 0\}$. Then there exists a plane S of the form $\{z_j = c_j = \mathrm{const} : \ j \in J\}$ (here J is a subset of $\{1, \ldots, n\}$ and depends on the point a) such that*

— the point a belongs to the internal part of the contour of the amoeba $\mathcal{A}_{V \cap S}$;

— the parts of the external contour of the amoeba $\mathcal{A}_{V \cap S}$ that do not contain the point a belong to the external part of the contour of the amoeba $\mathcal{A}_V$.

Proof. The critical set of the logarithmic projection $\mathrm{Log}\,|_V$ is defined by a solution $z(q)$ of the system of equations

$$\begin{cases} b_0 + b_1 z_1 + \cdots + b_n z_n = 0, \\ \dfrac{b_1 z_1}{q_1} = \cdots = \dfrac{b_n z_n}{q_n}, \end{cases}$$

where $q = (q_1 : \cdots : q_n) \in \mathbb{RP}_{n-1}$. This solution is given by the formula

$$z_i(q) = -\frac{b_0}{b_i} \frac{q_i}{q_1 + \cdots + q_n}, \quad i = 1, \ldots, n. \tag{12}$$

Let us find out for which values of q the image $\operatorname{Log} z(q)$ lies in the external and in the internal part of the contour of $\mathcal{A}_V$.

On the Reinhardt diagram, the external part of the contour corresponds to the boundary of the image of V. The boundary consists of $n+1$ connected components defined by the equations

$$|b_1 z_1| + \dots + |b_n z_n| - |b_0| = 0, \tag{13}$$

$$|b_1 z_1| + \dots - |b_j z_j| + \dots + |b_n z_n| + |b_0| = 0, \quad j = 1, \dots, n. \tag{14}$$

Indeed, on the Reinhardt diagram, in a neighbourhood of the image of each solution to, for example, the equations (13), there are both points belonging to the image of V, and points that do not have a preimage on V. Such points should be looked for on hyperplanes defined by the equations of the kind

$$|b_1||z_1| + \dots + |b_n||z_n| - r = 0. \tag{15}$$

For $r < |b_0|$ the solutions to (15) do not have a preimage on V, and for $r > |b_0|$ there are points from a neighbourhood of the solution having a preimage. Similarly, the equations (14) describe the external part of the contour.

Now we find the condition on the parameter q which guarantees that the image of the critical points of the logarithmic map lies in the boundary of V on the Reinhardt diagram.

We substitute (12) in (13) and (14) to see that the parameters corresponding to the connected components of the external part of the contour satisfy one of the equations

$$\left|\frac{q_1}{q_1 + \dots + q_n}\right| + \dots + \left|\frac{q_n}{q_1 + \dots + q_n}\right| - 1 = 0,$$

$$\left|\frac{q_1}{q_1 + \dots + q_n}\right| + \dots - \left|\frac{q_j}{q_1 + \dots + q_n}\right| + \dots + \left|\frac{q_n}{q_1 + \dots + q_n}\right| + 1 = 0, \quad j = 1, \dots, n.$$

Therefore, the image $\operatorname{Log} z(q)$ of the point $z(q)$ lies in the external part of the contour of $\mathcal{A}_V$ if the inequality

$$\frac{q_i}{q_1 + \dots + q_n} > 0$$

holds either for all i or only one.

Correspondingly, the image a of the point $z^0 = z(q^0)$ belongs to the internal part of the contour of $\mathcal{A}_V$ if

$$\begin{cases} \dfrac{q_j^0}{q_1^0 + \dots + q_n^0} < 0 & \text{for } j \in J \subset \{1, \dots, n\}, \\ \dfrac{q_i^0}{q_1^0 + \dots + q_n^0} > 0 & \text{for } i \in \{1, \dots, n\} \setminus J, \end{cases} \tag{16}$$

with the condition $1 \leqslant |J| \leqslant n-2$ on the cardinal number of the set J (for $n = 3$ see Fig. 4, where (q_1, q_2) are affine coordinates).

Consider the point a lying in the internal part of the contour of $\mathcal{A}_V$ and construct a suitable section S.

Let $a = \operatorname{Log} z^0$ and $z^0 = z(q^0)$. Determine the set J, for which the parameter $q^0 = (q_1^0, \dots, q_n^0)$ satisfies the system of equation (16) and consider the plane

$$S = \{z \in \mathbb{T}^n : \ z_j = c_j, \ j \in J\},$$

where $c_j = z_j^0$ are constants, which depend on the parameter q^0:

$$c_j = -\frac{b_0}{b_j} \frac{q_j^0}{q_1^0 + \dots + q_n^0}.$$

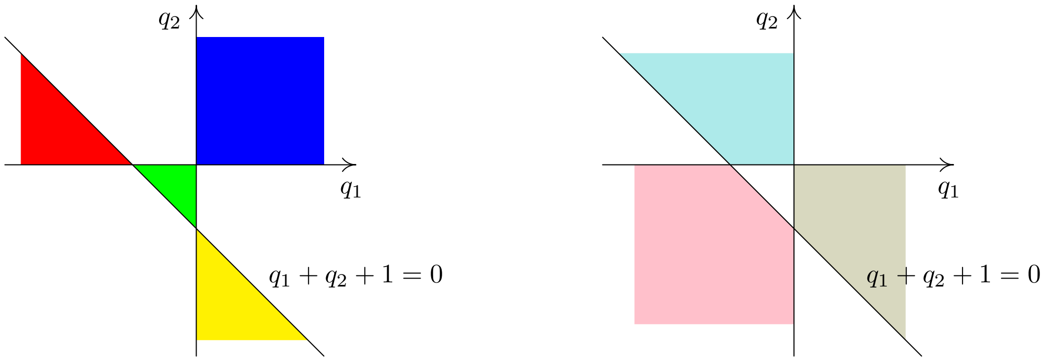

Fig. 4. The connected components of the definition domain for the parametrization of the contour of the amoeba of a plane in $\mathbb{C}^3$: for the external components (left) and for the internal components (right)

The intersection of V and S is a plane, which we see as a hyperplane in the space $\mathbb{T}^{|I|}$ of the remaining variables $'z=(z_i)$, $i\in I$, where $I=\{1,\dots,n\}\setminus J$:

$$V\cap S=\{'z\in\mathbb{T}^{|I|}:\ b_0+\sum_{j\in J}b_jc_j+\sum_{i\in I}b_iz_i=0\}.$$

The Log-image of the point $'z^0$ with the coordinates z_i^0, $i\in I$, belongs to the external part of the contour of $\mathcal{A}_{V\cap S}$. Indeed, denote $\widetilde{b_0}=b_0+\sum_{j\in J}b_jc_j$, then according to (12) the contour of amoeba $\mathcal{A}_{V\cap S}$ consists of the Log-images of points $'z\in\mathbb{T}^{|I|}$ with the coordinates

$$z_i(\widetilde{q})=-\frac{\widetilde{b}_0}{b_i}\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i},\quad i\in I,$$

where $\widetilde{q}=(\widetilde{q}_i)_{i\in I}$ runs over $\mathbb{RP}_{|I|-1}$. Furthermore, the point $'z^0$ corresponds to the value of $\widetilde{q}$ such that

$$-\frac{\widetilde{b}_0}{b_i}\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i}=-\frac{b_0}{b_i}\frac{q_i^0}{q_1^0+\dots+q_n^0},\quad i\in I.$$

So, according to (16), for all $i\in I$

$$\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i}=\frac{b_0}{\widetilde{b}_0}\frac{q_i^0}{q_1^0+\dots+q_n^0}=\frac{b_0}{b_0+\sum_{j\in J}b_j\left(-\dfrac{b_0}{b_j}\dfrac{q_j^0}{q_1^0+\dots+q_n^0}\right)}\frac{q_i^0}{q_1^0+\dots+q_n^0}=$$

$$=\frac{1}{1-\sum_{j\in J}\dfrac{q_j^0}{q_1^0+\dots+q_n^0}}\frac{q_i^0}{q_1^0+\dots+q_n^0}>0,$$

this means that the point $\mathrm{Log}('z^0)$ belongs to the external part of the contour of the amoeba $\mathcal{A}_{V\cap S}$.

Hence, the first statement of the lemma is proved.

Now consider the parts of the contour of $\mathcal{A}_{V\cap S}$ not containing the point a. It is obvious that the internal parts of the contour of $\mathcal{A}_{V\cap S}$, if they exist, do not intersect the external contour of $\mathcal{A}_V$.

Let the image of the point $'z = {}'z(\widetilde{q}) \in S$ lies in the external part of the contour of $\mathcal{A}_{V\cap S}$, then $\dfrac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i} > 0$ only for one i from I, because the external part of the contour, corresponding to the positive relations for all i, contains the point a. The point $z \in V$, corresponding to $'z \in V \cap S$, has the parametrization $z = z(q)$, related to the parameters used above by

$$\begin{aligned} z_i &= -\frac{b_0}{b_i}\frac{q_i}{q_1+\cdots+q_n} = -\frac{\widetilde{b}_0}{b_i}\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i} && \text{for } i \in I,\\ z_j &= -\frac{b_0}{b_j}\frac{q_j}{q_1+\cdots+q_n} = c_j = -\frac{b_0}{b_j}\frac{q_j^0}{q_1^0+\cdots+q_n^0} && \text{for } j \in J. \end{aligned}$$

Let us show that the image of the point z lies in the external part of the contour of $\mathcal{A}_V$. Indeed, for $i \in I$ the inequality

$$\frac{q_i}{q_1+\cdots+q_n} = \frac{\widetilde{b}_0}{b_0}\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i} = \frac{b_0+\sum_{j\in J} b_j c_j}{b_0}\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i} = \left(1-\sum_{j\in J}\frac{q_j^0}{q_1^0+\cdots+q_n^0}\right)\frac{\widetilde{q}_i}{\sum_{i\in I}\widetilde{q}_i} > 0$$

holds only for one value of i. But for $j \in J$

$$\frac{q_j}{q_1+\cdots+q_n} = \frac{q_j^0}{q_1^0+\cdots+q_n^0} < 0.$$

Thus, the second statement of the lemma is also proved. □

It follows that in the constructed sections we act exactly as in $\mathbb{C}^n$ taking into account that one of the *external* parts of the contour of the amoeba $\mathcal{A}_{V\cap S}$ of the section lies in the *internal* contour of the amoeba $\mathcal{A}_V$ of the given hyperplane.

We are now going to determine the asymptotic of the solutions of the type (11) for the cycles of dimension less then $n-1$ in the directions q corresponding to the internal part of the contour of $\mathcal{A}_V$. Let $z = z(q)$ be a point in the internal part of the contour, this means

$$\begin{aligned} \frac{q_j}{q_1+\cdots+q_n} &< 0 \quad \text{for } j \in J =\subset \{1,\ldots,n\},\\ \frac{q_i}{q_1+\cdots+q_n} &> 0 \quad \text{for } i \in I = \{1,\ldots,n\}\setminus J, \end{aligned}$$

and k is the cardinal number of I.

Consider a cycle $\gamma \in Z_{k-1}(V\cap S)$ and the corresponding integral

$$f(x) = \int\limits_\gamma z^x\,\omega(z) = \int\limits_\gamma \prod_{j\in J} z_j^{x_j}\prod_{i\in I} z_i^{x_i}\,\omega(z_J,z_I) = \prod_{j\in J} c_j^{x_j}\int\limits_\gamma\prod_{i\in I} z_i^{x_i}\omega(c_J,z_I),$$

where $z_J = (z_{j_1},\ldots,z_{j_{n-k}})$, $z_I = (z_{i_1},\ldots,z_{i_k})$ and $c_J = (c_{j_1},\ldots,c_{j_{n-k}})$.

For $x = ql$ it is a Laplace integral

$$f(ql) = \prod_{j\in J} c_j^{q_j l}\int\limits_\gamma \exp(l < q_I, \ln z_I >)\omega(c_J, z_I).$$

The critical phase points $\varphi(z_I) =< q_I, \ln z_I >|_{V\cap S}$ coincide with the critical points of the monomial ${z_I}^{q_I}$ and project to the contour of the amoeba $\mathcal{A}_{V\cap S}$ (see [6]). Moreover, $\max\limits_{z_I\in\gamma}\operatorname{Re}\varphi(z_I)$ is attained in the only point $z_I(q) \in V\cap S$, for which the image of the logarithmic Gauss map $\gamma(z_I)$ equals to q_I.

Therefore, following the saddle point method, we have the asymptotic

$$f(ql) \sim \prod_{j\in J} c_j^{q_j l} \cdot C \cdot l^{-\frac{k}{2}} z_I(q)^{q_I l} = C \cdot l^{-\frac{k}{2}} z(q)^{ql}, \quad l \to \infty,$$

where the coefficient C does not depend on l, therefore,

$$\lim_{l\to\infty} \log \left| \frac{f(ql+e)}{f(ql)} \right| = \log |z(q)|.$$

Note that it is possible to calculate the asymptotic of the solutions of the type (11) represented by integrals over any cycles $\gamma \in Z_k(V)$ and not only over cycles lying in the section $V \cap S$. Indeed, by the Bernshtein-Danilov-Khovanskii theorem [13] the elements of the group $H_k(V)$ are given by cycles in sections of V by complex planes of dimension k. One can do this the following way.

Every k-cycle on V is homologous to a sum of iterated Leray coboundaries over the intersection of the closure $\overline{V}$ in $\mathbb{C}^n$ with intersections of k coordinate planes $T_i = \{z \in \mathbb{C}^n : z_i = 0\}$ (see [14]):

$$H_k(V) = \bigoplus_{1\leqslant i_1<\cdots<i_k\leqslant n} \delta^k H_0(\overline{V} \cap T_{i_1} \cap \cdots \cap T_{i_k}),$$

where the sum is taken over all ordered sets of integers $1 \leqslant i_1 < \cdots < i_k \leqslant n$, and δ^k is an iterated Leray coboundary [15].

The construction of a Leray coboundary allows to consider a tube not just in $\mathbb{C}^n$ but in the section $V \cap S \subset \mathbb{T}^n$ over the intersection $\overline{V} \cap_{i\in I} T_i$. Therefore, any k-dimensional cycle on V is homologous to a sum of cycles lying in the section of V by some k-dimensional plane.

Consider on the chosen intersections $V \cap S$ the fundamental solutions

$$\mathcal{P}_{I,\nu}(x) = \frac{1}{(2\pi i)^k} \int\limits_{\Gamma_{I,\nu}} \frac{z^x}{P(z)} \frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_k}}{z_{i_k}}, \tag{17}$$

where $\Gamma_{I,\nu} = \mathrm{Log}^{-1} u_\nu$ and u_ν belongs to $E_{I,\nu}$, a connected component of the complement of the amoeba $\mathcal{A}_{V\cap S} \subset \mathbb{R}^k$.

This proves the following theorem.

Theorem 2. *The limit positions of the logarithmic Horn vector for the fundamental solutions (17) of the first-order scalar difference equation (9) fill the internal contour of the amoeba $\mathcal{A}_V$.*

Analogously to formula (7), a linear combination of solutions (17) gives some class of solutions to the equation (9):

$$f(x) = \sum_\nu m_{I,\nu} \mathcal{P}_{I,\nu}(x), \quad \sum_\nu m_{I,\nu} = 0.$$

Moreover, $f(x)$ can be represented in the form (11) due to the following proposition.

Proposition 1. *For $\nu \neq \mu$,*

$$\mathcal{P}_{I,\nu}(x) - \mathcal{P}_{I,\mu}(x) = \int\limits_{\gamma_{\nu,\mu}} z^x \,\mathrm{Res} \left[\frac{dz_I}{P(z) z_I} \right],$$

where $\mathrm{Res}\left[\frac{dz_I}{P(z)z_I}\right]$ *is a Leray residue form for* $\frac{dz_I}{P(z)z_I}$ *in* $(\mathbb{C} \setminus \{0\})^k$.

Proof. Let the cycles $\Gamma_{I,\nu}$ and $\Gamma_{I,\mu}$ be defined by points u_ν and u_μ from different connected components of the complement of the amoeba $\mathcal{A}_{V\cap S}$. Connect u_ν and u_μ by the segment h. The difference of the cycles $\Gamma_{I,\nu} - \Gamma_{I,\mu}$ is homeomorphic in $\mathbb{C}^k \setminus V \cap S$ to a tube over the cycle $V \cap S \cap \mathrm{Log}^{-1}(h)$.

Indeed, $\mathrm{Log}^{-1}(h)$ is a $(k+1)$-dimensional chain homeomorphic to a cylinder: $\mathrm{Log}^{-1}(h) \sim I \times \underbrace{S^1 \times \dots S^1}_{k}$. The tube over $V \cap S \cap \mathrm{Log}^{-1}(h)$ divides $\mathrm{Log}^{-1}(h)$ into two parts, one of which is bounded by this tube and the boundary of the chain $\mathrm{Log}^{-1}(h)$, therefore the difference $\Gamma_{I,\nu} - \Gamma_{I,\mu}$ is homological to a tube over the cycle $V \cap S \cap \mathrm{Log}^{-1}(h)$.

Now it remains to apply the Leray formula to the integral $\mathcal{P}_{I,\nu}(x) - \mathcal{P}_{I,\mu}(x)$

$$\mathcal{P}_{I,\nu}(x) - \mathcal{P}_{I,\mu}(x) = \frac{1}{(2\pi i)^k} \int\limits_{\Gamma_{I,\nu} - \Gamma_{I,\mu}} \frac{z^x}{P(z)} \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_k}}{z_{i_k}} =$$

$$= \frac{1}{(2\pi i)^k} \int\limits_{\delta\gamma_{\nu,\mu}} \frac{z^x}{P(z)} \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_k}}{z_{i_k}} = \int\limits_{\gamma_{\nu,\mu}} z^x \operatorname{Res}\left[\frac{dz_I}{P(z) z_I}\right],$$

that gives us the stated formula. □

Thus, we have a multidimensional analogue of Perron's theorem for the first order difference equations.

Theorem 3. *Each point of the contour of the amoeba for the characteristic set of the equation (10) is a limit position of the logarithmic Horn vector of some fundamental solution to this equation. The points in the external part of the contour of the amoeba correspond to fundamental solutions of the form (6), and for $k \geqslant 2$ the points in the internal part correspond to fundamental solution of the type (17).*

Proof. The proof follows from theorems 1 and 2. □

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References

[1] H.Poincaré, Sur les équations linéaires aux différentielles ordinaires et aux différences finies, *Amer. J. Math.*, **7**(1885), 203–258.

[2] O.Perron, Über die Poincarésche lineare Differenzengleichung, *J. Reine Angew. Math.*, **137**(1909), 6–64.

[3] A.O.Gelfond, Calculus of finite differences, Hindustan Publishing Corp., Delhi, 1971.

[4] V.M.Trutnev, A.K.Tsikh, On the structure of residue currents and functionals orthogonal to ideals in the space of holomorphic functions, *Izvestiya. Math.*, **59**(1995), no. 5, 1083–1107.

[5] M.V.Fedoryuk, The Saddle Point method, Nauka, Moscow, 1977 (in Russian).

[6] E.K.Leinartas, M.Passare, A.K.Tsikh, Multidimensional versions of Poincare's theorem for difference equations, *Sb. Math.*, **199**(2008), no. 10, 1505–1522.

[7] I.Gelfand, M.Kapranov, A.Zelevinsky, Discriminants, resultants and multidimensional determinants, Birkhäuser, Boston, 1994.

[8] M.Passare, A.Tsikh, Amoebas: their spines and contours, *Contemporary maths.*, **377**(2005), 275–288.

[9] N.A.Bushueva, A.K.Tsikh, On amoebas of algebraic sets of higher codimension, *Proceedings of the Steklov Institute of Mathematics*, **279**(2012), 52–62.

[10] G.Mikhalkin, Real algebraic curves, the moment map and amoebas, *Ann. Math.*, **151**(2000), 309–326.

[11] M.Forsberg, M.Passare, A.Tsikh, Laurent determinants and arrangements of hyperplane amoebas, *Adv. in Math.*, **151**(2000), 54–70.

[12] K.V.Kuzvesov, Contour of compactified amoeba of a hypersurface, *Vestnik KrasGU*, **7**(2006), 85–90 (in Russian).

[13] V.I.Danilov, A.G.Khovanskii, Newton polyhedra and an algorithm for computing Hodge–Deligne numbers, *Mathematics of the USSR-Izvestiya*, **29**(1987), no. 2, 279–298.

[14] N.A.Bushueva, On Isotopies and Homologies of Subvarieties of Toric Varieties, *Sib. Math. J.*, **51**(2010), no. 5, 776–788.

[15] J.Leray, Le calcul différentiel et intégral sur une variété analytique complexe (Problème de Cauchy III), *Bull. Math. Soc. France*, **87**(1959), 81–180.

Об асимптотике гомологических решений многомерных линейных разностных уравнений

Наталья А. Бушуева
Константин В. Кузвесов
Август К. Цих

Рассматривается многомерное линейное разностное уравнение с постоянными коэффициентами и пара (γ, ω), где γ — гомологический k-мерный цикл на характеристическом множестве уравнения, а ω — голоморфная форма степени k. Интеграл по γ формы ω, умноженной на экспоненциальное ядро, называется гомологическим решением. На примере уравнения первого порядка иллюстрируется многомерный вариант теоремы Перрона в классе гомологических решений.

Ключевые слова: разностное уравнение, асимптотика, амеба алгебраического множества, логарифмическое отображение Гаусса.