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Ladder type operators and recurrence relations for the radial wave functions of the N-th dimensional oscillators and hydrogenlike atoms

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Abstract

Using the method described in [11], we present some new *ladder type operators* and *recurrence relations* for the radial wave functions of the N-th dimensional isotropic harmonic oscillators and the hydrogenlike atoms.

KEYWORDS AND PHRASES: recurrence relations, ladder operators, wave functions, Laguerre polynomials
AMS SUBJECT CLASSIFICATION: 33C45

1 Introduction

The Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = H\psi(\vec{r},t)\,,$$

where $\hbar$ is the Planck's constant, $H = -\frac{\hbar}{2\mu}\nabla^2 + \mathcal{U}(\vec{r},t)$ the Hamiltonian operator, $\psi(\vec{r},t)$ the wave function of the variables $\vec{r}$ (spacial) and t (time), and ∇^2 the Laplace operator, describes the physical states of a particle or of a system, where $\mathcal{U}(\vec{r},t)$ and μ represents the potential energy and mass, respectively.

In the particular case where the potential $\mathcal{U}$ is time independent, one may write

$$\Psi(\vec{r},t) = \psi(\vec{r})e^{-i\frac{E}{\hbar}t}\,, \tag{1.1}$$

where the time independent wave function $\psi(\vec{r})$ is such that

$$H\psi(\vec{r}) = \left(-\frac{\hbar}{2\mu}\nabla^2 + \mathcal{U}\right)\psi(\vec{r}) = E\psi(\vec{r})\,, \tag{1.2}$$

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which means that the sum of the kinetic energy with the potential energy coincides with the total energy E. Hence, the energy E and the time independent wave function $\psi(\vec{r})$ are the corresponding eigenvalues and eigenfunctions, respectively, of the equation (1.2). They define, through (1.1), the wave function Ψ.

Numerous problems connected to the theory and applications of quantum and classic mechanics, correspond to potentials $\mathcal{U}$ which can be solved analytically [8, 17, 18, 20]. Its description by Schrödinger equation leads, in many situations, to a generalized differential equation of hypergeometric type

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0\,, \tag{1.3}$$

where σ, $\tilde{\sigma}$ and $\tilde{\tau}$ are polynomials such that $\deg[\sigma] \leq 2$, $\deg[\tilde{\sigma}] \leq 2$ and $\deg[\tilde{\tau}] \leq 1$, which, through a change of variables $u(z) = \phi(z)y(z)$, can be transformed [27] into a more simpler equation, known by *equation of hypergeometric type*

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0, \tag{1.4}$$

where σ and τ are polynomials of degree less or equal to two and one, respectively, and λ is a constant. The solutions of (1.4) are known as hypergeometric type functions, being the Bessel, Airy, Weber, Whittaker, Gauss, Kummer and the Hermite functions, as well as the classic orthogonal polynomials, some of the most relevant examples. Numerous of its properties can be found in [2, 13, 26, 27, 30, 34].

There exist many applications in modern physics that require the knowledge of the wave functions of hydrogenlike atoms and the isotropic harmonic oscillators. They are important, for instance, to find the corresponding matrix elements of several physical quantities (see e.g. [25, 31] and references therein). Among several methods for generating such wave functions, the so-called factorization method (see e.g. [19]) is particularly relevant (for more recent papers see e.g. [6, 20, 24, 32]). Moreover, the *ladder type operators* and the *recurrence relations* for these wave functions are useful for finding the transition probabilities and for the evaluation of certain integrals [25, 31]. In order to obtain such recurrence relations, several authors have developed different methods (see e.g. [9, 28, 29]). Most of them are based on the connection of such functions with the classical Laguerre polynomials. A Laplace-transform-based method has been developed recently [9, 33] but the calculation are cumbersome and requires inversion formulas.

An unified approach for generating *recurrence relations* and *ladder-type operators* for the N-th dimensional isotropic harmonic oscillators and the hydrogenlike atoms was presented in [11]: it explores the connection between the radial wave functions and the classical Laguerre polynomials together with a general theorem for the hypergeometric-type functions [27] that will enable to obtain several new relations for these polynomials and therefore for the wave functions. Its advantage, comparing to the others approach, it's not only because it can be easily extended to other exactly solvable models which involves hypergeometric functions or polynomials (see e.g. [8]), but also because it is a constructive method: one chooses the values of the parameters and one gets the corresponding coefficients. Using the same idea used to prove the above mentioned general theorem, many authors [14, 15, 35, 36] derived new recurrence relations for the hypergeometric polynomials and functions, i.e., the solutions of the second order differential equation (1.4).

In some of the most important applications, as the one that describes the hydrogen atom and the one of the isotropic harmonic oscillator, equation (1.2) can be solved by changing from cartesian $\vec{r} \equiv (x, y, z)$ to spherical coordinates $\vec{r} \equiv (r, \theta, \phi)$, admitting valid the separation of variables $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$. We then get two differential equations of type (1.3), one for $R(r)$ (the *radial* part) and the other for $Y(\theta, \phi)$ (the *angular* part), which, after being transformed into equation (1.4), admits polynomial solutions (classic orthogonal polynomials) [27, 33].

Here, we will apply the above mentioned technique to generate several new *ladder type operators* and *recurrence relations* for the radial wave functions of the hydrogen atom and the isotropic harmonic oscillator. With one exception for the theorems 3.1, 3.2, 4.1 and 4.2, all the recurrence relations and ladder-type operators of sections 3 and 4 are original and are not published elsewhere.

The structure of the paper is as follows: In section 2 the needed results and notations from the special function theory are introduced. In section 3 the isotropic harmonic oscillator is introduced and several recurrence and ladder-type relations are obtained. Similar results for the hydrogenlike atoms is presented in section 4. Finally, relevant references are quoted.

2 Preliminaries

In this section we deal with the hypergeometric functions y_ν, which are solutions of the hypergeometric type differential equation (1.4), where ν is such that $\lambda = -\nu\tau' - \nu(\nu-1)\sigma''/2$. This functions y_ν have the form [27]

$$y(z) = y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu+1}} ds, \tag{2.1}$$

where ρ is a solution of the Pearson equation $(\sigma\rho)' = \tau\rho$, σ and τ do not depend on ν, C is a contour in the complex plane such that its end points s_1 and s_2 satisfy the condition

$$\left. \frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}} \right|_{s_1}^{s_2} = 0, \tag{2.2}$$

and C_ν is a normalizing factor.

For the hypergeometric functions y_ν the following theorem holds [27, page 18]:

Theorem 2.1 *Let $y_{\nu_i}^{(k_i)}(z)$, $i = 1, 2, 3$, be any three derivatives of order k_i of the functions of hypergeometric type being $\nu_i - \nu_j$ an integer and such that*

$$\left. \frac{\sigma^{\nu_0+1}(s)\rho(s)}{(s-z)^{\mu_0}} s^m \right|_{s_1}^{s_2} = 0\,, \quad m = 0, 1, 2, \dots\,,$$

where ν_0 denotes the index ν_i with minimal real part and μ_0 the one with maximal real part. Then, there exist three non vanishing polynomials $B_i(z)$, $i = 1, 2, 3$, such that

$$\sum_{i=1}^{3} B_i(z) y_{\nu_i}^{(k_i)}(z) = 0. \tag{2.3}$$

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A special case of these functions are the polynomials of hypergeometric type, i.e., the polynomial solutions of the equation (1.4). They are defined by [27]

$$p_n(z) = \frac{C_n}{\rho(z)} \oint \frac{\sigma^n(s)\rho(s)}{(s-z)^{n+1}} ds\,, \quad n = 0, 1, 2, \dots\,,$$

i.e., the same function y_ν of the expression (2.1) but the contour C is closed and ν is a non-negative integer. Notice that, in this case, the condition (2.2) is automatically fulfilled, so the Theorem 2.1 holds for any family of polynomials of hypergeometric type. Notice also that Theorem 2.1 assures the existence of the non vanishing polynomials in (2.3) and its proof suggests a method to follow but it requires the computation of the polynomials coefficients B_i, $i = 1, 2, 3$. In general, it is not easy to find explicit expressions for these polynomials B_i but, in some cases [10, 14, 15, 16, 35, 36], these coefficients are obtained explicitly in terms of the coefficients of the polynomials σ and τ in (1.4).

An example of such polynomials are the Laguerre polynomials L_n^α defined by the hypergeometric series

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\, {}_1\mathrm{F}_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array} \middle| x \right) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!},$$

$$(a)_0 := 1, \quad (a)_k := a(a+1)(a+2)\cdots(a+k-1), \quad k = 1, 2, 3, \dots\,.$$

These polynomials satisfy the following recurrence and differential-recurrence relations useful in the next sections (see e.g. [1, 27, 34])

$$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x), \tag{2.4}$$

$$x\frac{d}{dx} L_n^\alpha(x) = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x) = (n+1)L_{n+1}^\alpha(x) - (n+\alpha+1-x)L_n^\alpha(x), \tag{2.5}$$

$$xL_n^{\alpha+1}(x) = (n+\alpha+1)L_n^\alpha(x) - (n+1)L_{n+1}^\alpha(x), \tag{2.6}$$

$$xL_n^{\alpha+1}(x) = (n+\alpha)L_{n-1}^\alpha(x) - (n-x)L_n^\alpha(x), \tag{2.7}$$

$$L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x), \tag{2.8}$$

$$(n+1)L_{n+1}^\alpha(x) - (2n+\alpha+1-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0. \tag{2.9}$$

Other instances of hypergeometric polynomials are the Jacobi, Bessel and Hermite polynomials [12, 27, 34].

3 Radial wave functions of the isotropic harmonic oscillator

The N-dimensional isotropic harmonic oscillator (I.H.O.) is described by the Shrödinger equation

$$\left(-\Delta + \frac{1}{2}\lambda^2 r^2 \right) \Psi = E\Psi, \qquad \Delta = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}, \quad r = \sqrt{\sum_{k=1}^{n} x_k^2}\,.$$

For solving it one uses the method of separation of variables that leads to a solution of the form $\Psi = R_{nl}^{(N)}(r) Y_{lm}(\Omega_N)$, where $R_{nl}^{(N)}(r)$ is the radial part, usually called the *radial wave functions*, defined by (see e.g. [7, 9])

$$R_{nl}^{(N)}(r) = \mathcal{N}_{nl}^{(N)}\, r^l e^{-\frac{1}{2}\lambda r^2} L_n^{l+\frac{N}{2}-1}(\lambda r^2), \qquad \mathcal{N}_{nl}^{(N)} = \sqrt{\frac{2n!\lambda^{l+\frac{N}{2}}}{\Gamma\left(n+l+\frac{N}{2}\right)}}, \tag{3.1}$$

being $n = 0, 1, 2, \dots$ and $l = 0, 1, 2, \dots$, the quantum numbers, and $N \geq 3$ the dimension of the space. The angular part $Y_{lm}(\Omega_N)$ are the so-called Nth-spherical or hyperspherical harmonics [7, 26]. In the following, we will assume that the parameters n, l, N are nonnegative integers.

3.1 Recurrence relations for the I.H.O. radial wave functions

The next theorem establishes a general recurrence relation for three different radial wave functions of the N-th dimensional isotropic harmonic oscillator.

Theorem 3.1 *Let $R_{nl}^{(N)}(r)$, $R_{n+n_1,l+l_1}^{(N)}(r)$ and $R_{n+n_2,l+l_2}^{(N)}(r)$ be three different radial wave functions of the N-th dimensional isotropic harmonic oscillator, were n_1, n_2 and l_1, l_2 are integers such that $\min(n+n_1, n+n_2, l+l_1, l+l_2) \geq 0$. Then, there exist non-vanishing polynomials in r, A_0, A_1, and A_2, such that*

$$A_0 R_{n,l}^{(N)}(r) + A_1 R_{n+n_1,l+l_1}^{(N)}(r) + A_2 R_{n+n_2,l+l_2}^{(N)}(r) = 0. \tag{3.2}$$

Its proof can be found in [11] and a crucial relation [11, formula (3.7), page 2058] to find explicitly the polynomials coefficients A_0, A_1 and A_2 corresponding to a given choice of the parameters n_1, n_2, l_1 and l_2, is

$$C_0 L_n^{l+\frac{N}{2}-1}(s) + C_1 L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) + C_2 L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(s) = 0, \tag{3.3}$$

where $s = \lambda r^2$. Then, after the constants C_i, $i = 0, 1, 2$, are determined, relation (3.2) is fulfilled with

$$A_0 = \left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} C_0 r^{l_1+l_2}, \quad A_1 = \left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} C_1 r^{l_2}, \quad A_2 = \left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} C_2 r^{l_1}. \tag{3.4}$$

In general, it is not easy to obtain the coefficients C_i in (3.3). Nevertheless, combining in a certain way the properties (2.4)–(2.9) they can be easily identified. We'll show how this technique works in the following examples. The idea is as follows: one should first decide which values of n_1, n_2, l_1 and l_2 in (3.2) one wants to consider. Then, combining in a certain way Eqs. (2.4)–(2.9), one transforms (3.3) into one of the formulas (2.4)–(2.9) or in a sum of linearly independent Laguerre polynomials from where the unknown coefficients easily follow.

- **Case $\mathbf{n_1 = -1}$, $\mathbf{l_1 = 0}$, $\mathbf{n_2 = 2}$, $\mathbf{l_2 = 0}$.**

The radial wave functions of the isotropic harmonic oscillator satisfy the following three term recurrence relation:

$$\begin{aligned}\sqrt{n\left(n+l+\tfrac{N}{2}-1\right)}\left(2n+l+\tfrac{N}{2}+2-\lambda r^2\right)R^{(N)}_{n-1,l}(r)+\\ \left[(n+1)\left(n+l+\tfrac{N}{2}\right)-\left(2n+l+\tfrac{N}{2}-\lambda r^2\right)\left(2n+l+\tfrac{N}{2}+2-\lambda r^2\right)\right]R^{(N)}_{n,l}(r)+\\ \sqrt{(n+1)(n+2)\left(n+l+\tfrac{N}{2}\right)\left(n+l+1+\tfrac{N}{2}\right)}\,R^{(N)}_{n+2,l}(r)=0.\end{aligned} \tag{3.5}$$

Proof. Considering $n_1=-1$, $n_2=2$, $l_1=l_2=0$ and $\alpha = l+\frac{N}{2}-1$ then (3.3) becomes

$$C_0L^{\alpha}_n(s)+C_1L^{\alpha}_{n-1}(s)+C_2L^{\alpha}_{n+2}(s)=0.$$

From (2.9) we may write

$$L^{\alpha}_{n+2}(s)=\frac{2n+\alpha+3-s}{n+2}\,L^{\alpha}_{n+1}(s)-\frac{n+1+\alpha}{n+2}\,L^{\alpha}_n(s),$$

which enable us to transform the above equation into

$$\frac{2n+\alpha+3-s}{n+2}\,C_2L^{\alpha}_{n+1}(s)+\left(C_0-\frac{n+1+\alpha}{n+2}\,C_2\right)L^{\alpha}_n(s)+C_1L^{\alpha}_{n-1}(s)=0.$$

Comparing with relation (2.9) one gets a solution

$$\begin{aligned}&C_0=(n+1)(n+\alpha+1)-(2n+\alpha+1-s)(2n+\alpha+3-s),\\ &C_1=(n+\alpha)(2n+\alpha+3-s),\qquad C_2=(n+1)(n+2).\end{aligned}$$

Introducing this expressions into (3.4) and using (3.1) gives

$$\left\{\begin{aligned}A_0&=\sqrt{\frac{\Gamma\left(n+l+\frac{N}{2}\right)}{2n!\lambda^{l+\frac{N}{2}}}}\left[(n+1)\left(n+l+\frac{N}{2}\right)-\left(2n+l+\frac{N}{2}-\lambda r^2\right)\left(2n+l+\frac{N}{2}+2-\lambda r^2\right)\right],\\ A_1&=\sqrt{\frac{\Gamma\left(n-1+l+\frac{N}{2}\right)}{2(n-1)!\lambda^{l+\frac{N}{2}}}}\left[\left(n+l+\frac{N}{2}-1\right)\left(2n+l+\frac{N}{2}+2-\lambda r^2\right)\right],\\ A_2&=\sqrt{\frac{\Gamma\left(n+2+l+\frac{N}{2}\right)}{2(n+2)!\lambda^{l+\frac{N}{2}}}}\,(n+1)(n+2),\end{aligned}\right.$$

which, through (3.2), proves (3.5). $\square$

In a similar way, by using formula (2.9), one can prove the following case.

- **Case $\mathbf{n_1 = -2}$, $\mathbf{l_1 = 0}$, $\mathbf{n_2 = 1}$, $\mathbf{l_2 = 0}$.**

$$\sqrt{n(n-1)\left(n+l+\tfrac{N}{2}-1\right)\left(n+l+\tfrac{N}{2}-2\right)}\, R_{n-2,l}^{(N)}(r)+ \\ \left[n\left(n+l+\tfrac{N}{2}-1\right)-\left(2n+l+\tfrac{N}{2}-\lambda r^2\right)\left(2n+l+\tfrac{N}{2}-2-\lambda r^2\right)\right] R_{n,l}^{(N)}(r)+ \\ \sqrt{(n+1)\left(n+l+\tfrac{N}{2}\right)}\left(2n+l+\tfrac{N}{2}-2-\lambda r^2\right) R_{n+1,l}^{(N)}(r)=0.$$

- **Case $\mathbf{n_1 = 0}$, $\mathbf{l_1 = -2}$, $\mathbf{n_2 = 0}$, $\mathbf{l_2 = 2}$.**

$$\lambda r^2\sqrt{\left(n+l+\tfrac{N}{2}-2\right)\left(n+l+\tfrac{N}{2}-1\right)}\left(l+\tfrac{N}{2}+\lambda r^2\right) R_{n,l-2}^{(N)}(r)- \\ \left\{\left(l+\tfrac{N}{2}-2+\lambda r^2\right)\left[\left(n-\lambda r^2\right)^2-n\left(n+l+\tfrac{N}{2}\right)\right]-\left(n+l+\tfrac{N}{2}-1\right)\left(l+\tfrac{N}{2}-2\right)\left(l+\tfrac{N}{2}+\lambda r^2\right)\right\} R_{n,l}^{(N)}(r)+ \\ +\lambda r^2\sqrt{\left(n+l+\tfrac{N}{2}\right)\left(n+l+\tfrac{N}{2}+1\right)}\left(l+\tfrac{N}{2}-2+\lambda r^2\right) R_{n,l+2}^{(N)}(r)=0. \tag{3.6}$$

Proof. In this case (3.3) becomes

$$C_0 L_n^{\alpha}(s)+C_1 L_n^{\alpha-2}(s)+C_2 L_n^{\alpha+2}(s)=0, \tag{3.7}$$

where $\alpha = l+\frac{N}{2}-1$.
On one hand, using twice (2.8) and (2.9) we have

$$L_n^{\alpha-2}(s)=\frac{\alpha-1}{n+\alpha-1}\, L_n^{\alpha}(s)+\frac{1-\alpha-s}{n+\alpha-1}\, L_{n-1}^{\alpha}(s). \tag{3.8}$$

On the other hand, using twice (2.7) we may write

$$L_n^{\alpha+2}(s)=\frac{n+\alpha+1}{s}\, L_{n-1}^{\alpha+1}(s)+\frac{(n+\alpha)(s-n)}{s^2}\, L_{n-1}^{\alpha}(s)+\frac{(n-s)^2}{s^2}\, L_n^{\alpha}(s). \tag{3.9}$$

Hence, introducing (3.8) and (3.9) into equation (3.7) it results

$$\tfrac{n+\alpha+1}{s}\, C_2\, L_{n-1}^{\alpha+1}(s)=-\left(C_0+\tfrac{\alpha-1}{n+\alpha-1}\, C_1+\tfrac{(n-s)^2}{s^2}\, C_2\right) L_n^{\alpha}(s)- \\ \left(\tfrac{1-\alpha-s}{n+\alpha-1}\, C_1+\tfrac{(n+\alpha)(s-n)}{s^2}\, C_2\right) L_{n-1}^{\alpha}(s). \tag{3.10}$$

Using (2.4) in the left member of (3.10) and comparing the resulting equation with (2.5) one is able to compute the coefficients

$$C_0=(1-\alpha-s)(s^2-2ns-\alpha n-n)-(\alpha-1)(n+\alpha)(\alpha+1+s), \\ C_1=(n+\alpha)(n+\alpha-1)(\alpha+1+s), \qquad C_2=-(1-\alpha-s)s^2.$$

Introducing the corresponding expressions into (3.4), one gets the coefficients A_0, A_1 and A_2. Using this last ones in (3.2) it results, after some simplifications, (3.6).

□

- **Case $\mathbf{n_1 = 0}$, $\mathbf{l_1 = -1}$, $\mathbf{n_2 = 0}$, $\mathbf{l_2 = 2}$.**

$$\begin{aligned}&\sqrt{\lambda\left(n+l+\tfrac{N}{2}-1\right)}\left(\lambda r^2+l+\tfrac{N}{2}\right) r R_{n,l-1}^{(N)}(r)- \\ &\quad\left[\lambda^2 r^4+\left(l+\tfrac{N}{2}-1-n\right)\lambda r^2+\left(l+\tfrac{N}{2}-1\right)\left(l+\tfrac{N}{2}\right)\right] R_{n,l}^{(N)}(r)+ \\ &\qquad\sqrt{\left(n+l+\tfrac{N}{2}\right)\left(n+l+\tfrac{N}{2}+1\right)}\,\lambda r^2 R_{n,l+2}^{(N)}(r)=0.\end{aligned} \tag{3.11}$$

Proof. Considering $n_1 = 0, l_1 = -1, n_2 = 0$ and $l_2 = 2$ in (3.3) one gets

$$C_0 L_n^{\alpha}(s) + C_1 L_n^{\alpha-1}(s) + C_2 L_n^{\alpha+2}(s) = 0,$$

where $\alpha = l + \frac{N}{2} - 1$. From (2.6) and (2.8) we may write

$$\tfrac{n+\alpha+1}{s} C_2 L_{n-1}^{\alpha+1}(s) = \left(-C_0 - C_1 - \tfrac{(n-s)^2}{s^2} C_2\right) L_n^{\alpha}(s) + \left(C_1 + \tfrac{(n+\alpha)(n-s)}{s^2} C_2\right) L_{n-1}^{\alpha}(s),$$

hence, in a similar way to the preceding case, first using (2.4) and then comparing with (2.5), one may consider the solution

$$C_0 = s^2 + (\alpha - n)s + \alpha(\alpha+1)\,, \qquad C_1 = -(n+\alpha)(s+\alpha+1)\,, \qquad C_2 = -s^2\,. \tag{3.12}$$

Computing the coefficients A_0, A_1 and A_2 in (3.4), substituting in (3.2) and simplifying the resulting equation one gets (3.11).

□

Remark 3.1 *We notice that the way to find C_0, C_1 and C_2 is not unique. For instance, one could have make use of (2.6) in the left member of (3.12), in order to obtain a linear combination of two linearly independent polynomials $L_n^{\alpha}(s)$ and $L_{n-1}^{\alpha}(s)$,*

$$\left(\tfrac{n(n+\alpha+1)-(n-s)^2}{s^2} C_2 - C_0 - C_1\right) L_n^{\alpha}(s) + \left(C_1 - \tfrac{(n+\alpha)(\alpha+1+s)}{s^2} C_2\right) L_{n-1}^{\alpha}(s) = 0,$$

from which we may obtain the same solution (3.12).

The following cases can be proved by a similar reasoning.

- **Case $\mathbf{n_1 = 0}$, $\mathbf{l_1 = 1}$, $\mathbf{n_2 = 0}$, $\mathbf{l_2 = -2}$.**

$$\begin{aligned}&\sqrt{\left(n+l+\tfrac{N}{2}-1\right)\left(n+l+\tfrac{N}{2}-2\right)}\,\lambda r^2 R_{n,l-2}^{(N)}(r)+ \\ &\quad\left[\left(2-l-\tfrac{N}{2}\right)\left(n+l+\tfrac{N}{2}-1\right)+\left(n-\lambda r^2\right)\left(\lambda r^2+l+\tfrac{N}{2}-2\right)\right] R_{nl}^{(N)}(r)+ \\ &\qquad\sqrt{\lambda\left(n+l+\tfrac{N}{2}\right)}\left(\lambda r^2+l+\tfrac{N}{2}-2\right) r R_{n,l+1}^{(N)}(r)=0.\end{aligned}$$

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- **Caso $\mathbf{n_1 = -1}$, $\mathbf{l_1 = -1}$, $\mathbf{n_2 = 2}$, $\mathbf{l_2 = 0}$.**

$$\sqrt{\lambda n\left(n+l+\tfrac{N}{2}-1\right)\left(n+l+\tfrac{N}{2}-2\right)}\left(2n+l+\tfrac{N}{2}+2-\lambda r^2\right) r R^{(N)}_{n-1,l-1}(r)+$$
$$\Big[\left(n+l+\tfrac{N}{2}-1\right)(n+1)(\lambda r^2-n)-n\left(n+l+\tfrac{N}{2}-1\right)\left(2n+l+\tfrac{N}{2}+2-\lambda r^2\right)-$$
$$\left(2n+l+\tfrac{N}{2}-\lambda r^2\right)\left(2n+l+\tfrac{N}{2}+2-\lambda r^2\right)\left(\lambda r^2-n\right)\Big] R^{(N)}_{n,l}(r)+$$
$$\sqrt{(n+1)(n+2)\left(n+l+\tfrac{N}{2}\right)\left(n+l+\tfrac{N}{2}+1\right)}AUI(\lambda r^2-n)R^{(N)}_{n+2,l}(r)=0.$$

- **Caso $\mathbf{n_1 = -1}$, $\mathbf{l_1 = -1}$, $\mathbf{n_2 = 1}$, $\mathbf{l_2 = -1}$.**

$$\sqrt{\lambda n\left(n+l+\tfrac{N}{2}-1\right)\left(n+l+\tfrac{N}{2}-2\right)}R^{(N)}_{n-1,l-1}(r)+\lambda r\left(\lambda r^2-l-\tfrac{N}{2}-2n+1\right)R^{(N)}_{n,l}(r)+$$
$$\sqrt{\lambda(n+1)}(\lambda r^2-n)R^{(N)}_{n+1,l-1}(r)=0.$$

- **Caso $\mathbf{n_1 = -1}$, $\mathbf{l_1 = 0}$, $\mathbf{n_2 = 2}$, $\mathbf{l_2 = -2}$.**

$$\sqrt{n\left(n+l+\tfrac{N}{2}-1\right)}\left(l+\tfrac{N}{2}-2-\lambda r^2\right)R^{(N)}_{n-1,l}(r)+$$
$$\Big[(n+1)\left(l+\tfrac{N}{2}-2\right)-\left(2n+l+\tfrac{N}{2}-\lambda r^2\right)\left(l+\tfrac{N}{2}-2-\lambda r^2\right)\Big]R^{(N)}_{n,l}(r)+$$
$$\sqrt{(n+1)(n+2)}\,\lambda r^2 R^{(N)}_{n+2,l-2}(r)=0.$$

- **Caso $\mathbf{n_1 = -1}$, $\mathbf{l_1 = 0}$, $\mathbf{n_2 = 0}$, $\mathbf{l_2 = 2}$.**

$$\sqrt{n\left(n+l+\tfrac{N}{2}-1\right)}\left(l+\tfrac{N}{2}+\lambda r^2\right)R^{(N)}_{n-1,l}(r)+\Big[\lambda^2 r^4-2\lambda n r^2-(l+\tfrac{N}{2})n\Big]R^{(N)}_{n,l}(r)-$$
$$\sqrt{\left(n+l+\tfrac{N}{2}\right)\left(l+n+\tfrac{N}{2}+1\right)}\lambda r^2 R^{(N)}_{n,l+2}(r)=0.$$

- **Caso $\mathbf{n_1 = 1}$, $\mathbf{l_1 = 0}$, $\mathbf{n_2 = 0}$, $\mathbf{l_2 = 2}$.**

$$\left(n+l+\tfrac{N}{2}\right)\left(l+\tfrac{N}{2}\right)R^{(N)}_{n,l}(r)-\sqrt{\left(l+\tfrac{N}{2}+n\right)\left(l+\tfrac{N}{2}+n+1\right)}\lambda r^2 R^{(N)}_{n,l+2}(r)-$$
$$\sqrt{(n+1)\left(n+l+\tfrac{N}{2}\right)}\left(l+\tfrac{N}{2}+\lambda r^2\right)R^{(N)}_{n+1,l}(r)=0.$$

3.2 Ladder-type relations for the I.H.O. radial wave functions

The next theorem establishes a linear relation with polynomials coefficients, involving two radial wave functions of the I.H.O. and the derivative of one of them. Some of these relations will define the so-called ladder operators for the radial wave functions and have important applications in the so-called factorization method (see e.g. [6, 19, 20, 25, 32]).

Theorem 3.2 *Let $R^{(N)}_{n,l}(r)$ and $R^{(N)}_{n+n_1,l+l_1}(r)$ be two radial functions of the N-th dimensional isotropic harmonic oscillator and let $\min(n+n_1, l+l_1) \geq 0$ and $(n_1)^2+(l_1)^2 \neq 0$, where n_1 and l_1 are integers. Then, there exist not vanishing polynomials in r, A_0, A_1, and A_2, such that*

$$A_0 R^{(N)}_{n,l}(r) + A_1 \frac{d}{d\,r} R^{(N)}_{n,l}(r) + A_2 R^{(N)}_{n+n_1,l+l_1}(r) = 0. \tag{3.13}$$

Its proof can be found in [11] and the fundamental relation to find the relation (3.13) that corresponds to a certain choice of the parameters n_1 and l_1 is the following:

$$B_0 L^{\alpha}_n(s) + B_1 L^{\alpha+1}_{n-1}(s) + B_2 L^{\alpha+l_1}_{n+n_1}(s) = 0, \tag{3.14}$$

where $s = \lambda r^2$, $\alpha = l + \frac{N}{2} - 1$ and the coefficients B_0, B_1, B_2 are non-vanishing polynomials. Then,

$$\left[B_1 \frac{d}{dr} + \lambda r\,(B_1 - 2B_0) - B_1 \frac{l}{r}\right] R^{(N)}_{nl}(r) = 2\lambda B_2 \frac{\mathcal{N}^{(N)}_{n,l}}{\mathcal{N}^{(N)}_{n+n_1,l+l_1}} r^{1-l_1} R^{(N)}_{n+n_1,l+l_1}(r) \tag{3.15}$$

is the equivalent operator form of equation (3.13).

By presenting some examples, we will show how one can obtain ladder-type relations for the radial wave function $R^{(N)}_{n,l}(r)$ of the I.H.O.. Again, we have only an existing theorem but its proof suggests partially the way. We proceed as follows: first, one fixes the relation (3.13) by choosing the values of the parameters n_1 and l_1. Then, after we introduce this parameters into (3.14), we combine in a certain way Eqs. (2.4)–(2.9) in order to transform (3.14) into one of the formulas (2.4)–(2.9) or in a sum of linearly independent Laguerre polynomials and solve the resulting equations for the unknown coefficients.

- **Case $\mathbf{n_1 = -2}$, $\mathbf{l_1 = 2}$.**

$$\left[\left(\lambda r^2 - l - \frac{N}{2}\right)\left(\frac{d}{dr} + \lambda r - \frac{l}{r}\right) - 2\lambda n r\right] R^{(r)}_{n,l}(r) = 2\lambda r\sqrt{n(n-1)}\, R^{(N)}_{n-2,l+2}(r). \tag{3.16}$$

Proof. Substituting $n_1 = -2$ and $l_1 = 2$ in (3.14) one gets

$$B_0 L^{\alpha}_n(s) + B_1 L^{\alpha+1}_{n-1}(s) + B_2 L^{\alpha+2}_{n-2}(s) = 0. \tag{3.17}$$

Using (2.6) and (2.8) in (3.17) it becomes

$$B_0 L^{\alpha+1}_n(s) + \left(-B_0 + B_1 - \frac{n-1}{s} B_2\right) L^{\alpha+1}_{n-1}(s) + \left(\frac{n+\alpha}{s} B_2\right) L^{\alpha+1}_{n-2}(s) = 0.$$

Comparing with (2.9) we may consider

$$B_0 = n, \quad B_1 = s - \alpha - 1, \quad B_2 = s.$$

Then, (3.16) follows by introducing the above coefficients, together with $n_1 = -2$ and $l_1 = 2$, into (3.15) and using, as well, (3.1). $\square$

The proof of the next cases of this section are similar to the previous ones so we will omit it.

- **Caso $\mathbf{n_1 = 0}$, $\mathbf{l_1 = 2}$**

$$\left[\left(\lambda r^2+l+\tfrac{N}{2}\right)\left(\tfrac{d}{dr}+\lambda r-\tfrac{l}{r}\right)+2\lambda r(r^2-n)\right]R^{(r)}_{n,l}(r) = -2\lambda r\sqrt{\left(n+l+\tfrac{N}{2}\right)\left(n+l+\tfrac{N}{2}+1\right)}\, R^{(N)}_{n,l+2}(r).$$

- $\mathbf{n_1 = 2}$, $\mathbf{l_1 = -1}$.

$$\left[\tfrac{1}{2}\left(n+l+\tfrac{N}{2}-\lambda r^2\right)\left(\tfrac{d}{dr}-\tfrac{l-\lambda r^2}{r}\right)-\tfrac{(n+1)\left(n+l+\frac{N}{2}\right)+\left(n+l+\frac{N}{2}-\lambda r^2\right)^2}{r}\right]R^{(N)}_{n,l}(r) =$$
$$\sqrt{\lambda(n+1)(n+2)\left(n+l+\tfrac{N}{2}\right)}\, R^{(N)}_{n+2,l-1}(r).$$

- $\mathbf{n_1 = 2}$, $\mathbf{l_1 = -2}$.

$$\left[\left(l+\tfrac{N}{2}-2-\lambda r^2\right)\left(\tfrac{1}{2}\left(\tfrac{d}{dr}+\lambda r-\tfrac{l}{r}\right)-\tfrac{n+l+\frac{N}{2}-\lambda r^2}{\lambda r^2}\right)-\tfrac{(n+1)(l+\frac{N}{2}-2)}{r}\right]R^{(N)}_{n,l}(r) =$$
$$\lambda\sqrt{(n+1)(n+2)}R^{(N)}_{n+2,l-2}(r).$$

4 Radial wave functions for the Hydrogen atom.

In this section we will provide a similar study for the $N-$dimensional Hydrogen atom described by the Shrödinger equation

$$\left(-\Delta-\frac{1}{r}\right)\Psi = E\Psi, \qquad \Delta=\sum_{k=1}^{n}\frac{\partial}{\partial x_k}, \quad r=\sqrt{\sum_{k=1}^{n}x_k^2}.$$

The solution is given by $\Psi = R^{(N)}_{nl}(r)Y_{lm}(\Omega_N)$, where the radial part $R^{(N)}_{nl}(r)$ is defined by [5, 21]

$$R^{(N)}_{nl}(r) = \mathcal{N}^{(N)}_{n,l}\left(\frac{r}{n+\frac{N}{2}-\frac{3}{2}}\right)^{l}\exp\left(-\frac{r}{2\left(n+\frac{N}{2}-\frac{3}{2}\right)}\right)L^{2l+N-2}_{n-l-1}\left(\frac{r}{n+\frac{N}{2}-\frac{3}{2}}\right).$$

Here $n = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ are the quantum numbers, $N \geq 3$ is the dimension of the space, and the normalizing constant $\mathcal{N}^{(N)}_{n,l}$ is

$$\mathcal{N}^{(N)}_{n,l} = \sqrt{\frac{(n-l-1)!}{(n+l+N-3)!}}\,\frac{2}{\left(n+\frac{N-3}{2}\right)^2}\,. \tag{4.1}$$

As before, $Y_{lm}(\Omega_N)$ denotes the hyperspherical harmonics.

Here it is important to notice that the Laguerre polynomials that appear in the expression of the radial wave functions are not the classical ones $L^{\alpha}_{n}(x)$ in the sense that the parameter

α as well as the variable x depend on the degree of the polynomials, n. Nevertheless, the algebraic properties of the classical Laguerre polynomials (2.4)–(2.9) can be used for deriving the algebraic relations of the radial wave functions as we will show in this section. When the parameters of the classical polynomials depend on n, the polynomials are orthogonal with respect to a *variant weights* [22, 23]. Using the theory of these variant classical polynomials the same recurrence relations can be derived as it is shown in [37]. For more details on this *varying classical* polynomials we refer the reader to the aforesaid works [22, 23, 37].

4.1 Recurrence relations for the radial wave functions of the Hydrogen atom

We begin this subsection with a general theorem involving three different radial wave functions of the Hydrogen atom. We notice that it's only an existent theorem.

Theorem 4.1 *Let the functions* $R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]$, $R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\frac{N-3}{2}\right)r\right]$ *and* $R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\frac{N-3}{2}\right)r\right]$ *be three different radial wave functions of the* N*-th Hydrogen atom and* n_1, n_2 *and* l_1, l_2 *integers such that* $\min\left(n+n_1, n+n_2, l+l_1, l+l_2\right) \geq 0$. *Then, there exist non-vanishing polynomials in* r, A_0, A_1, *and* A_2, *such that*

$$A_0 R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right]+A_1 R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\tfrac{N-3}{2}\right)r\right]+A_2 R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\tfrac{N-3}{2}\right)r\right]=0. \tag{4.2}$$

Its proof can be found in [11]. The corresponding relations that we are going to use in order to obtain the coefficients for the different choices of the parameters n_1, n_2, l_1 and l_2 are the following:

$$A_0^* L_m^{\alpha}(r)+A_1^* L_{m+n_1-l_1}^{\alpha+2l_1}(r)+A_2^* L_{m+n_2-l_2}^{\alpha+2l_2}(r)=0, \qquad \alpha=2l+N-2, \quad m=n-l-1, \tag{4.3}$$

and

$$A_0^*\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right]+A_1^*\left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{-l_1} R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\tfrac{N-3}{2}\right)r\right]+ A_2^*\left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{-l_2} R_{n+n_2,l+l_2}^{(N)}\left[\left(n+n_2+\tfrac{N-3}{2}\right)r\right]=0. \tag{4.4}$$

Relation (4.4) corresponds to (4.2) with

$$A_0=A_0^*\left(\mathcal{N}_{n,l}^{(N)}\right)^{-1} r^{l_1+l_2}, \quad A_1=A_1^*\left(\mathcal{N}_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{l_2}, \quad A_2=A_2^*\left(\mathcal{N}_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{l_1}.$$

Next we are going to present some examples involving recurrence relations with the radial wave functions of the Hydrogen atom. The technique works analogously to the corresponding ones of the isotropic harmonic oscillator, since as we said before, the formulas (2.4)–(2.9) remain valid for this Laguerre polynomials.

- **Case $\mathbf{n_1 = -1, l_1 = 0, n_2 = 2, l_2 = 0}$.**

$$\begin{aligned}
&[(n-l)(n+l+N-2)-(2n+N-3-r)(2n+N-1-r)]\,R_{n,l}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right]+\\
&\sqrt{(n+l+N-3)(n-l-1)}(2n+N-1-r)\left(\frac{n+\frac{N-5}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n-1,l}^{(N)}\left[\left(n+\tfrac{N-5}{2}\right)r\right]+\\
&+\sqrt{(n+l+N-1)(n+l+N-2)(n-l)(n-l+1)}\left(\frac{n+\frac{N+1}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n+2,l}^{(N)}\left[\left(n+\tfrac{N+1}{2}\right)r\right]=0.
\end{aligned} \tag{4.5}$$

Proof. Considering $n_1 = -1$, $l_1 = 0$, $n_2 = 2$ and $l_2 = 0$ in relation (4.3) one obtains

$$A_0^* L_m^\alpha(r) + A_1^* L_{m-1}^\alpha(r) + A_2^* L_{m+2}^\alpha(r) = 0.$$

By (2.9) the previous equation may be written in the form

$$A_2^* L_{m+2}^\alpha(r) + \left(-\frac{m+1}{m+\alpha}A_1^*\right) L_{m+1}^\alpha(r) + \left(A_0^* + \frac{2m+\alpha+1-r}{m+\alpha}A_1^*\right) L_m^\alpha(r) = 0,$$

which, again by (2.9), enables to write the solution

$$\begin{aligned}
A_0^* &= (m+1)(m+\alpha+1)-(2m+\alpha+1-r)(2m+\alpha+3-r),\\
A_1^* &= (m+\alpha)(2m+\alpha+3-r), \qquad A_2^* = (m+1)(m+2).
\end{aligned}$$

Introducing this expressions in (4.4), together with (4.1), one gets (4.5). □

The proof of next case is similar to the previous one so we will omit it.

- **Case $\mathbf{n_1 = -2, l_1 = 0, n_2 = 1, l_2 = 0}$.**

$$\begin{aligned}
&[(n+l+N-3)(n-l-1)-(2n+N-3-r)(2n+N-5-r)]\,R_{n,l}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right]+\\
&\sqrt{(n+l+N-3)(n+l+N-4)(n-l-2)(n-l-1)}\left(\frac{n+\frac{N-7}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n-2,l}^{(N)}\left[\left(n+\tfrac{N-7}{2}\right)r\right]+\\
&\sqrt{(n+l+N-2)(n-l)}\,(2n+N-5-r)\left(\frac{n+\frac{N+1}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n+1,l}^{(N)}\left[\left(n+\tfrac{N-1}{2}\right)r\right]=0.
\end{aligned}$$

- **Case $\mathbf{n_1 = -1,\ l_1 = -1,\ n_2 = 1,\ l_2 = -1}$.**

$$\begin{aligned}
&(2n+N-3-r)rR_{n,l}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right]+\\
&\sqrt{(n+l+N-3)(n+l+N-4)}(r-2l-N+3)\left(\frac{n+\frac{N-5}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n-1,l-1}^{(N)}\left[\left(n+\tfrac{N-5}{2}\right)r\right]+\\
&\sqrt{(n-l+1)(n-l)}(2l+N-3+r)\left(\frac{n+\frac{N-1}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n+1,l-1}^{(N)}\left[\left(n+\tfrac{N-1}{2}\right)r\right]=0.
\end{aligned} \tag{4.6}$$

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Proof. Considering $n_1 = -1$, $l_1 = -1$, $n_2 = 1$ and $l_2 = -1$ in (4.3) it follows

$$A_0^* L_m^{\alpha}(z) + A_1^* L_m^{\alpha-2}(z) + A_2^* L_{m+2}^{\alpha-2}(z) = 0,$$

which, by (2.6), may be transformed into

$$\left[\frac{(m+1)(m+2)}{r^2}\, A_0^* + A_2^*\right] L_{m+2}^{\alpha-2}(z) - \frac{2(m+1)(m+\alpha)}{r^2}\, A_0^*\, L_{m+1}^{\alpha-2}(z) +$$
$$\left[\frac{(m+\alpha)(m+\alpha-1)}{r^2}\, A_0^* + A_1^*\right] L_m^{\alpha-2}(z) = 0.$$

Comparing with (2.9) we may write the solution

$$A_0^* = (2m+\alpha+1-r)r^2, \qquad A_1^* = (m+\alpha)(m+\alpha-1)(r-\alpha+1),$$
$$A_2^* = (m+2)(m+1)(\alpha-1+r).$$

Introducing it in (4.4) and using (4.1) one gets (4.6).

□

The proof of next case is similar to the previous one so we will omit it.

- **Case $\mathbf{n_1 = -2}$, $\mathbf{l_1 = 0}$, $\mathbf{n_2 = 2}$, $\mathbf{l_2 = 0}$**

$$\Big[(2n+N-3-r)(2n+N-5-r)(2n+N-1-r) +$$
$$(n-l-1)(n+l+N-3)(2n+N-1-r) +$$
$$(n-l)(n+l+N-2)(2n+N-5-r)\Big] R_{n,l}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right] +$$

$$+\sqrt{(n-l-1)(n-l-2)(n+l+N-3)(n+l+N-4)}(2n+N-1-r)\times$$
$$\left(\frac{n+\frac{N-7}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n-2,l}^{(N)}\left[\left(n+\tfrac{N-7}{2}\right)r\right] +$$

$$\sqrt{(n-l-1)(n-l-2)(n+l+N-1)(n+l+N-2)}\times$$
$$(n-l)(n-l+1)(2n+N-5-r)\left(\frac{n+\frac{N+1}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n+2,l-1}^{(N)}\left[\left(n+\tfrac{N+1}{2}\right)r\right] = 0.$$

4.2 Ladder-type operators for the radial wave functions of the Hydrogen atom

Now we will state for the N-th Hydrogen atom a general theorem for ladder-type operators.

Theorem 4.2 *Let* $R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]$, *and* $R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\frac{N-3}{2}\right)r\right]$ *two different radial wave functions of the N-th Hydrogen atom and* $\frac{d}{dr}R_{nl}^{(N)}\left[\left(n+\frac{N-3}{2}\right)r\right]$, *the first derivative with* *respect to* r*, where* n_1 *and* l_1 *are integers such that* $\min(n+n_1, l+l_1) \geq 0$, $(n_1)^2 + (l_1)^2 \neq 0$. *Then, there exist not vanishing polynomials in* r*,* A_0*,* A_1 *and* A_2*, such that*

$$A_0 R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right] + A_1 \tfrac{d}{dr} R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right] + A_2 R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\tfrac{N-3}{2}\right)r\right] = 0. \tag{4.7}$$

The relevant identities of the corresponding proof [11] that are going to be used for the computation of the coefficients of the examples are

$$B_0 L_m^{\alpha}(z) + B_1 L_{m-1}^{\alpha+1}(z) + B_2 L_{m+n_1-l_1}^{\alpha+2l_1}(z) = 0 \tag{4.8}$$

where $z = \frac{2r}{2n+N-3}$, $\alpha = 2l+N-2$, $m = n-l-1$, and

$$r^{l_1}\left[B_1\left(\frac{d}{dr} - \frac{l}{r} + \frac{1}{2}\right) - B_0\right] R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right] = B_2 \frac{\mathcal{N}_{n,l}^{(N)}}{\mathcal{N}_{n+n_1,l+l_1}^{(N)}} R_{n+n_1,l+l_1}^{(N)}\left[\left(n+n_1+\tfrac{N-3}{2}\right)r\right]. \tag{4.9}$$

Relation (4.9) is the corresponding operator form of equation (4.7). Again, it is easy to obtain several ladder operators in n and l for the radial wave functions of the N-th Hydrogen atom.

- **Case $\mathbf{n_1 = -2,\ l_1 = 1}$**

$$\begin{aligned}&\left\{\left[(n-l-2)(n+l+N-3) - \left(n+l+N-3-\frac{2r}{2n+N-3}\right)^2\right]\left(\frac{d}{dr} - \frac{l}{r} + \frac{1}{2}\right) - \right.\\ &\qquad \left.(n-l-1)\left(n+l+N-3-\tfrac{2r}{2n+N-3}\right)\right\} R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right] = \\ &2\frac{\sqrt{(n-l-1)(n-l-2)(n-l-3)(n+l+N-3)}}{2n+N-3}\left(\frac{n+\frac{N-7}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n-2,l+1}^{(N)}\left[\left(n+\frac{N-7}{2}\right)r\right].\end{aligned} \tag{4.10}$$

Proof. Considering $n_1 = -2$ and $l_1 = 1$ in formula (4.8) it results

$$B_0 L_m^{\alpha}(z) + B_1 L_{m-1}^{\alpha+1}(z) + B_2 L_{m-3}^{\alpha+2}(z) = 0.$$

Using relation (2.8) for L_m^{α}, and relations (2.6) and (2.9) for $L_{m-3}^{\alpha+2}$, one may write the above equation in the form

$$B_0 L_m^{\alpha+1}(z) + \left(B_1 - B_0 - \frac{m-1}{z}B_2\right) L_{m-1}^{\alpha+1}(z) + \frac{m+\alpha-z}{z} B_2 L_{m-2}^{\alpha+1}(z) = 0.$$

Comparing with (2.9), we may consider the solution

$$B_0 = m(m+\alpha-z), \quad B_1 = (m-1)(m+\alpha) - (m+\alpha-z)^2, \quad B_2 = (m+\alpha)z,$$

which, together with (4.1), transform relation (4.9) into (4.10). □

The proof of the following case can be done in a similar way to the previous one.

- **Case $\mathbf{n_1 = 2}$ e $\mathbf{l_1 = 1}$.**

$$\frac{1}{2}\Big\{\Big[\Big(\tfrac{2r}{2n+N-3}-n+l\Big)^2-(n-l)(n+l+N-1)\Big]\Big(\tfrac{d}{dr}-\tfrac{l}{r}+\tfrac{1}{2}\Big)-$$
$$\Big(\tfrac{2r}{2n+N-3}-2n-N+2\Big)\Big(\tfrac{2r}{2n+N-3}-n+l+1\Big)+(n-l)(n+l+N-1)\Big\}R_{nl}^{(N)}\left[\left(n+\tfrac{N-3}{2}\right)r\right]=$$
$$\frac{\sqrt{(n-l)(n+l+N)(n+l+N-1)(n+l+N-2)}}{2n+N-3}\left(\frac{n+\frac{N+1}{2}}{n+\frac{N-3}{2}}\right)^2 R_{n+2,l+1}^{(N)}\left[\left(n+\tfrac{N+1}{2}\right)r\right].$$

Concluding remarks

In this paper we present some new examples of *recurrence relations* and *ladder-type operators* both for the radial wave functions of the ISOTROPIC HARMONIC OSCILLATOR and for the HYDROGEN ATOM. The main aim is not only to exhibit the representative coefficients but to show its simplicity and the naturalness of the corresponding technique. It is a very simple and constructive approach: the existence of the coefficients is based on a general result for functions of hypergeometric type, due to A.F. Nikiforov and V.B. Uvarov, and its specific computation derives from the relation between the corresponding radial wave functions and the Laguerre polynomials. We notice that the set of formulas (2.4)–(2.9) allows one to choose without any restrictions the quantum parameters n_i and l_i that figure in the radial wave functions. This way, comparing to others approach [9, 33], we hope to point out that the approach described in [11] is not only more attainable but also more embracing and unified since one can choose arbitrarily the parameters and the same technique works for the *recurrence relations* and for the *ladder operators* of the radial wave functions of both the isotropic harmonic oscillator and the Hydrogen atom. Beyond that, it remains valid for other quantum systems like, for instance, the Morse problem [28] and the relativistic Hydrogen atom [33] since the corresponding wave functions are proportional to the Laguerre polynomials. Obviously, this method for finding recurrence relations can be extended to any quantum system whose (radial) wave function are proportional to hypergeometric-type functions (see e.g. [8]).

To conclude this paper let us mention that a numerical study of several recurrence relations as well as ladder-type relations were studied in [4]. The extension of the method for the discrete case was done in [3].

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