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Special Version of the Collocation Method for a Class of Integral Equations of the Third Kind Based on Hermite–Fejer Interpolation Polynomials

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In this document we propose and justify special direct method for the approximate solution of equations of the third kind in the space of distributions.

Keywords: third-kind integral equation, approximate solution, space of distributions, theoretical substantiation.

We consider the following integral equation of the third-kind (ETK):

$$(Ax)(t) \equiv (Ux)(t) + (Kx)(t) = y(t), \tag{1}$$

where

$$(Ux)(t) \equiv x(t)t^{p_1}(1-t)^{p_2}\prod_{j=1}^{q}(t-t_j)^{m_j},\ (Kx)(t) \equiv \int_0^1 K(t,s)x(s)ds,\ t \in I \equiv [0,1],$$

$p_1,\ p_2 \in \mathbb{R}^+$, $t_j \in (0,1)$, $m_j \in \mathbb{N}$ $(j \in \overline{1,q})$; K and y are known continuous functions with certain "pointwise smoothness" properties, and x is the desired function.

Such equations arise in a number of problems in the elasticity theory, neutron transfer, particles dispersion (e.g., [1] and references therein; [2]). As a rule, natural classes of solutions to ETK are special spaces of distributions. Since the ETK under consideration can be solved exactly only in some particular cases, both in the theory and in applications one needs approximate solutions methods with a proper theoretical justification. Some relevant results are obtained in papers [3–8]. In [3, 4] one proposes and justifies several special direct methods for solving ETK (1). These methods are based on splines of the first and second orders and on polynomials in the space $D\{p_1, p_2; \overline{m}, \overline{\tau}\}$ of distributions and in the V-type based on Hadamard integral space in particular cases of zeros of the coefficient an the desired function outside the integral (in what follows for brevity we just say "the coefficient"). In [5–8] one considers close issues in the space $V\{p_1, p_2; \overline{m}, \overline{\tau}\}$.

In this present paper, using results obtained in [3–10] we develop and justify in the sense of [11] (Chap. 1) special variant of the collocation method for ETK (1) (i.e., in the general case of zeros of the coefficient) based on Hermite–Fejer interpolation polynomials in the space $D\{p_1, p_2; \overline{m}, \overline{\tau}\}$.

1. Spaces of basic functions and distributions

Let $C \equiv C(I)$ be the space of continuous functions on I with the ordinary max-norm, and let $m \in \mathbb{N}$. Following [12], we call the Taylor derivative of order m at the point t_0 the limit (if

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present)

$$y^{\{m\}}(t_0) \equiv m! \lim_{t\to t_0} \frac{y(t) - \sum\limits_{i=0}^{m-1} y^{\{i\}}(t_0)(t-t_0)^i/i!}{(t-t_0)^m};$$

by definition $y^{\{m\}}(t_0) \equiv y(t_0)$. We denote by $C\{m; t_0\}$ the class of functions $y \in C$ which at the point $t_0 \in (0,1)$ have the Taylor derivatives of order m.

Let $t_1, t_2, ..., t_q$ be arbitrarily fixed pairwise distinct points in the interval $(0,1)$. We associate each point t_j with a number $m_j \in \mathbb{N}\,(j = \overline{1,q})$. Introduce the vector space

$$C\{\overline{m}; \overline{\tau}\} \equiv \bigcap_{j=1}^{q} C\{m_j; t_j\},$$

where $\overline{m} \equiv (m_1, m_2, ..., m_q)$, $\overline{\tau} \equiv (t_1, t_2, ..., t_q)$ are finite collections of the corresponding values.

Let $p_1 \in \mathbb{R}^+$. Denote by $C\{p_1; 0\}$ the space of functions $y \in C$ which have right Taylor derivatives $y^{\{i\}}(0)$ $(i = \overline{1,[p_1]})$ at the point $t = 0$; we assume that in the case $p_1 \neq [p_1]$ (the symbol $[\cdot]$ stands for the integer part) the finite limit

$$\lim_{t\to 0+0} \left\{ \left[y(t) - \sum_{i=0}^{[p_1]} y^{\{i\}}(0) t^i/i! \right] t^{-p_1} \right\}$$

exists. The class $C\{p_2; 1\}$ $(p_2 \in \mathbb{R}^+)$ is defined analogously.

Let us now form the basic vector space

$$Y \equiv C\{p_1, p_2; \overline{m}, \overline{\tau}\} \equiv C\{\overline{m}; \overline{\tau}\} \bigcap C\{p_1; 0\} \bigcap C\{p_2; 1\}.$$

We assume that $C\{0, 0; \overline{0}, \overline{\tau}\} \equiv C$. Following [3], we define in it the norm

$$\|y\|_Y \equiv \|Ty\|_C + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j - 1} |y^{\{i\}}(t_j)|, \tag{2}$$

where T is the "characteristic" operator of class Y, namely,

$$(Ty)(t) \equiv \left[y(t) - \sum_{j=1}^{q+2} \sum_{i=0}^{m_j-1} y^{\{i\}}(t_j) R_{ji}(t) \right] /u(t) \equiv \Phi(t), \tag{3}$$

$u(t) \equiv t^{p_1}(1-t)^{p_2} \prod\limits_{j=1}^{q} (t-t_j)^{m_j}$, $\Phi \in C$, $\Phi(t_j) \equiv \lim\limits_{t\to t_j} \Phi(t)\,(j = \overline{1,q+2})$; $t_j \in (0,1)$ $(j = \overline{1,q})$, $t_{q+1} \equiv 0$, $t_{q+2} \equiv 1$, and $R_{ji}(t)$ are Hermite fundamental polynomials of degree $m-1$ by nodes $\{t_j\}_1^{q+2}$. Here $m \equiv \sum\limits_{j=1}^{q+2} m_j$, $m_{q+1} \equiv \lambda_1 + 1$, $m_{q+2} \equiv \lambda_2 + 1$, $\lambda_k \equiv \lambda(p_k)$ $(k = \overline{1,2})$, $\lambda(p) \equiv [p] - (1 + \mathrm{sign}([p] - p))$.

It is known (e.g., [3]) that elements of the space Y are functions in the form

$$y(t) = u(t)\Phi(t) + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j-1} a_{ji} R_{ji}(t), \tag{4}$$

where $\Phi(t) = (Ty)(t) \in C$, $a_{ji} = y^{\{i\}}(t_j) \in \mathbb{R}$ $(i = \overline{0,m_j-1},\ j = \overline{1,q+2})$, and Y in norm (2) is complete and embedded in C.

Let a linear operator $F_n^T \equiv F_{2n+m}^T : Y \to U(H_{2n-1}) \bigoplus H_{m-1}$, is defined by the rule

$$(F_n^T y)(t) \equiv (UF_nTy)(t) + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j-1} y^{\{i\}}(t_j)R_{ji}(t) \ (y \in Y), \tag{5}$$

where $F_n : C \to H_{2n-1} \equiv \text{span}\{t^k\}_0^{2n-1}$ is a Hermite–Fejer operator (e.g. [9], Part 3) by the system of Chebyshev nodes of the first kind

$$\xi_k \equiv \xi_k^{(n)} \equiv \frac{1}{2}\cos\frac{(2k-1)\pi}{2n} + \frac{1}{2} \ (k = \overline{1,n}). \tag{6}$$

The following lemma is valid [8].

Lemma. *If $y \in Y$ and $Ty \in Lip\alpha$ $(0 < \alpha \leqslant 1)$, then*

$$\|y - F_n^T\|_Y = O\{n^{-\alpha/2}\}.$$

Let us now consider on the basis space Y the set $X \equiv D\{p_1, p_2; \overline{m}, \overline{\tau}\}$ of distributions in the form

$$x(t) \equiv z(t) + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j-1} c_{ji}\delta^{\{i\}}(t_j), \tag{7}$$

where $t \in I$, $z \in C$, $c_{ji} \in \mathbb{R}$ are arbitrary constants, while $\delta^{\{i\}}(t_j)$ are distributions defined on Y by the rule

$$(\delta^{\{i\}}(t_j), \varphi) \equiv (-1)^i \varphi^{\{i\}}(t_j)$$
$$(\varphi \in Y, \, i = \overline{0, m_j-1}, \ j = \overline{1, q+2}).$$

One can easily prove that the space X is complete with respect to the norm

$$\|x\|_X \equiv \|z\|_C + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j-1} |c_{ji}|. \tag{8}$$

2. Generalized collocation method (GCM) based on Hermite-Fejer interpolation polynomials

Assume that the initial data in ETK (1) satisfy the conditions

$$K \in C^{\{p_1,p_2;\overline{m},\overline{\tau}\}}(I^2),\ K_t^{\{i\}}(t_j,s),\ K_s^{\{i\}}(t,t_j),\ y \in Y \ (i = \overline{0, m_j-1}, \ j = \overline{1, q+2}), \tag{9}$$

and x is the desired distribution in the form (7). In [3] one establishes the Fredholm property and sufficient conditions for the continuous invertibility of the operator $A : X \to Y$ and proposes an exact solution method for ETK (1) in the space X.

We approximate the solution to Eq. (1) by the element

$$x_n(t) \equiv z_n(t) + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j-1} c_{ji}\delta^{\{i\}}(t_j), \tag{10}$$

$$z_n(t) \equiv \sum_{k=0}^{2n-1} c_k t^k. \tag{11}$$

We find unknown coefficients $c_k \equiv c_k^{(n)}$ $(k = \overline{0, 2n-1})$ and c_{ji} $(i = \overline{0, m_j - 1},\ j = \overline{1, q+2})$ in accordance with the method under consideration from the system of linear algebraic equations (SLAE)

$$(TAx_n - Ty)(\xi_k) = 0 \ (k = \overline{1,n});\ (d(TUx_n)/dt)(\xi_k) = 0 \ (k = \overline{1,n});$$

$$(Ax_n - y)^{\{i\}}(t_j) = 0 \ \ (i = \overline{0, m_j - 1},\ j = \overline{1, q+2}), \tag{12}$$

where ξ_k are defined by formula (6).

The computational scheme (1), (9)–(12) is justified in the following assertion.

Theorem. *Let Eq. (1) have a unique solution in X with any right-hand side $y \in Y$, while functions $\theta(t,s) \equiv (T_t K)(t,s)$ (in t), $g_{ji}(t) \equiv (T\psi_{ji})(t)$, $\psi_{ji}(t) \equiv K_s^{\{i\}}(t, t_j)$ $(i = \overline{0, m_j - 1},\ j = \overline{1, q+2})$, and $(Ty)(t)$ satisfy the Lipschitz condition with some $\alpha\,(0 < \alpha \leqslant 1)$. Then with sufficiently large $n \in \mathbb{N}$ approximate solutions $x_n^*(t)$ constructed on the base of conditions (10)–(12) exist, are unique, and converge in the norm of the space X to the exact solution $x^*(t)$ of Eq. (1) with the rate*

$$\|x^* - x_n^*\| = O\{n^{-\alpha/2}\}. \tag{13}$$

Proof. We denote by $X_n \subset X$ the $(2n+m)$-dimensional subspace of elements in the form (10)–(11) such that

$$(d(TUx_n)/dt)(\xi_k) = 0 \ (k = \overline{1,n}),$$

while the subspace $Y_n \subset Y$ consists of all functions $y_n \in U(H_{2n-1}) \oplus H_{m-1}$ such that

$$(d(Ty_n)/dt)(\xi_k) = 0 \ (k = \overline{1,n}).$$

Let us prove that system (10)–(12) is equivalent to the operator equation

$$A_n x_n \equiv Ux_n + F_n^T K x_n = F_n^T y \ \ (x_n \in X_n,\ F_n^T y \in Y_n). \tag{14}$$

Really, let x_n^* be a solution to Eq. (14), i.e.,

$$Ux_n^* + F_n^T(Kx_n^* - y) \equiv 0.$$

Then in view of formulas (5) and (10) we get

$$u(t)(z_n^* + F_n T(Kx_n^* - y))(t) + \sum_{j=1}^{q+2} \sum_{i=0}^{m_j - 1} (Kx_n^* - y)^{\{i\}}(t_j) R_{ji}(t) \equiv 0. \tag{15}$$

It is clear that (15) is equivalent to the system

$$z_n^*(t) + (F_n T(Kx_n^* - y))(t) = 0,\ (Kx_n^* - y)^{\{i\}}(t_j) = 0 \ (i = \overline{0, m_j - 1},\ j = \overline{1, q+2}). \tag{16}$$

Using definition Hermite–Fejer operator (e.g., [9], Part 3) one can easily verify

$$(Ty - TKx_n^*)(\xi_k) = z_n^*(\xi_k) \ (k = \overline{1,n}),\ \ (dz_n^*/dt)(\xi_k) = 0 \ (k = \overline{1,n}),$$

$$(Kx_n^* - y)^{\{i\}}(t_j) = 0 \ (i = \overline{0, m_j - 1},\ j = \overline{1, q+2}).$$

Then in view of formulas

$$TU\varphi = \varphi \ (\varphi \in C),$$

and

$$(\delta^i(t_j), U\varphi) = 0 \ (\varphi \in C,\ i = \overline{0, m_j - 1},\ j = \overline{1, q+2})$$

we get

$$(Ty - TKx_n^*)(\xi_k) = (TUx_n^*)(\xi_k)\ (k = \overline{1,n}),\ \ (d(TUx_n^*)/dt)(\xi_k) = 0\ (k = \overline{1,n}),$$

$$(Kx_n^* - y)^{\{i\}}(t_j) = 0\ (i = \overline{0, m_j - 1},\ j = \overline{1, q+2}).$$

Moreover, the correlations

$$(Ux_n^*)^{\{i\}}(t_j) = 0\ (i = \overline{0, m_j - 1},\ j = \overline{1, q+2})$$

are evident. Consequently, system (16) takes the form (12).

We prove the converse assertion by repeating the same reasoning in a reverse order.

Let us now demonstrate the closeness of operators A and A_n on X_n. In view of (1), (14), (4), (5), (2), (7), estimates [10]

$$\|\varphi - F_n\varphi\|_C = O\{n^{-\alpha/2}\},$$

and (8) for any element $x_n \in X_n$ we have

$$\|Ax_n - A_nx_n\|_Y = \|Kx_n - F_n^TKx_n\|_Y = \|TKx_n - F_nTKx_n\|_C =$$

$$= \max_{t\in I}\left|\int_0^1 (\theta - F_n\theta)(t,s)z_n(s)ds + \sum_{j=1}^{q+2}\sum_{i=0}^{m_j-1}(-1)^i c_{ji}(g_{ji}(t) - (F_ng_{ji}))(t)\right| \leqslant$$

$$\leqslant dn^{-\alpha/2}\left(\|z_n\|_C + \sum_{j=1}^{q+2}\sum_{i=0}^{m_j-1}|c_{ji}|\right) = dn^{-\alpha/2}\|x_n\|_X,$$

where d are some constant. Hence we get

$$\varepsilon^{(n)} \equiv \|A - A_n\|_{X_n\to Y} = O\{n^{-\alpha/2}\}. \tag{17}$$

Based on formula (17) and Lemma, from theorem 7 in [11] (Chap. 1) we obtain the desired assertion with bound (13). □

Remark 1. Since $C\{0,0;\overline{0},\overline{\tau}\} \equiv C(I)$ and $D\{0,0;\overline{0},\overline{\tau}\} \equiv C(I)$, with $p_1 = p_2 = m_j = 0$ $(j = \overline{1,q})$ from ETK (1) we obtain a Fredholm integral equation of the second kind in the space C, and the GCM on the base of Hermite-Fejer interpolation polynomials turns into the known variant of the collocation method [4] (Chap. 4), while $(Ty)(t) \equiv y(t)$, and $h(t,s) \equiv K(t,s)$. Therefore bound (13) agrees with the bound that corresponds to te second-order equation [4] (Chap. 4).

Remark 2. Since under the assumptions of Theorem the corresponding approximating operators A_n satisfy the condition

$$\|A_n^{-1}\| = O(1)\ (A_n^{-1} : Y_n \to X_n,\ n \geqslant n_0),$$

we conclude [11] (Chap. 1) that the direct method for Eq. (1) are stable with respect to small perturbations of initial data. Moreover, if ETK (1) is well-posed, then so are SLAE (12).

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Специальный вариант метода коллокации для одного класса интегральных уравнений третьего рода, основанный на интерполяционных полиномах Эрмита–Фейера

Светлана А. Соловьева

В статье предложен и обоснован специальный прямой метод приближенного решения уравнений третьего рода в пространстве обобщенных функций.

Ключевые слова: интегральные уравнения третьего рода, приближенное решение, пространство обобщенных функций, теоретическое обоснование.