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Construction of Measure by Given Projections

Shaturgun Sharakhmetov*
Tashkent State University of Economic,
Uzbekistantion, 49, Tashkent, 100003,
Uzbekistan

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In the paper the general form of absolutely continuous with respect to the Lebesgue measure charges with given projections together with an algorithm for the construction of such charges have been obtained.

Keywords: measure, absolutely continuous, algorithm of the construction.

In the work "Some unsolved problems in analysis and probability theory" the following problem has been posed: To describe the class of all compatible finite-dimensional distributions with given marginal distributions (see [1], problem 7). This problem was solved in [2–5] (see also [6–8]). This subject is closely connected also with the main problem of tomography and the theory of measure concerning construction of a measure by its given projections. In this paper we give a method of construction of all absolutely continuous and discrete charges (measures) by their given projections. To explain the obtained results we give a version of the theorem in the two-dimensional case:

Theorem. *Any absolutely continuous charge $m(A_1 \times A_2)$ given on the $\sigma-$algebra of subsets of $R \times R$ with projections $m_1(A_1),\ m_2(A_2)\ \ (m_1(R) = m_2(R) = 1)$ has the following form:*

$$m(A_1 \times A_2) = \int_{A_1}\int_{A_2} G_1(x,\ y)\,dxdy + \int_{A_1}\int_{A_2} G_2(x,\ y)\,dm_1(x)\,dm_2(y) -$$
$$-m_1(A_1)\left(\int_R\int\_{A_2} G_1(x,\ y)dxdy + \int_R\int_{A_2} G_2(x,\ y)\,dm_1(x)\,dm_2(y)\right) -$$
$$-m_2(A_2)\left(\int_{A_1}\int_R G_1(x,\ y)dxdy + \int_{A_1}\int_R G_2(x,\ y)\,dm_1(x)\,dm_2(y)\right) +$$
$$+m_1(A_1)\cdot m_2(A_2)\left(1 + \int_R\int_R G_1(x,\ y)dxdy + \int_R\int_R G_2(x,\ y)\,dm_1(x)\,dm_2(y)\right),$$

where G_1, G_2 are integrable functions (with respect to the Lebesgue measure and $m_1 \cdot m_2$, respectively.)

Thus, for a couple of functions G_1, G_2 we give a rule for construction of a charge with given projections. Conversely, any charge with projections m_1, m_2 can be represented in the form as above with some functions G_1, G_2.

This representation of charges can be used for construction and the description of the class of all measures with given projections, to recover measures possessing certain properties, and in problems of probability theory and tomography.

Assume that $f_i(x_i)$, are Lebesgue measurable and integrable functions on sets X_i, $i = \overline{1,\ ,n}$.

*sh_sharakhmetov@.mail.ru

On the Cartesian product $X_1 \times X_2 \times ... \times X_n$ introduce the following classes of functions:

$$L_1 = \left\{ G : \int_{X_1} \dots \int_{X_n} |G(x_1, \dots, x_n)| \, dx_1, \dots, dx_n < \infty \right\},$$

$$L_1(f_1, \dots, f_n) = \left\{ G : \int_{X_1} \dots \int_{X_n} \left| G(x_1, \dots, x_n) \prod_{k=1}^{n} f_k(x_k) \right| dx_1, \dots, dx_n < \infty \right\}.$$

For functions $G_1 \in L_1$ and $G_2 \in L_1(f_1, \dots, f_n)$, respectively, define the functions

$$\begin{aligned} U_1(x_1, \dots, x_n) = G_1(x_1, \dots, x_n) - \sum_{i=1}^{n} \prod_{\substack{k=1 \\ k \neq i}}^{n} f_k(x_k) \underset{\neq i}{\int \cdots \int} G_1(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} dx_k + \\ + (n-1) \prod_{k=1}^{n} f_k(x_k) \int_{X_1} \dots \int_{X_n} G_1(x_1, \dots, x_n) \prod_{k=1}^{n} dx_k, \\ U_2(x_1, \dots, x_n) = G_2(x_1, \dots, x_n) - \sum_{i=1}^{n} \underset{\neq i}{\int \cdots \int} G_2(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} f_k(x_k) \, dx_k + \\ + (n-1) \int_{X_1} \dots \int_{X_n} G_2(x_1, \dots, x_n) \prod_{k=1}^{n} f_k(x_k) \, dx_k, \end{aligned} \tag{1}$$

where $\underset{\neq i}{\int \cdots \int}$ denotes the integration with respect to $x_1, \dots x_{i-1}, x_{i+1} \dots x_n$ (i.e. the integral by x_i is absent).

Theorem 1. *Assume that on the σ–algebra of subsets of X_i, $i = 1 \dots, n$ $n \geqslant 2$, there are given absolute continuous charges*

$$m_i(A_i) = \int_{A_i} f_i(x_i) \, dx_i; \quad m_i(X_i) = 1.$$

Then for any set functions $G_1 \in L_1$, $G_2 \in L_1(f_1, \dots, f_n)$ defined on the σ–algebra of subsets of $X_1 \times X_2 \times \dots \times X_n$ the following charge

$$m(A_1 \times \dots \times A_n) = \int_{A_1} \cdots \int_{A_n} \left[U_1(x_1, \dots, x_n) + \prod_{k=1}^{n} f_k(x_k) (1 + U_2(x_1, \dots, x_n)) \right] \prod_{k=1}^{n} dx_k \tag{2}$$

is an absolutely continuous charge with projections m_i, $i = 1 \dots, n$. We denote the set of such charges by M. Let T be the set of all absolutely continuous charges with projections m_i, $i = 1 \dots, n$. It is clear that $M \subset T$.

The following theorem gives the converse inclusion $M \supset T$.

Theorem 2. *Let*

$$m(A_1 \times \dots \times A_n) = \int_{A_1} \cdots \int_{A_n} f(x_1, \dots, x_n) \prod_{k=1}^{n} dx_k$$

be an arbitrary charge given on the σ–algebra of subsets of $X_1 \times X_2 \times \dots \times X_n$ with projections

$$m_i(A_i) = \int_{A_i} f_i(x_i) \, dx_i; \quad m_i(X_i) = 1, i = 1, \dots, n.$$

Then there are functions (explicitly given by f) $G_1 \in L_1$, $G_2 \in L_1(f_1, \dots, f_n)$ such that m has the form (2). Thus, the set of all absolutely continuous charges with projections m_i, $i = 1, n$ coincides with the class M.

Summarizing Theorems 1 and 2, we get a general form of absolutely continuous charges, which gives an improvement of the Radon–Nikodim theorem.

Theorem 3. *Any absolutely continuous charge m, given on the $\sigma-$algebra of subsets of $X_1 \times X_2 \times \ldots \times X_n$ with projections*

$$m_i(A_i) = \int_{A_i} f_i(x_i)\, dx_i; \quad m_i(X_i) = 1,\ i = 1, \ldots, n,$$

has the form (2), where $G_1 \in L_1,\ G_2 \in L_1(f_1, \ldots, f_n)$.

For non-negative charges (measures) we obtain the following

Corollary 1. *Any absolutely continuous measure given on the $\sigma-$algebra of subsets of $X_1 \times X_2 \times \ldots \times X_n$ with projections $m_i(A_i) = \int_{A_i} f_i(x_i)\, dx_i;\ f_i(x_i) \geqslant 0,\ m_i(X_i) = 1$ has the following form*

$$m(A_1 \times \ldots \times A_n) = \int_{A_1} \cdots \int_{A_n} (1 + U_2(x_1, \ldots, x_n)) \prod_{k=1}^{n} f_k(x_k)\ dx_k$$

if and only if $U_2 \geqslant -1$.

Proofs of theorems.
The proof of Theorem 1 follows from the following lemmas.

Lemma 1. *Assume that on the $\sigma-$algebra of subsets of X_i there are given absolutely continuous charges*

$$m_i(A_i) = \int_{A_i} f_i(x_i)\, dx;$$

$$m_i(X_i) = 1,\ \ i = 1, \ldots, n, \quad n \geqslant 2.$$

Then for any $g_1 \in L_1,\ g_2 \in L_1(f_1, \ldots, f_n)$ satisfying the following conditions $i = 1, \ \ldots, \ n:$

$$\underbrace{\int \cdots \int}_{\neq i} g_1(x_1, \ \ldots, \ x_n) \prod_{\substack{k=1\\k\neq i}}^{n} dx_k = 0\,, \tag{3}$$

$$\underbrace{\int \cdots \int}_{\neq i} g_2(x_1, \ \ldots, \ x_n) \prod_{\substack{k=1\\k\neq i}}^{n} f_k(x_k)\, dx_k = 0 \tag{4}$$

the following set function on the $\sigma-$algebra of subsets of $X_1 \times X_2 \times \ldots \times X_n$;

$$m(A_1 \times \ldots \times A_n) = \int_{A_1} \cdots \int_{A_n} \left[g_1(x_1, \ldots, x_n) + \prod_{k=1}^{n} f_k(x_k)(1 + g_2(x_1, \ldots, x_n))\right] \prod_{k=1}^{n} dx_k$$

is an absolutely continuous charge with projections $m_i,\ i = 1 \ldots, n$.

Proof of the Lemma 1. We note that any set function given in the form of the Lebesgue integral with respect to some measure from a summable function is an absolutely continuous charge in relation to the same measure. Therefore the set function $m(A_1 \times \ldots \times A_n)$ is an absolutely continuous charge with respect to the Lebesgue measure. Let us prove now that the projections of this charge coincide with the given charges m_i. Indeed, for all $i = 1\ldots, n$ we have

$$m\left(X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n\right) =$$

$$= \int_{X_1} \cdots \int_{A_i} \cdots \int_{X_n} \left[g_1\left(x_1, \ldots, x_n\right) + \prod_{k=1}^{n} f_k\left(x_k\right)\left(1 + g_2\left(x_1, \ldots, x_n\right)\right)\right] \prod_{k=1}^{n} dx_k =$$

$$= \int_{A_1} \left[\underset{\neq i}{\int \cdots \int} g_1\left(x_1, \ldots, x_n\right) \prod_{\substack{k=1 \\ k\neq i}}^{n} dx_k + f_i\left(x_i\right) \underset{\neq i}{\int \cdots \int} \prod_{\substack{k=1 \\ k\neq i}}^{n} f_k\left(x_k\right) dx_k + \right.$$

$$\left. + f_i\left(x_i\right) \underset{\neq i}{\int \cdots \int} g_2\left(x_1, \ldots, x_n\right) \prod_{\substack{k=1 \\ k\neq i}}^{n} f_k\left(x_k\right) dx_k \right] dx_i =$$

$$= \int_{A_i} f_i\left(x_i\right) dx_i \underset{\neq i}{\int \cdots \int} \prod_{\substack{k=1 \\ k\neq i}}^{n} f_k\left(x_k\right) dx_k = m_i\left(A_i\right) \prod_{\substack{k=1 \\ k\neq i}}^{n} m_k\left(X_k\right) = m_i\left(A_i\right).$$

Lemma is proved. □

Lemma 2. *For any $G_1 \in L_1,\ G_2 \in L_1\left(f_1, \ldots, f_n\right)$ the functions U_1 and U_2 satisfy the conditions (3) and (4), respectively, for all $i = 1\ldots, n$.*

Proof of Lemma 2. We shall give the proof for $i = 1$; the remaining cases $i = 2, \ldots, n$, can be proved similarly. Integrating U_1 by x_n, we obtain

$$\int_{X_n} U_1\left(x_1, \ldots, x_n\right) dx_n = \int_{X_n} \left[G_1\left(x_1, \ldots,, x_n\right) - \right.$$

$$- \sum_{i=t}^{n-1} \prod_{\substack{k=1 \\ k\neq i}}^{n} f_k\left(x_k\right) \underset{\neq i}{\int \cdots \int} G_1\left(x_1, \ldots, x_n\right) \prod_{\substack{k=1 \\ k\neq i}}^{n} dx_k -$$

$$- \prod_{\substack{k=1 \\ k\neq n}}^{n} f_k\left(x_k\right) \underset{\neq i}{\int \cdots \int} G_1\left(x_1, \ldots, x_n\right) \prod_{\substack{k=1 \\ k\neq n}}^{n} dx_k +$$

$$\left. + (n-1) \prod_{k=1}^{n} f_k\left(x_k\right) \int_{X_1} \ldots \int_{X_n} G_1\left(x_1, \ldots, x_n\right) \prod_{k=1}^{n} dx_k \right] dx_n =$$

$$= \int_{X_n} G_1\left(x_1, \ldots, x_n\right) dx_n - \sum_{i=1}^{n-1} \prod_{\substack{k=1 \\ k\neq i}}^{n-1} f_k\left(x_k\right) \underset{\neq i}{\int \cdots \int} G_1\left(x_1, \ldots, x_n\right) \prod_{\substack{k=1 \\ k\neq i}}^{n} dx_k \times$$

$$\times \int_{X_n} f_n\left(x_n\right) dx_n - \prod_{k=1}^{n-1} f_n\left(x_n\right) \cdot \int_{X_1} \ldots \int_{X_n} G_1\left(x_1, \ldots, x_n\right) \prod_{k=1}^{n} dx_k +$$

$$+ (n-1) \prod_{k=1}^{n-1} f_k\left(x_k\right) \int_{X_1} \ldots \int_{X_n} G_1\left(x_1, \ldots, x_n\right) \prod_{k=1}^{n} dx_k \int_{X_n} f_n\left(x_n\right) dx_n =$$

$$= \int_{X_n} G_1(x_1, \ldots, x_n)\, dx_n - \sum_{i=1}^{n-1} \prod_{\substack{k=1\\k\neq i}}^{n-1} f_k(x_k) \int \cdots \int_{\neq i} G_1(x_1, \ldots, x_n) \prod_{\substack{k=1\\k\neq i}}^{n} dx_k +$$

$$+ (n-2) \prod_{k=1}^{n-1} f_k(x_k) \int_{X_1} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=1}^{n} dx_n.$$

Now integrating the last expression by x_{n-1}, we get

$$\int_{X_{n-1}} \int_{X_n} U_1(x_1, \ldots, x_n)\, dx_{n-1} dx_n =$$

$$= \int_{X_{n-1}} \left\{ \int_{X_n} G_1(x_1, \ldots, x_n)\, dx_n - \sum_{i=1}^{n-2} \left[\prod_{\substack{k=1\\k\neq i}}^{n-1} f_k(x_k) \int \cdots \int_{\neq i} G_1(x_1, \ldots, x_n) \prod_{\substack{k=1\\k\neq i}}^{n} dx_k - \right.\right.$$

$$\left. - \prod_{k=1}^{n-2} f_k(x_k) \int \cdots \int_{\neq i} G_1(x_1, \ldots, x_n) \prod_{\substack{k=1\\k\neq n-1}}^{n} dx_k \right] +$$

$$\left. + (n-2) \prod_{k=1}^{n-1} f_k(x_k) \int_{X_1} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=1}^{n} dx_k \right\} dx_{n-1} =$$

$$= \int_{X_{n-1}} \int_{X_n} G_1(x_1, \ldots, x_n)\, dx_{n-1} dx_n - \sum_{i=1}^{n-2} \prod_{\substack{k=1\\k\neq i}}^{n-2} f_k(x_k) \int \cdots \int_{\neq i} G_1(x_1, \ldots, x_n) \prod_{\substack{k=1\\k\neq i}}^{n} dx_k +$$

$$+ (n-3) \prod_{k=1}^{n-2} f_k(x_k) \int_{X_1} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=1}^{n} dx_n.$$

Further, successively integrating by $x_{n-2}, \ldots, x_3$, we get the following

$$\int_{X_3} \cdots \int_{X_n} U_1(x_1, \ldots, x_n) \prod_{k=3}^{n} dx_n = \int_{X_3} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=3}^{n} dx_k -$$

$$- f_2(x_2) \int_{X_2} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=2}^{n} dx_k - f_1(x_1) \int_{X_1} \int_{X_3} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{\substack{k=1\\k\neq 2}}^{n} dx_k +$$

$$+ f_1(x_1) f_2(x_2) \int_{X_1} \cdots \int_{Xn} G_1(x_1, \ldots, x_n) \prod_{k=1}^{n} dx_k.$$

Finally, integrating by x_2, we obtain

$$\int_{X_2} \cdots \int_{X_n} U_1(x_1, \ldots, x_n) \prod_{k=2}^{n} dx_k =$$

$$= \int_{X_2} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=2}^{n} dx_k - \int_{X_2} f_2(x_2)\, dx_2 \int_{X_2} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=2}^{n} dx_k -$$

$$- f_1(x_1) \int_{X_1} \cdots \int_{X_n} G_1(x_1, \ldots, x_n) \prod_{k=1}^{n} dx_k +$$

$$+f_1(x_1)\int_{X_n} f_2(x_2)\,dx_2\int_{X_1}\ldots\int_{X_n} G_1(x_1,\ \ldots,\ x_n)\prod_{k=1}^{n} dx_k = 0.$$

Thus the equality (3) is proved. We are going to prove (4). Since this proof is similar to the proof of (3), we omit some calculations.

$$\int_{X_n} U_2(x_1,\ldots,x_n)\,f_n(x_n)\,dx_n = \int_{X_n}\Big[G_2(x_1,\ldots,x_n) -$$

$$-\sum_{i=1}^{n-1}\int\underset{\neq i}{\cdots}\int G_2(x_1,\ \ldots,\ x_n)\prod_{\substack{k=1\\k\neq i}}^{n} f_k(x_k)\,dx_k - \int\underset{\neq i}{\cdots}\int G_2(x_1,\ \ldots,\ x_n)\prod_{\substack{k=1\\k\neq n}}^{n} f_k(x_k)dx_k +$$

$$+(n-1)\int_{X_1}\ldots\int_{X_n} G_2(x_1,\ \ldots,\ x_n)\prod_{k=1}^{n} f_k(x_k)dx_k\Big]\,f_n(x_n)\,dx_n =$$

$$= \int_{X_n} G_2(x_1,\ldots,x_n)\,f_n(x_n)\,dx_n -$$

$$-\int_{X_n} f_n(x_n)dx_n\sum_{i=1}^{n-1}\int\underset{\neq i}{\cdots}\int G_2(x_1,\ \ldots,\ x_n)\prod_{\substack{k=1\\k\neq i}}^{n} f_k(x_k)\,dx_k -$$

$$-\int_{X_1}\ldots\int_{X_n} G_2(x_1,\ \ldots,\ x_n)\prod_{k=1}^{n} f_n(x_n)\,dx_k +$$

$$+(n-1)\int_{X_n} f_n(x_n)\,dx_n\int_{X_1}\ldots\int_{X_n} G_2(x_1,\ \ldots,\ x_n)\prod_{k=1}^{n} f_k(x_k)\,dx_k =$$

$$= \int_{X_n} G_2(x_1,\ldots,x_n)\,f_n(x_n)\,dx_n - \sum_{i=1}^{n-1}\int\underset{\neq i}{\cdots}\int G_2(x_1,\ \ldots,\ x_n)\prod_{\substack{k=1\\k\neq i}}^{n} f_k(x_k)\,dx_k +$$

$$+(n-2)\int_{X_1}\ldots\int_{X_n} G_2(x_1,\ \ldots,\ x_n)\prod_{k=1}^{n} f_k(x_k)dx_k.$$

As above, integrating successively by $x_{n-1},\ \ldots,\ x_3$, we obtain

$$\int_{X_3}\ldots\int_{X_n} U_2(x_1,\ \ldots,\ x_n)\prod_{k=3}^{n} f_k(x_k)\,dx_k =$$

$$= \int_{X_3}\ldots\int_{X_n} G_2(x_1,\ \ldots,\ x_n)\prod_{k=3}^{n} f_k(x_k)\,dx_k - \sum_{i=1}^{2}\int\underset{\neq i}{\cdots}\int G_2(x_1,\ \ldots,\ x_n)\prod_{\substack{k=1\\k\neq i}}^{n} f_k(x_k)\,dx_k -$$

$$-\int_{X_1}\ldots\int_{X_n} G_2(x_1,\ \ldots,\ x_n)\prod_{k=1}^{n} f_k(x_k)\ dx_k.$$

Finally, integrating the last equality by x_2, we complete the proof of the equality (4). Lemma 2 is proved, consequently, Theorem 1 is proved as well. □

Proof of Theorem 2. Let

$$m(A_1\times\ldots\times A_n) = \int_{A_1}\cdots\int_{A_n} f(x_1,\ldots,x_n)\prod_{k=1}^{n} dx_k$$

be an absolutely continuous charge with projections

$$m_i\left(A_i\right)=\int_{A_i} f_i\left(x_i\right) d x_i ; m_i\left(X_i\right)=1.$$

Construct functions U_1, U_2 as explained above. Taking in Theorem 1 (in definition of U_2)

$$G_2\left(x_1, \ldots, x_n\right)=\left(\frac{f\left(x_1, \ldots, x_n\right)}{\prod\limits_{k=1}^n f_k\left(x_k\right)}-1\right) \prod_{k=1}^n I\left(f_k\left(x_k\right) \neq 0\right), \tag{5}$$

i.e.

$$G_2=\begin{cases}\dfrac{f}{\prod\limits_{k=1}^n f_k}-1, \ if \prod\limits_{k=1}^n f_k\left(x_k\right) \neq 0, \\ 0, \ if \prod\limits_{k=1}^n f_k\left(x_k\right)=0,\end{cases}$$

where $I(\cdot)$ is the indicator function. Then

$$\underset{\neq i}{\int \cdots \int} G_2\left(x_1, \ \ldots, \ x_n\right) \prod_{\substack{k=1 \\ k \neq i}}^n f_k\left(x_k\right) d x_k=$$

$$=\left(\frac{I\left(f_i\left(x_i\right) \neq 0\right)}{f_i\left(x_i\right)}\right) \underset{\neq i}{\int \cdots \int} f\left(x_1, \ \ldots, \ x_n\right) \prod_{\substack{k=1 \\ k \neq i}}^n I\left(f_k\left(x_k\right) \neq 0\right) d x_k-$$

$$-I\left(f_i\left(x_i\right) \neq 0\right) \underset{\neq i}{\int \cdots \int} \prod_{\substack{k=1 \\ k \neq i}}^n f_k\left(x_k\right) I\left(f_k\left(x_k\right) \neq 0\right) d x_k .$$

We note that the expression $\dfrac{I\left(f_i \neq 0\right)}{f_i}$ is correct by definition of G_2 and

$$\frac{I\left(f_i \neq 0\right)}{f_i}=\begin{cases}\dfrac{1}{f_i}, if \quad f_i \neq 0, \\ 0, if \quad f_i=0 .\end{cases}$$

From conditions $\int_{X_k} f_k\left(x_k\right) d x_k=1, \ k=1, \ldots, n$, it follows that

$$\int_{X_k} f_k\left(x_k\right) I\left(f_k \neq 0\right) d x_k=1 .$$

Hence

$$\underset{\neq i}{\int \cdots \int} G_2\left(x_1, \ \ldots, \ x_n\right) \prod_{\substack{k=1 \\ k \neq i}}^n f_k\left(x_k\right) d x_k=$$

$$=\left(\frac{I\left(f_i \neq 0\right)}{f_i\left(x_i\right)}\right) \underset{\neq i}{\int \cdots \int} f\left(x_1, \ \ldots, \ x_n\right) \prod_{\substack{k=1 \\ k \neq i}}^n I\left(f_i \neq 0\right) d x_k-I\left(f_i \neq 0\right) .$$

Substituting these expressions in the definition of U_2, we obtain

$$U_2(x_1, \dots, x_n) = \left(\frac{f(x_1, \dots, x_n)}{\prod_{k=1}^{n} f_k(x_k)} - 1 \right) \prod_{k=1}^{n} I(f_k(x_k) \neq 0) -$$

$$- \sum_{i=1}^{n} \left[\frac{I(f_i \neq 0)}{f_i} \int \underset{\neq i}{\cdots} \int f(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} I(f_k(x_k) \neq 0) dx_k - I(f_i \neq 0) \right] +$$

$$+ (n-1) \left[\int_{X_1} \cdots \int_{X_n} f(x_1, \dots, x_n) \prod_{k=1}^{n} I(f_k(x_k) \neq 0) dx_k - 1 \right],$$

$$(1 + U_2(x_1, \dots, x_n)) \prod_{k=1}^{n} f_k(x_k) = \tag{6}$$

$$= \prod_{k=1}^{n} f_k + f(x_1, \dots, x_n) \prod_{k=1}^{n} I(f_k \neq 0) - \prod_{k=1}^{n} f_k I(f_k \neq 0) -$$

$$- \sum_{i=1}^{n} I(f_i \neq 0) \prod_{\substack{k=1 \\ k \neq i}}^{n} f_k(x_k) \int \underset{\neq i}{\cdots} \int f(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} I(f_k(x_k) \neq 0) dx_k +$$

$$+ \sum_{i=1}^{n} I(f_i \neq 0) f_i \prod_{\substack{k=1 \\ k \neq i}}^{n} f_k(x_k) +$$

$$+ (n-1) \prod_{k=1}^{n} f_k(x_k) \cdot \int_{X_1} \cdots \int_{X_n} f(x_1, \dots, x_n) \prod_{k=1}^{n} I(f_k(x_k) \neq 0) dx_k - (n-1) \prod_{k=1}^{n} f_k.$$

Now taking into account the equality $f_i I(f_i(x_i) \neq 0) = f_i(x_i)$, $i = 1, \dots, n$, we get

$$(1 + U_2(x_1, \dots, x_n)) \prod_{k=1}^{n} f_k = \prod_{k=1}^{n} f_k + f(x_1, \dots, x_n) \prod_{k=1}^{n} I(f_k \neq 0) -$$

$$- \sum_{i=1}^{n} I(f_i \neq 0) \prod_{\substack{k=1 \\ k \neq i}}^{n} f_k(x_k) \int \underset{\neq i}{\cdots} \int f(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} I(f_k(x_k) \neq 0) dx_k +$$

$$+ (n-1) \prod_{k=1}^{n} f_k(x_k) \cdot \int_{X_1} \cdots \int_{X_n} f(x_1, \dots, x_n) \prod_{k=1}^{n} I(f_k(x_k) \neq 0) dx_k.$$

Now we choose

$$G_1 = f - (1 + U_2) \prod_{k=1}^{n} f_k. \tag{7}$$

According to the equality

$$m(X_1 \times X_2 \times \ldots \times A_i \times \ldots \times X_n) = m(A_i) = \int_{A_i} f_i(x_i) dx_i$$

and Lemma 2, we have

$$\int \underset{\neq i}{\cdots} \int G_1(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} dx_k = \int \underset{\neq i}{\cdots} \int f(x_1, \dots, x_n) \prod_{\substack{k=1 \\ k \neq i}}^{n} dx_k -$$

$$-\int \cdots \int_{\neq i} \prod_{k=1}^{n} f(x_k)\, dx_k - \int \cdots \int_{\neq i} U_2(x_1,\ \ldots,\ x_n) \prod_{\substack{k=1 \\ k\neq i}}^{n} f_k(x_k)\, dx_k = f_i - f_i - 0 = 0.$$

Then by definition of U_1 we get

$$U_1 = G_1. \tag{8}$$

Consequently, by construction (see (7), (8))

$$U_1 + (1+U_2)\prod_{k=1}^{n} f_k = \left(f - (1+U_2)\prod_{k=1}^{n} f_k\right) + (1+U_2)\prod_{k=1}^{n} f_k = f. \tag{9}$$

Thus, given the charge

$$m(A_1 \times \ldots \times A_n) = \int_{A_1} \cdots \int_{A_n} f(x_1,\ \ldots,\ x_n) \prod_{k=1}^{n} dx_k$$

we have constructed functions G_2, G_1 respectively, and U_2, U_1 (see (5)–(9)) such that

$$f = U_1 + (1+U_2)\prod_{k=1}^{n} f_k.$$

The proof of Theorem 2 is completed. □

The proof of Corollary in a simplified form ($U_1 = 0$) repeats the proofs of Theorems 1, 2 and therefore we omit it.

Remark 1. Let $G_1', G_1'' \in L_1$, and $G_2', G_2'' \in L_1(f_1,\ \ldots,\ f_n)$. Then each of the following set functions is an absolute continuous charge

$$m' = \int_{A_1} \ldots \int_{A_n} \left(U'_1 + \prod_{k=1}^{n} f_k\left(1 + U_2{}'\right)\right) dx_1 \ldots dx_n$$

and

$$m'' = \int_{A_1} \ldots \int_{A_n} \left(U'_1 + \prod_{k=1}^{n} f_k\left(1 + U''_2\right)\right) dx_1 \ldots dx_n.$$

Moreover, it is easy to see that

$$m' = m'' \Leftrightarrow \left(U'_1 + U'_2 \prod_{k=1}^{n} f_k = U''_1 + U''_2 \prod_{k=1}^{n} f_k\right).$$

Consequently, the same charge can correspond to distinct couples $(U_1,\ U_2)$, but if in the vectors $(U'_1,\ U'_2)$ and $(U''_1,\ U''_2)$ at least one component is the same, for example, in the case $U_1' = U_1'' = 0$ (see Corollary), then the charge is uniquely defined by the second components.

Remark 2 (Discrete case). If X_i, $i = 1, \ldots, n$, are finite or countable sets, then in the theorems above replacing the integration by corresponding sums

$$m_i(A_i) = \sum_{x_1 \in A_1} f_i(x_i); \quad A_i \subset X_i, \quad m_i(X_i) = 1, \quad i = 1, \ldots, n$$

we get the general form of discrete charges with given projections.

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Построение меры по заданным проекциям

Шотургун Шарахметов

В статье дан способ построения всех абсолютно непрерывных и дискретных зарядов (мер) с заданными проекциями.

Ключевые слова: мера, абсолютная непрерывность, алгоритм построения.