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On Distance-Regular Graphs with $\lambda = 2$

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V.P. Burichenko and A.A. Makhnev have found intersection arrays of distance-regular graphs with $\lambda = 2$, $\mu > 1$, having at most 1000 vertices. Earlier, intersection arrays of antipodal distance-regular graphs of diameter 3 with $\lambda \leqslant 2$ and $\mu = 1$ were obtained by the second author. In this paper, the possible intersection arrays of distance-regular graphs with $\lambda = 2$ and the number of vertices not greater than 4096 are obtained.

Keywords: distance-regular graph, nearly n-gon.

Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices, that are at a distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighborhood of the vertex a*.

We denote by k_a the *degree of a vertex a*, i. e. the number of vertices in $[a]$. A graph Γ is said to be *regular with degree k*, if $k_a = k$ for every vertex a of Γ. A graph Γ is called a *strongly regular graph with parameters* (v, k, λ, μ), if Γ is regular with degree k on v vertices, in which every edge is placed in precisely λ triangles, and for any two non-adjacent triangles and any non-adjacent vertices a, b one has $|[a] \cap [b]| = \mu$. A graph with a diameter d is called *antipodal*, if the relation on the set of its vertices – to coincide or to be at a distance d – is an equivalence relation. Classes of this relation are called the *antipodal classes*.

If vertices u, w are at a distance i in Γ, then we denote by $b_i(u, w)$ (by $c_i(u, w)$) the number of vertices in the intersection of $\Gamma_{i+1}(u)$ (of $\Gamma_{i-1}(u)$) with $[w]$. A graph Γ of diameter d is said to be *distance-regular with the intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}$, if the values of $b_i(u, w)$, $c_i(u, w)$ do not depend on the choice of vertices u and w separated by a distance i in Γ, and are equal to b_i, c_i for $i = 0, ..., d$. Let $a_i = k - b_i - c_i$. Note that a distance-regular graph is amply regular with $k = b_0$, $\lambda = k - b_1 - 1$ and $\mu = c_2$, by definition $c_1 = 1$. Further, we denote by $p^l_{ij}(x, y)$ the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x, y that are at a distance l in the graph Γ. In a distance-regular graph, the numbers $p^l_{ij}(x, y)$ are independent of the choice of the vertices x, y; they are denoted by p^l_{ij} and are called the intersection numbers of the graph Γ.

V. P. Burichenko and A. A. Makhnev found [1] the intersection arrays for distance-regular graphs with $\lambda = 2$, $\mu > 1$, such that the number of vertices is not greater than 1000.

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Note here that the arrays $\{9,6,3;1,2,3\}$ of Hemming's graph $H(3,4)$ with $v = 64$, and $\{19,16,15,9;1,2,3,4\}$ of Hemming's graph $H(4,4)$ with $v = 256$, and the array $\{45,42,1;1,14,45\}$ were omitted from the consideration of [1]. However, there is an additional array $\{13,10,7;1,2,7\}$ (according to [2], a graph with such an intersection array should not exist). In [3], there were found intersection arrays for antipodal distance-regular graphs of diameter 3 with $\lambda \leqslant 2$ and $\mu = 1$. In the present paper, the possible intersection arrays of distance-regular graphs with $\lambda = 2$ and 4096 vertices at most are obtained.

Theorem. *Let Γ be a distance-regular graph with $\lambda = 2$, $\mu = 1$, having 4096 vertices at most. Then Γ has one of the following intersection arrays:*

(1) $\{21,18;1,1\}$ *($v = 400$);*

(2) $\{6,3,3,3;1,1,1,2\}$ *(Γ is a generalized octagon of order $(3,1)$, $v = 160$)*, $\{6,3,3;1,1,2\}$ *(Γ is a generalized hexagon of order $(3,1)$, $v = 52$)*, $\{12,9,9;1,1,4\}$ *(Γ is a generalized hexagon of order $(3,3)$, $v = 364$)*, $\{6,3,3,3,3,3;1,1,1,1,1,2\}$ *(Γ is a generalized dodecagon of order $(3,1)$, $v = 1456$);*

(3) $\{18,15,9;1,1,10\}$ *($v = 1+18+270+243 = 532$, Γ_3 is a strongly regular graph);* $\{21,18,12,4;1,1,6,21\}$ *($v = 1+21+378+756+144 = 1300$, $q^4_{3,4} = 0$).*

Corollary. *Let Γ be a distance-regular graph of diameter greater than 2, with $\lambda = 2$, and having at most 4096 vertices. Then one of the following assertions holds:*

(1) *Γ is a primitive graph with the intersection array* $\{6,3,3;1,1,2\}$, $\{9,6,3;1,2,3\}$, $\{12,9,9;1,1,4\}$, $\{15,12,6;1,2,10\}$, $\{18,15,9;1,1,10\}$, $\{19,16,8;1,2,8\}$, $\{24,21,3;1,3,18\}$, $\{33,30,15;1,2,15\}$, $\{35,32,8;1,2,28\}$, $\{42,39,1;1,1,42\}$, $\{51,48,8;1,4,36\}$;

(2) *Γ is an antipodal graph with $\mu = 2$ and the intersection array* $\{2r+1,2r-2,1;1,2,2r+1\}$, $r \in \{3,4,...,44\} - \{10,16,28,34,38\}$ *and* $v = 2r(r+1)$;

(3) *Γ is an antipodal graph with $\mu \geqslant 3$ and the intersection array* $\{15,12,1;1,4,15\}$, $\{18,15,1;1,5,18\}$, $\{27,24,1;1,8,27\}$, $\{35,32,1;1,4,35\}$, $\{45,42,1;1,6,45\}$, $\{42,39,1;1,3,42\}$, $\{63,60,1;1,4,63\}$, $\{75,72,1;1,12,75\}$, $\{99,96,1;1,4,99\}$, $\{108,105,1;1,5,108\}$, $\{143,140,1;1,20,143\}$, $\{147,144,1;1,16,147\}$, $\{171,168,1;1,12,171\}$;

(4) *Γ is a primitive graph with the intersection array* $\{6,3,3,3;1,1,1,2\}$, $\{19,16,15,9;1,2,3,4\}$, $\{21,18,12,4;1,1,6,21\}$, $\{15,12,9,6,3;1,2,3,4,5\}$, $\{6,3,3,3,3,3;1,1,1,1,1,2\}$, $\{18,15,12,9,6,3;1,2,3,4,5,6\}$.

We note that only arrays of some generalized polygons, Hemming's graphs $H(n,4)$, two graphs with $\mu = 1$, the array $\{33,30,15;1,2,15\}$, and arrays of antipodal graphs of diameter 3 have been added to the list of Burichenko and Makhnev.

Now we prove the Theorem. Let Γ be a distance-regular graph of diameter d with $\lambda = 2$, $\mu = 1$, having 4096 vertices at most. Let a be a vertex in the graph Γ and $k_i = |\Gamma_i(a)|$. Then $[a]$ is the union of $t+1$ isolated 3-cliques, $k = 3(t+1)$ and $t \leqslant 20$. Otherwise, $v > 1+66+66\cdot 63$, a contradiction.

Lemma 1. *The following assertions hold:*

(1) *if the diameter of Γ is 2, then Γ possesses the parameters $(400,21,2,1)$;*

(2) *if Γ is a generalized $2n$-gon, then Γ has the intersection array from the Corollary.*

Proof. If the diameter of Γ is equal to 2, then, according to [5], Γ has the parameters $(400, 21, 2, 1)$. Assume that the diameter of Γ is greater than 2.

Let Γ be a regular almost n-gon. Then $s = 3$, and in accordance with [4, Theorem 6.4.1] we have $b_i = k - 3c_i$ for $i = 0, 1, ..., d-1$, $k \geqslant 3c_d$, here $n = 2d$ if $k = 3c_d$, and $n = 2d+1$ if not. If Δ is a pointwise graph of a generalized polygon of order (s, t), then $k_i = s^i t^{i-1}(t+1)/c_i$. In the case of $n = 6$, the number of its vertices is $(s+1)(s^2t^2 + st + 1)$. Therefore $v = 4(9t^2 + 3t + 1)$ and $t \leqslant 10$. If $t > 1$, then, in view of [4, Theorem 6.5.1], the number st is a square, hence $t = 3$. If $n = 8$ and $t > 1$, then, according to [4, Theorem 6.5.1], the number $2st$ is a square, and so $t \geqslant 6$ and $v > 4096$, a contradiction. If $n = 12$, then $t = 1$ and $v = 1 + 6 + 18 + \cdots = 1456$. $\square$

Lemma 2. *Let Γ be not a generalized $2n$-gon. Then the following assertions hold:*

(1) *if the diameter of Γ is 3, then Γ has the intersection array $\{18, 15, 9; 1, 1, 10\}$;*

(2) *if the diameter of Γ is greater than 4, then $k \leq 45$.*

Proof. Let the diameter of Γ be equal to 3.

If $k = 63$, then Γ has the intersection array $\{63, 60, b_2; 1, 1, c_3\}$, $b_2 \leqslant 4$ and c_3 divides $3^3 140 b_2$. In any case, there is no valid intersection array. In a similar way one considers the cases $57 \leqslant k \leqslant 30$.

If $k = 27$, then Γ has the intersection array $\{27, 24, b_2; 1, 1, c_3\}$, c_3 divides $3^4 8 b_2$. Here arise interesting intersection arrays $\{27, 24, 8; 1, 1, 16\}$, $v = 1000$ with integer eigenvalues 7, 2, -5, but 2 and -5 have fractional multiplicity, and $\{27, 24, 4; 1, 1, 24\}$, $v = 784$ with integer eigenvalues 6, -1, -5, where 6 and -5 have fractional multiplicity. In all cases, there is no admissible intersection array.

If $k = 24$, then Γ has intersection array $\{24, 21, b_2; 1, 1, c_3\}$, c_3 divides $3^2 56 b_2$. Interesting intersection array $\{24, 21, 11; 1, 1, 18\}$, $v = 837$ with integer eigenvalues 6, -3, -7 arise, but 6 and -7 have fractional multiplicity, and there is also $\{24, 21, 7; 1, 1, 18\}$, $v = 725$ with integer eigenvalues 6, -1, -5, but 6 and -5 have fractional multiplicity. In any case, there is no admissible intersection array.

If $k = 21$, then Γ has the intersection array $\{21, 18, b_2; 1, 1, c_3\}$, c_3 divides $3^3 14 b_2$. There arises an interesting intersection array $\{21, 18, 10; 1, 1, 12\}$, $v = 715$ with integer eigenvalues 6, -1, -5, but -1 and -5 have fractional multiplicity. In any case, there is no admissible intersection array.

If $k = 18$, then Γ has the intersection array $\{18, 15, b_2; 1, 1, c_3\}$, c_3 divides $3^3 10 b_2$. There arise interesting intersection arrays $\{18, 15, 13; 1, 1, 6\}$, $v = 874$ with integer eigenvalues 6, -1, -5, having fractional multiplicity, $\{18, 15, 5; 1, 1, 18\}$, $v = 364$ with integer eigenvalues 5, -3, -6, but 5 and -6 have fractional multiplicity, and the array $\{18, 15, 9; 1, 1, 10\}$ with the spectrum 18^1, $(1+\sqrt{105})/2^{171}$, -1^{189}, $(1-\sqrt{105})/2^{171}$. There are no other admissible intersection arrays.

If $k = 15$, then Γ has the intersection array $\{15, 12, b_2; 1, 1, c_3\}$, c_3 divides $3^2 20 b_2$. There arise interesting intersection arrays $\{15, 12, 8; 1, 1, 10\}$, $v = 340$ with integer eigenvalues 5, -2, -5, but where 5 and -2 have fractional multiplicity, and $\{15, 12, 6; 1, 1, 10\}$, $v = 304$ with integer eigenvalues 5, -1, -4, but 5 and -4 have fractional multiplicity. In any case, there are no admissible intersection arrays.

If $k = 12$, then Γ has the intersection array $\{12, 9, b_2; 1, 1, c_3\}$, c_3 divides $3^3 4 b_2$. There arise interesting intersection arrays $\{12, 9, 3; 1, 1, 6\}$, $v = 175$ with integer eigenvalues 5, 2, -3, but 5 and -3 are with fractional multiplicity, and $\{12, 9, 1; 1, 1, 12\}$, $v = 130$ with integer eigenvalues 4, -1, -3, but 4 and -3 have fractional multiplicity. In any case, there are no admissible intersection arrays.

If $k = 9$, then Γ has the intersection array $\{9, 6, b_2; 1, 1, c_3\}$, c_3 divides $3^3 2 b_2$. There arises an interesting intersection array $\{9, 6, 4; 1, 1, 6\}$, $v = 100$ with integer eigenvalues 4, -1, -3, but 4 and -3 have fractional multiplicity. In any case, there are no admissible intersection arrays.

If $k = 6$, then Γ has the intersection array $\{6, 3, b_2; 1, 1, c_3\}$, c_3 divides $3^2 2b_2$. An interesting intersection array $\{6, 3, 1; 1, 1, 6\}$, $v = 28$ with integer eigenvalues 3, -1, -2 arises here, but 3 and -2 have fractional multiplicity. In any case, there are no admissible intersection arrays.

Assertion (1) is proved.

Let now the diameter of Γ be greater than 4. Then $b_i \geqslant c_{5-i}$ and $k_3 \geqslant k_2$. It follows that $4096 \geqslant v \geqslant 2(1 + k + k(k-3))$, and taking into account the divisibility of k by 3, we see that $k \leqslant 45$. The Lemma is proved. □

Let the diameter of Γ be greater than 3, and Γ be not a generalized $2n$-gon. Considering admissible intersection arrays with $\lambda = 2$ from [4], we obtain only the array $\{21, 18, 12, 4; 1, 1, 6, 21\}$. The Theorem is thus proved. □

Let us prove the Corollary. If Γ is not an antipodal graph of diameter 3, then considering admissible intersection arrays with $\lambda = 2$ from [4], we obtain only the arrays from the Corollary.

Lemma 3. *If Γ is an antipodal graph of diameter 3 with $\lambda = \mu = 2$, then Γ has the intersection array $\{2r + 1, 2r - 2, 1; 1, 2, 2r + 1\}$, $r \in \{3, 4, ..., 44\} - \{10, 16, 28, 34, 38\}$.*

Proof. By the assumption, Γ has the intersection array $\{2r + 1, 2r - 2, 1; 1, 2, 2r + 1\}$ and $v = r(2r + 2)$ vertices. If $r \geqslant 45$, then $v \geqslant 4 \cdot 45 \cdot 23$, a contradiction with $v \leqslant 4096$. In view of [4, Proposition 1.10.5], if r is even, then $k = 2r + 1$ is the sum of squares of two integers, therefore $r \in \{3, 4, ..., 44\} - \{10, 16, 28, 34, 38\}$. The Lemma is proved. □

In Lemmata 4–9 it is supposed that Γ is an antipodal graph of diameter 3 with $\lambda = 2 < \mu$. Therefore, Γ has the spectrum $k^1, n^f, -1^k, -m^g$, where $n, -m$ are integers, that are the roots of the equation $x^2 - (\lambda - \mu)x - k = 0$, $f = m(r-1)(k+1)/(m+n)$, $g = n(r-1)(k+1)/(m+n)$ and $r = (k + \mu - 3)/\mu$. If $r = 2$, then Γ is Taylor's graph and $\mu = k - 3$. In this case, $k = 6$, $n = 2$, $m = 3$, a contradiction with the fact that $f = 3 \cdot 7/5$. Consequently, $r > 2$, and the condition $q^3_{33} \geqslant 0$ gives $m \leqslant n^2$.

Lemma 4. *If $\mu \leqslant 5$, then Γ has one of the following intersection arrays:*

(1) $\{42, 39, 1; 1, 3, 42\}$*;*

(2) $\{4u^2 - 1, 4u^2 - 4, 1; 1, 4, 4u^2 - 1\}$*, $u \in \{2, 3, 4, 5\}$;*

(3) $\{18, 15, 1; 1, 5, 18\}$ *or* $\{108, 105, 1; 1, 5, 108\}$*.*

Proof. Let $\mu = 3$. Then $4k + 1 = (2n + 1)^2$, and so, $k = n(n + 1)$, $m = n + 1$ and $r = k/3$. If $n = 3s$, then $f = (3s + 1)(3s^2 + s - 1)(9s^2 + 3s + 1)/(6s + 1)$. In this case, $(6s + 1, 9s^2 + 3s + 1)$ divides 3 and $(6s + 1, 3s^2 + s - 1) = (6s + 1, s - 2)$ divides 13, therefore, $s = 2$ and Γ has the intersection array $\{42, 39, 1; 1, 3, 42\}$.

If $n = 3s - 1$, then $f = 3s(3s^2 - s - 1)(9s^2 - 3s + 1)/(6s - 1)$. In this case, $(6s - 1, 9s^2 - 3s + 1)$ divides 3 and $(6s - 1, 3s^2 - s - 1) = (6s - 1, s + 2)$ divides 13, consequently, $s = 11$, a contradiction with the fact that 5 does not divide $33 \cdot 351 \cdot 1057$.

Let $\mu = 4$. Then $r = (k + 1)/4$, $k + 1 = 4u^2$, and so, $k = 4u^2 - 1$, $n = 2u - 1$ and $m = 2u + 1$. Further, $f = (2u + 1)4u^2(u^2 - 1)/(4u)$, $g = (2u - 1)u(u^2 - 1)$ and $v = 4u^4 \leqslant 4096$, therefore, Γ has the intersection array $\{4u^2 - 1, 4u^2 - 4, 1; 1, 4, 4u^2 - 1\}$, $u \in \{2, ..., 5\}$.

Let $\mu = 5$. Then $r = (k + 2)/5$, $4k + 9 = (2u + 1)^2$, and hence, $k = u^2 + u - 2$, $n = u - 1$ and $m = u + 2$. Further, $f = (u + 2)((u^2 + u)/5 - 1)(u^2 + u - 1)/(2u + 1)$, $(2u + 1, u + 2)$ divides 3 and $(u^2 + u - 1, 2u + 1) = (u - 2, 2u + 1)$ divides 5.

If $u = 5s$, then $(10s + 1, 5s^2 + s - 1) = (10s + 1, s - 2)$ divides 21. In this case, $10s + 1$ divides 63, therefore, $s = 2$ and Γ has the intersection array $\{108, 105, 1; 1, 5, 108\}$.

If $u = 5s - 1$, then $(10s - 1, 5s^2 - s - 1) = (10s - 1, s + 2)$ divides 21. In this case $10s - 1$ divides 63, hence $s = 1$, and Γ has the intersection array $\{18, 15, 1; 1, 5, 18\}$. □

Lemma 5. *If $6 \leqslant \mu \leqslant 8$, then Γ has one of the following intersection arrays:*

(1) $\{45, 42, 1; 1, 6, 45\}$;

(2) $\{27, 24, 1; 1, 8, 27\}$.

Proof. Let $\mu = 6$. Then $r = (k+3)/6$, $k+4 = (2u+1)^2$, and so, $k = 4u^2+4u-3$, $n = 2u-1$ and $m = 2u+3$. Further, $f = (2u+3)(2u^2+2u-1)((4u^2+4u)/6-1)/(2u+1)$, $(2u+1, 4u^2+4u-2) = (2u+1, 2u-2)$ divides 3.

If $u = 3s$, then $f = (6s+3)(18s^2+12s-2)(6s^2+2s-1)/(6s+1)$. In this case, $(6s+1, 6s^2+2s-1) = (6s+1, s-1)$ divides 7, therefore $6s+1$ divides 21, $s = 1$ and Γ has the intersection array $\{45, 42, 1; 1, 6, 45\}$.

If $u = 3s-1$, then $f = (6s+1)(18s^2-6s-1)(6s^2-2s-1)/(6s-1)$. In this case $(6s-1, 6s^2-2s-1) = (6s-1, s+1)$ divides 7 and $6s-1$ divides 21, a contradiction.

Let $\mu = 7$. Then $r = (k+4)/7$, $4k+25 = (2u+1)^2$, hence $k = u^2+u-6$, $n = u-2$ and $m = u+3$. Further, $f = (u+3)((u^2+u-2)/7-1)(u^2+u-5)/(2u+1)$, $(2u+1, u+3)$ divides 5 and $(2u+1, u^2+u-5) = (2u+1, u-5)$ divides 11.

If $u = 7s+1$, then $(14s+3, 7s^2+3s-1) = (14s+3, 3s-2)$ divides 37. In this case, $14s+3$ divides $5 \cdot 11 \cdot 37$, a contradiction.

If $u = 7s+5$, then $(14s+11, 7s^2+11s+3) = (14s+11, 11s+6)$ divides 37. In this case $14s+11$ divides $5 \cdot 11 \cdot 37$, a contradiction.

Let $\mu = 8$. Then $r = (k+5)/8$, $k+9 = 4u^2$, therefore $k = 4u^2-9$, $n = 2u-3$ and $m = 2u+3$. Further, $f = (2u+3)((u^2-1)/2-1)(u^2-2)/u$, $(u, 2u+3)$ divides 3 and (u^2-2, u) divides 2. Consequently, $u = 2s+1$, $(2s+1, 2s^2+2s-1)$ divides 3 and $2s+1$ divides 9, and so, $s = 1$ and Γ has the intersection array $\{27, 24, 1; 1, 8, 27\}$. □

Lemma 6. *If $9 \leqslant \mu \leqslant 11$, then there is no admissible intersection array.*

Proof. Let $\mu = 9$. Then $r = (k+6)/9$, $4k+49 = (2u+1)^2$, therefore $k = u^2+u-12$, $n = u-3$ and $m = u+4$. Further, $f = (u+4)(u^2+u-11)((u^2+u-6)/9-1)/(2u+1)$, $(2u+1, u+4)$ divides 7, and $(2u+1, u^2+u-11) = (2u+1, u-22)$ divides 45.

If $u = 9s+2$, then $(18s+5, 9s^2+5s-1) = (18s+5, 5s-2)$ divides 61. In this case, $18s+5$ divides $35 \cdot 61$, a contradiction.

If $u = 9s-3$, then $(18s-5, 9s^2-5s-1) = (18s-5, 5s+2)$ divides 61. In this case, $18s-5$ divides $35 \cdot 61$, a contradiction.

Let $\mu = 10$. Then $r = (k+7)/10$, $k+16 = (2u+1)^2$, therefore $k = 4u^2+4u-15$, $n = 2u-3$ and $m = 2u+5$. Further, $f = (2u+5)(2u^2+2u-7)((2u^2+2u-4)/5-1)/(2u+1)$, $(2u+1, 2u+5)$ divides 4, and $(2u+1, 2u^2+2u-7)$ divides 15.

If $u = 5s+1$, then $(10s+3, 10s^2+6s-1) = (10s+3, 3s-1)$ divides 19. In this case, $10s+3$ divides 57, a contradiction.

If $u = 5s+3$, then $(10s+7, 10s^2+14s+3) = (10s+7, 7s+3)$ divides 19. In this case, $10s+7$ divides 57, $s = 5$, $u = 28$, a contradiction with $v \leqslant 4096$.

Let $\mu = 11$. Then $r = (k+8)/11$, $4k+81 = (2u+1)^2$, therefore $k = u^2+u-20$, $n = u-4$ and $m = u+5$. Further, $f = (u+5)((u^2+u-12)/11-1)(u^2+u-19)/(2u+1)$, $(2u+1, u+5)$ divides 9 and $(u^2+u-19, 2u+1) = (u-38, 2u+1)$ divides 77.

If $u = 11s+3$, then $(22s+7, 11s^2+7s-1) = (22s+7, 7s-2)$ divides 93, and $22s+7$ divides $27 \cdot 7 \cdot 31$, a contradiction with the fact that $v \leq 4096$.

If $u = 11s-4$, then $(22s-7, 11s^2-7s-1) = (22s-7, 7s+2)$ divides 93, and so, $22s-7$ divides $27 \cdot 7 \cdot 31$, that contradicts with $v \leqslant 4096$. □

Lemma 7. *If $12 \leqslant \mu \leqslant 14$, then Γ has either the intersection array $\{75, 72, 1; 1, 12, 75\}$ or the intersection array $\{171, 168, 1; 1, 12, 171\}$.*

Proof. Let $\mu = 12$. Then $r = (k+9)/12$, $k+25 = 4u^2$, therefore $k = 4u^2-25$, $n = 2u-5$ and $m = 2u+5$. Further, $f = (2u+5)((u^2-4)/3-1)(u^2-6)/u$, $(2u+5, u)$ divides 5, and (u^2-6, u) divides 6.

If $u = 3s+1$, then $(3s+1, 3s^2+2s-2) = (3s+1, s-2)$ divides 7 and $3s+1$ divides 70, hence $s = 2$ and Γ has the intersection array $\{171, 168, 1; 1, 12, 171\}$.

If $u = 3s-1$, then $(3s-1, 3s^2-2s-2) = (3s-1, s+2)$ divides 7 and $3s-1$ divides 70, and so, $s = 2$ and Γ has the intersection array $\{75, 72, 1; 1, 12, 75\}$.

Let $\mu = 13$. Then $r = (k+10)/13$, $4k+121 = (2u+1)^2$, therefore $k = u^2+u-30$, $n = u-5$ and $m = u+6$. Further, $f = (u+6)((u^2+u-20)/13-1)(u^2+u-29)/(2u+1)$, $(2u+1, u+6)$ divides 11, and $(2u+1, u^2+u-29) = (2u+1, u-58)$ divides 117.

If $u = 13s+4$, then $(26s+9, 13s^2+9s-1) = (26s+9, 9s-2)$ divides 133 and $26s+9$ divides $99 \cdot 133$, a contradiction with that $v \leq 4096$.

If $u = 13s-5$, then $(26s-9, 13s^2-9s-1) = (26s-9, s+2)$ divides 61, and $26s-9$ divides $99 \cdot 61$, a contradiction with $v \leqslant 4096$.

Let $\mu = 14$. Then $r = (k+11)/14$, $k+36 = (2u+1)^2$, therefore $k = 4u^2+4u-35$, $n = 2u-5$ and $m = 2u+7$. Further, $f = (2u+7)((2u^2+2u-12)/7-1)(2u^2+2u-17)/(2u+1)$, $(2u+1, 2u+7)$ divides 6 and $(2u^2+2u-17, 2u+1) = (u-17, 2u+1)$ divides 35.

If $u = 7s+2$, then $(14s+5, 14s^2+10s-1) = (14s+5, 5s-1)$ divides 39 and $14s+5$ divides $15 \cdot 39$, a contradiction with the condition $v \leq 4096$.

If $u = 7s-3$, then $(14s-5, 14s^2-4s-1) = (14s-5, s-1)$ divides 9, and so, $14s-5$ divides 135, $s = 1$, $n = 3$, $m = 15$, a contradiction with $m \leq n^2$. □

Lemma 8. *If* $15 \leqslant \mu \leqslant 17$*, then* Γ *has the intersection array* $\{147, 144, 1; 1, 16, 147\}$.

Proof. Let $\mu = 15$. Then $r = (k+12)/15$, $4k+169 = (2u+1)^2$, therefore $k = u^2+u-42$, $n = u-6$ and $m = u+7$. Further, $f = (u+7)((u^2+u-30)/15-1)(u^2+u-41)/(2u+1)$, $(2u+1, u+7)$ divides 13 and $(u^2+u-41, 2u+1) = (u-82, 2u+1)$ divides 165.

If $u = 15s$, then $(30s+1, 15s^2+s-3) = (30s+1, s-6)$ divides 181 and $30s+1$ divides $11 \cdot 13 \cdot 181$, a contradiction with the condition $v \leqslant 4096$.

If $u = 15s-1$, then $(30s-1, 15s^2-s-3) = (30s-1, s+6)$ divides 181 and $30s-1$ divides $11 \cdot 13 \cdot 181$, a contradiction with $v \leqslant 4096$.

If $u = 15s+5$, then $(30s+11, 15s^2+11s-1) = (30s+11, 11s-2)$ divides 181 and $30s+11$ divides $11 \cdot 13 \cdot 181$, a contradiction with the condition $v \leqslant 4096$.

If $u = 15s-6$, then $(30s-11, 15s^2-11s-1) = (30s-11, 11s+2)$ divides 181 and $30s-11$ divides $11 \cdot 13 \cdot 181$, a contradiction with $v \leqslant 4096$.

Let $\mu = 16$. Then $r = (k+13)/16$, $k+49 = 4u^2$, therefore $k = 4u^2-49$, $n = 2u-7$ and $m = 2u+7$. Further, $f = (2u+7)((u^2-9)/4-1)(u^2-12)/u$, $(2u+7, u)$ divides 7 and (u, u^2-12) divides 12. Consequently, $u = 2s+1$, $(2s+1, s^2+s-3) = (2s+1, s-6)$ divides 13 and $2s+1$ divides $21 \cdot 13$, hence, $s = 3$ and Γ has the intersection array $\{147, 144, 1; 1, 16, 147\}$.

Let $\mu = 17$. Then $r = (k+14)/17$, $4k+225 = (2u+1)^2$, therefore $k = u^2+u-56$, $n = u-7$ and $m = u+8$. Further, $f = (u+8)((u^2+u-42)/7-1)(u^2+u-55)/(2u+1)$, $(2u+1, u+8)$ divides 15 and $(u^2+u-55, 2u+1) = (u-110, 2u+1)$ divides 221. Hence, $u = 7s-1$, $(14s-1, 7s^2-s-7) = (14s-1, s+14)$ divides 197 and $14s-1$ divides $15 \cdot 221 \cdot 197$, a contradiction with the condition $v \leqslant 4096$. □

Lemma 9. *If* $18 \leqslant \mu \leqslant 20$*, then* Γ *has the intersection array* $\{143, 140, 1; 1, 20, 143\}$.

Proof. Let $\mu = 18$. Then $r = (k+15)/18$, $k+256 = (2u+1)^2$, hence $k = 4u^2+4u-255$, $n = 2u-6$ and $m = 2u+8$. Further, $f = (u+4)((2u^2+2u-120)/9-1)(4u^2+4u-255)/(2u+1)$, $(2u+1, u+4)$ divides 7 and $(4u^2+4u-255, 2u+1) = (2u-255, 2u+1)$ divides 256.

If $u = 9s+2$, then $(18s+5, 18s^2+10s-13) = (18s+5, 5s-13)$ divides 259 and $18s+5$ divides $7 \cdot 259$, a contradiction with $v \leqslant 4096$.

If $u = 9s-3$, then $(18s-5, 18s^2-10s-13) = (18s-5, 5s+13)$ divides 259 and $18s-5$ divides $7^2 \cdot 37$, a contradiction with $v \leqslant 4096$.

Let $\mu = 19$. Then $r = (k+16)/19$, $4k+289 = (2u+1)^2$, therefore $k = u^2+u-72$, $n = u-8$ and $m = u+9$. Further, $f = (u+9)((u^2+u-56)/19-1)(u^2+u-71)/(2u+1)$, $(2u+1, u+9)$ divides 17 and $(2u+1, u^2+u-71)$ divides 285.

If $u = 19s+7$, then $(38s+15, 19s^2+15s-1) = (38s+15, 15s-2)$ divides 301 and $38s+15$ divides $15 \cdot 17 \cdot 301$, a contradiction with $v \leqslant 4096$.

If $u = 19s-8$, then $(38s-15, 19s^2-15s-1) = (38s-15, 15s+2)$ divides 301 and $38s-15$ divides $15 \cdot 17 \cdot 301$, a contradiction with $v \leqslant 4096$.

Let $\mu = 20$. Then $r = (k+17)/20$, $k+81 = 4u^2$, hence $k = 4u^2+4u-81$, $n = 2u-9$ and $m = 2u+9$. Further, $f = (2u+9)((u^2+u-16)/5-1)(u^2+u-20)/u$, $(2u+9, u)$ divides 9 and (u^2+u-20, u) divides 20. Consequently, $u = 5s+2$, $(5s+2, 5s^2+5s-3) = (5s+2, 3s-3)$ divides 21 and $5s+2$ divides $21 \cdot 36$, therefore $s = 1$ and Γ has the intersection array $\{143, 140, 1; 1, 20, 143\}$. The Lemma is proven. □

Computer calculations show that there is no admissible intersection array in the case $\mu \geqslant 21$. The Theorem, and also the corresponding Corollary with it, are thus proven.

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О дистанционно–регулярных графах с $\lambda = 2$

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В.П. Буриченко и А.А. Махнев нашли массивы пересечений дистанционно регулярных графов с $\lambda = 2$, $\mu > 1$ и числом вершин не большим 1000. Ранее вторым автором найдены массивы пересечений антиподальных дистанционно-регулярных графов диаметра 3 с $\lambda \leqslant 2$ и $\mu = 1$. В данной статье найдены возможные массивы пересечений дистанционно-регулярных графов с $\lambda = 2$ и не более 4096 вершинами.

Ключевые слова: дистанционно-регулярный граф, почти n-угольник.