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Groups with Given Element Orders

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This paper is a survey of some results and open problems about the structure of (mostly infinite) periodic groups with a given set of element orders. It is based on a talk of authors given on the conference "Algebra and Logic: Theory and Application" dedicated to the 80-th anniversary of V. P. Shunkov (Krasnoyarsk, July 21–27, 2013).

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To the memory of Vladimir Petrovich Shunkov

Introduction

In late October 1900, William Burnside wrote a letter to Robert Fricke which contained in particular the following lines:

"I take the opportunity of asking you, whether the following question has ever presented itself to you; and if it has, whether you have come to any conclusion about it.

Can a group, generated by a *finite* number of operations, and such that the order of every one of its operations is finite and less than an assigned integer, consist of an infinite number of operations.

E.g. as a very particular case: — If S_1 & S_2 are operations, and if Σ, representing in turn any and every combination or repetition of S_1 & S_2, such as $S_1^a S_2^b S_1^c ... S_2^e$, is such that $\Sigma^m = 1$, where m is a given integer, is the group generated by S_1 & S_2 a group of finite order or not. Of course if m is 2, the group is of order 4 and if m is 3 the group is of order 27; but for values of m greater than 3, the question seems to me to present serious difficulties however one looks at it." (Cited by [1].)

In the subsequent letter to Fricke dated June 20, 1901 Burnside wrote:

"I have recently returned to the question I wrote you about in the winter, viz. that of the discontinuous group defined by $S^m = 1$ when S represents any and every combination of n independent generating operations $A_1, A_2, ... A_n$; and m, n are given integers. I find that when $m = 3$ and n is given, the order can be determined by a kind of recurring formula... For $m = 4, n = 2$ I find 2^{12} for the order... So far I am quite baffled by the case $n = 2, m = p$, a prime greater than 3; but it is easy to shew that the order cannot be less than p^{p+2} and I think it is probably greater, if finite at all." (Cited by [1].)

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We see that in these letters, speaking contemporary language, Burnside is interested in two questions:

Question 1. Let n be a natural number and G a group such that the order of every element in G is less than or equal to n. Is G locally finite?

Question 2. Let n be a natural number and G a group satisfying the identity $x^n = 1$. Is G locally finite?

Of course, these questions are not the same: Question 2 is a particular case of Question 1, but, for example, for $n = 6$ the answer to Question 2 is positive [2] and Question 1 has not been resolved so far, since it includes in particular Question 2 for $n = 5$ which is open.

Question 1 includes also the following

Question 3. Let ω be a fixed finite set of natural numbers. Suppose that the set of element orders of G coincides with ω. Is G locally finite?

In 1902 Burnside published the paper [3] in which he discussed Question 2. Over time it became known as Burnside Problem on groups of exponent n. To this problem vast literature is devoted (see references in [4]). Question 1 is mentioned in the book [5] with a reference to Burnside.

We will focus on Question 3, touching Question 1 and Question 2 only to the extent which is consistent with our main goal.

1. Burnside problem on groups of given exponent

A group G is said to be *periodic* if, for every $g \in G$ there exists a natural n (depending on g) such that $g^n = 1$; if there exists a common n with $g^n = 1$ for all $g \in G$ then n is called a *period* of G and the smallest such n is said to be the *exponent* of G. A group G is *locally finite* if every finite set of its elements is contained in a finite subgroup. A group G is *nilpotent* of class at most n is for all $x_1, \ldots, x_n \in G$ the identity $[x_1, x_2, \ldots, x_n] = 1$ holds where $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$ and $[x_1, x_2, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ for $n \geqslant 3$.

The class C_n of groups of period n is a *variety*, i.e. it is closed under taking subgroups, factor groups and cartesian products. Let $B(d, n)$ be the free group of C_n with d generators.

1.1. Groups of exponents 2 and 3

In [3] Burnside noted an obvious fact that a group of exponent 2 is locally finite, and showed that this is also true for groups of exponent 3. He also proved that in groups of exponent 3 every couple of conjugate elements commute, i. e. in such groups the 2-Engel identity $[[y, x], x] = 1$ holds. In a sequel [6] of work [3], which was issued a year later, Burnside proved that every 2-Engel group satisfies the identities $[[x, y], z] = [[y, z], x]$ and $[[x, y], z]^3 = 1$, thus it is nilpotent of nilpotency class 2, in the case when it has no elements of order 3. Apparently, C. Hopkins [7] was first to prove that a 2-Engel group is nilpotent of class 3, in particular such are groups of exponent 3. This result is usually ascribed to F. Levi [8], although he published his work 13 years later than Hopkins.

In 1932 Levi and B. van der Waerden [9] repeated Burnside's result that a group of exponent 3 is 2-Engel. They also gave an estimate for the order of a group of exponent 3 with d generators.

The final result is the following

Theorem 1.1. *1. A group of exponent 2 is elementary Abelian and hence locally finite;*
$|B(d, 2)| = 2^d$.

  2. Let G be a group of exponent 3. Then

   (a) G is 2-Engel and nilpotent of class at most 3;

(b) $B(2,3)$ is nilpotent of class 2 and $|B(2,3)| = 27$;

(c) for $d \geqslant 3$ the nilpotency class of $B(d,3)$ is exactly 3 and $|B(d,3)| = 3^k$ where

$$k = \frac{6d + 3d(d-1) + d(d-1)(d-2)}{6}.$$

1.2. Groups of exponent 4

Burnside showed in [3] that $|B(2,4)| \leqslant 2^{12}$. I. Sanov [10] proved that every group of exponent 4 is locally finite and gave a (crude) estimate for $|B(d,4)|$. With the increase of the number d of generators the derived length of $B(d,4)$ grows with no limit [11].

More information on groups of exponent 4 can be gleaned from the comments to the book of A. I. Kostrikin [12].

1.3. Groups of exponent 5

The question on local finiteness of groups of exponent 5 is not resolved yet.

Conjecture 1.1. $B(2,5)$ *is infinite.*

An informtaion on finite groups of exponent 5 see in [12].

1.4. Groups of exponent 6

In 1956 P. Hall and G. Higman published their famous article [13], which equipped mathematicians with new powerfull tools for the study of finite groups. In particular, it moved M. Hall to write down his work [2], where he proved that a group of exponent 6 is locally finite. This result together with results from the article by Hall and Higman imply the following

Theorem 1.2. *Let G be a group of exponent 6. Then*

1. *A Sylow 2-subgroup of $G/O_3(G)$ is normal in $G/O_3(G)$.*

2. *A Sylow 3-subgroup of $G/O_2(G)$ is normal in $G/O_2(G)$.*

3. *The derived length of G is at most 4.*

4. *If G is d-generated then*

$$|G| \leqslant 2^a 3^{b+b(b-1)/2+b(b-1)(b-2)/6},$$

where

$$a = 1 + (d-1)3^{d+d(d-1)/2+d(d-1)(d-2)/6},$$

$$b = 1 + (d-1)2^d,$$

and this bound is exact.

Note that later M. Newman [14] shortened essentially Hall's proof, reducing it to some statement which he checked with the help of a computer. I. G. Lysenok [15] in 1987 cleaned Newman's proof of computer computations.

1.5. Negative solution

In 1959 P. S. Novikov [16] announced the existence of infinite, finitely generated periodic groups of finite period. On the ground of this note Novikov and S. I. Adian in 1968 wrote a large article [17–19] with a proof that there exists an infinite m-generated group of period n for every $m \geqslant 2$ and every odd $n \geqslant 4381$. In the book of Adian [20], the bound $n \geqslant 4381$ was decreased to $n \geqslant 665$.

Existence of non locally finite groups of finite exponent of the form 2^t was announced in 1992 independantly by S.V.Ivanov and Lysenok. Their works [21] and [22] with full proofs were issued in 1994 and 1996 respectively. In particular, Lysenok's work contains a proof of existence of infinite m-generated groups of period n for every $m \geqslant 2$ and $n \geqslant 8000$.

2. Groups with given set of element orders

Let G be a periodic group. The set $\omega(G)$ consisting of all element orders of G is called the *order spectrum* or, simply, *spectrum* of G. If $\omega(G)$ is finite then it is uniquely defined by its subset $\mu(G)$ consisting of maximal under divisibility condition elements of $\omega(G)$.

2.1. Groups with $\omega(G) = \{1, 2, 3\}$

B. H. Neumann [23] was the first who turned to Question 3. His result states the following.

Theorem 2.1. *Suppose that $\omega(G) = \{1, 2, 3\}$. Then G is locally finite and one of the following conditions holds:*

1. *G is a semidirect product of normal elementary Abelian 2-subgroup V and a group of order 3 acting freely on V;*

2. *G is a semidirect product of normal elementary Abelian 3-subgroup V and a group of order 2 inverting every element of V by conjugation in G.*

Remind that if a group G acts on a non-trivial group V then this action is called *free* action (or G acts *freely* on V) if, for every non-trivial $g \in G$ and every non-trivial $v \in V$, the image v^g of v under action of g is not equal to v. A group $\langle t \rangle$ of order 2 inverts a group V on which it acts, if $v^t = v^{-1}$ for every $v \in V$.

Corollary 2.1 (An answer to Question 1 for $n = 3$). *Let G be a group in which the order of every element is at most 3. Then G is locally finite.*

Indeed, this follows from Theorems 1.1, 1.2 and 2.1.

2.2. Groups with elements of order at most 4

In his work about groups of exponent 4, Sanov [10] proved that a group G with $\omega(G) = \{1, 2, 3, 4\}$ is locally finite. D. V. Lytkina [24] later found out the structure of such groups.

Theorem 2.2. *Let G be a group with $\mu(G) = \{3, 4\}$, then G is soluble of length at most 3 and only one of the following cases is possible:*

1. *$G = VQ$, where V is a non-trivial normal elementary abelian 3-subgroup, Q is a 2-group, which acts freely on V and is isomorphic either to a cyclic group of order 4, or to quaternion group of order 8;*

2. $G = T\langle a \rangle$, where T is a normal nilpotent 2-subgroup of nilpotency class 2, and the order of a equals 3;

3. $G = TS$, where T is an elementary abelian normal 2-subgroup, and S is isomorphic to a symmetric group of degree 3.

2.3. Free action

Let G be an automorphism group of a non-trivial group V. An automorphism $a \in G$ of order n is said to be *splitting* if $vv^a \cdots v^{a^{n-1}} = 1$ for every $v \in V$. In other words, a is a splitting automorphism of V if $(va^{-1})^n = 1$ for all $v \in V$ in the natural semidirect product VG.

In this subsection we collect results on groups which admit a splitting automorphism of small order, or contain such an automorphism.

Theorem 2.3 ([25,26]). *Let V be a group admitting an automorphism a of order 3.*

1. *If $\langle a \rangle$ acts freely on V and $V = \{v^a v^{-1} \mid v \in V\}$ then V is nilpotent of class at most 2 [25];*

2. *If a is splitting then V is nilpotent of class at most 3 [26].*

Theorem 2.4 ([26]). *Let G be a non-trivial periodic group acting freely on an Abelian group and $x \in G$ is an element of order 3. Then either x lies in the center of G or $\langle x^G \rangle$ is isomorphic to $SL_2(3)$ or $SL_2(5)$. In every case, the center of G is non-trivial.*

Theorem 2.5 ([27,28]). *Let G be a group of exponent 5 acting freely on an Abelian group. Then G is cyclic.*

2.4. $\{2,3\}$-groups acting freely on Abelian groups

In this subsection we list the results of [29]. The interest in the relevant subject was raised by the paper [30] which contains a proof of the fact that a $\{2,3\}$-group G of finite exponent $2^m 3^2$ which acts freely on Abelian group is locally finite. Later on Lytkina [31] showed that this result is also true without presupposition that the exponent of a Sylow 2-subgroup of G is finite.

Locally cyclic group G is a group such that every finite subset of G is contained in a cyclic subgroup. *Locally cyclic primary group* is either a finite cyclic p-group, where p is a prime, or isomorphic to a group of the type p^∞, i.e. a group

$$C(p) = \langle a_0, a_1, \ldots \mid a_0^p = 1, a_{i+1}^p = a_i \text{ for } i \geqslant 0 \rangle.$$

Quaternion group is either the quaternion group Q_8 of order 8 or generalized quaternion group, i.e. a group isomorphic to the group

$$Q_{2^{m+1}} = \langle a, b \mid a^{2^m} = b^4 = 1, \ a^b = a^{-1}, \ b^2 = a^{2^{m-1}} \rangle, \quad m \geqslant 3.$$

Locally quaternion group is a 2-group G such that every finite subset of G is contained in a quaternion subgroup. By a known result of V. P. Shunkov [32] locally quaternion group either is a quaternion group or is isomorphic to a group

$$Q(2) = \langle C(2), b \mid b^2 = a_0, \ a_i^b = a_i^{-1} \rangle.$$

Locally quaternion group contains exactly one involution (i.e. an element of order 2), which generates the centre of this group.

The quaternion group Q_8 possesses an automorphism of order 3. The corresponding semidirect product of order 24 is isomorphic to the group $SL_2(3)$ of 2×2 matrices with determinant 1 over a field of order 3. Denote by $\tilde{S}_4$ the extension of a group of order 2 with the symmetric

group S_4 of degree 4, which has a quaternion Sylow 2-subgroup. Up to isomorphism there is one and only one group $\tilde{S}_4$ (e.g. in [33]). Note that all prime order elements of $\tilde{S}_4$ generate a subgroup of index 2, which is isomorphic to $SL_2(3)$.

A group G is called a *central product* of its subgroups A and B with the union by a subgroup C, if $G = AB$, $[A, B] = 1$ and $C = A \cap B$. Central product AB is said to be *non-trivial*, iff $A \neq G \neq B$. Obviously, $C \leqslant Z(G)$ and central product AB with the union by C is isomorphic to a quotient group of $A \times B$ by $C_1 = \{(c, c^{-1}) \mid c \in C\}$.

Let p be an odd prime and a be a positive integer. We say that an infinite p-group P is a *group of the type $Q(p^a)$* (or a $Q(p^a)$-type group), if P possesses the following properties:

1. $Z(P)$ is a cyclic group of order p^a.

2. Every finite subgroup of P is cyclic.

Note that groups of the type $Q(p^a)$ are not locally finite.

Let p be an odd prime and a be a positive integer. We say that a $\{2, p\}$-group H with an involution is a *group of the type $Q(p^a, d)$* (or a $Q(p^a, d)$-type group) if H possesses the following properties:

1. $Z(H)$ is locally cyclic.

2. Every 2-element of H belongs to $Z(H)$; all 2-elements form a group T of order 2^d (it is possible that $d = \infty$).

3. H/T is a group of the type $Q(p^a)$ for some positive integer a.

Lemma 2.1.   *1. For every odd prime p and a positive integer a there exists a $Q(p^a)$-type group.*

   2. For any odd prime p and integers a and d there exists a $Q(p^a, d)$-type group.

   3. For every odd prime p and a positive integer a there exists a $Q(p^a, \infty)$-type group.

Proof. Suppose that $n = p^s > 665$, $m \geqslant 2$ and $G = A(m, n)$ is a group from [20, ch. VII, §1]. As it is shown in this chapter, $Z(G) = \langle z \rangle$ is an infinite cyclic group, $Z(G) \leqslant G'$, $G/Z(G) \simeq B(m, n)$, and every two cyclic subgroups of G have non-trivial intersection.

1. Put $P = G/\langle z^{p^a} \rangle$ and show that P is a group of desired type. By [20, ch. VII, §1, Theorem 1.8] all finite subgroups of $B(m, n)$ which is not locally finite and is a m-generated free group of exponent $n \geqslant 665$ are cyclic. Hence, if K is a non-trivial finite subgroup of P, then its full preimage A in G is generated by z and some $x \in G \setminus Z(G)$. Due to the properties of the group $A(m, n)$ its subgroup $A = \langle z, x \rangle$ is an Abelian torsion-free group. Since every non-identity subgroup of A has non-trivial intersection with $\langle z \rangle$, then by the main theorem on finitely generated Abelian groups A is cyclic. Therefore, $A/\langle z^{p^a} \rangle$ is a cyclic p-group of order p^r, where $a < r \leqslant a + s$, and a subgroup $A/\langle z^{p^a} \rangle$ is generated by $x\langle z^{p^a} \rangle$ and contains a subgroup $\langle z \rangle/\langle z^{p^a} \rangle$. Thus $Z(P) = \langle z \rangle/\langle z^{p^a} \rangle$ is a cyclic group of order p^a and every finite subgroup of P is cyclic. This proves that P is a group of the type $Q(p^a)$.

2. Put $H = G/\langle z^{p^a 2^d} \rangle$. Then the same arguments as in item 1 show that H is a group of the type $Q(p^a, d)$.

3. Suppose that $n = p^t > 10^{10}$, $m \geqslant 2$, and H is a free group of rank m with the identity $x^n y = y x^n$. It follows from [34, Corollary 31.1 and Theorem 31.2] that there exists a quotient group G of H with the following properties:

1. $Z(G) \leqslant G'$, and $Z(G)$ is a free Abelian group of countable rank.

2. $G/Z(G) \simeq B(m, n)$.

3. There exists a subgroup A of $Z(G)$ such that $G/A \simeq A(m,n)$.

The last property implies that there exists $N \leq Z(G)$ such that G/N is a group of the type $Q(p^a)$. Now, $Z(G)$ is a free Abelian group of countable rank, therefore there exists $M \leq Z(G)$ such that $Z(G)/M \simeq C(2)$. Put $H = G/N \cap M$. It is clear that P is a group of the type $Q(p^a, \infty)$. $\qquad\square$

Theorem 2.6. *Let G be a $\{2,3\}$-group acting freely on an Abelian group. Then one of the following statements is true.*

1. *G is locally finite and isomorphic to one of the following groups: locally cyclic group; direct product of a locally cyclic 3-group and a locally quaternion group; semidirect product of a locally cyclic 3-group R and a cyclic 2-group $\langle b \rangle$, where $b^2 \neq 1$ and $a^b = a^{-1}$ for every $a \in R$; semidirect product of a locally cyclic 3-group R and a locally quaternion group Q, where $|Q : C_Q(R)| = 2$; semidirect product of the quaternion group $Q_8 = \langle x, y \rangle$ of order 8 and a cyclic 3-group $\langle a \rangle$, where $x^a = y$; group $\tilde{S}_4$.*

2. *G is not locally finite and all prime order elements of G generate a cyclic subgroup.*

Any of the groups mentioned can act frelly on some Abelian group.

The explicit description of groups from item 2 of Theorem 2.6 is given in the following theorem.

Theorem 2.7. *Let G be a not locally finite $\{2,p\}$-group, where p is an odd prime. All prime order elements of G generate a cyclic subgroup iff all 2-elements of G generate a 2-subgroup S, which is locally cyclic or locally quaternion. Besides, one of the following conditions is true*:

1. *$G = P \times S$, where P is a $Q(p^a)$-type group for some positive integer a (it is possible that $S = 1$);*

2. *S is a non-trivial locally cyclic group, and G is $Q(p^a, d)$-type for some positive integer a;*

3. *S is a locally cyclic group, and G is a non-trivial central product of S and a $Q(p^a, d)$-type group for some positive integers a and d with the union by a subgroup of order 2^d;*

4. *S is a locally quaternion group, and G is a central product of S and a $Q(p^a, 1)$-type group or some positive integer a with the union by a subgroup of order 2;*

5. *$S = Q_8$, $p = 3$, $|G : C_G(S)| = 3$ and G/S is a $Q(p^a)$-type group for some positive integer a. Here $C_G(S)$ is either a $Q(p^a, 1)$-group, or a central product of a $Q(p^a)$-type group and a group of order 2.*

3. $\{2,3\}$-groups without elements of order 6

In this section G allways denotes a $\{2,3\}$-group without elements of order 6. The following results are obtained in [35].

Theorem 3.1. *If G is locally finite then one of the following conditions holds:*

1. *$G = O_3(G)T$ where $O_3(G)$ is Abelian and T is a locally cyclic or locally quaternion group acting freely on $O_3(G)$.*

2. *$G = O_2(G)R$ where $O_2(G)$ is nilpotent of class at most 2 and R is a locally cyclic 3-group acting freely on $O_2(G)$.*

3. $G = O_2(G)D$ where D contains a subgroup R of index 2 and $O_2(G)R$ satisfies (2).

4. G is a 2-group or a 3-group.

In (1)–(3) G is soluble of derived length at most 4.

Here $O_p(G)$ where p is a prime is the largest normal p-subgroup of G. A 2-group Q is called locally quaternion if every finite subset of Q is contained in a finite subgroup of Q isomorphic to the generalized quaternion group. Note that an infinite locally quaternion group is isomorphic to a group

$$\langle a_i, b, i = 0, 1, 2, \cdots \mid a_0^2 = 1, a_{i+1}^2 = a_i \text{ for } i \geqslant 0, b^2 = a_0, a_i^b = a_i^{-1} \text{ for } i > 0\rangle.$$

Note also that there exist not locally finite groups G with $\mu(G) = \{2, 3^n\}$, where $n \geqslant 7$ [36] and $\mu(G) = \{2^m, 3\}$, where $m \geqslant 54$ [37]. In general case we manage to prove local finiteness under some additional conditions.

Theorem 3.2.   *1. If G contains a non-trivial normal p-subgroup ($p = 2$ or 3) then either G is locally finite, or G is a p-group, or G is an extension of a class 2 nilpotent 2-group by a 3-group which contains a unique subgroup of order 3.*

2. *If G contains an involution and no elements of order 27 then either G is locally finite, or G is an extension of a simple group by a 2-group.*

We say that, for a prime p, a periodic group H is p-isolated if and only if $\omega(G) = \{p\} \cup w'$, where none of the numbers from w' is divisible by p.

Theorem 3.3. *Let H be an arbitrary periodic 3-isolated group. Then one of the following holds:*

1. *H is an extension of a non-abelian simple group by a $3'$-group.*

2. *H is an extension of a non-trivial 3-group of period 3 by a $3'$-group.*

3. *H is an extension of a nilpotent group of class at most 2 by a group of order 3 or 6.*

4. *$H = NA$ where A is isomorphic to $A_5 \simeq SL_2(4)$ and N is the direct product of subgroups of order 16 every of which is invariant in G and isomorphic to the natural $SL_2(4)$-module of dimension 2 over a field of order 4.*

Theorem 3.4. *If period of G is equal to $72 = 2^3 \cdot 3^2$ then G is locally finite.*

This theorem generalizes previous results of B. Neumann [23], I. N. Sanov [10], V. D. Mazurov [37], A. Kh. Zhurtov and V. D. Mazurov [38] and E. Jabara and D. V. Lytkina [39] where local finiteness is shown of groups with $\mu(G)$ equal to $\{2, 3\}$, $\{4, 3\}$, $\{8, 3\}$, $\{2, 9\}$ and $\{4, 9\}$ respectively.

In fact, Theorem 3.4 can be generalized as follows.

Theorem 3.5. *Suppose that the order of the product of every two elements of orders at most 4 from G is at most 9. Assume also that, for every subgroup $H \leq G$ generated by an element of order 2 and an element of order 3, a maximal 2-subgroup of H is not an infinite locally cyclic subgroup.*

Then one of the following holds:

1. *$G = O_3(G)T$ where $O_3(G)$ is Abelian and T is a locally cyclic 2-group or a quaternion 2-group of order at most 16 and T acts freely on $O_3(G)$.*

2. *$G = O_2(G)R$ where $O_2(G)$ is nilpotent of class at most 2 and R is a 3-group having a unique subgroup of order 3 and acting freely on $O_2(G)$.*

3. *$G = O_2(G)D$ where D contains a locally cyclic subgroup R of index 2 and $O_2(G)R$ satisfies (2).*

4. *G is a 2-group or a 3-group.*

4. 2-isolated groups

Theorem 4.1 ([40]). *Suppose that G is a periodic 2-isolated group, i. e. $\omega(G) = \{2\} \cup \omega'$ and none of the elements of ω' is even. Then one of the following conditions holds:*

1. *G contains an Abelian subgroup A of index 2, $\omega(A) = \omega'$, and $a^x = a^{-1}$ for all $a \in A$ and $x \in G \setminus A$.*

2. *G contains an elementary Abelian normal 2-subgroup A and G/A acts freely on A by conjugation in G.*

3. *$G \simeq L_2(P)$ where P is a locally finite field of characteristic two.*

Conversely, every periodic group satisfying (1), (2) or (3) is 2-isolated.

Corollary 4.1.   1. *Suppose that $\mu(G) = \{2, m\}$ where $m = 5$ or 9. Then G is locally finite and either G contains an Abelian subgroup of index 2 and exponent m, or G is a semidirect product of elementary Abelian normal 2-subgroup A and a cyclic group of order m acting freely on A.*

2. *If $\mu(G) = \{2, 2^m - 1, 2^m + 1\}$ then $G \simeq L_2(2^m)$.*

Proof. In every case G is 2-isolated and hence satisfies the conditions of Theorem 4.1. Now Theorems 1.1, 2.4 and 2.5 show that conclusion of Corollary 4.1 is true.   □

Notice that the structure of groups with spectrum $\{1, 2, 5\}$ was studied in [14].

5. Groups with element orders at most 5

Let G be a periodic group.

Theorem 5.1 ([27, 41]). *If $\mu(G) = \{3, 5\}$ then one of the following holds:*

1. *$G = FT$ where F is a normal 5-subgroup of exponent 5 and nilpotency class at most 2 and T is a group of order 3 acting freely on F.*

2. *$G = TF$ where T is a normal 3-subgroup which is nilpotent of class at most 3 and F is a group of order 5 acting freely on T.*

In particular, G is locally finite.

Theorem 5.2 ([27, 41]). *If $\mu(G) = \{4, 5\}$ then one of the following holds:*

1. *$G = TD$ where T is a normal non-trivial elementary Abelian 2-subgroup and U is a non-Abelian subgroup of order 10.*

2. *$G = FT$ where F is an elementary Abelian normal 5-subgroup of G, T acts freely on F and is isomorphic to a subgroup of the quaternion group of order 8.*

3. *$G = TF$ where T is a normal 2-subgroup of nilpotency class at most 6 and F is a subgroup of order 5 acting freely on T.*

Theorem 5.3 ([42]). *If $\mu(G) = \{3, 4, 5\}$ then either $G \simeq A_6$ or $G = VC$ where $C \simeq A_5$ and V is an elementary Abelian normal 2-subgroup which is a direct product of C-invariant groups of order 16.*

Corollary 5.1. *Let G be a periodic group all of whose elements are of orders at most 5. Then either G is a 5-group or G is locally finite.*

6. Groups all of whose elements have orders at most 6

In this section G denotes a periodic group.

Theorem 6.1 ([43]). *If $\mu(G) = \{5, 6\}$ then G is a soluble locally finite group and one of the following holds:*

1. $G = FC$ *where F is a normal elementary Abelian 5-group and C is a cyclic group of order 6 acting freely on F.*

2. $G = TD$ *where T is a normal subgroup of exponent 3 and D is non-abelian of order 10.*

3. $G = (T \times U)F$ *where T is a normal subgroup of exponent 3, U is a normal subgroup of exponent 2 and F is a group of order 5 acting freely on $T \times U$.*

Theorem 6.2 ([44]). *If $\mu(G) = \{4, 6\}$ then G is locally finite.*

Theorem 6.3 ([45]). *If $\mu(G) = \{4, 5, 6\}$ then G is locally finite and one of the following conditions holds:*

1. $G = NA$ *where N is a normal non-trivial elementary Abelian 2-subgroup, A acts freely on N by conjugation and A is isomorphic to $SL_2(3)$ or to a non-abelian group of order 12 with a cyclic Sylow 2-subgroup.*

2. G *contains a non-trivial normal elementary Abelian 2-subgroup V and $G/V \simeq A_5$.*

3. G *is isomorphic to S_5 or S_6.*

Corollary 6.1. *If the order of every element in G is at most 6 then G is locally finite or a 5-group.*

7. Various results and conjectures

Let G be a periodic group.

Theorem 7.1 ([46]). *Suppose that $\mu(G) = \{3, 4, 7\}$. Then $G \simeq L_2(7)$.*

Let M_{10} be a group isomorphic to a point stabilizer of sporadic Mathieu group in its permutation representation of degree 11.

Theorem 7.2 ([47]). *If $\mu(G) = \{3, 5, 8\}$ then $G \simeq M_{10}$.*

Theorem 7.3 ([43, 48]). *If G is of exponent 12 and one of the conditions (a), (b) holds then G is locally finite.*
   (a) The order of the product of every two involutions is distinct from 4.
   (b) The order of the product of every two involutions is distinct from 6.

Conjecture 7.1. *Groups of exponent 12 are locally finite.*

Conjecture 7.2.   *1. If $\mu(G) = \{4, 5, 6, 7\}$ then $G \simeq A_7$.*

2. *If $\mu(G) = \{5, 6, 11\}$ then $G \simeq L_2(11)$.*

3. *If $\mu(G) = \{8, 9, 17\}$ then $G \simeq L_2(17)$.*

4. *If $\mu(G) = \{6, p\}$ where p is a prime, $p \geqslant 7$, then G is locally finite.*

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О группах с заданными порядками элементов

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В статье приводится обзор результатов и некоторых открытых проблем, связанных со структурой (как правило, бесконечных) периодических групп с заданным набором порядков элементов. Работа основана на докладе, представленном на конференции "Алгебра и логика: теория и приложения", посвящённой 80-летию В. П. Шункова (Красноярск, 21–27 июля 2013 г.).

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