Journal of Siberian Federal University. Mathematics & Physics 2013, 6(3), 349–356

УДК 517.958 + 534-18

Spectrum of One-dimensional Vibrations of a Layered Medium Consisting of a Kelvin-Voigt Material and a Viscous Incompressible Fluid

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Received 24.12.2012, received in revised form 11.02.2013, accepted 11.03.2013

The paper considers a mathematical model for natural vibrations of a periodic layered medium. The medium consists of a viscoelastic Kelvin-Voigt material and a viscous incompressible fluid. For the given model, two homogenized models are derived. They correspond to the cases of transverse and longitudinal vibrations of the layered medium. It is shown that the spectrum of each homogenized model is the union of roots of the corresponding quadratic equations.

Keywords: spectrum, layered medium, homogenized model, viscoelasticity, viscous fluid.

Introduction

In this paper the study initiated in [1] and [2] is continued. The paper is concerned with spectral properties of homogenized models of strongly inhomogeneous layered media. The motivation to study spectral properties of such models is one interesting experimental fact obtained in [3]. It was found that even a small amount of viscous fluid in pores of an elastic solid leads to a qualitative different spectral properties of a continuous elastic solid and an elastic solid saturated with fluid (see [3] for details). Therefore, it would appear natural that media consisting of viscoelastic and fluid components also have some interesting mechanical properties.

In the present paper we consider a mathematical model for natural vibrations of a periodic medium consisting of alternating layers of an isotropic viscoelastic Kelvin-Voigt material and a viscous incompressible fluid. For this medium two homogenized models are derived. They correspond to the cases of transverse and longitudinal vibrations of the layered medium. These homogenized models describe one-dimensional natural vibrations of some viscoelastic Kelvin-Voigt materials. We also show that the spectrum of each homogenized model is the union of roots of the corresponding quadratic equations. In order to compare results obtained for incompressible and compressible fluid layers, we briefly review the homogenized problems and their spectra given in [1], where fluid was supposed to be compressible.

The paper is organized as follows. In Section 1 we formulate an original mathematical model and derive the corresponding homogenized model. In Sections 2 and 3 we construct homogenized models corresponding to the cases of transverse and longitudinal vibrations, respectively. Then we study the spectral properties of these models.

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1. Mathematical models

Let $\Omega = (0,l)^3$ and d is a constant such that $0 < d < 1$. Let us denote

$$I^h = (0,(1-d)/2) \cup ((1+d)/2,1), \quad I^s = ((1-d)/2,(1+d)/2).$$

Then for a sufficiently small $\varepsilon > 0$ we define

$$I^h_\varepsilon = (0,l) \cap \left(\cup_{k\in\mathbb{Z}}(\varepsilon I^h + \varepsilon k)\right), \quad I^s_\varepsilon = (0,l) \cap \left(\cup_{k\in\mathbb{Z}}(\varepsilon I^s + \varepsilon k)\right),$$

$$\Omega^h_\varepsilon = I^h_\varepsilon \times (0,l) \times (0,l), \quad \Omega^s_\varepsilon = I^s_\varepsilon \times (0,l) \times (0,l).$$

Obviously, $\Omega = \Omega^h_\varepsilon \cup \Omega^s_\varepsilon \cup S_\varepsilon$ with $S_\varepsilon = \partial\Omega^h_\varepsilon \cap \partial\Omega^s_\varepsilon$. We assume that the set Ω^s_ε is occupied by a viscous incompressible fluid whereas the set Ω^h_ε is occupied by an isotropic viscoelastic Kelvin-Voigt material. In the sequel, the sets Ω^h_ε and Ω^s_ε are called the viscoelastic and the fluid parts (or layers) of Ω, respectively. Note that all viscoelastic and fluid layers of Ω are parallel to the x_2x_3-plane. Denoting $Y = (0,1)^3$ we see that the cube εY is the cell of periodicity of the combined medium Ω. In fact, the set $Y^h = I^h \times (0,1) \times (0,1)$ is the viscoelastic part of Y, and the set $Y^s = I^s \times (0,1) \times (0,1)$ is the fluid part of Y.

We now turn to the formulation of mathematical model for the cooperative motion of viscoelastic and fluid layers of Ω. Let us assume that positive constants ρ^h and ρ^s are the densities of the viscoelastic material and the fluid, respectively. Assume also that $f(x,t)$ is the force vector and $u^\varepsilon(x,t)$ is the displacement vector. The equations of motion in the viscoelastic part Ω^h_ε are as follows

$$\rho^h \frac{\partial^2 u^\varepsilon_i}{\partial t^2} = \frac{\partial \sigma^\varepsilon_{ij}}{\partial x_j} + f_i(x,t), \quad x \in \Omega^h_\varepsilon, \quad t > 0. \tag{1}$$

Here σ^ε_{ij} are the components of the stress tensor,

$$\sigma^\varepsilon_{ij} = a_{ijkh} e_{kh}(u^\varepsilon) + b_{ijkh} e_{kh}\left(\frac{\partial u^\varepsilon}{\partial t}\right), \quad x \in \Omega^h_\varepsilon,$$

and $e_{kh}(u^\varepsilon)$ are the components of the strain tensor,

$$e_{kh}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u^\varepsilon_k}{\partial x_h} + \frac{\partial u^\varepsilon_h}{\partial x_k}\right).$$

Since the viscoelastic material is isotropic the coefficients a_{ijkh} and b_{ijkh} are defined by

$$a_{ijkh} = \lambda_e \delta_{ij}\delta_{kh} + \mu_e(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \quad b_{ijkh} = \lambda_v \delta_{ij}\delta_{kh} + \mu_v(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}),$$

$$1 \leqslant i,j,k,h \leqslant 3,$$

where λ_e and μ_e are the elastic Lamé constants, λ_v and μ_v are their viscoelastic counterparts and δ_{ij} is the Kronecker symbol.

In the fluid part Ω^s_ε the equations of motion are the Stokes equations

$$\rho^s \frac{\partial^2 u^\varepsilon_i}{\partial t^2} = \frac{\partial \sigma^\varepsilon_{ij}}{\partial x_j} + f_i(x,t), \quad \operatorname{div} u^\varepsilon = 0, \quad x \in \Omega^s_\varepsilon, \quad t > 0, \tag{2}$$

with

$$\sigma^\varepsilon_{ij} = -\delta_{ij} p^\varepsilon + 2\mu \delta_{ik}\delta_{jh} e_{kh}\left(\frac{\partial u^\varepsilon}{\partial t}\right), \quad x \in \Omega^s_\varepsilon.$$

Here $p^\varepsilon(x,t)$ is the fluid pressure and μ is the fluid viscosity.

Besides, at the interface S_ε between viscoelastic and fluid parts of Ω the conditions of continuity of displacement and normal stress are imposed:

$$[u^\varepsilon]_{S_\varepsilon} = 0, \quad [\sigma^\varepsilon_{i1}]_{S_\varepsilon} = 0, \tag{3}$$

where $[\cdot]_{S_\varepsilon}$ denotes the jump across the boundary S_ε.

Finally, the problem is supplemented by homogeneous initial and Dirichlet boundary conditions:

$$u^\varepsilon(x,0) = 0, \quad \frac{\partial u^\varepsilon}{\partial t}(x,0) = 0, \quad x \in \Omega, \tag{4}$$

$$u^\varepsilon(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{5}$$

Remark 1.1. *In general, the continuity of the normal stress takes the form* $[\sigma^\varepsilon_{ij} n_j]_{S_\varepsilon} = 0$, *where* n_j, $j = 1,2,3$ *are the components of the unit normal to* S_ε. *Since every layer of* Ω *is parallel to the* x_2x_3*-plane, the unit normal to* S_ε *is ether* $n = (1,0,0)$ *or* $n = (-1,0,0)$. *This explains the form of the second boundary condition in* (3).

To formulate the homogenized problem that corresponds to the original problem (1)–(5) we define the pairs $\{Z^{kh}(y), B^{kh}(y)\}$, $\{D^{kh}(y), A^{kh}(y)\}$, and $\{W^{kh}(y,t), S^{kh}(y,t)\}$. They are solutions of the following auxiliary problems:

$$\begin{cases} \dfrac{\partial \sigma^{(1)}_{ij}}{\partial y_j} = 0, \ y \in Y; \quad \operatorname{div} Z^{kh} = -\delta_{kh}, \quad y \in Y^s; \\ \displaystyle\int_Y Z^{kh} dy = 0; \quad [Z^{kh}]_S = 0; \quad [\sigma^{(1)}_{i1}]_S = 0; \end{cases} \tag{6}$$

$$\begin{cases} \dfrac{\partial \sigma^{(2)}_{ij}}{\partial y_j} = 0, \ y \in Y; \quad \operatorname{div} D^{kh} = 0, \ y \in Y^s; \\ \displaystyle\int_Y D^{kh} dy = 0; \quad [D^{kh}]_S = 0; \quad [\sigma^{(2)}_{i1}]_S = 0; \end{cases} \tag{7}$$

$$\begin{cases} \dfrac{\partial \sigma^{(3)}_{ij}}{\partial y_j} = 0, \quad W^{kh}(y,0) = D^{kh}(y), \ y \in Y; \quad \operatorname{div}_y W^{kh} = 0, \ y \in Y^s; \\ \displaystyle\int_Y W^{kh} dy = 0, \quad [W^{kh}]_S = 0; \quad [\sigma^{(3)}_{i1}]_S = 0. \end{cases} \tag{8}$$

Here $Z^{kh}(y)$, $D^{kh}(y)$ and $W^{kh}(y,t)$ are Y-periodic vector functions, $B^{kh}(y)$, $A^{kh}(y)$ and $S^{kh}(y,t)$ are Y-periodic scalar functions, $S = \partial Y^h \cap \partial Y^s$ and

$$\sigma^{(1)}_{ij} = b_{ijlm} e_{lm}(Z^{kh}) + b_{ijkh}, \quad y \in Y^h;$$
$$\sigma^{(1)}_{ij} = 2\mu e_{ij}(Z^{kh}) + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) - \delta_{ij} B^{kh}, \quad y \in Y^s;$$
$$\sigma^{(2)}_{ij} = b_{ijlm} e_{lm}(D^{kh}) + a_{ijlm} e_{lm}(Z^{kh}) + a_{ijkh}, \quad y \in Y^h;$$
$$\sigma^{(2)}_{ij} = 2\mu e_{ij}(D^{kh}) - \delta_{ij} A^{kh}, \quad y \in Y^s;$$
$$\sigma^{(3)}_{ij} = a_{ijlm} e_{lm}(W^{kh}) + b_{ijlm} e_{lm}\left(\frac{\partial W^{kh}}{\partial t}\right), \quad y \in Y^h;$$
$$\sigma^{(3)}_{ij} = 2\mu e_{ij}\left(\frac{\partial W^{kh}}{\partial t}\right) - \delta_{ij} S^{kh}, \quad y \in Y^s.$$

Then under some additional assumptions on $f(x,t)$ (see [5]) the homogenized problem corresponding to (1)–(5) takes the form

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i(x,t), \quad x \in \Omega, \quad t > 0; \tag{9}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0; \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega; \tag{10}$$

where $\rho_0 = \rho^h|Y^h| + \rho^s|Y^s|$,

$$\sigma_{ij} = \alpha_{ijkh} e_{kh}(u) + \beta_{ijkh} e_{kh}\left(\frac{\partial u}{\partial t}\right) + g_{ijkh}(t) * e_{kh}(u), \quad g_1(t) * g_2(t) = \int_0^t g_1(t-s) g_2(s) ds,$$

$$\alpha_{ijkh} = \int_{Y^h} \left(a_{ijkh} + a_{ijlm} e_{lm}(Z^{kh}) + b_{ijlm} e_{lm}(D^{kh})\right) dy + \int_{Y^s} (2\mu e_{ij}(D^{kh}) - \delta_{ij} A^{kh}) dy, \tag{11}$$

$$\begin{aligned} \beta_{ijkh} = \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})|Y^s| + \int_{Y^h} \left(b_{ijkh} + b_{ijlm} e_{lm}(Z^{kh})\right) dy + \\ + \int_{Y^s} \left(2\mu e_{ij}(Z^{kh}) - \delta_{ij} B^{kh}\right) dy, \end{aligned} \tag{12}$$

$$\begin{aligned} g_{ijkh}(t) = \int_{Y^h} \left(a_{ijlm} e_{lm}(W^{kh}) + b_{ijlm} e_{lm}\left(\frac{\partial W^{kh}}{\partial t}\right)\right) dy + \\ + \int_{Y^s} \left(2\mu e_{ij}\left(\frac{\partial W^{kh}}{\partial t}\right) - \delta_{ij} S^{kh}\right) dy. \end{aligned} \tag{13}$$

Remark 1.2. *To obtain the homogenized problem* (9) *and* (10) *we modify the results given in [4]. Namely, the auxiliary problems* (6) *and* (8) *have the same form as in [4], but we change auxiliary problems which define the initial conditions for* $W^{kh}(y,t)$. *Nevertheless, setting* $P^{kh}(y,t) = B^{kh}(y)\delta'(t) + A^{kh}(y)\delta(t) + S^{kh}(y,t)$ *in formula (5.3) from [4], we can easily derive problems* (7).

In what follows we suppose that $f(x,t) \equiv 0$. Then the homogenized problem (9), (10) describes natural vibrations of the homogeneous viscoelastic medium. In order to define the spectrum of the homogenized problem we apply the Laplace transform to equations (9), (10). We have

$$\lambda^2 \rho_0 \hat{u}_i = \frac{\partial}{\partial x_j}\left((\alpha_{ijkh} + \lambda\beta_{ijkh} + \hat{g}_{ijkh}(\lambda)) \frac{\partial \hat{u}_k}{\partial x_h}\right), \quad x \in \Omega, \tag{14}$$

$$\hat{u}(x,\lambda) = 0, \quad x \in \partial\Omega, \tag{15}$$

where $\hat{u}(x,\lambda)$ and $\hat{g}_{ijkh}(\lambda)$ are the Laplace transforms of $u(x,t)$ and $g_{ijkh}(t)$, respectively. Taking λ for a spectral parameter, the spectrum of the homogenized problem (9), (10) is the set $S = \{\lambda \in \mathbb{C} : \hat{u}(x,\lambda) \not\equiv 0\}$, where $\hat{u}(x,\lambda)$ is a solution of (14), (15).

It should be noted that if the set Ω^s_ε is occupied by a viscous compressible fluid, then the condition $\operatorname{div} u^\varepsilon = 0$ in (2) is replaced by the condition $p^\varepsilon = -\gamma \operatorname{div} u^\varepsilon$, where $\gamma = c^2\rho^s$ (here c is the speed of sound in the fluid). However, in this case the corresponding homogenized model is also described by system (9), (10) (see [1]) while the periodic auxiliary problems differ from (6)–(8). It is clear that incompressible fluid models can be considered as a limiting case of compressible fluid models when the acoustic speed c goes to infinity.

2. The case of transverse vibrations

In this section we consider the displacement vectors $u^\varepsilon(x,t)$ and $u(x,t)$ such that $u^\varepsilon(x,t) = (u_1^\varepsilon(x_1,t),0,0)$ and $u(x,t) = (u_1(x_1,t),0,0)$. Then it is easy to see that the homogenized system (9) contains only one integro-differential equation:

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} = \alpha_1 u_1'' + \beta_1 \frac{\partial u_1''}{\partial t} + g_1(t) * u_1''.$$

Hereinafter, the following notation is used: $\alpha_i = \alpha_{iiii}$, $\beta_i = \beta_{iiii}$, $g_i(t) = g_{iiii}(t)$, $i = 1, 2$.

To determine the constants α_1, β_1 and the kernel of convolution $g_1(t)$ we solve the auxiliary problems (6)–(8) for $k = h = 1$ and find

$$Z^{11}(y) = (z(y_1), 0, 0), \quad D^{11}(y) = (0, 0, 0), \quad W^{11}(y, t) = (0, 0, 0), \quad y \in Y;$$

$$B^{11}(y) = -\frac{b_1}{1-d}, \quad A^{11}(y) = -\frac{a_1}{1-d}, \quad S^{11}(y, t) = 0, \quad y \in Y^s,$$

where $a_1 = a_{1111} = \lambda_e + 2\mu_e$, $b_1 = b_{1111} = \lambda_v + 2\mu_v$, and

$$z(y_1) = \begin{cases} \dfrac{y_1 d}{1-d}, & \text{for } y_1 \in (0, (1-d)/2], \\ -y_1 + \dfrac{1}{2}, & \text{for } y_1 \in I^s, \\ \dfrac{(y_1 - 1)d}{1-d}, & \text{for } y_1 \in [(1+d)/2, 1). \end{cases}$$

Using formulas (11)–(13) we obtain

$$\alpha_1 = \frac{a_1}{1-d}, \quad \beta_1 = \frac{b_1}{1-d}, \quad g_1(t) = 0.$$

Finally, we find that the homogenized problem takes the form

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} = \alpha_1 u_1'' + \beta_1 \frac{\partial u_1''}{\partial t}, \quad x_1 \in (0, l), \quad t > 0; \tag{16}$$

$$u_1(0, t) = u_1(l, t) = 0, \quad t > 0; \quad u_1(x_1, 0) = \frac{\partial u_1}{\partial t}(x_1, 0) = 0, \quad x_1 \in (0, l). \tag{17}$$

It follows from (16), (17) that in the case of transverse vibrations the homogenized problem does not contain long-term memory and describes one-dimensional vibrations of the viscoelastic Kelvin-Voigt material.

By definition, the spectrum of problem (16), (17) is the union of all $\lambda \in \mathbb{C}$ so that the corresponding spectral problem

$$\rho_0 \lambda^2 \hat{u}_1 = (\alpha_1 + \beta_1 \lambda) \hat{u}_1'', \quad x_1 \in (0, l), \tag{18}$$

$$\hat{u}_1(0, \lambda) = \hat{u}_1(l, \lambda) = 0 \tag{19}$$

has a non-trivial solution $\hat{u}_1(x_1, \lambda)$. In order to define the values of λ we seek a solution of problem (18) and (19) in the form

$$\hat{u}_1(x_1, \lambda) = \sum_{k=1}^{\infty} \hat{v}_k(\lambda) \sin \frac{\pi k}{l} x_1. \tag{20}$$

Substituting (20) into (18) gives

$$\sum_{k=1}^{\infty} \left(\lambda^2 + \beta_1 C_k \lambda + \alpha_1 C_k \right) \hat{v}_k(\lambda) \sin \frac{\pi k}{l} x_1 = 0$$

with $C_k = \pi^2 k^2 / (\rho_0 l^2)$. The spectrum of problem (16), (17) is the union of roots of the quadratic equations

$$\lambda^2 + \beta_1 C_k \lambda + \alpha_1 C_k = 0 \tag{21}$$

for all $k \in \mathbb{N}$. It is clear that for every fixed value of $k \in \mathbb{N}$ the roots of equation (21) lie in the left half-plane $\{\lambda : \mathrm{Re}\lambda < 0\}$. Let us denote

$$k_1 = \max\left\{k : \ k \in \mathbb{N} \cup \{0\}, \quad k < \frac{2l}{\pi\beta_1}\sqrt{\rho_0\alpha_1}\right\}.$$

Since defining the spectrum of problem (16), (17) is reduced to finding the roots of the quadratic equations (21), the following statement is valid.

Theorem 2.1. *The spectrum S_1 of problem* (16), (17) *has the form*

$$S_1 = \{\lambda_{1k}\}_{k=1}^{\infty} \cup \{\lambda_{2k}\}_{k=1}^{\infty},$$

where $\lambda_{1k,2k} = \frac{1}{2}\left(-\beta_1 C_k \pm \sqrt{\beta_1^2 C_k^2 - 4\alpha_1 C_k}\right), \quad k=1,2,\dots .$

In particular, $\lambda_{1k}, \lambda_{2k} \notin \mathbb{R}$ *for* $k = 1, ..., k_1$, *and* $\lambda_{1k}, \lambda_{2k} \in \mathbb{R}$ *for all* $k > k_1$. *Moreover, the following asymptotic relations are valid:*

$$\lambda_{1k} \to -\frac{\alpha_1}{\beta_1} + O\left(\frac{1}{k^2}\right), \quad \lambda_{2k} \to \frac{\alpha_1}{\beta_1} - \beta_1 C_k + O\left(\frac{1}{k^2}\right) \quad as \quad k \to \infty.$$

Therefore, in the case of transverse vibrations the spectrum of the homogenized model contains k_1 pairs of complex conjugate eigenvalues and infinite number of real eigenvalues. In particular, if $l \leqslant \pi b_1/(2\sqrt{(1-d)\rho_0 a_1})$ then the spectrum S_1 contains only real eigenvalues.

Note that if we change α_1 and β_1 in Theorem 2.1 for a_1 and b_1, respectively, then this theorem gives the spectral properties of the problem that describes one-dimensional vibrations (along the x_1-axes) of the original Kelvin-Voigt material. Moreover, the equality $\alpha_1/\beta_1 = a_1/b_1$ means that eigenvalues λ_{1k} of the latter problem and of problem (16), (17) have identical asymptotic behavior as $k \to \infty$.

To conclude this section we suppose that the original fluid is compressible with the large enough value of γ. Our aim now is to study the behavior of the spectrum of the corresponding homogenized problem as $\gamma \to \infty$. It is known (see [1]) that this homogenized problem has the form

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} = A_1 u_1'' + B_1 \frac{\partial u_1''}{\partial t} + G_1(t) * u_1'', \quad x_1 \in (0,l), \quad t > 0; \tag{22}$$

$$u_1(0,t) = u_1(l,t) = 0, \quad t > 0; \quad u_1(x_1,0) = \frac{\partial u_1}{\partial t}(x_1,0) = 0, \quad x_1 \in (0,l), \tag{23}$$

where

$$A_1 = p_1^2(4\mu^2 a_1(1-d) + \gamma b_1^2 d), \quad B_1 = 2\mu b_1 p_1, \quad G_1(t) = -Q_1 e^{-\xi t},$$

$$Q_1 = p_1^3 d(1-d)(\gamma b_1 - 2\mu a_1)^2, \quad \xi = p_1 p_2, \quad p_1 = \frac{1}{2\mu(1-d) + b_1 d}, \quad p_2 = \gamma(1-d) + a_1 d.$$

We see that problem (22), (23) describes one-dimensional vibrations of the viscoelastic material with long-term memory. Furthermore, it was shown in [1] that the spectrum S_2 of problem (22), (23) takes the form

$$S_2 = \{\lambda_{1k}\}_{k=1}^{\infty} \cup \{\lambda_{2k}\}_{k=1}^{\infty} \cup \{\lambda_{3k}\}_{k=1}^{\infty},$$

where λ_{ik}, $i = 1, 2, 3$ are the roots of the cubic equation

$$\lambda^3 + (\xi + B_1 C_k)\lambda^2 + (B_1\xi + A_1)C_k\lambda + (A_1\xi - Q_1)C_k = 0. \tag{24}$$

Now we divide the left-hand side of (24) by γ and consider the limit as $\gamma \to \infty$. Since

$$\frac{1}{\gamma}(\xi + B_1 C_k) \to (1-d)p_1, \quad \frac{1}{\gamma}(B_1\xi + A_1) \to b_1 p_1, \quad \frac{1}{\gamma}(A_1\xi - q_1) \to a_1 p_1,$$

two roots of (24) approach the roots of the quadratic equation (21) as $\gamma \to \infty$. Furthermore, as it follows from Vieta's theorem the last root of (24) approaches $-\infty$ because other two roots of (24) are bounded as $\gamma \to \infty$. Therefore, we observe the following interesting fact: there is a qualitative difference in the form of problems (16), (17) and (22), (23) and we cannot obtain the first problem as a limiting case of the second problem as $\gamma \to \infty$. However, if we study eigenvalues of these two problems then eigenvalues of problem (16), (17) are the finite limits of eigenvalues of problem (22), (23) as $\gamma \to \infty$.

3. The case of longitudinal vibrations

In this section we assume that $u^\varepsilon(x,t) = (0, u_2^\varepsilon(x_2,t), 0)$ and $u(x,t) = (0, u_2(x_2,t), 0)$. In order to obtain the corresponding homogenized problem, we need to determine the constants α_2 and β_2, and the kernel of convolution $g_2(t)$. To do this, we solve the auxiliary problems (6)–(8) for $k = h = 2$ and find

$$Z^{22}(y) = (z(y_1), 0, 0), \quad D^{22}(y) = (0,0,0), \quad W^{22}(y,t) = (0,0,0), \quad y \in Y;$$

$$B^{22}(y) = -2\mu - b_{12} - \frac{b_1 d}{1-d}, \quad A^{22}(y) = -a_{12} - \frac{a_1 d}{1-d}, \quad S^{22}(y,t) = 0, \quad y \in Y^s,$$

where $a_{12} = a_{1122} = \lambda_e$, $b_{12} = b_{1122} = \lambda_v$. Using (11)–(13) we obtain

$$\alpha_2 = a_1(1-d) + 2a_{12}d + \frac{a_1 d^2}{1-d}, \quad \beta_2 = b_1(1-d) + 4\mu d + 2b_{12}d + \frac{b_1 d^2}{1-d}, \quad g_2(t) = 0.$$

Therefore, the homogenized problem takes the form

$$\rho_0 \frac{\partial^2 u_2}{\partial t^2} = \alpha_2 u_2'' + \beta_2 \frac{\partial u_2''}{\partial t}, \quad x_2 \in (0,l), \quad t > 0; \tag{25}$$

$$u_2(0,t) = u_2(l,t) = 0, \quad t > 0; \quad u_2(x_2,0) = \frac{\partial u_2}{\partial t}(x_2,0) = 0, \quad x_2 \in (0,l). \tag{26}$$

We see that problem (25), (26) has the same form as problem (16), (17). Hence, to describe the spectral properties of problem (25), (26) one needs to use Theorem 2.1 and change α_1 and β_1 for α_2 and β_2, respectively. However, since $a_{2222} = a_1$, $b_{2222} = b_1$ and $\alpha_2/\beta_2 \neq a_1/b_1$, eigenvalues λ_{1k} of problem (25), (26) and eigenvalues of the problem describing one-dimensional vibrations (along the x_2-axes) of the original Kelvin-Voigt material have different asymptotic behavior as $k \to \infty$.

It should be noted that if original fluid is assumed to be compressible with large enough value of γ then the corresponding homogenized problem, as in the case of transverse vibrations, describes one-dimensional vibrations of the viscoelastic material with long-term memory (see [1]). Moreover, the spectrum of this problem is the union of roots of the cubic equations (24) with the subscript 1 changed for 2 and constants in the equations are

$$A_2 = a_1(1-d) + d(\gamma + (a_{12} - \gamma)c_3 + b_{12}c_4), \quad B_2 = b_1(1-d) + d(2\mu + b_{12}c_3),$$

$$Q_2 = p_1 d(1-d)(\gamma - a_{12} + b_{12}p_1 p_2)^2,$$

where $c_3 = -b_{12}p_1(1-d)$, $c_4 = p_1(1-d)(\gamma - a_{12} + b_{12}p_1p_2)$. Finally, assuming $\gamma \to \infty$ and repeating the above-mentioned arguments we can easily obtain results that are similar to the results obtained in the previous section.

This work was supported by the Russian Foundation for Basic Research, grant 13-01-00384 a.

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Спектр одномерных колебаний слоистой среды, состоящей из материала Кельвина-Фойгта и вязкой несжимаемой жидкости

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Рассмотрена математическая модель, описывающая собственные колебания периодической слоистой среды, составленной из вязкоупругого материала Кельвина-Фойгта и вязкой несжимаемой жидкости. Для данной модели построены две усредненные модели, соответствующие поперечным и продольным колебаниям слоистой среды. Показано, что спектр каждой усредненной модели есть объединение корней соответствующих квадратных уравнений.

Ключевые слова: спектр, слоистая среда, усредненная модель, вязкоупругость, вязкая жидкость.