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On Error Estimates for Weighted Quadrature Formulas Exact for Haar Polynomials

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On the spaces S_p, estimates are found for the norm of the error functional of weighted quadrature formulas. For quadrature formulas exact for constants a lower estimate of $\|\delta_N\|_{S_p^*}$ is proved, and for quadrature formulas possessing the Haar d-property upper estimates of the $\|\delta_N\|_{S_p^*}$ are obtained.

Keywords: Haar d-property, error functional of a quadrature formula, function spaces S_p.

Introduction

The problem of constructing and analyzing cubature formulas that integrate a given collection of functions exactly has been earlier considered mainly in the cases when these functions are algebraic or trigonometric polynomials. The approximate integration formulas exact for finite Haar sums can be found in the monograph [1], the accuracy of approximate integration formulas for finite Haar sums was used there for deriving the error of these formulas.

A description of all minimal weighted quadrature formulas possessing the Haar d–property, i.e., formulas exact for Haar polynomials of degree at most d, was given in [2]. In [3] the estimates of the norm of the error functional for quadrature formulas exact for Haar polynomials were proved on the spaces S_p and H_α in the case of the weight function $g(x) \equiv 1$. In [4] the estimates of the norm of the error functional for minimal quadrature formulas exact for Haar polynomials were obtained on the spaces S_p in the case of the weight function $g \in L_\infty[0, 1]$.

In the two-dimensional case, the problem of constructing cubature formulas possessing the Haar d-property was considered in [5–9]. Estimates of the norm of the error functional for cubature formulas on the spaces S_p and H_α were obtained in [10, 11].

The results obtained in [3] are extended in this paper to the case of weighted quadrature formulas on the spaces S_p. In the case of the weight function $g \in L_1[0, 1]$, a lower estimate of the norm of the error functional δ_N is derived for quadrature formulas exact for any constant. In the cases of the weight function $g \in L_\infty[0, 1]$ and $g \in L_q[0, 1]$ ($p^{-1} + q^{-1} = 1$), upper estimates of the norm of the error functional δ_N are proved for quadrature formulas possessing the Haar d-property. It is shown that in the case of $N \asymp 2^d$ with $d \to \infty$, the value $\|\delta_N\|_{S_p^*}$ for the formulas under study has the best convergence rate to zero, which is equal to $N^{-\frac{1}{p}}$ with $N \to \infty$.

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1. Basic definitions

In this paper, we use the original definition of the functions $\chi_{m,j}(x)$ introduced by A. Haar [12], which differs from the definition of these functions used in [1].

Dyadic intervals $l_{m,j}$ are the intervals with their endpoints at $(j-1)/2^{m-1}$, $j/2^{m-1}$, $m = 1, 2, \ldots,$ $j = 1, 2, \ldots, 2^{m-1}$. If the left endpoint of a dyadic interval coincides with 0, then we consider this interval to be closed on the left. If the right endpoint of a dyadic interval coincides with 1, then we consider this interval as closed on the right. The remaining intervals are considered open. The left (right) half of $l_{m,j}$ (without its midpoint) we denote by $l_{m,j}^-$ ($l_{m,j}^+$).

It is convenient to construct the Haar system of functions in groups: the mth group contains 2^{m-1} functions $\chi_{m,j}(x)$, where $m = 1, 2, \ldots, j = 1, 2, \ldots, 2^{m-1}$. The Haar functions $\chi_{m,j}(x)$ are defined as:

$$\chi_{m,j}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & \text{if } x \in l_{m,j}^-, \\ -2^{\frac{m-1}{2}}, & \text{if } x \in l_{m,j}^+, \\ 0, & \text{if } x \in [0,1] \setminus \overline{l_{m,j}}, \\ \dfrac{1}{2}[\chi_{m,j}(x-0) + \chi_{m,j}(x+0)], & \text{if } x \text{ is an interior point of discontinuity of the} \\ & \text{function } \chi_{m,j} \end{cases}$$

with $\overline{l_{m,j}} = \left[\dfrac{j-1}{2^{m-1}}, \dfrac{j}{2^{m-1}} \right]$, $m = 1, 2, \ldots, j = 1, 2, \ldots, 2^{m-1}$. The Haar system of functions includes the function $\chi_1(x) \equiv 1$ too, which is outside of any group.

The Haar polynomials of degree d are by definition the functions

$$P_d(x) = a_0 \chi_1(x) + \sum_{m=1}^{d} \sum_{j=1}^{2^{m-1}} a_m^{(j)} \chi_{m,j}(x),$$

where $d = 1, 2, \ldots,$ $a_0, a_m^{(j)} \in \mathbb{R}$, $m = 1, 2, \ldots, d$, $j = 1, 2, \ldots, 2^{m-1}$, and

$$\sum_{j=1}^{2^{d-1}} \left(a_d^{(j)} \right)^2 \neq 0.$$

By the 0-degree Haar polynomials we understand real constants.

We consider the following quadrature formula

$$I[f] = \int_0^1 g(x)f(x)dx \approx \sum_{i=1}^{N} C_i f(x^{(i)}) = Q_N[f], \tag{1}$$

where $x^{(i)} \in [0,1]$ are the nodes, the coefficients C_i at the nodes are real and satisfy the inequalities

$$C_i > 0, \tag{2}$$

$i = 1, 2, \ldots, N$, the functions $g(x)$, $f(x)$ are defined and summable on $[0,1]$.

We denote the error functional of the quadrature formula (1) by $\delta_N[f]$ so that

$$\delta_N[f] = \sum_{i=1}^{N} C_i f(x^{(i)}) - \int_0^1 g(x)f(x)\,dx. \tag{3}$$

The quadrature formula (1) is said to possess the Haar d–property (or just the d-property) if it is exact for any Haar polynomial $P_d(x)$ of degree at most d, i.e., $Q_N[P_d] = I[P_d]$.

2. Estimates for the norm of the error functional for weighted quadrature formulas

We recall the definition of the classes S_p introduced by I.M. Sobol' [1].

Let p be a fixed number with $1 \leqslant p < +\infty$. For an arbitrary positive A one can define the class $S_p(A)$ as the set of functions $f(x)$ on $[0,1]$ that can be represented by a Fourier–Haar series

$$f(x) = c_0^{(1)} + \sum_{m=1}^{\infty} \sum_{j=1}^{2^{m-1}} c_m^{(j)} \chi_{m,j}(x) \tag{4}$$

with real coefficients $c_0^{(1)}$, $c_m^{(j)}$ $(m = 1, 2, \ldots, j = 1, 2, \ldots, 2^{m-1})$ satisfying the condition

$$A_p(f) = \sum_{m=1}^{\infty} 2^{\frac{m-1}{2}} \Big[\sum_{j=1}^{2^{m-1}} |c_m^{(j)}|^p \Big]^{\frac{1}{p}} \leqslant A. \tag{5}$$

It is proved in [1] that $\sum\limits_{A>0} S_p(A)$ with the norm

$$\|f\|_{S_p} = A_p(f), \tag{6}$$

forms a linear normed space, which is denoted by S_p. All the functions $f(x)$ that differ by constant terms are regarded as a single function.

To establish an estimate for the norm of the error functional of the quadrature formula (1) we need to recall the definition of the space $L_\infty[0,1]$ [13]. It consists of all measurable almost everywhere finite functions $g(x)$, and for each function there exists a number C_g such that $|g(x)| \leqslant C_g$ almost everywhere. We call such functions essentially bounded on $[0,1]$. For a function $g \in L_\infty[0,1]$ one defines the proper (essential) supremum of its absolute value

$$\operatorname*{ess\,sup}_{x \in [0,1]} |g(x)|$$

as the infimum of the set of numbers $\alpha \in \mathbb{R}$ such that the measure of the set

$$\{x \in [0,1] : |g(x)| > \alpha\}$$

is zero. $L_\infty[0,1]$ is a linear subset in the set of measurable almost everywhere finite functions. The norm on $L_\infty[0,1]$ can be introduced as

$$\|g\|_{L_\infty[0,1]} = \operatorname*{ess\,sup}_{x \in [0,1]} |g(x)|. \tag{7}$$

Lemma 1([2]). *Let m be a fixed positive integer. The functions*

$$\kappa_{m,j}(x) = \begin{cases} 2^m, & \text{if } x \in l_{m+1,j}, \\ 2^{m-1}, & \text{if } x \in \overline{l_{m+1,j}} \setminus l_{m+1,j}, \\ 0, & \text{if } x \in [0,1] \setminus \overline{l_{m+1,j}}, \end{cases} \tag{8}$$

$j = 1, 2, \ldots, 2^m$, are Haar polynomials of degree m and they form a basis in the linear space of Haar polynomials of degree at most m.

Lemma 2. *For all $m = 2, 3, \ldots,$ $j = 1, 2, \ldots, 2^{m-1}$,*

$$\chi_{m,j}(x) = 2^{-\frac{m+1}{2}} \left[\kappa_{m,2j-1}(x) - \kappa_{m,2j}(x) \right], m = 1, 2, \ldots, j = 1, 2, \ldots, 2^{m-1}, \tag{9}$$

$$\kappa_{m,2j-1}(x) + \kappa_{m,2j}(x) = 2\kappa_{m-1,j}(x). \tag{10}$$

The definition of the Haar functions and relation (8) imply (9), (10).

Fix $p > 1$, then let q be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Lemma 3. *For all nonnegative integer $k \leqslant m$ $(m \in \mathbb{N})$,*

$$\sum_{j=1}^{2^m} \Big(\int_{l_{m+1,j}} |g(x)|\, dx + 2^{-m} Q[\kappa_{m,j}] \Big)^q \leqslant \sum_{j=1}^{2^{m-k}} \Big(\int_{l_{m-k+1,j}} |g(x)|\, dx + 2^{-m+k} Q[\kappa_{m-k,j}] \Big)^q. \quad (11)$$

Proof. Inequality (11) is proved by induction on k.

For $k = 0$ it becomes an equality.

Based on the induction hypothesis that

$$\sum_{j=1}^{2^m} \Big(\int_{l_{m+1,j}} |g(x)|\, dx + 2^{-m} Q[\kappa_{m,j}] \Big)^q \leqslant \sum_{j=1}^{2^{m-k+1}} \Big(\int_{l_{m-k+2,j}} |g(x)|\, dx + 2^{-m+k-1} Q[\kappa_{m-k+1,j}] \Big)^q, \quad (12)$$

we prove (11). For $a, b > 0$ and $q > 1$, it is easy to see that

$$a^q + b^q \leqslant (a+b)^q. \quad (13)$$

In view of (13) and (10), it follows from (12) that

$$\sum_{j=1}^{2^{m-k+1}} \Big(\int_{l_{m-k+2,j}} |g(x)|\, dx + \frac{Q[\kappa_{m-k+1,j}]}{2^{m-k+1}} \Big)^q \leqslant \sum_{j=1}^{2^{m-k}} \Bigg[\Big(\int_{l_{m-k+2,2j}} |g(x)|\, dx + \frac{Q[\kappa_{m-k+1,2j}]}{2^{m-k+1}} \Big)^q +$$

$$+ \Big(\int_{l_{m-k+2,2j-1}} |g(x)|\, dx + \frac{Q[\kappa_{m-k+1,2j-1}]}{2^{m-k+1}} \Big)^q \Bigg] \leqslant \sum_{j=1}^{2^{m-k}} \Big(\int_{l_{m-k+1,j}} |g(x)|\, dx + \frac{Q[\kappa_{m-k,j}]}{2^{m-k}} \Big)^q. \quad (14)$$

Inequalities (12) and (14) imply (11).   □

Let

$$\Psi_q(m) = \Bigg[\sum_{j=1}^{2^{m-1}} \Big| -\int_{l_{m,j}^-} g(x)\, dx + \int_{l_{m,j}^+} g(x)\, dx + 2^{-\frac{m-1}{2}} Q[\chi_{m,j}] \Big|^q \Bigg]^{\frac{1}{q}}, \quad (15)$$

where $m = 1, 2, 3, \ldots$

Lemma 4. *If $f \in S_p$, $g \in L_1[0,1]$, then for the quadrature formula (1) all $\Psi_q(m)$ are bounded, and*

$$|\Psi_q(m)| \leqslant 2 \Bigg[\sum_{j=1}^{2^d} \Big(\int_{l_{d+1,j}} |g(x)|\, dx \Big)^q \Bigg]^{\frac{1}{q}}, \quad (16)$$

where $m = d+1, d+2, \ldots$

Proof. In view of (9) and (10), it follows from (15) that

$$[\Psi_q(m)]^q \leqslant \sum_{j=1}^{2^{m-1}} \Big[\int_{l_{m,j}^-} |g(x)|\, dx + \int_{l_{m,j}^+} |g(x)|\, dx + 2^{-m} \big| Q[\kappa_{m,2j-1}] - Q[\kappa_{m,2j}] \big| \Big]^q \leqslant$$

$$\leqslant \sum_{j=1}^{2^{m-1}} \Big[\int_{l_{m,j}} |g(x)|\, dx + 2^{-m} Q[\kappa_{m,2j-1} + \kappa_{m,2j}] \big| \Big]^q = \sum_{j=1}^{2^{m-1}} \Big[\int_{l_{m,j}} |g(x)|\, dx + 2^{-m+1} Q[\kappa_{m-1,j}] \Big]^q. \quad (17)$$

By Lemma 1 and relations (2), we have

$$\int_{l_{d+1,j}} g(x)\, dx = 2^{-d} \int_0^1 g(x)\kappa_{d,j}(x)\, dx = 2^{-d} \sum_{i=1}^N C_i \kappa_{d,j}(x^{(i)}) \geqslant 0, \quad j = 1, 2, \ldots, 2^d.$$

Therefore

$$\int_{l_{d+1,j}} g(x)\, dx \geqslant 0. \tag{18}$$

Using Lemma 3, Lemma 1 and inequality (18), for $m = d+1, d+2, \ldots$ we obtain

$$\sum_{j=1}^{2^{m-1}} \left[\int_{l_{m,j}} |g(x)|\, dx + 2^{-m+1} Q[\kappa_{m-1,j}] \right]^q \leqslant \sum_{j=1}^{2^d} \left[\int_{l_{d+1,j}} |g(x)|\, dx + 2^{-d} Q[\kappa_{d,j}] \right]^q \leqslant$$

$$\leqslant \sum_{j=1}^{2^d} \left[\int_{l_{d+1,j}} |g(x)|\, dx + \int_{l_{d+1,j}} g(x)\, dx \right]^q \leqslant \sum_{j=1}^{2^d} \left[2 \int_{l_{d+1,j}} |g(x)|\, dx \right]^q. \tag{19}$$

Inequalities (17) and (19) imply (16).

It follows from inequalities

$$|\Psi_q(m)| \leqslant \max \left\{ \Psi_q(1), \Psi_q(2), \ldots, \Psi_q(d), 2 \left[\sum_{j=1}^{2^d} \left(\int_{l_{d+1,j}} |g(x)|\, dx \right)^q \right]^{\frac{1}{q}} \right\}, \quad m = 1, 2, \ldots$$

that all $\Psi_q(m)$ are bounded.   □

Lemma 5. *If $g \in L_1[0,1]$, then for the norm of the error functional of the quadrature formula (1) exact for any constant we have*

$$\|\delta_N\|_{S_p^*} = \sup_{1 \leqslant m < \infty} \Psi_q(m). \tag{20}$$

If, in addition, the quadrature formula (1) possesses the Haar d–property, then

$$\|\delta_N\|_{S_p^*} = \sup_{d < m < \infty} \Psi_q(m). \tag{21}$$

Proof. The series (4) is substituted into (3). Since the quadrature formula (1) is exact for any constant, it follows that

$$\delta_N[f] = \sum_{m=1}^\infty \sum_{j=1}^{2^{m-1}} c_m^{(j)} \left[\sum_{i=1}^N C_i \chi_{m,j}(x^{(i)}) - \int_0^1 g(x)\chi_{m,j}(x)\, dx \right]. \tag{22}$$

By virtue of the definition of Haar functions, it follows from (22) that

$$\delta_N[f] = \sum_{m=1}^\infty 2^{\frac{m-1}{2}} \sum_{j=1}^{2^{m-1}} \left\{ c_m^{(j)} \left[-\int_{l_{m,j}^-} g(x)\, dx + \int_{l_{m,j}^+} g(x)\, dx + 2^{-\frac{m-1}{2}} Q[\chi_{m,j}] \right] \right\}. \tag{23}$$

Since the series (23) converges uniformly, it is true that

$$|\delta_N[f]| \leqslant \sum_{m=1}^\infty 2^{\frac{m-1}{2}} \sum_{j=1}^{2^{m-1}} \left\{ |c_m^{(j)}| \left| -\int_{l_{m,j}^-} g(x)\, dx + \int_{l_{m,j}^+} g(x)\, dx + 2^{-\frac{m-1}{2}} Q[\chi_{m,j}] \right| \right\}. \tag{24}$$

Applying the Hölder inequality to the expression in (24), we obtain

$$|\delta_N[f]| \leqslant \sum_{m=1}^{\infty} \left\{ 2^{\frac{m-1}{2}} \left[\sum_{j=1}^{2^{m-1}} |c_m^{(j)}|^p \right]^{\frac{1}{p}} \Psi_q(m) \right\}. \tag{25}$$

Then, in view of (5) and (6), it follows from (25) that

$$|\delta_N[f]| \leqslant \|f\|_{S_p} \sup_{1 \leqslant m < \infty} \Psi_q(m). \tag{26}$$

In order to establish that inequality (26) can not be improved, we use the technique applied in [1]. Let $m = m_0 \in \mathbb{N}$ be a fixed number. We consider the function

$$f_{m_0}(x) = \sum_{j=1}^{2^{m_0-1}} \left\{ \left| -\int_{l_{m_0,j}^-} g(x)\,dx + \int_{l_{m_0,j}^+} g(x)\,dx + 2^{-\frac{m_0-1}{2}} Q[\chi_{m_0,j}] \right|^{q-1} \times \right.$$
$$\left. \times \operatorname{sign}\left[-\int_{l_{m_0,j}^-} g(x)\,dx + \int_{l_{m_0,j}^+} g(x)\,dx + 2^{-\frac{m_0-1}{2}} Q[\chi_{m_0,j}] \right] \chi_{m_0,j}(x) \right\}.$$

The Fourier–Haar coefficients of this function are

$$c_{m_0}^{(j)} = \left| -\int_{l_{m_0,j}^-} g(x)\,dx + \int_{l_{m_0,j}^+} g(x)\,dx + 2^{-\frac{m_0-1}{2}} Q[\chi_{m_0,j}] \right|^{q-1} \times$$
$$\times \operatorname{sign}\left[-\int_{l_{m_0,j}^-} g(x)\,dx + \int_{l_{m_0,j}^+} g(x)\,dx + 2^{-\frac{m_0-1}{2}} Q[\chi_{m_0,j}] \right],\ j = 1, 2, \ldots, 2^{m_0-1},$$

$c_0^{(0)} = 0$, $c_m^{(j)} = 0$, $m \in \mathbb{N} \setminus \{m_0\}$, $j = 1, 2, \ldots, 2^{m-1}$. Therefore, (23) implies

$$|\delta_N[f_{m_0}]| = 2^{\frac{m_0-1}{2}} \sum_{j=1}^{2^{m_0-1}} \left| -\int_{l_{m_0,j}^-} g(x)\,dx + \int_{l_{m_0,j}^+} g(x)\,dx + 2^{-\frac{m_0-1}{2}} Q[\chi_{m_0,j}] \right|^q = \|f_{m_0}\|_{S_p} \Psi_q(m_0). \tag{27}$$

Since the function $f_{m_0}(x)$ exists for any $m_0 \in \mathbb{N}$, relation (27) implies (20). It follows from Lemma 4 that the norm of the error functional δ_N is finite.

Let the quadrature formula (1) possess the Haar d–property. Since it is exact for Haar polynomials of degree at most d, in the relations (22), (23) and (25) the lower index in the sum over m is equal to $d+1$. Therefore, inequality (26) turns into $|\delta_N[f]| \leqslant \|f\|_{S_p} \sup_{d < m < \infty} \Psi_q(m)$. For any $m_0 \in \mathbb{N}$ there exists a function $f_{m_0}(x)$ satisfying (27), this implies (21). $\qquad \square$

Let

$$G = \int_0^1 g(x)\,dx. \tag{28}$$

Theorem 1. *If $g \in L_1[0,1]$, then for the norm of the error functional of the quadrature formula (1) we have the lower estimate*

$$\|\delta_N\|_{S_p^*} \geqslant 2^{-\frac{1}{p}} G N^{-\frac{1}{p}}, \tag{29}$$

where the constant G is defined by (28).

Proof. Let ε be a number satisfying the condition

$$0 < \varepsilon < \min_{1 \leqslant i \leqslant N} \left\{ \frac{C_i}{2} \right\}. \tag{30}$$

There exists a number $m_0 \in \mathbb{N}$ such that for any $m \geqslant m_0$ the following inequalities hold:

$$\left| \int_{l_{m+1,j}} g(x)\, dx \right| \leqslant \frac{\varepsilon}{2}, \quad j = 1, 2, \ldots, 2^{m-1}.$$

Let m_1 be the minimal number among all numbers m satisfying the following condition: each of the segments $\overline{l_{m+1,1}}, \ldots, \overline{l_{m+1,2^m}}$ contains at most one node of formula (1).

If every node of the quadrature formula (1) differs from the points $\dfrac{2j-1}{2^{m_1}}$, $j = 1, 2, \ldots, 2^{m_1-1}$, we set $m_2 = m_1$. Otherwise, we set

$$m_2 = 1 + \max\{ m : \text{ there exists } x_r = \frac{2j_r - 1}{2^m}, j_r \in \{1, 2, \ldots, 2^{m-1}\} \}.$$

Let $m' = \max\{m_0, m_2\}$. Then for all $m \geqslant m'$ the following three conditions are satisfied:
– for all $j = 1, 2, \ldots, 2^{m-1}$

$$\left| \int_{l_{m,j}^-} g(x)\, dx - \int_{l_{m,j}^+} g(x)\, dx \right| \leqslant \varepsilon, \tag{31}$$

– each of the segments $\overline{l_{m+1,j}}$ contains at most one node of the quadrature formula (1), $j = 1, 2, \ldots, 2^m$,
– every node of the quadrature formula (1) differs from the points $\dfrac{2j-1}{2^m}$, $j = 1, 2, \ldots, 2^{m-1}$.

In view of (9), it follows from (15) that

$$\Psi_q(m') = \left\{ \sum_{j=1}^{2^{m'-1}} \left| -\int_{l_{m',j}^-} g(x)\, dx + \int_{l_{m',j}^+} g(x)\, dx + 2^{-m'} \sum_{i=1}^{N} C_i \left(\kappa_{m',2j-1}(x^{(i)}) - \kappa_{m',2j}(x^{(i)}) \right) \right|^q \right\}^{\frac{1}{q}}. \tag{32}$$

It follows from the definition of the number m' that the nodes of the quadrature formula (1) differ from the points

$$\left\{ \frac{2j-1}{2^{m'}} \right\} = \operatorname{supp} \{\kappa_{m',2j-1}\} \cap \operatorname{supp} \{\kappa_{m',2j}\},$$

and each of the segments $\overline{l_{m'+1,j}}$ contains at most one node of formula (1). Then relation (32) can be rewritten as

$$\Psi_q(m') = \left\{ \sum_{j=1}^{2^{m'}} \left| \pm 2^{-m'} \sum_{i=1}^{N} C_i \kappa_{m',j}(x^{(i)}) - \left(\int_{l_{m',j}^-} g(x)\, dx - \int_{l_{m',j}^+} g(x)\, dx \right) \right|^q \right\}^{\frac{1}{q}}. \tag{33}$$

In this formula we choose the plus sign if $x^{(i)} \in l_{m',j}^-$, and the minus sign if $x^{(i)} \in l_{m',j}^+$.

Taking into account (30) and (31), it follows from (33) that

$$\Psi_q(m') = \left\{ \sum_{j=1}^{2^{m'}} \left| 2^{-m'} \sum_{i=1}^{N} C_i \kappa_{m',j}(x^{(i)}) - \left[\pm \left(\int_{l_{m',j}^-} g(x)\, dx - \int_{l_{m',j}^+} g(x)\, dx \right) \right] \right|^q \right\}^{\frac{1}{q}} \geqslant$$
$$\geqslant \left\{ \sum_{j=1}^{2^{m'}} \left| 2^{-m'} \sum_{i=1}^{N} C_i \kappa_{m',j}(x^{(i)}) \right|^q \right\}^{\frac{1}{q}}. \tag{34}$$

Let N_1 be the number of nodes of the quadrature formula (1) that coincide with the points $\left\{ \dfrac{j}{2^{m'}} \right\}$ ($j = 1, \ldots, 2^{m'} - 1$). To be specific, we denote them by $x^{(1)}, \ldots, x^{(N_1)}$. Since every

segment $\overline{l_{m'+1,j}}$ contains at most one node of the formula (1), it follows from (34) that

$$\Psi_q(m') \geqslant \left\{ \sum_{i=N_1+1}^{N} C_i{}^q + 2\sum_{i=1}^{N_1} \left(\frac{C_i}{2}\right)^q \right\}^{\frac{1}{q}}. \tag{35}$$

Since the quadrature formula (1) is exact for any constant, we have

$$C_1 + C_2 + \ldots + C_N = G, \tag{36}$$

where the constant G is defined by (28). Because of (2), it follows from (36) that $G > 0$.

If the quadrature formula (1) satisfies the conditions (36) and (2), it is easy to show that the function

$$\varphi(C_1, C_2, \ldots, C_N) = 2\sum_{i=1}^{N_1}\left(\frac{C_i}{2}\right)^q + \sum_{i=N_1+1}^{N} C_i{}^q$$

attains its infimum, which is equal to $G^q(N + N_1)^{1-q}$, when

$$C_1 = C_2 = \ldots = C_{N_1} = 2G(N + N_1)^{-1}, \ C_{N_1+1} = C_{N_1+2} = \ldots = C_N = G(N + N_1)^{-1}.$$

It follows from (35) that

$$\Psi_q(m') \geqslant G(N + N_1)^{-\frac{1}{p}} \geqslant 2^{-\frac{1}{p}}GN^{-\frac{1}{p}}. \tag{37}$$

The relations (20) and (37) imply (29). $\qquad\square$

Let

$$G_0 = \int_0^1 |g(x)|\,dx. \tag{38}$$

Theorem 2. *If $g \in L_\infty[0,1]$, then the norm of the error functional of the quadrature formula (1) possessing the Haar d-property satisfies the estimate*

$$\|\delta_N\|_{S_p^*} \leqslant 2\left(2^{-d}\right)^{\frac{1}{p}}G_0^{\frac{1}{q}}\|g\|_{L_\infty[0,1]}^{\frac{1}{p}}, \tag{39}$$

where the constant G_0 is defined by (38).

Proof. In view of (7) and (38),

$$\sum_{j=1}^{2^d}\left(\int_{l_{d+1,j}}|g(x)|\,dx\right)^q = \sum_{j=1}^{2^d}\left[\left(\int_{l_{d+1,j}}|g(x)|\,dx\right)^{q-1}\int_{l_{d+1,j}}|g(x)|\,dx\right] \leqslant$$
$$\leqslant \left(2^{-d}\|g\|_{L_\infty[0,1]}\right)^{q-1}\sum_{j=1}^{2^d}\int_{l_{d+1,j}}|g(x)|\,dx \leqslant \left(2^{-d}\|g\|_{L_\infty[0,1]}\right)^{q-1}G_0. \tag{40}$$

The relations (21), (16) and (40) imply (39). $\qquad\square$

Theorem 3. *If $g \in L_q[0,1]$, then the norm of the error functional of the quadrature formula (1) possessing the Haar d-property satisfies the estimate*

$$\|\delta_N\|_{S_p^*} \leqslant 2\left(2^{-d}\right)^{\frac{1}{p}}\|g\|_{L_q[0,1]}. \tag{41}$$

Proof. Using the Hölder integral inequality, we obtain

$$\int_{l_{d+1,j}} |g(x)|\, dx \leqslant \left(2^{-d}\right)^{\frac{1}{p}} \left(\int_{l_{d+1,j}} |g(x)|^q\, dx\right)^{\frac{1}{q}}, \quad j = 1, 2, \ldots, 2^d. \tag{42}$$

It follows from (16) and (42) that

$$|\Psi_q(m)| \leqslant 2\left[\sum_{j=1}^{2^d} \left(2^{-d}\right)^{\frac{q}{p}} \int_{l_{d+1,j}} |g(x)|^q\, dx\right]^{\frac{1}{q}} = 2\left(2^{-d}\right)^{\frac{1}{p}} \left[\int_0^1 |g(x)|^q\, dx\right]^{\frac{1}{q}} = 2\left(2^{-d}\right)^{\frac{1}{p}} \|g\|_{L_q[0,1]}. \tag{43}$$

The relations (43) and (21) imply (41). □

Conclusions

In [14] I.M.Sobol' establishes a quadrature formula with the weight function $g(x) \geqslant 0$ and positive coefficients at the nodes that satisfy the following condition

$$\sum_{i=1}^N C_i = \int_0^1 g(x)\, dx = G.$$

The error functional δ_N of that formula can be estimated as follows [14]:

$$GN^{-\frac{1}{p}} \leqslant \|\delta_N\|_{S_p^*} \leqslant 2GN^{-\frac{1}{p}},$$

and, therefore, $\|\delta_N\|_{S_p^*} \asymp N^{-\frac{1}{p}}$ when $N \to \infty$.

It follows from Theorems 1, 2 and 3 that for $g \in L_\infty[0,1]$ and $g \in L_q[0,1]$ in the case of $N \asymp 2^d$ with $d \to \infty$ the quadrature formula (1) possessing the Haar d-property has the best rate of convergence $\|\delta_N\|_{S_p^*}$ to zero, which is equal to $N^{-\frac{1}{p}}$ when $N \to \infty$.

In particular, the minimal weight quadrature formulas constructed in [7] satisfy the condition $N \asymp 2^d$ when $d \to \infty$. At the same time these formulas, being the minimal formulas of approximate integration, provide the best pointwise convergence of $\delta_N[f]$ to zero as $N \to \infty$.

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Об оценках погрешности весовых квадратурных формул, точных для полиномов Хаара

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Для весовых квадратурных формул получены оценки нормы функционала погрешности на пространствах S_p — нижняя оценка величины $\|\delta_N\|_{S_p^*}$ для формул, точных на константах, и верхние оценки $\|\delta_N\|_{S_p^*}$ для формул, обладающих d-свойством Хаара.

Ключевые слова: d-свойство Хаара, функционал погрешности квадратурной формулы, пространства функций S_p.