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Decidability of Multi-modal Logic LTK of Linear Time and Knowledge

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The paper investigates modal (temporal-model) logics based at a semantic approach with models combining knowledge and time. We introduce multi-modal logics LTK_r and LTK_{ir} containing modalities for knowledge and time as the sets of all LTK_r-valid, and LTK_{ir}-valid formulae for a class of special LTK_r-frames, LTK_{ir}-frames, respectively. The main results of this paper are theorems stating that LTK_r and LTK_{ir} are decidable; we also give an explicit solving algorithm.

Keywords: multi-modal logic, temporal logic, epistemic logic, decidability, effective finite model property

Introduction

Nowadays modal and multi-modal propositional logics describing human reasoning and agents' behavior is an active area [1–3]. It is well known that the combination of temporal and knowledge modalities provides an highly expressive and efficient language (cf. [1, 4]). Multi-modal logics generated by adjoining operators representing time and knowledge to the classical propositional calculus **PC** are very effective for modeling reasoning, where agents, who possess a certain knowledge, are operating in the processes of computation in a flow of time (see [1, 3, 4]).

In this paper we study a semantic approach for construction models combining knowledge and time. Some linear logics using the knowledge and time are investigated in a papers of V.Rybakov and E.Calardo [5–7]. Complete axiomatization of a number of different logics involving conditions for knowledge and time can be founded in [3]. The aim of this article is to prove decidability of multi-modal logics LTK_r, LTK_{ir} with intransitive, reflexive/irreflexive relation of time.

We consider time as a linear and discrete sequence of states. Each state consists of a set of information points connected by modal relations R_i. To bring it more exact, R_i says which information points are effectively available for the agent i: it specifies the piece of information that the agent may access at given moment. Agents operating synchronously and each agent knows what time it is and distinguishes present from future time. So logic is semantically defined as the set of all LTK_r-valid (LTK_{ir}-valid) formulae, where LTK_r-frames (LTK_{ir}-frames) are multi-modal Kripke-frames combining a linear and discrete representation of the flow of time with special $S5$-like modalities, defined at each time cluster and representing knowledge. The main result of our paper are theorems stating that LTK_r and LTK_{ir} have the effective finite model property and hence, that they are decidable.

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1. Preliminaries

First we recall basic definitions, notation and known results used in this article (for more detailed information about the subject see [8, 9]).

The language $\mathcal{L}^{LTK}$ consists of a countable set of propositional letters $P := \{p_1, \dots, p_n, \dots\}$, the standard boolean operations and the set of modal operations $\{\Box_T, \Box_\sim, \Box_i\ (i \in I)\}$. *Well formed formulae* (wff's) are defined in the standard way, in particular, if A is a wff, than $\Box_T A, \Box_\sim A, \Box_i A (i \in I)$ are wff's. We denote by $Fma(\mathcal{L}^{LTK})$ the set of all the wff's of $\mathcal{L}^{LTK}$ (in the sequel, in an expression *formula* we always refer to a formula from $Fma(\mathcal{L}^{LTK})$). The intended meaning of the modal operations is: (a) $\Box_T A$ for logic LTK_r means that the formula A true in the current state and will be true in the next state. $\Box_T A$ in logic LTK_{ir} means that the formula A will be true in the next state. (b) $\Box_\sim A$ means that A is known everywhere in the present time-cluster (i.e. A is part of the environmental knowledge); (c) $\Box_i A (i \in I)$ stands for the agent i (operating in the system) knows A in the current state. Semantics for the language $\mathcal{L}^{LTK}$ is based on a linear and discrete flow of time, associating a time point with any natural number n.

Definition 1.1. A k-modal Kripke-frame is a tuple $\mathcal{F} = \langle W_\mathcal{F}, R_1, \dots, R_k \rangle$ where $W_\mathcal{F}$ is a nonempty set of worlds and each R_i is some binary relation on $W_\mathcal{F}$.

Definition 1.2. Given a Kripke-frame $\mathcal{F} = \langle W_\mathcal{F}, R_1, \dots, R_k \rangle$, for any $R_i (1 \leqslant i \leqslant k)$ an R_i-cluster of worlds is a subset C_{R_i} of $W_\mathcal{F}$ s.t.: $\forall w \forall z \in C_{R_i}(wR_i z\ \&\ zR_i w)$ and $\forall z \in W_\mathcal{F} \forall w \in C_{R_i}(((wR_i z\ \&\ zR_i w) \Rightarrow z \in C_{R_i})$. For any $R_i, C_{R_i}(w)$ is the R_i-cluster s.t. $w \in C_{R_i}$. Given two R_i-clusters C_m and C_j the expression $C_m R_i C_j$ is an abbreviation for $\forall w \in C_m, \forall z \in C_j (wR_i z)$.

Definition 1.3. An LTK_r*-frame* is a multi-modal frame $\mathcal{F} = \langle W_\mathcal{F}, R_T, R_\sim, R_1, \dots, R_k \rangle$, where:

(a) $W_\mathcal{F}$ is the disjoint union of certain nonempty sets C_n: $W_\mathcal{F} := \bigcup\limits_{n \in J} C_n$ where $J = [0, L]$ and $L \in N$ or $J = N$.

(b) R_T is the linear, reflexive and intransitive relation on $W_\mathcal{F}$ such that: $\forall w \forall z \in W_\mathcal{F}(wR_T z \Leftrightarrow [\exists\, n \in J((w \in C_n)\&(z \in C_n))] \vee [\exists\, n+1 \in J((w \in C_n)\&(z \in C_{n+1}))])$.

(c) $R_\sim$ is a universal relation on any $C_n \in W_\mathcal{F}$: $\forall w \forall z \in W_\mathcal{F}(wR_\sim z \Leftrightarrow \exists n \in J((w \in C_n)\&(z \in C_n)))$.

(d) $\forall i \in I, R_i$ is some equivalence relation on C_n.

Let $\mathsf{LTK_r}$ be the class of all LTK_r-frames.

Definition 1.4. An LTK_{ir}*-frame* is a multi-modal frame $\mathcal{F} = \langle W_\mathcal{F}, R_T, R_\sim, R_1, \dots, R_k \rangle$, where:

(a) $W_\mathcal{F}$ is the disjoint union of certain nonempty sets C_n: $W_\mathcal{F} := \bigcup\limits_{n \in J} C_n$ where $J = [0, L]$ and $L \in N$ or $J = N$.

(b) R_T is the linear, irreflexive and intransitive relation on $W_\mathcal{F}$ such that: $\forall w \forall z \in W_\mathcal{F}(wR_T z \Leftrightarrow \exists\, n+1 \in J((w \in C_n)\&(z \in C_{n+1})))$.

(c) $R_\sim$ is a universal relation on any $C_n \in W_\mathcal{F}$: $\forall w \forall z \in W_\mathcal{F}(wR_\sim z \Leftrightarrow \exists n \in J((w \in C_n)\&(z \in C_n)))$.

(d) $\forall i \in I, R_i$ is some equivalence relation on C_n.

Let $\mathsf{LTK_{ir}}$ be the class of all LTK_{ir}-frames.

Such frames simulate the situation in which agents, having a certain knowledge background at any moment, are operating in the linear flow of time. Each time-cluster (i.e. an R_T-cluster) C_n consists of a set of information points that are available at the moment n. The relation R_T

is the connection of such information points by the flow of time. That is, given two information points w and z, the expression wR_Tz means either that w and z are both available at a moment n, or that z will be available in the moment $n+1$ with respect to w. Since the relation $R_\sim$ connects all the information-points available at the same moment, it is intended to represent a sort of environmental knowledge, that is, the whole information potentially available for the agent at a given time. The relation R_i says which information points are effectively available for the agent i at any given moment.

Definition 1.5. Given a Kripke-frame $\mathcal{F}$, a *model* $\mathcal{M}_\mathcal{F}$ on $\mathcal{F}$ is a tuple $\mathcal{M}_\mathcal{F} = \langle \mathcal{F}, V\rangle$ where V is a valuation of a set P of propositional letters in $\mathcal{F}$. That is $\forall p \in P, \quad V(p) \subseteq W_\mathcal{F}$.

Given a model $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is an LTK_r-frame (LTK_{ir}-frame), the valuation V can be extended in the standard way from the set P onto all well formed formulae constructed from P. In particular, $\forall w \in W_\mathcal{F}$:

(a) $\langle \mathcal{F}, w\rangle \models_V p \Leftrightarrow w \in V(p)$;

(b) $\langle \mathcal{F}, w\rangle \models_V \Box_T A \Leftrightarrow \forall\, z \in W_\mathcal{F}(wR_Tz \Rightarrow \langle \mathcal{F}, z\rangle \models_V A)$;

(c) $\langle \mathcal{F}, w\rangle \models_V \Box_\sim A \Leftrightarrow \forall\, z \in W_\mathcal{F}(wR_\sim z \Rightarrow \langle \mathcal{F}, z\rangle \models_V A)$;

(d) $\forall\, i \in I, \langle \mathcal{F}, w\rangle \models_V \Box_i A \Leftrightarrow \forall\, z \in W_\mathcal{F}(wR_iz \Rightarrow \langle \mathcal{F}, z\rangle \models_V A)$.

A formula A is said to be true in the model $\mathcal{M}$ at the world w if $\langle \mathcal{F}, w\rangle \models_V A$. A formula A is true in the model $\mathcal{M}$, notation $\mathcal{F} \models_V A$, if $\forall\, w \in W_\mathcal{F}$, $\langle \mathcal{F}, w\rangle \models_V A$. A is valid in the frame $\mathcal{F}$, notation $\mathcal{F} \models A$ if, for any valuation V for $\mathcal{F}$, $\mathcal{F} \models_V A$. The expression $V(A)$ is an abbreviation for the set $\{w | w \models_V A\}$.

Definition 1.6. *Logic* LTK_r is the set of all LTK_r-valid formulae:

$$LTK_r := \{A \in Fma(\mathcal{L}^{LTK_r}) | \forall \mathcal{F} \in \mathsf{LTK_r}(\mathcal{F} \models A)\}.$$

If A belongs to LTK_r, then A is said to be a theorem of LTK_r.

Definition 1.7. *Logic* LTK_{ir} is the set of all LTK_{ir}-valid formulae:

$$LTK_{ir} := \{A \in Fma(\mathcal{L}^{LTK_{ir}}) | \forall \mathcal{F} \in \mathsf{LTK_{ir}}(\mathcal{F} \models A)\}.$$

If A belongs to LTK_{ir}, then A is said to be a theorem of LTK_{ir}.

Definition 1.8. Logic L has *effective finite model property (efmp)* if there is a computable function f such that for every formula $A \notin L$, there is a finite model $\langle \mathcal{F}, V\rangle$: $\mathcal{F} \not\models_V A$ and $\mathcal{F} \models_V B$, for any formula $B \in L$, where $||W_\mathcal{F}|| \leqslant f||A||$.

Definition 1.9. Logic L is *decidable* if there is an algorithm which, for any formula A, determines whether $A \in L$ holds.

Definition 1.10. Let $A \in Fma(\mathcal{L}^{LTK})$. *Modal time degree* $td(A)$ *of* A is defined as follows: $td(p) = td(T) = td(\bot) = 0$; $td(\neg\alpha) = td(\alpha)$; $td(\alpha \to \beta) = td(\alpha \vee \beta) = td(\alpha \wedge \beta) = max(td(\alpha), td(\beta))$; $td(\Box_\sim\alpha) = td(\Box_i\alpha) = td(\alpha)$; $td(\Box_T\alpha) = td(\alpha) + 1$.

Definition 1.11. Let $A \in Fma(\mathcal{L}^{LTK})$. *Modal time-static degree* $m(A)$ *of* A is defined as: $m(p) = m(T) = m(\bot) = 0$; $m(\neg\alpha) = m(\alpha)$; $m(\alpha \to \beta) = m(\alpha \vee \beta) = m(\alpha \wedge \beta) = max(m(\alpha), m(\beta))$; $m(\Box_T\alpha) = m(\alpha)$; $m(\Box_\sim\alpha) = m(\Box_i\alpha) = m(\alpha) + 1$.

2. Effective finite model property

The main question we will give an answer is: whether LTK_r and LTK_{ir} have the effective finite model property (efmp). We will first prove below that LTK_r has the efmp and hence it is decidable.

Theorem 2.1. *The logic LTK_r has the effective finite model property.*

Proof. Take a formula A such that $A \notin LTK_r$ and let $td(A) = n$. Then there are an LTK_r-frame $\mathcal{F} = \langle\{\bigcup_{l\in J} C_l\}, R_T, R_\sim, R_1, \dots, R_k\rangle$ ($J = [0, L], L \in N$ or $J = N$), valuation V and a world $x \in C_r$ such that $\langle\mathcal{F}, x\rangle \not\models_V A$. Denote the basic set $\{\bigcup_{l\in J} C_l\}$ as $W_\mathcal{F}$.

We start by reducing the number of worlds belonging to each R_T-cluster C of worlds from $W_\mathcal{F}$ using the standard filtration technique. We briefly sketch it below. Let $Sub(A)$ be the set of all the sub-formulae of A. Define the equivalence relation $\approx$ on $\mathcal{F}$ as follows:

$\forall w \forall z \in W_\mathcal{F}(w \approx z \Leftrightarrow wR_\sim z \;\&\; zR_\sim w \;\&\; \forall B \in Sub(A)\ (\langle\mathcal{F}, w\rangle \models_V B \Leftrightarrow \langle\mathcal{F}, z\rangle) \models_V B)$.

Then we define the quotient set of the original model: $\forall w \in W_\mathcal{F}$ ($[w] := \{z | w \approx z\}$, $[C] := \bigcup_{[w]\in C}[w]$). The filtrated model M_1 is define as $M_1 := \langle\mathcal{F}_1, V_1\rangle$ where:

(a) $W_{\mathcal{F}_1} := \{\bigcup_{l\in J}[C_l]\}$;

(b) $[w]R^1_\sim[z] \Leftrightarrow wR_\sim z$;

(c) $[w]R^1_T[z] \Leftrightarrow wR_T z$;

(d) $[w]R^1_i[z](1 \leqslant i \leqslant k) \Leftrightarrow ([w] \in C \;\&\; [z] \in C \;\&\; \forall B \in Sub(A)\ (\langle\mathcal{F}, w\rangle \models_V \Box_i B \Leftrightarrow \langle\mathcal{F}, z\rangle \models_V \Box_i B)$;

(e) $\forall p \in Sub(A)(V_1(p) := \{[w] | w \in V(p)\})$.

Since the model described is the result of a filtration, the standard filtration lemma holds:

Lemma 2.1. *For any formula $B \in Sub(A)$ and for any world $w \in W_\mathcal{F}$, $\langle\mathcal{F}, w\rangle \models_V B \Leftrightarrow \langle\mathcal{F}_1, [w]\rangle \models_{V_1} B$.*

Corollary 2.1. $\langle\mathcal{F}_1, [x]\rangle \not\models_{V_1} A$.

We define now the model $\mathcal{M}$ as follows:

$$\mathcal{M} = \langle\{[C_r], [C_{r+1}], [C_{r+2}], ..., [C_j]\}, R^1_T, R^1_\sim, R^1_1, \dots, R^1_k, V_1\rangle$$

where $j = \left\{ \begin{array}{ll} r + n, & \text{if } \mathcal{F}_1 \text{ is infinite or } r + n \leqslant L; \\ L, & \text{if } r + n > L. \end{array} \right\}$

It is easy to see, that $\mathcal{M}$ consists of at most $n + 1$ clusters.

Now we are going to show that if the model $\langle\mathcal{F}_1, V_1\rangle$ refutes a formula A, then the model $\mathcal{M}$ also refutes A.

Lemma 2.2. *For any formula α s.t. $td(\alpha) = 0$ and for any world $[y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$*

$$\langle\mathcal{F}_1, [y]\rangle \models_{V_1} \alpha \Longleftrightarrow \langle\mathcal{M}, [y]\rangle \models \alpha, \tag{2.1}$$

i.e. truth of time-static formulae on model $\mathcal{M}$ remains the same.

Proof. The proof can be given by induction on modal time-static degree $m(\alpha)$ of formula α.

If $m(\alpha) = 0$, then α consists of only propositional variables and the standard boolean operations. In this case the truth of α is determined by the valuation of propositional variables at the world $[y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$. Thus, by construction of a model $\mathcal{M}$, we have $\langle\mathcal{F}_1, [y]\rangle \models_{V_1} \alpha \Longleftrightarrow \langle\mathcal{M}, [y]\rangle \models \alpha$.

Let $m(\alpha) = l + 1$ and for all other α s.t. $m(\alpha) \leqslant l$ condition (2.1) holds. By definition wff, formula α can be obtained from the sub-formulae by is several ways specified below. We will show that condition (2.1) is true in all the cases.

1) Let $\alpha = \Box_\sim \alpha_1$ where $m(\alpha_1) = l$. If $\langle\mathcal{F}_1, [y]\rangle \models_{V_1} \alpha$ and $[y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$, then $\forall [z] \in [C(y)], \langle\mathcal{F}_1, [z]\rangle \models_{V_1} \alpha_1$. By the inductive hypothesis it follows $\forall [z] \in [C(y)]$, $\langle\mathcal{M}, [z]\rangle \models \alpha_1$. Hence $\langle\mathcal{M}, [y]\rangle \models \alpha$. The proof of converse case is similar.

2) Now let $\forall\, i \in \{1,\dots,k\}, \alpha = \Box_i \alpha_1$ and $m(\alpha_1) = l$. If $\langle \mathcal{F}_1, [y]\rangle \models_{V_1} \alpha$ with $[y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$, then for all worlds z s.t. $[y]R_i^1[z]$ we have $\langle \mathcal{F}_1, [z]\rangle \models_{V_1} \alpha_1$. By the inductive hypothesis it follows $\forall [z] : [y]R_i^1[z]$, $\langle \mathcal{M}, [z]\rangle \models \alpha_1$. i.e. $\langle \mathcal{M}, [y]\rangle \models \alpha$. The proof of converse case is similar.

3) Suppose that $\alpha = \alpha_1 \circ \alpha_2$ with $\circ \in \{\wedge, \vee\}$, besides $max(m(\alpha_1), m(\alpha_2)) = l+1$, $\alpha_1 = \Box_\xi \alpha_1'$ and $\alpha_2 = \Box_\xi \alpha_2'$ ($\xi \in \{\sim, 1, ..., k\}$). Then for any world $[y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$ holds $\langle \mathcal{F}_1, [y]\rangle \models_{V_1} \alpha_1 \circ \alpha_2 \iff \langle \mathcal{F}_1, [y]\rangle \models_{V_1} \alpha_1$ and/or $\langle \mathcal{F}_1, [y]\rangle \models_{V_1} \alpha_2 \iff \langle \mathcal{M}, [y]\rangle \models \alpha_1$ and/or $\langle \mathcal{M}, [y]\rangle \models \alpha_2 \iff \langle \mathcal{M}, [y]\rangle \models \alpha_1 \circ \alpha_2$.

Also if $\alpha = \neg\alpha_1$, $m(\alpha_1) = l+1$ and $\alpha_1 = \Box_\xi \alpha_2$, then $\forall [y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$ by previous proof we have $\langle \mathcal{F}_1, [y]\rangle \models_{V_1} \neg\alpha_1 \iff \langle \mathcal{F}_1, [y]\rangle \not\models_{V_1} \alpha_1 \iff \langle \mathcal{M}, [y]\rangle \not\models \alpha_1 \iff \langle \mathcal{M}, [y]\rangle \models \neg\alpha_1$.

Hence applying boolean operations does not change the truth of relation (2.1). Thus, composition of formulae of the forms 1), 2), and 3), by using a finite number of boolean operations holds (2.1).

Thus, by induction at the last step we have the following statement: $\forall\alpha$ s.t. $td(\alpha) = 0$ and $\forall [y] \in \{[C_r] \cup [C_{r+1}] \cup [C_{r+2}] \cup ... \cup [C_j]\}$ holds $\langle \mathcal{F}_1, [y]\rangle \models_{V_1} \alpha \iff \langle \mathcal{M}, [y]\rangle \models \alpha$.

So if formula α contains only time-static modalities, then the truth of α is determined only by the valuation of propositional variables at the worlds of cluster $[C(y)]$, so the condition (2.1) is true and Lemma 2.2 is proved. □

Lemma 2.3. *For any formula $B \in Sub(A)$ and for any world $[x] \in [C_r]$,*

$$\langle \mathcal{F}_1, [x]\rangle \models_{V_1} B \iff \langle \mathcal{M}, [x]\rangle \models B$$

Proof. If the length of generated subframe $C_r^{\leqslant}$ of frame $\mathcal{F}_1$ less then n, then model $\mathcal{M}$ coincides with the submodel $\langle C_r^{\leqslant}, V_1\rangle$, which already refuted the formula B and is already finite (has length $< n$). It remains to consider the case when the length of the subframe $C_r^{\leqslant}$ is longer then $n+1$, or frame $\mathcal{F}_1$ is infinite. In this case the length of model $\mathcal{M}$ is $n+1$ (i.e. $j - r = n$).

By definition of wff, formula B is constructed from sub-formulas α and β one of the following ways:

a) $B = \Box_T \alpha, td(\alpha) = l$, $(0 \leqslant l \leqslant n-1)$;

b) $B = \Box_\sim \alpha, td(\alpha) = l$, $(0 \leqslant l \leqslant n)$;

c) $B = \Box_i \alpha, td(\alpha) = l$, $(1 \leqslant i \leqslant k)$ and $(0 \leqslant l \leqslant n)$;

d) $B = \alpha \circ \beta$, $\circ \in \{\wedge, \vee\}$, $\max(td(\alpha), td(\beta)) = l$ and $(0 \leqslant l \leqslant n)$;

e) $B = \neg\alpha, td(\alpha) = l$, $(0 \leqslant l \leqslant n)$.

Our next step is to show that for any formula B such that $td(B) \leqslant s$ ($s \in [0, n]$) and for all m such that $r \leqslant j - m - s$, $\langle \mathcal{F}_1, [x]\rangle \models_{V_1} B \iff \langle \mathcal{M}, [x]\rangle \models B$ where $[x] \in [C_{j-m-s}]$. We conduct an induction on the modal time degree $td(B)$ of formula B.

The inductive base: if $td(B) = 0$ using Lemma 2.2 we have $\langle \mathcal{F}_1, [x]\rangle \models_{V_1} B \iff \langle \mathcal{M}, [x]\rangle \models B$.

The inductive hypothesis: suppose that for formula B s.t. $td(B) \leqslant s$ with $s \in [0, n-1]$, for all m s.t. $r \leqslant j - m - s$ and for worlds $[x] \in [C_{j-m-s}]$ assertion $\langle \mathcal{F}_1, [x]\rangle \models_{V_1} B \iff \langle \mathcal{M}, [x]\rangle \models B$ is true.

The inductive step: let $B = \Box_T \alpha$ $(td(\alpha) = s)$. If $\langle \mathcal{F}_1, [x]\rangle \models_{V_1} B$ where $[x] \in [C_{j-m-(s+1)}]$, then $\langle \mathcal{F}_1, [x]\rangle \models_{V_1} \alpha$ and for all worlds $[y]$ s.t $[x]R_T^1[y]$ the following is carried out: $\langle \mathcal{F}_1, [y]\rangle \models_{V_1} \alpha$. By the inductive hypothesis it implies $\langle \mathcal{M}, [x]\rangle \models \alpha$ & $\forall [y] : [x]R_T^1[y], \langle \mathcal{M}, [y]\rangle \models \alpha$. That is $\langle \mathcal{M}, [x]\rangle \models B$ and hence $\langle \mathcal{F}_1, [x]\rangle \models_{V_1} B \Longrightarrow \langle \mathcal{M}, [x]\rangle \models B$. The proof of converse case is similar.

Now show that the application of boolean operations $\Box_\sim$ and $\Box_i$ to a formula α of the form a) holds the case true. Note that in cases b)–e) the modal time degree of sub-formulas α and β does not increase and does not require checking the truth of B in the neighboring clusters.

Let $B = \Box_\sim\alpha$ with $td(\alpha) = s+1$ and $\alpha = \Box_T\alpha'$. If $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} B$ with $[x] \in [C_{j-m-(s+1)}]$, then $\forall[z] \in [C(x)], \langle\mathcal{F}_1, [z]\rangle \models_{V_1} \alpha$. By previous proof we have $\forall[z] \in [C(x)]$, $\langle\mathcal{M}, [z]\rangle \models \alpha$, hence $\langle\mathcal{M}, [x]\rangle \models B$. The proof of converse case is similar.

Let $B = \Box_i\alpha$ with $i \in \{1, \ldots, k\}$, $td(\alpha) = s+1$ and $\alpha = \Box_T\alpha'$. If $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} B$ where $[x] \in [C_{j-m-(s+1)}]$, then for all worlds z s.t. $[x]R_i^1[z]$ the following $\langle\mathcal{F}_1, [z]\rangle \models_{V_1} \alpha$ holds. By previous proof we have $\forall[z] : [x]R_i^1[z]$, $\langle\mathcal{M}, [z]\rangle \models \alpha$ or, in other words, $\langle\mathcal{M}, [x]\rangle \models B$. The proof of converse case is similar.

Let $B = \alpha \circ \beta$ with $\circ \in \{\wedge, \vee\}$, $\max(td(\alpha), td(\beta)) = s+1$, $\alpha = \Box_T\alpha'$ and $\beta = \Box_T\beta'$. Then for any world $[x] \in [C_{j-m-(s+1)}]$ we have $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} \alpha \circ \beta \Longleftrightarrow \langle\mathcal{F}_1, [x]\rangle \models_{V_1} \alpha$ and/or $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} \beta \Longleftrightarrow \langle\mathcal{M}, [x]\rangle \models \alpha$ and/or $\langle\mathcal{M}, [x]\rangle \models \beta \Longleftrightarrow \langle\mathcal{M}, [x]\rangle \models \alpha \circ \beta$.

Now let $B = \neg\alpha, td(\alpha) = s+1$ and $\alpha = \Box_T\alpha'$. Then $\forall[x] \in [C_{j-m-(s+1)}]$, $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} \neg\alpha \Longleftrightarrow \langle\mathcal{F}_1, [x]\rangle \not\models_{V_1} \alpha \Longleftrightarrow \langle\mathcal{M}, [x]\rangle \not\models \alpha \Longleftrightarrow \langle\mathcal{M}, [x]\rangle \models \neg\alpha$.

So applying time-static operations does not change the truth of relation $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} B \Longleftrightarrow \langle\mathcal{M}, [x]\rangle \models B$. Thus, composition of sub-formulas α and β of the form a) with time degree s $(0 \leq s \leq n)$, by using a finite number of boolean operations, $\Box_\sim$ or $\Box_i$, holds this relation true.

Thus, by induction at step $n+1$ we have the following statement: $\forall B \in Sub(A)$ s.t. $tb(B) \leqslant n$ and $\forall[x] \in [C_r]$ holds $\langle\mathcal{F}_1, [x]\rangle \models_{V_1} B \Longleftrightarrow \langle\mathcal{M}, [x]\rangle \models B$. Thus Lemma 2.3 is proved. □

Corollary 2.2. $\langle\mathcal{M}, [x]\rangle \not\models A$.

Note that the model $\mathcal{M}$ is a linear chain of at most $n+1$ clusters, and each R_T-cluster contains a finite number of worlds, bounded by the size of A, namely $||C|| \leqslant 2^{|Sub(A)|}$ for each R_T-cluster C. So we can conclude that $||\mathcal{M}|| \leqslant (n+1)2^{|Sub(A)|}$. Thus, if the formula A does not belong to LTK_r, then there is a finite model $\mathcal{M}$ (of effectively computable size) which refutes A. Hence, the logic LTK_r has effective finite model property. Therefore we immediately derive:

Theorem 2.2. *The logic LTK_r is decidable.*

Proof. The deciding algorithm for arbitrary formula α looks as follows:

1. Define the modal-time degree for α: $td(\alpha) = n$.

2. Construct all possible linear LTK_r-frames, which contain at most $n+1$ time-clusters and bounded by the size of α.

3. Consider all valuations of variables of α such that the resulting models are non-isomorphic. Since the number of variables of formula α is finite, the number of non-isomorphic models is also finite. If α is true in all such models, then it is the theorem of LTK_r, if at least one model refutes α, then it is not the theorem of LTK_r. □

Now consider the logic LTK_{ir}.

Theorem 2.3. *The logic LTK_{ir} has the effective finite model property and hence it is decidable.*

Proof. The proving schemata almost verbatim follows the proof of the case for LTK_r, but the model $\mathcal{M}$ in this case is defined as follows:

$$\mathcal{M} = \langle\{[C_r], [C_{r+1}], [C_{r+2}], ..., [C_j]\}, R_T^1, R_\sim^1, R_1^1, \ldots, R_k^1, V\rangle$$

where $j = \left\{ \begin{array}{ll} r+n+1, & \text{if } \mathcal{F} \text{ is infinite or } r+n+1 \leqslant L; \\ \text{L}, & \text{if } r+n+1 > L, \end{array} \right\}$ and for time-cluster C_{r+n+1} the relation R_T is defined to be reflexive. The other steps of the proof for this case may be easy adjusted from the previous proof.

Since there is a effectively finite model which refutes any formula A which does not belong to LTK_{ir}, then LTK_{ir} has the effective finite model property, and hence LTK_{ir} is decidable. □

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Разрешимость многомодальной линейной логики знания и времени LTK

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В представленной статье используется семантический подход к построению моделей, комбинирующих модальности знания и времени. Семантически вводятся многомодальные логики LTK_r и LTK_{ir}, содержащие модальности знания и времени как множество формул, истинных на фреймах специального вида. Главным результатом работы являются теоремы об эффективной финитной аппроксимируемости и, как следствие, разрешимости данных логик.

Ключевые слова: многомодальная логика, линейная временная логика, разрешимость, эффективная финитная аппроксимируемость.