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On the Spectral Properties of a Non-coercive Mixed Problem Associated with $\overline{\partial}$-operator

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We consider a non-coercive Sturm–Liouville boundary value problem in a bounded domain D of the complex space $\mathbb{C}^n$ for the perturbed Laplace operator. More precisely, the boundary conditions are of Robin type on ∂D while the first order term of the boundary operator is the complex normal derivative. We prove that the problem is Fredholm one in proper spaces for which an Embedding Theorem is obtained; the theorem gives a correlation with the Sobolev-Slobodetskii spaces. Then, applying the method of weak perturbations of compact self-adjoint operators, we show the completeness of the root functions related to the boundary value problem in the Lebesgue space. For the ball, we present the corresponding eigenvectors as the product of the Bessel functions and the spherical harmonics.

Keywords: Sturm-Liouville problem, non-coercive problems, the multidimensional Cauchy-Riemann operator, root functions.

Introduction

Non-coercive boundary value problems for elliptic differential operators attract attention of mathematicians since the middle of XX-th century (see, for instance, [1, 2]). One of the typical problems of this type is the famous $\overline{\partial}$-Neumann problem for the Dolbeault complex (see [3]). The investigation of the problem resulted in the discovery of the subellipticity phenomenon which greatly influenced to the development of the Theory of Partial Differential Equations (cf. [4]).

As it is known (under reasonable assumptions) the Spectral Theory gives both the solvability conditions and the formulae for the exact and the approximate solutions to boundary value problems via expansions over (generalized) eigenfunctions related to the corresponding linear operators (see, for instance, [5] and elsewhere). This is well understood for the coercive boundary value problems in smooth domains for both self-adjoint and non-selfadjoint cases (see [6–8]). For the Spectral Theory related to the elliptic problems in Lipschitz domains we refer to the survey [9] and its bibliography (see also [10, 11] for the domains with the conic and edge singularities). Recently Agranovich [12] noted that the use of the negative Sobolev spaces gives an additional advantage proving the completeness of the root functions related to the coercive boundary value problems in non-smooth domains.

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The aim of the present paper is to extend the results to the non-coercive boundary value problem for the weakly perturbed Laplace operator in the complex space $\mathbb{C}^n$ ($\cong \mathbb{R}^{2n}$). First, using the standard methods of the Functional Analysis (see [13,14] and elsewhere) we prove that the problem is a Fredholm one in the proper Sobolev type spaces. Then, applying the method of weak perturbations of compact self-adjoint operators (see [6]), we prove the completeness of the generalized eigenvectors related to the boundary value problem in the Lebesgue space. Examples of the eigenfunctions related to the problem in the ball are constructed.

1. The mixed problem

Let D be a bounded domain in the complex space $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with a Lipschitz boundary, i.e., the surface ∂D is locally the graph of a Lipschitz function. In particular, the boundary ∂D possesses a tangent hyperplane almost everywhere.

Let the complex structure in $\mathbb{C}^n$ be given by $z_j = x_j + \sqrt{-1}x_{n+j}$ with $j = 1, \dots, n$ and $\bar\partial$ stand for the Cauchy-Riemann operator corresponding to this structure, i.e., the column of n complex derivatives $\dfrac{\partial}{\partial \bar z_j} = \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_j} + \sqrt{-1}\dfrac{\partial}{\partial x_{n+j}}\Big)$. The formal adjoint $\bar\partial^*$ of $\bar\partial$ with respect to the usual Hermitian structure in the space $L^2(\mathbb{C}^n)$ is the line of n operators $-\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x_j} - \sqrt{-1}\dfrac{\partial}{\partial x_{n+j}}\Big) =: -\dfrac{\partial}{\partial z_j}$. Then an easy computation shows that $\bar\partial^*\bar\partial$ just amounts to the $-1/4$ multiple of the Laplace operator $\Delta_{2n} = \displaystyle\sum_{j=1}^{2n} \left(\frac{\partial}{\partial x_j}\right)^2$ in $\mathbb{R}^{2n}$.

We consider complex-valued functions defined in the domain D and its closure $\overline{D}$. We write $L^q(D)$ for the Lebesgue space, i.e. the set of all measurable functions u in D, such that the integral of $|u|^q$ over D is finite. We also write $H^s(D)$, $s \in \mathbb{N}$, for the corresponding Sobolev space of functions with all the weak derivatives up to order s belonging to $L^2(D)$. For non-negative non-integer s we denote by $H^s(D)$ the Sobolev-Slobodetskii space, see, for instance, [14].

Consider the second order linear partial differential operator A in the domain D associated with the Cauchy-Riemann operator:

$$Au = -\Delta_{2n} + \sum_{j=1}^{n} a_j(z)\frac{\partial u}{\partial \bar z_j} + a_0(z)u,$$

the coefficients a_j and a_0 being of class $L^\infty(D)$. Consider also a first order boundary operator

$$B = b_1(z)\overline{\partial}_\nu + b_0(z)$$

where $\overline{\partial}_\nu = \displaystyle\sum_{j=1}^{n} \big(\nu_j(z) - \sqrt{-1}\nu_{j+n}(z)\big)\ \frac{\partial}{\partial \bar z_j}$ is the complex normal derivative and $\nu(z) = (\nu_1(z), \dots \nu_{2n}(z))$ is the unit normal vector to ∂D at the point z (cf. with the usual normal derivative $\dfrac{\partial}{\partial\nu} = \displaystyle\sum_{j=1}^{2n} \nu_j(z)\ \frac{\partial}{\partial x_j}$). The coefficients $b_0(z)$ and $b_1(z)$ are assumed to be bounded measurable functions on ∂D satisfying $|b_0|^2 + |b_1|^2 \neq 0$. We allow the function $b_1(z)$ to vanish on an open connected subset S of ∂D with piecewise smooth boundary ∂S.

Consider the following boundary value problem with the Robin-type condition on the surface ∂D. Given a distribution f in D, find a distribution u in D which satisfies in a proper sense

$$\left\{ \begin{array}{lllll} Au & = & f & \text{in} & D, \\ Bu & = & 0 & \text{on} & \partial D. \end{array} \right. \tag{1}$$

Note that in general the Shapiro-Lopatinskii condition is violated on the smooth part of $\partial D \setminus S$ for the pair (A, B) because if $S = \varnothing$, $a_j \equiv 0$ for all $j = 0, \dots, n$ and $b_0 \equiv 0$ problem (1) is a version of the famous $\overline{\partial}$-Neumann problem (cf. [3]).

Denote by $H^1(D, S)$ the subspace of $H^1(D)$ consisting of those functions whose restriction to the boundary vanishes on S. This space is Hilbert under the induced norm. It is easily seen that smooth functions on $\overline{D}$ vanishing in a neighborhood of $\overline{S}$ are dense in $H^1(D, S)$; then the space $H^1(D, \partial D)$ is usually denoted $H^1_0(D)$. Since on S the boundary operator reduces to $B = b_0$ and $b_0(z) \neq 0$ for $z \in S$, the functions of $H^1(D)$ satisfying $Bu = 0$ on ∂D belong to $H^1(D, S)$.

As we want to study perturbations of self-adjoint operators we split both a_0 and b_0 into two parts $a_0 = a_{0,0} + \delta a_0$, $b_0 = b_{0,0} + \delta b_0$, where $a_{0,0}$ is a non-negative bounded function in D and $b_{0,0}$ a bounded function on ∂D satisfying $b_{0,0}/b_1 \geqslant 0$. Consider now the Hermitian form

$$(u, v)_+ = 4 \sum_{j=1}^{n} \left(\frac{\partial u}{\partial \bar{z}_j}, \frac{\partial v}{\partial \bar{z}_j} \right)_{L^2(D)} + (a_{0,0} u, v)_{L^2(D)} + 4 (b_{0,0} b_1^{-1} u, v)_{L^2(\partial D \setminus S)}.$$

on the space $H^1(D, S)$. It follows from the Uniqueness Theorem for holomorphic functions that the form defines a scalar product on $H^1(D, S)$ if one of the following conditions holds true:

1) the open set $S \subset \partial D$ is not empty;

2) $a_{0,0} \geqslant c_0$ in $\overline{U}$ with some constant $c_0 > 0$ on an open non-empty set $U \subset D$;

3) $b_{0,0} \geqslant c_1$ in $\overline{V}$ with some constant $c_0 > 1$ on an open non-empty set $V \subset \partial D \setminus S$.

Then we denote by $H^+(D)$ the completion of $H^1(D, S)$ with respect to the norm $\| \cdot \|_+$ coherent with the scalar product $(\cdot, \cdot)_+$.

From now on we assume that the space $H^+(D)$ is continuously embedded into the Lebesgue space $L^2(D)$, i.e.,

$$\|u\|_{L^2(D)} \leq c \, \|u\|_+ \text{ for all } u \in H^+(D), \tag{2}$$

where c is a constant independent of u. It is true under rather weak assumptions (see Theorem 1 below). Now we need the continuous inclusion

$$\iota : H^+(D) \hookrightarrow L^2(D), \tag{3}$$

to specify the dual space of $H^+(D)$ via the pairing in $L^2(D)$. More precisely, let $H^-(D)$ be the completion of $H^1(D, S)$ with respect to the negative norm (cf. [15])

$$\|u\|_- = \sup_{\substack{v \in H^1(D,S) \\ v \neq 0}} \frac{|(v, u)_{L^2(D)}|}{\|v\|_+}.$$

Lemma 1. *The space $L^2(D)$ is continuously embedded into $H^-(D)$. If inclusion (3) is compact then the space $L^2(D)$ is compactly embedded into $H^-(D)$.*

Proof. By definition and estimate (2) we get

$$\|u\|_- \leqslant \sup_{\substack{v \in H^1(D,S) \\ v \neq 0}} \frac{\|u\|_{L^2(D)} \|v\|_{L^2(D)}}{\|v\|_+} \leqslant c \, \|u\|_{L^2(D)}$$

for all $u \in L^2(D)$, i.e., the space $L^2(D)$ is continuously embedded into $H^-(D)$ indeed.

Suppose (3) is compact. Then the Hilbert space adjoint $\iota^* : L^2(D) \hookrightarrow H^+(D)$ is compact, too. As $H^1(D,S)$ is dense in $H^+(D)$ and the norm $\| \cdot \|_+$ majorizes $\| \cdot \|_{L^2(D)}$ we conclude that

$$\|u\|_- = \sup_{\substack{v \in H^1(D,S) \\ v \neq 0}} \frac{|(\iota(v), u)_{L^2(D)}|}{\|v\|_+} = \sup_{\substack{v \in H^+(D) \\ v \neq 0}} \frac{|(v, \iota^*(u))_+|}{\|v\|_+} = \|\iota^*(u)\|_+ \tag{4}$$

for all $u \in L^2(D)$. Therefore, any weakly convergent sequence in $L^2(D)$ converges in $H^-(D)$, which shows the second part of the lemma. □

Since $C^\infty_{\mathrm{comp}}(D)$ is dense in $L^2(D)$ and the norm $\| \cdot \|_{L^2(D)}$ majorizes the norm $\| \cdot \|_-$, we conclude that $C^\infty_{\mathrm{comp}}(D)$ is dense in $H^-(D)$, too.

Lemma 2. *The Banach space $H^-(D)$ is topologically isomorphic to the dual space $H^+(D)'$ and the isomorphism is defined by the sesquilinear pairing*

$$\langle v, u \rangle = \lim_{\nu \to \infty} (v, u_\nu)_{L^2(D)} \tag{5}$$

for $u \in H^-(D)$ and $v \in H^+(D)$ where $\{u_\nu\}$ is any sequence in $H^1(D,S)$ converging to u.

Proof. See, for instance, [16, Theorem 1.4.28]. □

Note that $H^+(D)$ is reflexive, since it is a Hilbert space. Hence it follows that $(H^-(D))' = H^+(D)$, i.e., the spaces $H^+(D)$ and $H^-(D)$ are dual to each other with respect to (5).

From now on the Sobolev space $H^{-s}(D)$, $s > 0$, stands for the dual to $H^s(D)$ via the pairing induced by the scalar product $(\cdot,\cdot)_{L^2(D)}$ as in Lemma 2 above. Similarly, let $\tilde H^{-s}(D)$, $s > 0$, stands for the dual to $H^s_0(D)$. Obviously $H^{-s}(D) \subset \tilde H^{-s}(D)$. We also denote by $h^s(D)$ the space of all the harmonic functions in the domain D belonging to the Sobolev space $H^s(D)$.

Theorem 1. *Let ∂D be a Lipschitz surface. Then*

1) the space $H^1(D,S)$ is continuously embedded into $H^+(D)$ if $b_{0,0} b_1^{-1} \in L^\infty(\partial D \setminus S)$;

2) the elements of $H^+(D)$ belong to $H^1_{\mathrm{loc}}(D \cup S, S)$; in particular, if $S = \partial D$ then the space $H^+(D)$ is continuously embedded into $H^1_0(D)$;

3) the space $H^+(D)$ is continuously embedded into $L^2(D)$ if

$$a_{0,0} \geqslant c_0 \text{ in } \overline{D} \text{ with some constant } c_0 > 0; \tag{6}$$

4) the space $H^+(D)$ is continuously embedded into $h^{1/2-\varepsilon}(D) \oplus H^1_0(D)$ with any $\varepsilon > 0$ if

$$b_{0,0} b_1^{-1} \geqslant c_1 \text{ in } \partial D \setminus S \text{ with some constant } c_1 > 0. \tag{7}$$

Moreover, if $\partial D \in C^2$ then, under (7), the space $H^+(D)$ is continuously embedded into the space $h^{1/2}(D) \oplus H^1_0(D)$. In particular, estimate (7) implies that ι is compact.

Proof. If $b_{0,0} b_1^{-1} \in L^\infty(\partial D \setminus S)$ then, according to the Trace Theorem for the Sobolev spaces, we obtain

$$\|u\|^2_+ \leqslant \|\sqrt{a_{0,0}}\|_{L^\infty(D)} \|u\|^2_{L^2(D)} + \|\sqrt{b_{0,0}} b_1^{-1}\|_{L^\infty(\partial D \setminus S)} \|u\|^2_{L^2(\partial D)} + \|u\|^2_{H^1(D)} \leqslant c\, \|u\|^2_{H^1(D)}$$

for all $u \in H^1(D,S)$ with some positive constant c independent on u. This proves 1).

The statement 2) follows from the fact that the Dirichlet problem for the Helmholtz operator $(a_{0,0} - \Delta_{2n})$ is coercive.

Now using the definition of the norm $\|\cdot\|_+$ we see that

$$\|u\|_+ \geqslant \|\sqrt{a_{0,0}}u\|_{L^2(D)} \geqslant \sqrt{c_0}\,\|u\|^2_{L^2(D)}$$

i.e. estimate (2) holds true and the embedding ι is continuous under estimate (6).

Further, let (7) holds. Then the norm $\|\cdot\|_+$ is not weaker than the norm $\|\cdot\|_h$ on $H^1(D,S)$ defined by

$$\|u\|_h = \Big(4\sum_{j=1}^n \Big\|\frac{\partial u}{\partial \bar z_j}\Big\|^2_{L^2(D)} + 4\|u\|^2_{L^2(\partial D\setminus S)}\Big)^{1/2},\ u \in H^1(D,S).$$

Fix a number $\varepsilon > 0$. Let us show that the norm $\|\cdot\|_h$ is not weaker than the norm $\|\cdot\|_{H^{1/2-\varepsilon}(D)}$ on $H^1(D,S)$. Indeed, let ϕ_{2n} denote the two-sided fundamental solution of the convolution type for the Laplace operator Δ_{2n} in $\mathbb{R}^{2n}$. Then the volume potential

$$\Phi v(x) = \int_D \phi_{2n}(x-y)v(y)dx,\ v \in L^2(D), \tag{8}$$

induces the bounded linear operator $\Phi : L^2(D) \to H^2(X)$ for any bounded domain X containing D. It is clear that any element $u \in H^{-s}(D)$ extends up to an element $U \in H^{-s}(\mathbb{R}^{2n})$ via

$$\langle U, v\rangle_{\mathbb{R}^{2n}} = \langle u, v\rangle_D \text{ for all } v \in H^s(\mathbb{R}^{2n});$$

here $\langle\cdot,\cdot\rangle_D$ is the pairing on $H \times H'$ for a space H of distributions over D. It is natural to denote it by $\chi_D u$. Thus, the defined in this way linear operator $\chi_D : H^{-s}(D) \to H^{-s}(\mathbb{R}^{2n})$, $s \in \mathbb{R}_+$, is obviously bounded. The distribution $\chi_D u$ is supported in $\overline{D}$, so we actually may reduce our consideration to a smooth closed manifold. This allows us to conclude that the volume potential (8) induces the bounded linear operator

$$\Phi \circ \chi_D : H^{\varepsilon-1/2}(D) \to H^{\varepsilon+3/2}(X), \quad 0 < \varepsilon \leqslant 1/2,$$

for any bounded domain X containing D (see, [17]). Hence, the operators

$$\frac{\partial}{\partial \bar z_j} \circ \Phi \circ \chi_D : H^{\varepsilon-1/2}(D) \to H^{\varepsilon+1/2}(X) \text{ and } \overline{\partial}_\nu \circ \Phi \circ \chi_D : H^{\varepsilon-1/2}(D) \to H^{\varepsilon}(\partial D)$$

are bounded, too, if $0 < \varepsilon \leqslant 1/2$ (the last one is bounded because of the Trace Theorem for the Sobolev spaces). For $\varepsilon = 0$ the arguments fail because the elements of the space $H^{1/2}(X)$ may have no traces on $\partial D \subset X$.

Now the integration by parts with $u \in H^1(D,S)$ and $v \in L^2(D)$ yields

$$(v,u)_{L^2(D)} = (\Delta_{2n}\Phi v, u)_{L^2(D)} = 4\sum_{j=1}^n \Big(\frac{\partial \Phi v}{\partial \bar z_j}, \frac{\partial u}{\partial \bar z_j}\Big)_{L^2(D)} + 4(\overline{\partial}_\nu \Phi v, u)_{L^2(\partial D\setminus S)}. \tag{9}$$

Take a sequence $\{v_k\} \subset C^\infty(\overline{D})$ converging to v in the space $H^{\varepsilon-1/2}(D)$, $0 < \varepsilon < 1/2$. As the space $H^s(D)$ is reflexive for each s, using (9) and the continuity of the operators $\dfrac{\partial}{\partial \bar z_j} \circ G \circ \chi_D$, $\overline{\partial}_\nu \circ G \circ \chi_D$ above, we obtain for $u \in H^1(D,S)$:

$$\|u\|_{H^{1/2-\varepsilon}(D)} = \sup_{\substack{v \in H^{\varepsilon-1/2}(D)\\ v \neq 0}} \frac{|\langle v, u\rangle|}{\|v\|_{H^{\varepsilon-1/2}(D)}} = \sup_{\substack{v \in H^{\varepsilon-1/2}(D)\\ v \neq 0}} \frac{\lim_{k\to+\infty} |(v_k, u)_{L^2(D)}|}{\|v\|_{H^{\varepsilon-1/2}(D)}} =$$

$$= 4 \sup_{\substack{v \in H^{\varepsilon-1/2}(D)\\ v \neq 0}} \frac{|\sum_{j=1}^n (\frac{\partial}{\partial \bar z_j} \circ \Phi \circ \chi_D v, \frac{\partial u}{\partial \bar z_j})_{L^2(D)} + (\overline{\partial}_\nu \circ \Phi \circ \chi_D v, u)_{L^2(\partial D\setminus S)}|}{\|v\|_{H^{\varepsilon-1/2}(D)}} \leqslant$$

$$\leqslant c \left(\sum_{j=1}^{n} \left\| \frac{\partial}{\partial \bar{z}_j} \circ \Phi \circ \chi_D \right\| \left\| \frac{\partial u}{\partial \bar{z}_j} \right\|_{L^2(D)} + \left\| \overline{\partial}_\nu \circ \Phi \circ \chi_D \right\| \|u\|_{L^2(\partial D \setminus S)} \right).$$

with a constant $c > 0$ being independent on u. Thus, there are constant $C_1 > 0$, $C_2 > 0$ such that

$$\|u\|_{H^{1/2-\varepsilon}(D)} \leqslant C_1 \|u\|_h \leqslant C_2 \|u\|_+ \text{ for all } u \in H^1(D, S).$$

This proves the continuous embedding $H^+(D) \hookrightarrow H^{1/2-\varepsilon}(D)$ with any $\varepsilon > 0$.

Further, let G and P stand for the Green function and the Poisson integral of the Dirichlet Problem for the Laplace operator Δ_{2n} in D respectively. Then they induce the bounded operators (cf. [13,15])

$$G_1 : \tilde{H}^{-1}(D) \to H^1_0(D), \quad P_1 : H^{1/2}(D) \to h^1(D).$$

As the operator Δ_{2n} extends to the continuous linear operator $\Delta_{2n} : H^1(D) \to \tilde{H}^{-1}(D)$ via

$$\langle \Delta_{2n} u, v \rangle = 4(\overline{\partial} u, \overline{\partial} v)_{L^2(D)}, u \in H^1(D), v \in H^1_0(D),$$

we see that $u = P_1 u + G_1 \Delta_{2n} u$ for each $u \in H^1(D)$. Hence, for $u, v \in H^1(D, S)$, we obtain:

$$(u, v)_h = (Pu, Pv)_{L^2(\partial D \setminus S)} + (\overline{\partial} G \Delta_{2n} u, (\overline{\partial} G \Delta_{2n} v)_{L^2(D)}. \tag{10}$$

In particular,

$$\|u\|^2_+ \geqslant \|u\|^2_h = \|P_1 u\|^2_{L^2(\partial D \setminus S)} + \|\overline{\partial} G_1 \Delta_{2n} u\|^2_{L^2(D)} \text{ for all } u \in H^1(D, S).$$

On the other hand, the Gårding inequality yields

$$\|v\|_{H^1(D)} \leqslant \|\overline{\partial} v\|_{L^2(D)} \text{ for all } u \in H^1_0(D). \tag{11}$$

Therefore, using (10) and (11) we conclude that any sequence $\{u_k\} \subset H^1(D, S)$ converging to $u \in H^+(D)$ in the space $H^+(D)$ can be presented as

$$u_\nu = P_1 u_k + G_1 \Delta_{2n} u_k$$

where the sequence $\{G_1 \Delta_{2n} u_k\}$ converges in $H^1_0(D) \subset H^1(D, S)$ to an element w_1. Now the already proved part of the theorem yields that $\{P_1 u_k\}$ converges to an element w_2 in $H^{1/2-\varepsilon}(D)$. According to the Stiltjes–Vitali Theorem the element w_2 is harmonic in D. Hence

$$u = w_1 + w_2, \quad \Delta_{2n} u = \Delta_{2n} w, \quad u = Pu + G_1 \Delta_{2n} u; \tag{12}$$

here Pu is the Poisson integral of the trace $u_{|\partial D} \in L^2(\partial D)$ related to $u \in H^+(D)$. This proves the continuous embedding $H^+(D) \hookrightarrow h^{1/2-\varepsilon}(D) \oplus H^1_0(D)$.

Finally, if $\partial D \in C^2$ then we may use the regularity of the solutions to the Dirichlet Problem for the Laplace operator in D. More precisely, in this case we have the bounded linear operators

$$G_2 : L^2(D) \to H^2(D), \quad \overline{\partial}_\nu \circ G_2 : L^2(D) \to H^{1/2}(\partial D), \quad P_2 : H^{3/2}(\partial D) \to H^2(D);$$

for a Lipschitz boundary these may be not true in general.

To finish the proof we will show that the Poisson integral P induces the bounded linear operator $P_{1/2} : L^2(\partial D) \to H^{1/2}(D)$. With this aim, for $u_0 \in H^{-1/2}(\partial D)$ take a sequence $\{u_{0_k}\} \subset H^{1/2}(\partial D)$ converging to u_0 in $H^{-1/2}(\partial D)$. Then, integrating by parts we obtain:

$$\|P_1 u_{0_k}\|_{L^2(D)} = \sup_{\substack{v \in L^2(D) \\ v \neq 0}} \frac{|(v, P_1 u_{0_k})_{L^2(D)}|}{\|v\|_{L^2(D)}} = \sup_{\substack{v \in L^2(D) \\ v \neq 0}} \frac{|(\Delta_{2n} G_2 v, P_1 u_{0_k})_{L^2(D)}|}{\|v\|_{L^2(D)}} \leqslant$$

$$\leqslant \sup_{\substack{v\in L^2(D)\\ v\neq 0}} \frac{|(\overline{\partial}_\nu G_2 v, u_{0_k})_{L^2(\partial D)}|}{\|v\|_{L^2(D)}} \leqslant \sup_{\substack{v\in L^2(D)\\ v\neq 0}} \frac{\|\overline{\partial}_\nu G_2 v\|_{H^{1/2}(\partial D)}\|u_{0_k}\|_{H^{-1/2}(\partial D)}}{\|v\|_{L^2(D)}} \leqslant$$

$$\leqslant \|\overline{\partial}_\nu G_2\| \|u_{0_k}\|_{H^{-1/2}(\partial D)}.$$

Hence the sequence $\{P_1 u_{0_k}\}$ converges in $L^2(D)$ and the Poisson integral P induces the bounded linear operator $P_0 : H^{-1/2}(\partial D) \to L^2(D)$. Now we may use the interpolations arguments (see [14], [18]). Indeed, by the interpolation, the Poisson integral P induces the bounded linear operators

$$P_\theta : [H^{-1/2}(\partial D), H^{1/2}(\partial D)]_\theta \to [L^2(D), H^1(D)]_\theta, \quad 0 < \theta < 1,$$

where $[H_0, H_1]_\theta$ means the interpolation between the pair H_0 and H_1 of Hilbert spaces. But

$$[L^2(D), H^1(D)]_\theta = H^\theta(D), \quad [H^{-1/2}(\partial D), H^{1/2}(\partial D)]_\theta = H^{1/2-\theta}(\partial D),$$

see, for instance, [14, Ch. I, Theorems 9.6 and 12.5]. Therefore, choosing $\theta = 1/2$ we conclude that the Poisson integral P induces the bounded linear operator $P_{1/2} : L^2(\partial D) \to H^{1/2}(D)$. Hence (12) implies the continuous embedding $H^+(D) \hookrightarrow h^{1/2}(D) \oplus H^1_0(D)$ if $\partial D \in C^2$. □

We emphasize that the space $H^+(D)$ is not continuously embedded into $H^1(D)$ unless $S = \partial D$, because the Shapiro-Lopatinskii condition is violated on the smooth part of $\partial D \setminus S$. Actually the embeddings described in Theorem 1 are sharp at least for the ball (see Examples 1 and 2 below).

Further, on integrating by parts we see that

$$(Au, v)_{L^2(D)} = 4\sum_{j=1}^n \left(\frac{\partial u}{\partial \bar z_j}, \frac{\partial v}{\partial \bar z_j}\right)_{L^2(D)} + 4\left(b_1^{-1} b_0 u, v\right)_{L^2(\partial D\setminus S)} + \Big(\sum_{j=1}^n a_j \frac{\partial u}{\partial \bar z_j} + a_0 u, v\Big)_{L^2(D)}$$

for all $u \in H^2(D)$ and $v \in H^1(D)$ satisfying the boundary condition of (1). Suppose that

$$|\delta b_0| \leqslant \hat c_1 |\, b_{0,0}| \text{ on } \partial D \setminus S \text{ with a positive constant } \hat c_1. \tag{13}$$

Then, if

$$|\delta a_0| \leqslant \hat c_2 |\, a_{0,0}| \text{ on } \overline{D} \text{ with a positive constant } \hat c_2. \tag{14}$$

or (7) is fulfilled, we have

$$\Big| \left(b_1^{-1} \delta b_0 u, v\right)_{L^2(\partial D\setminus S)} + \Big(\sum_{j=1}^n a_j \frac{\partial u}{\partial \bar z_j} + \delta a_0\, u, v\Big)_{L^2(D)} \Big| \leqslant c\, \|u\|_+ \|v\|_+ \tag{15}$$

for all $u, v \in H^1(D, S)$, where c is a positive constant independent of u and v. Therefore, in these cases for each fixed $u \in H^+(D)$, the sesquilinear form

$$Q(u,v) = 4\left(\frac{\partial u}{\partial \bar z_j}, \frac{\partial v}{\partial \bar z_j}\right)_{L^2(D)} + 4\left(b_1^{-1} b_0 u, v\right)_{L^2(\partial D\setminus S)} + \Big(\sum_{j=1}^n a_j \frac{\partial u}{\partial \bar z_j} + a_0 u, v\Big)_{L^2(D)}$$

determines a continuous linear functional f on $H^+(D)$ by $f(v) := \overline{Q(u,v)}$ for $v \in H^+(D)$. By Lemma 2, there is a unique element in $H^-(D)$, which we denote by Lu, such that

$$f(v) = \langle v, Lu\rangle$$

for all $v \in H^+(D)$. We have thus defined a linear operator $L : H^+(D) \to H^-(D)$. From (15) it follows that L is bounded. The bounded linear operator $L_0 : H^+(D) \to H^-(D)$ defined in the same way via the sesquilinear form $(\cdot, \cdot)_+$, i.e.,

$$(v, u)_+ = \langle v, L_0 u \rangle \tag{16}$$

for all $u, v \in H_+(D)$, corresponds to the case $a_j \equiv 0$ for all $j = 1, \dots, n$, $a_0 = a_{0,0}$, and $b_0 = b_{0,0}$.

We are thus lead to a weak formulation of problem (1). Given $f \in H^-(D)$, find $u \in H^+(D)$, such that

$$\overline{Q(u, v)} = \langle v, f \rangle \text{ for all } v \in H^+(D). \tag{17}$$

Now one can handle problem (17) by standard techniques of functional analysis, see for instance [13, Ch. 3, §§ 4–6]) for the coercive case. As the properties of the Dirichlet problem are well known, we will be concentrated on the study of the mixed problem under condition (6) or condition (7) of Theorem 1 in the case $S \neq \partial D$.

Lemma 3. *Assume that $a_j \equiv 0$ for all $j = 1, \dots, n$, $\delta a_0 = 0$, and $\delta b_0 = 0$. If (6) or (7) hold then for each $f \in H^-(D)$ there is a unique solution $u \in H^+(D)$ to problem (17), i.e., the operator $L_0 : H^+(D) \to H^-(D)$ is continuously invertible. Moreover, the norms of both L_0 and its inverse L_0^{-1} are equal to* 1.

Proof. Under the hypotheses of the lemma, (17) is just a weak formulation of problem (1) with A and B replaced by $A_0 = -\Delta_{2n} + a_{0,0}$, $B_0 = b_1 \overline{\partial}_\nu + b_{0,0}$, respectively. The corresponding bounded operator in Hilbert spaces just amounts to $L_0 : H^+(D) \to H^-(D)$ defined by (16). Its norm equals 1, for, by Lemma 2, we get

$$\|L_0 u\|_- = \sup_{\substack{v \in H^+(D) \\ v \neq 0}} \frac{|(\langle v, L_0 u \rangle|}{\|v\|_+} = \sup_{\substack{v \in H^+(D) \\ v \neq 0}} \frac{|(v, u)_+|}{\|v\|_+} = \|u\|_+ \tag{18}$$

whenever $u \in H^+(D)$.

The existence and uniqueness of solutions to problem (17) follows immediately from the Riesz theorem on the general form of continuous linear functionals on Hilbert spaces. From (18) we conclude that L_0 is actually an isometry of $H^-(D)$ onto $H^+(D)$, as desired. □

Corollary 1. *Let estimates (7), (13) be fulfilled and the constant $\hat{c}$ in (13) satisfy $0 < \hat{c}_1 < 1$. Then problem (17) is Fredholm.*

Proof. If $a_j = 0$ for all $1 \leq j \leqslant n$ and $\delta a_0 = 0$ then, under the hypothesis of the corollary, estimate (15) holds with $0 < c < 1$. In this case the operator $L_1 : H^+(D) \to H^-(D)$ corresponding to problem (17) is easily seen to differ from L_0 by a bounded operator $\delta L_1 : H^+(D) \to H^-(D)$ whose norm does not exceed $0 < c < 1$. As L_0 is invertible according to Lemma 3 and the inverse operator L_0^{-1} has norm 1, a familiar argument shows that L_1 is invertible, too.

On the other hand, as δa_0 and a_j, $1 \leqslant j \leqslant n$ belong to $L^\infty(D)$, the term $\delta a_0 + \sum\limits_{j=1}^{n} a_j(z) \dfrac{\partial u}{\partial \bar{z}_j}$ induces the bounded linear operator $\delta L_2 : H^+(D) \to L^2(D)$. Then Theorem 1 and Lemma 1 imply that the operator $\delta L_2 : H^+(D) \to H^-(D)$ is compact. This means that the operator $L_2 : H^+(D) \to H^-(D)$ corresponding to problem (17) differs from the invertible operator L_1 by the compact operator $\delta L_2 : H^+(D) \to H^-(D)$, i.e. L_2 is a Fredholm operator. □

2. Spectral properties of the problem

As estimate (6) does not provide the compactness of the embedding ι, we are to study the spectral properties of problem (17) under condition (7) of Theorem 1 in the case $S \neq \partial D$. With this aim we consider the sesquilinear form on $H^-(D)$ given by

$$(u, v)_{H^-(D)} := \langle L_0^{-1} u, v\rangle \text{ for } u, v \in H^-(D).$$

Since

$$\langle L_0^{-1} u, v\rangle = \langle L_0^{-1} u, L_0 L_0^{-1} v\rangle = (L_0^{-1} u, L_0^{-1} v)_+ \text{ for all } u, v \in H^-(D), \tag{19}$$

the last equality being due to (16), this form is Hermitian. Combining (18) and (19) yields

$$\sqrt{(u, u)_-} = \|u\|_- \text{ for all } u \in H^-(D).$$

From now on we endow the space $H^-(D)$ with the scalar product $(\cdot, \cdot)_-$.

We recall that a compact self-adjoint operator C is said to be of finite order if there is $0 < p < \infty$, such that the series $\sum_\nu |\lambda_\nu|^p$ converges where $\{\lambda_\nu\}$ is the system of eigenvalues of the operator C (its existence is provided by Hilbert-Schmidt Theorem, see, for instance, [5] and elsewhere).

Lemma 4. *Suppose that (6) or (7) is fulfilled. Then the inverse L_0^{-1} of the operator given by (16) induces positive self-adjoint operators*

$$\iota' \iota\, L_0^{-1} : H^-(D) \to H^-(D), \quad \iota\, L_0^{-1}\, \iota' : L^2(D) \to L^2(D), \quad L_0^{-1}\, \iota' \iota : H^+(D) \to H^+(D)$$

which have the same systems of eigenvalues and eigenvectors; besides, the eigenvalues are positive. Moreover, if (7) holds true then they are compact operators of finite orders and there are orthonormal bases in $H^+(D)$, $L^2(D)$ and $H^-(D)$ consisting of the eigenvectors.

Proof. According to Theorem 1 the embedding ι is continuous. As ι', L_0^{-1} are bounded, all the operators $(\iota' \iota\, L_0^{-1})$, $(\iota\, L_0^{-1}\, \iota')$, $(L_0^{-1}\, \iota' \iota)$ are bounded, too. Then, by (19),

$$(\iota' \iota\, L_0^{-1} u, v)_- = \overline{(v, \iota' \iota\, L_0^{-1} u)_-} = \overline{\langle L_0^{-1} v, \iota' \iota\, L_0^{-1} u\rangle} = (\iota L_0^{-1} u, \iota L_0^{-1} v)_{L^2(D)}, \tag{20}$$

$$(u, \iota' \iota\, L_0^{-1} v)_- = \overline{(\iota' \iota\, L_0^{-1} v, u)_-} = (\iota L_0^{-1} u, \iota L_0^{-1} v)_{L^2(D)}$$

for all $u, v \in H^-(D)$, i.e., the operator $(\iota' \iota\, L_0^{-1})$ is self-adjoint.

Using (16) we get

$$(\iota\, L_0^{-1}\, \iota' u, v)_{L^2(D)} = (\iota(L_0^{-1}(\iota' u)), v)_{L^2(D)} = \langle L_0^{-1}(\iota' u), \iota' v\rangle = (L_0^{-1}(\iota' u), L_0^{-1}(\iota' v))_+,$$

$$(u, \iota\, L_0^{-1}\, \iota' v)_{L^2(D)} = \overline{(\iota\, L_0^{-1}\, \iota' v, u)_{L^2(D)}} = (L_0^{-1}(\iota' u), L_0^{-1}(\iota' v))_+$$

for all $u, v \in L^2(D)$, i.e., the operator $(\iota\, L_0^{-1}\, \iota')$ is self-adjoint.

On applying (16) once again we obtain

$$(L_0^{-1}\, \iota' \iota u, v)_+ = (L_0^{-1}(\iota' \iota\, u), v)_+ = \langle \iota' \iota\, u, v\rangle = (\iota u, \iota v)_{L^2(D)}, \tag{21}$$

$$(u, L_0^{-1}\, \iota' \iota v)_+ = \overline{(v, u)_+} = (\iota u, \iota v)_{L^2(D)}$$

for all $u, v \in H^+(D)$, which establishes the self-adjointness of $(L_0^{-1}\, \iota' \iota)$.

Finally, as the operator L_0^{-1} is injective, so are the operators $(\iota' \iota\, L_0^{-1})$, $(\iota\, L_0^{-1}\, \iota')$ and $(L_0^{-1}\, \iota' \iota)$. Hence, all their eigenvectors $\{u_\nu\}$ (if exist !) belong to the space $H^+(D)$, for $L_0^{-1} u_\nu$ lies in $H^+(D)$

and all the eigenvalues are positive. From the injectivity of ι and ι' we also conclude that the systems of eigenvalues and eigenvectors of $(\iota'\iota\, L_0^{-1})$, $(\iota\, L_0^{-1}\,\iota')$ and $(L_0^{-1}\,\iota'\iota)$ coincide.

If (7) holds true then Theorem 1 implies that the embedding ι is compact. Then all the operators $(\iota'\iota\, L_0^{-1})$, $(\iota\, L_0^{-1}\,\iota')$, $(L_0^{-1}\,\iota'\iota)$ are compact, too. Now we refer to [7] (see also Proposition 5.4.1 in [17]) that if there is $\delta > 0$ such that a compact operator C maps $H^s(D)$ continuously to $H^{s+\delta}(D)$, then it has a finite order (actually, one may choose $p = n/\delta + \tau$ for each $\tau > 0$). But, under estimate (7), Theorem 1 implies that the operator $(\iota\, L_0^{-1}\iota')$ actually maps $L^2(D)$ to $H^{1/2-\varepsilon}(D)$ with any $\varepsilon > 0$. Hence it has a finite order. As the operators $(\iota'\iota\, L_0^{-1})$ and $(L_0^{-1}\,\iota'\iota)$ have the same eigenvalues, their orders are finite, too.

The last part of the lemma follows from Hilbert–Schmidt Theorem. □

Our next goal is to apply Keldysh's Theorem (see [6] or [5, Ch. 5, § 8]) for studying the completeness of root functions of weak perturbations of the finite order compact self-adjoint operators.

Theorem 2. *Let estimate (7) be fulfilled and $\delta b_0 = 0$. Then, for any invertible operator $L : H^+(D) \to H^-(D)$ related to problem (17) the system of root functions of the compact operator $(\iota'\iota\, L^{-1}) : H^-(D) \to H^-(D)$ is complete in the spaces $H^-(D)$, $L^2(D)$ and $H^+(D)$.*

Proof. By assumption there is a bounded inverse $L^{-1} : H^-(D) \to H^+(D)$. Since $I - L_0 L^{-1} = (L - L_0)\, L^{-1}$, we conclude that

$$(\iota'\iota\, L_0^{-1}) - (\iota'\iota\, L^{-1}) = (\iota'\iota\, L_0^{-1})\left((L - L_0)\, L^{-1}\right). \tag{22}$$

As $\delta b_0 = 0$, the operator $(L - L_0) : H^+(D) \to H^-(D)$ is induced by the term $\delta a_0 + \sum\limits_{j=1}^{n} a_j(z)\frac{\partial u}{\partial \bar z_j}$. Then, as we have seen in the proof of Corollary 1, this operator is compact. Since L^{-1} is bounded, it follows that the operator $(L - L_0)\, L^{-1} : H^-(D) \to H^-(D)$ is compact, too.

Hence, $(\iota'\iota\, L^{-1})$ is an injective weak perturbation of the compact self-adjoint operator $(\iota'\iota\, L_0^{-1})$ of finite order (see Lemma 4). Then Keldysh's Theorem [6] or [5, Ch. 5, § 8]) implies that the countable system $\{u_\nu\}$ of root functions related to the operator $(\iota'\iota\, L^{-1})$ is complete in the Hilbert space $H^-(D)$.

Pick a root function u_ν of the operator $(\iota'\iota\, L^{-1})$ corresponding to an eigenvalue λ_ν. Note that $\lambda_\nu \neq 0$, for the operator L^{-1} is injective. By definition there is a natural number m, such that $((\iota'\iota\, L^{-1}) - \lambda_\nu I)^m u_\nu = 0$. Using the binomial formula yields

$$u_\nu = \sum_{j=1}^{m} \binom{m}{j} \lambda_\nu^{-j} (\iota'\iota\, L^{-1})^j u_\nu.$$

Hence, $u_\nu \in H^+(D)$ because the range of the operator L^{-1} lies in the space $H^+(D)$.

We have thus proved that $\{u_\nu\} \subset H^+(D)$. Our next concern will be to show that the linear span $\mathcal{L}(\{u_\nu\})$ of the system $\{u_\nu\}$ is dense in $H^+(D)$ (cf. Proposition 6.1 of [9] and [12, p. 12]). For this purpose, pick $u \in H^+(D)$. As L maps $H^+(D)$ continuously onto $H^-(D)$, we get $Lu \in H^-(D)$. Hence, there is a sequence $\{f_k\} \subset \mathcal{L}(\{u_\nu\})$ converging to Lu in $H^-(D)$. On the other hand, the inverse L^{-1} maps $H^-(D)$ continuously to $H^+(D)$, and so the sequence

$$L^{-1} f_k = L^{-1}\iota'\iota\, f_k$$

converges to u in $H^+(D)$.

If now $u_{\nu_0} \in \mathcal{L}(\{u_\nu\})$ corresponds to an eigenvalue λ_0 of multiplicity m_0 then the vector $v_{\nu_0} = (\iota'\iota\, L^{-1}) u_{\nu_0}$ satisfies

$$((\iota'\iota\, L^{-1}) - \lambda_0 I)^{m_0} v_{\nu_0} = ((\iota'\iota\, L^{-1}) - \lambda_0 I)^{m_0+1} u_{\nu_0} + \lambda_0 ((\iota'\iota\, L^{-1}) - \lambda_0 I)^{m_0} u_{\nu_0} = 0.$$

Thus, the operator $(\iota'\iota\, L^{-1})$ maps $\mathcal{L}(\{u_\nu\})$ to $\mathcal{L}(\{u_\nu\})$ itself. Therefore, the sequence $\{\iota'\iota\, L^{-1} f_k\}$ still belongs to $\mathcal{L}(\{u_\nu\})$ and we can think of $\{L^{-1} f_k\}$ as sequence of linear combinations of root functions of $\iota'\iota\, L^{-1}$ converging to u. These arguments show that the subsystem $L^{-1}\,\mathcal{L}(\{u_\nu\}) \subset \mathcal{L}(\{u_\nu\})$ is dense in $H^+(D)$.

Finally, since the space $C^\infty_0(D)$ of the functions with compact supports is included into $H^+(D)$ and $C^\infty_0(D)$ is dense in the Lebesgue space $L^2(D)$, the space $H^+(D)$ is dense in $L^2(D)$ as well. This proves the completeness of the system of root functions in $L^2(D)$. □

If $\delta b_0 \neq 0$ then the corresponding perturbation may be non-compact (see Example 2 below). In this case one may use another methods to study the root functions (see, for instance, [9, 11]). However these methods are beyond the scope of this paper.

3. Examples for the unit ball

Let $S = \varnothing$ and $a_j = 0$ for all $1 \leqslant j \leqslant n$, $b_1 = 1$ and a_0, b_0 be constants. Then we obtain the mixed problem for the Helmholtz equation. In the generalized setting the corresponding spectral problem reads as

$$4\sum_{j=1}^n \left(\frac{\partial u}{\partial \bar z_j}, \frac{\partial v}{\partial \bar z_j}\right)_{L^2(D)} + 4b_0(u,v)_{L^2(\partial D)} + (a_0-\lambda)(u,v)_{L^2(D)} = 0 \text{ for all } v \in H^+(D). \tag{23}$$

In particular, applying the last identity with $u = v$ we conclude that $\lambda \geqslant a_{0,0}$ if $\delta a_0 = \delta b_0 = 0$.

We are going to study the Sturm-Liouville problem on the unit ball $D = \mathbb{B}$ in $\mathbb{C}^n$. Actually, the matter is quite similar to the coercive mixed problem for the Laplace operator in the ball (see [19, Suppl. II, P. 1, §2]).

To this end, we pass to spherical coordinates $x = r\,S(\varphi)$ where φ are coordinates on the unit sphere $\partial D = \mathbb{S}$ in $\mathbb{C}^n$. The Laplace operator Δ in the spherical coordinates takes the form

$$\Delta_{2n} = \frac{1}{r^2}\Big(\Big(r\frac{\partial}{\partial r}\Big)^2 + (2n-2)\Big(r\frac{\partial}{\partial r}\Big) - \Delta_{\mathbb{S}}\Big), \tag{24}$$

where $\Delta_{\mathbb{S}}$ is the Laplace-Beltrami operator on the unit sphere.

On the other hand, in the unit ball we have

$$\frac{\partial}{\partial \nu} = r\frac{\partial}{\partial r}, \quad \overline{\partial}_\nu = \sum_{j=1}^n \bar z_j \frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + B_{\mathbb{S}}\right)$$

where the operator $B_{\mathbb{S}}$ depends on the coordinates on the sphere $\mathbb{S}$ only. If, for instance, $n = 1$ then, in polar coordinates, $\overline{\partial}_\nu = \dfrac{1}{2}\left(r\dfrac{\partial}{\partial r} + \sqrt{-1}\,\dfrac{\partial}{\partial \varphi}\right)$.

To solve the homogeneous equation $(-\Delta_{2n} + a)u = 0$ we make use of the Fourier method of separation of variables. Writing $u(r,\varphi) = g(r)h(\varphi)$ we get two separate equations for g and h, namely

$$\begin{array}{rcl} \Big(-\Big(r\dfrac{\partial}{\partial r}\Big)^2 + (2-2n)\Big(r\dfrac{\partial}{\partial r}\Big) + ar^2\Big)g &=& c\,g \\ \Delta_{\mathbb{S}} h &=& c\,h, \end{array}$$

c being an arbitrary constant.

The second equation has non-zero solutions if and only if c is an eigenvalue of $\Delta_{\mathbb{S}}$. These are well known to be $c = k(2n+k-2)$, for $k = 0, 1, \ldots$ (see for instance [19]). The corresponding eigenfunctions of $\Delta_{\mathbb{S}}$ are spherical harmonics $h_k(\varphi)$ of degree k, i.e.,

$$\Delta_{\mathbb{S}} h_k = k(2n+k-2)\, h_k. \tag{25}$$

The number of the linearly independent spherical harmonics of the degree k is finite and equals to $J(k) = \dfrac{(2n+2k-2)(2n+k-3)!}{k!(2n-2)!}$. In the complex space $\mathbb{C}^n$ we may choose the harmonics h_k in accordance with the complex structure. Namely, it possible to find an orthonormal basis $\{H^{(j)}_{p,q}\}$ in $L^2(\mathbb{S})$ of consisting on the polynomials of the form

$$H^{(j)}_{p,q}(z,\overline{z}) = \sum_{|\alpha|=p, |\beta|=q} c^{(j)}_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta}$$

with complex coefficients $c^{(j)}_{\alpha,\beta}$ (see, [20]). Let $J(p,q)$ stands for the number of the polynomials of the bi-degree (p,q) in the basis; of course $J(p,q) \leqslant J(p+q)$. Clearly,

$$\overline{\partial}_{\nu} H^{(j)}_{p,q} = q H^{(j)}_{p,q}, \quad B_{\mathbb{S}} H^{(j)}_{p,q} = (q-p) H^{(j)}_{p,q}. \tag{26}$$

Consider now the following Sturm-Liouville Problem for ordinary differential equation with respect to the variable $0 < r < 1$ (see [19, Suppl. II, P. 1, §2])

$$\Big(-\frac{1}{r^2}\Big(r\frac{\partial}{\partial r}\Big)^2 + (2-2n)\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big) + \frac{(p+q)(2n+p+q-2)}{r^2} + a_0 \Big) g(r) = \lambda\, g(r), \tag{27}$$

$$\frac{\partial g}{\partial r}(1) + \Big(2b_0 + (q-p)\Big) g(1) = 0 \text{ and } g(r) \text{ is bounded at the point } r = 0. \tag{28}$$

Actually, if $a_0, \lambda \in \mathbb{R}$ then (27) is a version of the Bessel equation, and its (real-valued) solution $g(r)$ is a Bessel function defined on $(0, +\infty)$ while the space of all the solutions is two-dimensional. For example, if $\lambda = a_0$ then $g(r) = \alpha r^{p+q} + \beta r^{2-p-q-n}$ with arbitrary constants α and β is a general solution to (27). In the general case the space of solutions to (27) contains a one-dimensional subspace of functions bounded at the point $r = 0$, cf. [19].

For a triple (p,q,j), fix a non-trivial solution $g^{(j,i)}_{p,q}(r)$ to (27), (28) corresponding to an eigenvalue $\lambda^{(j,i)}_{p,q}$. Then the function $u^{(j,i)}_{p,q} = g^{(j,i)}_{p,q}(r) H^{(j)}_{p,q}(\varphi)$ satisfies

$$\Big(-\Delta_{2n} + (a_0 - \lambda^{(j)}_{p,q}) \Big)\, u^{(j,i)}_{p,q} = 0 \text{ on } \mathbb{C}^n, \tag{29}$$

$$(b_0 + \overline{\partial}_{\nu}) u^{(j,i)}_{p,q} = 0 \text{ on } \partial D. \tag{30}$$

Indeed, by (24), (25), (27) and the discussion above we conclude that this equality holds in $\mathbb{C}^n \setminus \{0\}$. We now use the fact that $u^{(j,i)}_{p,q}$ is bounded at the origin to see that (29) holds. On the other hand, (30) follows from (26) immediately.

Theorem 3. *Let $\delta a_0 = \delta b_0 = 0$ and $a^2_{0,0} + b^2_{0,0} \neq 0$. The system $\{u^{(j,i)}_{p,q}\}$, $i \in \mathbb{N}$, $p,q \in \mathbb{Z}_+$, $1 \leq j \leqslant J(p,q)$, coincides with system of all the eigenvectors of the Sturm-Liouville problem (17) in the ball $\mathbb{B}$. In particular, it is an orthogonal basis in $H^+(\mathbb{B})$, $L^2(\mathbb{B})$ and $H^-(\mathbb{B})$.*

Proof. As $a^2_{0,0} + b^2_{0,0} \neq 0$, Theorem 1 implies that $H^+(\mathbb{B})$ is continuously embedded to $L^2(\mathbb{B})$.

Now we note that the system $\{u^{(j,i)}_{p,q}\}$ consists of eigenvectors of the Sturm-Liouville problem (17) in the ball $\mathbb{B}$. Moreover, according to [21, Lemma 7.1], the system $\{u^{(j,i)}_{p,q}\}$ is orthogonal with respect to the Hermitian forms $(\cdot,\cdot)_{L^2(\mathbb{S})}$ $(\cdot,\cdot)_{L^2(\mathbb{B})}$ and $(\overline{\partial}\cdot, \overline{\partial}\cdot)_{[L^2(\mathbb{B})]^n}$. In particular, it is orthogonal in $H^+(\mathbb{B})$. The orthogonality of the system in $H^-(\mathbb{B})$ is fulfilled because (20) and Lemma 4 imply

$$(u^{(j,i)}_{p,q}, u^{(\tilde{j},\tilde{i})}_{\tilde{p},\tilde{q}})_- = (\lambda^{(j,i)}_{p,q})^{-1} (\iota' \iota L_0^{-1} u^{(j,i)}_{p,q}, u^{(\tilde{j},\tilde{i})}_{\tilde{p},\tilde{q}})_- = \lambda^{(\tilde{j},\tilde{i})}_{\tilde{p},\tilde{q}} (u^{(j,i)}_{p,q}, u^{(\tilde{j},\tilde{i})}_{\tilde{p},\tilde{q}})_{L^2(\mathbb{B})}.$$

By the very construction, the system $\{H^{(j)}_{p,q}\}$, $p,q \in \mathbb{Z}_+$, $1 \leqslant j \leqslant J(p,q)$, is an orthonormal basis in $L^2(\mathbb{S})$. As it is known, if $\delta a_0 = 0$ then $\lambda^{(j,i)}_{p,q} \geqslant a_{0,0}$ and the countable system $\{g^{(j,i)}_{p,q}(r)\}_{i\in\mathbb{N}}$ of eigenfunctions is an orthogonal basis in the weighted space $L^2_{\mathbb{R}}([0,1],r)$ of real valued functions with the scalar product $(\sqrt{r}\cdot, \sqrt{r}\cdot)_{L^2([0,1])}$ (see [19, Suppl. II, P. 1, §2]) for each fixed triple (p,q,j) with $p,q \in \mathbb{Z}_+$, $1 \leq j \leqslant J(p,q)$. Easily, it is also is an orthogonal basis in the weighted space $L^2([0,1],r)$ (consisting of complex-valued functions). Hence, by the familiar arguments, the system $\{u^{(j,i)}_{p,q} = g^{(j,i)}_{p,q}(r)H^{(j)}_{p,q}(\varphi)\}$, $p,q \in \mathbb{Z}_+$, is an orthogonal basis in $L^2(\mathbb{B}) = L^2(\mathbb{S}\times[0,1])$, see, for instance, [23, Ch. VII, §3.5, Theorem 1].

Now, as the system $\{u^{(j,i)}_{p,q}\}$ is an orthogonal basis in $L^2(\mathbb{B})$ there are no other eigenvalues of the problem (17) besides the already mentioned $\lambda^{(j,i)}_{p,q}$. Hence there are no eigenvectors corresponding to a value λ_0 besides the linear combinations of the already constructed eigenfunctions related to this value.

As we already mentioned, the space $L^2(\mathbb{B})$ is dense in $H^-(\mathbb{B})$. Hence the system $\{u^{(j,i)}_{p,q}\}$ is complete in $H^-(\mathbb{B})$, too. Finally, let a function $u \in H^+(D)$ is orthogonal to each vector $u^{(j,i)}_{p,q}$ with respect to $(\cdot,\cdot)_+$. Then, using Lemma 4 and (21) we conclude that $(u, u^{(j,i)}_{p,q})_{L^2(\mathbb{B})} = (u, L_0^{-1}\iota'\iota u^{(j,i)}_{p,q})_+ = \lambda^{(j,i)}_{p,q}(u, u^{(j,i)}_{p,q})_+ = 0$ i.e. u is orthogonal to each vector $u^{(j,i)}_{p,q}$ in $L^2(\mathbb{B})$. Therefore $u = 0$ in $L^2(\mathbb{B})$ and, consequently in the space $H^+(D)$. This exactly means that the system $\{u^{(j,i)}_{p,q}\}$ is complete in $H^+(\mathbb{B})$. □

We note that, as opposed to the coercive case, in this way we can not provide that the multiplicities of the eigenvalues of problem (17) are finite (cf. Example 1 below).

Example 1. Let $a_0 = a_{0,0} = 1$, $b_0 = b_{0,0} = 0$. Then the space $H^+(D)$ is continuously embedded to $L^2(D)$ (see Theorem 1 above). It follows from (23) that the eigenvalues (if exists) are equal or more than 1; moreover the eigenvalue $\lambda = 1$ corresponds to the space $\mathcal{O}^2(D)$ of holomorphic functions from the Lebesgue space $L^2(D)$. The dimension of the eigenspace $\mathcal{O}^2(D)$ (i.e. the multiplicity of the eigenvalue $\lambda = 1$) is not finite and hence the embedding ι is not compact. However, Theorem 3 allows us to construct an orthogonal basis in $H^+(\mathbb{B})$, $L^2(\mathbb{B})$ and $H^-(\mathbb{B})$ consisting of the eigenvectors of problem (17).

Let us see that the corresponding embedding in Theorem 1 is sharp for the ball $\mathbb{B}$. Indeed, if $D = \mathbb{B}$ and $n = 1$ then the series $u_\varepsilon(z) = \sum\limits_{k=0}^{\infty} \dfrac{z^k}{(k+1)^{\varepsilon/2}}$, $\varepsilon > 0$, converges in $H^+(D)$ and $\|u_\varepsilon\|^2_+ = \|u_\varepsilon\|^2_{L^2(\mathbb{B})} = \pi \sum\limits_{k=0}^{\infty} \dfrac{1}{(k+1)^{1+\varepsilon}}$. According to [22, Lemma 1.4], $\Big\|u_\varepsilon\Big\|^2_{H^s(\mathbb{B})} \geqslant \pi \sum_{k=0}^{\infty} \frac{k^{2s}}{(k+1)^{1+\varepsilon}}$, $0 < s \leqslant 1$, i.e. for each $s \in (0,1)$ there is $\varepsilon > 0$ such that $u_\varepsilon \notin H^s(\mathbb{B})$. Therefore $H^+(\mathbb{B})$ can not be continuously embedded to $H^s(\mathbb{B})$ for any $s > 0$. □

Actually, the embedding corresponding to (7) in Theorem 1 is sharp for the ball $\mathbb{B}$, too.

Example 2. Let first $a_{0,0} = 0$ and $b_{0,0} = b_0 = 1$. Then $H^+(\mathbb{B})$ is continuously embedded to $H^{1/2}(\mathbb{B})$ and the corresponding operator L_0 is of finite order (see Theorems 1 and Lemma 3). If $\delta a_0 \in \mathbb{C}$ then, according to Theorem 2, the system of the root functions related to problem (17) is complete in $H^+(\mathbb{B})$, $L^2(\mathbb{B})$ and $H^-(\mathbb{B})$. On the other hand, problem (27), (28) may be treated in a similar way as (17) with $H_0 = L^2([0,1],r)$, i.e. the term δa_0 induces a weak perturbation of the self-adjoint problem (27), (28) with $a_0 = a_{0,0} \geqslant 0$. Hence, by Keldysh's Theorem the corresponding system $\{g^{(j,i)}_{p,q}\}$ of its (complex-valued) root functions is complete in the weighted space $L^2([0,1],r)$. Thus we conclude that the system $\{u^{(j,i)}_{p,q}\}$ of the root functions related to problem (17) is complete in the Lebesgue space $L^2(\mathbb{B})$ (and then in $H^-(\mathbb{B})$) for every $a_0 \in \mathbb{C}$.

If $n = 1$ then the series $u_\varepsilon(z) = \sum\limits_{k=0}^{\infty} \dfrac{z^k}{(k+1)^{(1+\varepsilon)/2}}$, $\varepsilon > 0$, converges in $H^+(\mathbb{B})$ and $\|u_\varepsilon\|^2_+ =$

$\|u_\varepsilon\|^2_{L^2(\mathbb{S})} = 2\pi \sum\limits_{k=0}^{\infty} \dfrac{1}{(k+1)^{1+\varepsilon}}$, $\Big\|u_\varepsilon\Big\|^2_{H^s(\mathbb{B})} \geqslant \pi \sum\limits_{k=0}^{\infty} \frac{k^{2s-1}}{(k+1)^{1+\varepsilon}}$, $0 < s \leqslant 1$, i.e. for each $s \in (1/2, 1)$ there is $\varepsilon > 0$ such that $u_\varepsilon \notin H^s(\mathbb{B})$. Therefore $H^+(\mathbb{B})$ can not be continuously embedded to $H^s(\mathbb{B})$ for any $s > 1/2$.

Let now $0 \neq |\delta b_0| < b_{0,0} = 1$. Then problem (17) is still a Fredholm one (see Corollary 1). Take $n = 1$ and the sequence $\{z^p\}$. It is bounded in $H^+(\mathbb{B})$ because $\|z^p\|_+ = \|z^p\|_{L^2(\mathbb{S})} = \sqrt{2\pi}$. As $\|z^p - z^k\|^2_+ = 4\pi$ for every $k, p \in \mathbb{Z}_+$ we conclude that the sequence contains no fundamental subsequences. On the other hand, for the corresponding bounded operator δL_0 we have

$$\|\delta L_0(z^p - z^k)\|_- = 4 \sup_{\substack{v \in H^1(D)\\ v \neq 0}} \frac{|(v, \delta b_0(z^p - z^k))_{L^2(\mathbb{S})}|}{\|v\|_+} \geqslant 4|\delta b_0| \|z^p - z^k\|_{L^2(\mathbb{S})} = 8|\delta b_0|\sqrt{\pi},$$

i.e. the sequence $\{\delta L_0 z^p\}$ contains no fundamental subsequences, too. Hence the operator δL_0 can not be compact. $\square$

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О спектральных свойствах одной некоэрцитивной смешанной задачи, ассоциированной $\overline{\partial}$-оператором

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Мы рассматриваем некоэрцитивную задачу Штурма-Лиувилля в некоторой ограниченной области D комплексного пространства $\mathbb{C}^n$ для возмущенного оператора Лапласа. Более точно, мы ставим на границе условия Робиновского типа, в которых член первого порядка пропорционален комплексной нормальной производной. Доказывается фредгольмовость задачи в подходящих пространствах, для которых получена теорема вложения, дающая соотношения со шкалой пространств Соболева-Слободецкого. Затем, используя метод слабого возмущения компактных самосопряженных операторов, мы доказываем полноту корневых функций, ассоциированных с краевой задачей в пространстве Лебега. Для шара соответствующие собственные векторы представлены как произведение функций Бесселя и сферических гармоник.

Ключевые слова: задача Штурма-Лиувилля, некоэрцитивные задачи, многомерный оператор Коши-Римана, корневые функции.