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# Discriminant and Singularities of Logarithmic Gauss Map, Examples and Application

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*The study of hypersurfaces in a torus leads to the beautiful zoo of amoebas and their contours, whose possible configurations are seen from combinatorical data. There is a deep connection to the logarithmic Gauss map and its critical points. The theory has a lot of applications in many directions.*

*In this report we recall basic notions and results from the theory of amoebas, show some connection to algebraic singularity theory and consider some consequences from the well known classification of singularities to this subject. Moreover, we have tried to compute some examples using the computer algebra system* Singular *and discuss different possibilities and their effectivity to compute the critical points. Here we meet an essential obstacle: Relevant examples need real or even rational solutions, which are found only by chance. We have tried to unify different views to that subject.*

*Keywords: logarithmic Gauss map, singularities, discriminant, asymptotics, hypersurface amoeba.*

## 1. Toric Hypersurface and Logarithmic Gauss Map

Let $V^*(f)$ be an algebraic hypersurface in the algebraic torus $\mathbb{T}^n$, $\mathbb{T} := \mathbb{C}^*$, i.e.

$$V^*(f) = \{z \in \mathbb{T}^n \mid f(z) = 0\},$$

where $f(z) = \sum\limits_{A \subset \mathbb{Z}^n} a_\alpha z^\alpha$ is the Laurent polynomial.

Recall that the *Newton polyhedron* $\mathcal{N}_f \subset \mathbb{R}^n$ of $f$ is the convex hull in $\mathbb{R}^n$ of $A_f := \mathrm{supp}(f)$. Let $\mathbb{X}_\Sigma$ be the smooth toric variety associated to the fan $\Sigma$, which is a refinement of the fan dual to the Newton polyhedron $\mathcal{N}_f$. We denote by $\overline{V}(f) \subset \mathbb{X}_\Sigma$ the closure of $V^*(f)$ in $\mathbb{X}_\Sigma$. The polynomial $f$ is called *non-singular for its Newton polyhedron* if $V^*(f)$ is smooth and for any face $\Delta \subset \mathcal{N}_f$ one has

$$(z_1 \partial f^{(\Delta)}/\partial z_1, \ldots, z_n \partial f^{(\Delta)}/\partial z_n) \neq 0$$

for all $z \in \overline{V}(f) \cap \mathbb{X}_{\Sigma'}$, where $f^{(\Delta)}$ is the truncation of $f$ to the face $\Delta$, and $X_{\Sigma'}$ is the toric variety associated to $\Delta$. In accordance with singularity resolution theorem, cf. [6, page 291], a generic polynomial $f$ is non-singular for its Newton polyhedron, and, therefore, $\overline{V}(f)$ is non-singular.

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Next we introduce the so-called *logarithmic Gauss map* $\gamma_f : V^*(f) \to \mathbb{P}^{n-1}$. Let $\mathbf{t}^n$ denote the Lie algebra of $\mathbb{T}^n$ which is identified with the tangent space of $\mathbb{T}^n$ at the unit point $\mathbf{e}$. For any point $z \in V^*$ shift the tangent space $T_z(V^*)$ by the torus multiplication (with $z^{-1}$) to a hyperplane $h_z \subset \mathbf{t}^n$, inducing a point in the projective space of the dual $\mathbf{t}^{n*}$, which we define to be $\gamma_f(z) := h_z^* \in \mathbb{P}^{n-1} := \mathbb{P}(\mathbf{t}^{n*})$. In coordinates of $\mathbb{T}^n$ the map $\gamma_f$ is given by

$$\gamma_f(z) = (z_1 f_{z_1} : \ldots : z_n f_{z_n}) \in \mathbb{P}^{n-1}.$$

Described in more geometric terms we have: Let $U \subset \mathbb{T}^n$ be a neighbourhood of a regular point $z$ on $V^*(f)$. Choose a branch of the logarithmic map (restricted to $U$) $\log : U \to \mathbb{C}^n$ then the direction of the normal line at $\log(z)$ to transformed hypersurface $\log(V^*(f) \cap U)$ has components $(z_1 f_{z_1}, \ldots, z_n f_{z_n})$. This construction does not depend on the choice of the branch of log.

In [11, Section 3.2] one can find the idea of a construction how to extend $\gamma_f$ in the non-singular case to a finite map

$$\overline{\gamma}_f : \overline{V}(f) \to \mathbb{P}^{n-1}.$$

Having a finite map $\gamma$ to a smooth variety, one can associate the *ramification locus* or the *discriminant* as image of the critical locus: $\mathcal{D} := \gamma(\mathcal{C}_\gamma)$, which is usually a hypersurface. An analytic structure which is compatible with base change was introduced by Teissier, cf. [14]: The structure sheaf $\mathcal{O}_\mathcal{D}$ is defined to be the quotient by the 0-th fitting ideal of $\gamma_*(\mathcal{O}_\mathcal{C})$. In local coordinates the defining equation is obtained as the (classical) discriminat of the polynomial, that generates the finite extension of the structure sheafs over an open affine subsets.

From the well-known theorem of Kouchnirenko, cf. [7, Th. 3], Mikhalkin obtains:

**Proposition 1** ( [11]). *If the polynomial $f$ is non-singular for its Newton polyhedron, then the degree of $\overline{\gamma}_f$ is obtained as*

$$\deg(\overline{\gamma_f}) = n! \cdot \mathrm{Vol}(\mathcal{N}_f).$$

For later calculation we give a description of the logarithmic Gauss map $\gamma_f$ in local coordinates. Since $V^*(f)$ is smooth, we assume w.l.o.g. that locally $f_{z_n} := \partial f / \partial z_n \neq 0$. Then there exists a function $g(z')$, $z' := (z_1, \ldots, z_{n-1})$ such that $f(z', g(z')) \equiv 0$

Since $g_{z_i} = -f_{z_i}(z', g)/f_{z_n}(z', g)$ and $(\log(g(z'))_{z_i} = g_{z_i}/g$ hold, one obtains the formula

$$\gamma_f(z') = \left( -z_1 \frac{\partial \log g(z')}{\partial z_1} : \ldots : -z_{n-1} \frac{\partial \log g(z')}{\partial z_{n-1}} : 1 \right). \tag{1}$$

Then the fiber $\gamma_f^{-1}(y)$, $y = (y_1 : \ldots : y_n) \in \mathbb{P}^{n-1}$ is given by the zeros of the local complete intersection ideal generate by $f$ and the 2-minors of

$$\begin{pmatrix} z_1 f_{z_1} & \ldots & z_n f_{z_n} \\ y_1 & \ldots & y_n \end{pmatrix},$$

i.e. (in case of $y_n \neq 0$) $\gamma^{-1}(y)$ is defined by the complete intersection ideal

$$I_y := (f, h_1, \ldots h_{n-1}), \tag{2}$$

where $h_i = y_n z_i f_{z_i} - y_i z_n f_{z_n}$. There are at most $n! \cdot \mathrm{Vol}(\mathcal{N}_f)$ zeros in the torus by Proposition 1.

## 2. Amoeba and its Contour versus Laurent Series

Consider a rational function $F(z) = h(z)/f(z)$ of $n$ complex variables and different Laurent expansions

$$\sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \tag{3}$$

of $F$ centered at $z = 0$. The most natural way to describe these expansions uses the amoeba of polar hypersurface $V^* = V^*(f)$.

Recall that the *amoeba* $\mathcal{A}_{V^*}$ of a toric hypersurface $V^* = V^*(f)$ is the image of $V^*$ by the logarithmic map $\mathrm{Log} : \mathbb{T}^n \to \mathbb{R}^n$,

$$\mathrm{Log} : (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|).$$

The complement $\mathbb{R}^n - \mathcal{A}_{V^*}$ to the amoeba consists of a finite number of connected components $E_i$, which are open and convex, cf. [4, Section 6.1]. These components are characterized in the following Theorem, which is a summary of Propositions 2.5, 2.6 in [3], Theorem 10 and Corollary 6 in [13, Section I.5].

**Theorem 1.** *There exists an open subset $U_{\mathcal{N}}$ in the set of polynomials with fixed Newton polyhedron $\mathcal{N}$ that satisfies the following property:*

*If $f \in U_{\mathcal{N}}$, then there is a bijection from the set of lattice points of $\mathcal{N} \cap \mathbb{Z}^n$ to the set of connected components of $\mathbb{R}^n - \mathcal{A}_{V^*(f)}$ : $\nu \mapsto E_\nu$ such that the normal cone $C^\vee(\nu)$ to $\mathcal{N}_f$ at the point $\nu$ is the recession cone of the component $E_\nu$.*

A recession cone is the maximal cone which can be put inside $E_\nu$ by a translation. If $f \notin U_{\mathcal{N}}$, the expected component $E_\nu$ may not exist for some non-vertice lattice points $\nu \in \mathcal{N}$ because the associated Laurent series below does not converge.

Given a component $E_\nu$ one obtains a Laurent series of $F$ centered at $z = 0$ using the term $a_\nu z^\nu$ of $f$ as denominator in a corresponding geometric progression

$$\frac{1}{f} = \sum_{k=0}^{\infty} \frac{(a_\nu z^\nu - f)^k}{(a_\nu z^\nu)^{k+1}}. \tag{4}$$

The set $\{\mathrm{Log}^{-1}(E_\nu)\}$ contains the domain of convergence for this Laurent series. The support of expansion (4) is the minimal cone $K_\nu$ that after a translation by $\nu$ contains the face $\Delta \subset \mathcal{N}_f$, which has $\nu$ as an interior point.

A non-zero vector $q \in \mathbb{Z}^n \cap K_\nu$ defines a so-called *diagonal subsequence* $\{c_{k\cdot q}\}_{k\in\mathbb{N}}$ of the set of coefficients of expansion (3). We will discuss its asymptotics in the next section.

The set of critical values of the map Log restricted to $V^*$ is called the *contour* $\mathcal{C}_{V^*}$ of the amoeba $\mathcal{A}_{V^*}$ (see [12]). The contour is closely related to the logarithmic Gauss map $\gamma_f$. Recall Lemma 3 from [11].

**Lemma 1.** *The preimage of the real points under the logarithmic Gauss map is mapped by* Log *to the contour:*

$$\mathcal{C}_{V^*} = \mathrm{Log}\left(\gamma_f^{-1}(\mathbb{P}^{n-1}_{\mathbb{R}})\right).$$

*Proof.* Let $z$ be a regular point on $V^*$ and $U$ its neighbourhood. Since the map $\mathrm{Log}|_{V^*}$ is a composition of $\log : z \mapsto (\log(z_1), \ldots, \log(z_n))$ and the projection $\mathrm{Re} : \mathbb{C}^n \to \mathbb{R}^n$, the point $z$ is critical for $\mathrm{Log}|_{V^*}$ if the projection $d\,\mathrm{Re} : T_z \log(V^* \cap U) \to \mathbb{R}^n$ is not surjective at $z$. A fiber $T_z \log(V^* \cap U)$ of the tangent bundle of the image by log of the hypersurface $V^*$ is the hyperplane

$$\{t \in \mathbb{C}^n : \langle \gamma_f(z), t\rangle = 0\}.$$

For real $\gamma_f(z)$ the projection $d\,\mathrm{Re}$ is not surjective. If $\gamma_f(z)$ is not real one can consider $\langle \gamma_f(z), t\rangle = 0$ as a system of linear equations with fixed real part $\mathrm{Re}\, t$, and solve it with respect to $\mathrm{Im}\, t$. Hence, $z$ is not critical for $\mathrm{Log}|_{V^*}$. □

Therefore, the contour $\mathcal{C}_{V^*}$ can be computed as the Log-image of the zeros of the ideal

$$(f, q_n z_1 f_{z_1} - q_1 z_n f_{z_n}, \ldots, q_n z_{n-1} f_{z_{n-1}} - q_{n-1} z_n f_{z_n}), \tag{5}$$

where $q$ runs through all real points $(q_1 : \ldots : q_n) \in \mathbb{P}^{n-1}_{\mathbb{R}}$, (here w.l.o.g $q_n \neq 0$).

# 3. Singularities of Phase Function

Consider the function

$$\Phi : \mathbb{P}^{n-1} \times V^* \longrightarrow \mathbb{C},\ \Phi(y,z) = \langle y, \log z \rangle .$$

Introduce the *phase function* $\varphi_q := \Phi(q,-)$, later we show that it is indeed a phase function of some oscilllating integral. Denote by $\mathrm{Crit}(\varphi_q) \subset V^*$ the set of critical (or stationary) points of function $\varphi_q$. It coincides with the preimage of the logarithmic Gauss map $\gamma_f$:

**Proposition 2.** *The relative critical locus of $\Phi$ coincides with the graph of $\gamma_f$:*

$$\mathrm{Crit}_{\mathbb{P}^{n-1}}(\Phi) = \Gamma_{\gamma_f}.$$

*Proof.* Assume $f_{z_n} \neq 0$ then we use local coordinates $z'$ on $V^*$ and consider the function $g(z')$ such that $f(z', g(z')) \equiv 0$. We obtain

$$\partial\Phi(z,y)/\partial z_i = \frac{y_i}{z_i} + \frac{y_n}{g(z')}\partial g(z')/\partial z_i,\ i = 1, \ldots, n-1.$$

Up to a non-zero constant multiple the components of the gradient $\partial\Phi(z,y)/\partial z$ together with the defining polymonial $f(z)$ of $V^*$ give us the defining ideal (5) of the fiber of $y$ by the logarithmetic Gauss map $\gamma$. □

The last statement shows us that the Log-image of $\mathrm{Crit}(\varphi_q)$ is contained in the contour $\mathcal{C}_{V^*}$ of the amoeba $\mathcal{A}_{V^*}$, and the tangent hyperplane to $\mathcal{C}_{V^*}$ at a point $\mathrm{Log}(z_0)$, $z_0 \in \mathrm{Crit}\,\varphi_q$, has normal vector $q \in \mathbb{Z}^n - \{0\}$.

Another consequence of the above formula concerns the connection between the singularities in the fibers of the phase function and the fibers of the logarithmic Gauss map:

**Proposition 3.** *Let $(z_0, y_0) \in \Gamma_{\gamma_f}$ be a point of the graph of $\gamma_f$. Then the Jacobian matrix of $\gamma_f$ at $z_0$ coincides with the Hesse matrix of $\varphi_{y_0}$ at $z_0$ up to multiplication with a regular constant diagonal matrix $D$:*

$$\mathrm{Hess}(\varphi_{y_0})(z_0) = D \cdot \mathrm{Jac}(\gamma_f)(z_0).$$

*Proof.* As before we assume $f_{z_n}(z_0) \neq 0$ and use local coordinates $z'$. From (1) we obtain the entries of the Jacobian matrix $\mathrm{Jac}(\gamma_f)$ of the map $\gamma_f$

$$\mathrm{Jac}(\gamma_f)_{(i,j)} = -\left( z_i \frac{\partial^2 \log(g(z'))}{\partial z_i \partial z_j} + \delta_{ij} \frac{\partial \log(g(z'))}{\partial z_j} \right), i, j = 1, \ldots, n-1,$$

where $\delta_{ij}$ is the Kronecker symbol. Moreover,

$$\frac{\partial^2 \varphi_y}{\partial z_i \partial z_j} = y_n \frac{\partial^2 \log(g(z'))}{\partial z_i \partial z_j} - \delta_{ij} \frac{y_i}{z_i^2}$$

holds for the second derivatives of $\varphi_y$. Since $\dfrac{y_{0,i}}{z_{0,i}} = -y_{0,n} \dfrac{\partial \log(g(z_0'))}{\partial z_i}$ at a critical point $z_0$ of $\varphi_{y_0}$, we obtain the statement by putting the $i$-th entry of $D$ to be $d_i = -\dfrac{y_{0,n}}{z_i}$. □

We obtain as corollary of the last proposition that for directions $y = q$ outside the ramification locus of the logarithmic Gauss map $\gamma_f$ the phase function $\varphi_q$ has only Morse critical points.

**Corollary 1.** *The logaritimic Gauss map $\gamma_f$ is unramified at $q \in \mathbb{Z}^n - \{0\}$ iff the phase function $\varphi_q$ has only Morse critical points, e.g. non-degenerated singularities.*

*Proof.* The map $\gamma_f$ is not ramified over $y$ iff its Jacobian has full rank at all points of the fiber $\gamma^{-1}(y)$. From Proposition 2 we get

$$\det\left(\frac{\partial^2\varphi_q}{\partial z_i\partial z_j}(z_0)\right) = \frac{q_n^{n-1}}{z_{0,1}\cdots z_{0,n-1}}\det(Jac(\gamma_f)(z_0)).$$

Hence, the Jacobian determinant does not vanish iff the Hessian is not zero at corresponding points. □

Next we want to discuss degenerated critical points of the phase function. By Mather-Yao type theorem the $\mathcal{R}$-class (right-equivalence) of an analytic function $h(z) \in \mathbb{C}\{z\} =: \mathcal{O}_n$ at an isolated critical point $z_0 = 0$ is equivalent to the isomorphy type of the Milnor algebra $Q_h := \mathcal{O}_n/(\partial h/\partial z)$, but as $\mathbb{C}[t]$-algebra, the action of $t$ on $Q_h$ induced by multiplication with $h$, cf. [10]. The isomorphy type of the associated singularity $(V(h), z_0)$, i.e. the $\mathcal{K}$-class (contact-equivalence) of $h(z)$, is equivalent to the isomorphy class of the Tjurina algebra $T_h = Q_h/(h)$ itself, cf. [9]. Obviously, these equivalence classes coincide for quasi-homogeneous functions (because $\mu(h) = \tau(h)$, $T_h = Q_h$, $hQ = 0$). The Milnor algebra of the phase function at $z_0$ coincides with the local algebra of the fiber $\gamma_f^{-1}(\gamma_f(z_0))$ at $z_0$.

**Corollary 2.** *If $(z_0, y_0) \in \Gamma_{\gamma_f}$, denote by $Q_\varphi$ the Milnor algebra of the function $(\varphi_{y_0}(z) - y_0)$ at $z_0$, then we have*

$$Q_\varphi = \mathcal{O}_{\gamma_f^{-1}(y_0),z_0} \quad \textit{and} \quad Q_\varphi/Ann(\mathbf{m}_Q) = \mathcal{O}_{Sing(\gamma_f^{-1}(y_0)),z_0}.$$

*Proof.* By Proposition 2 the germs coincide: $(Crit(\varphi_{y_0}), z_0) = (\gamma_f^{-1}(y_0), z_0)$. The algebra of the critical locus is the Milnor algebra of $(\varphi_{y_0}(z) - y_0)$. By Proposition 3 the Jacobi determinant of $\gamma_f$ at $z_0$ equals up to a constant multiple to the Hessian of $\varphi_{y_0}$ at $z_0$, which generates the annulator of the maximal ideal in the local complete intersection algebra $Q_\varphi$. □

A function $h \in \mathcal{O}_n$ with isolated critical point is called *almost quasihomogeneous*, if $\mu = \tau + 1$. This is equivalent to $hQ_h = Ann(\mathbf{m}_Q)$. Assume that the singularities in a fiber of a phase function are quasihomogeneous or almost quasihomogeneous, then in spite of Mather-Yao type theorems these singularities are determined by the fiber germs of the logarithmic Gauss map because $Q_\varphi = T_\varphi$ or $Q_\varphi/(Ann(\mathbf{m})) = \mathbf{T}_\varphi$, respectively. Note, that all simple or unimodal critical points belong to these singularities. The singularities of a phase function on their part determine the asymptotic of corresponding oscillating intergrals.

All degenerated critical points are lying over the singularities of the discriminant $\mathcal{D} \subset \mathbb{P}^{n-1}$ of $\gamma_f$. Many results could be found concerning the connection between singularites of discrimant and singularities in the fiber. We try to discuss some consequnces with respect to our setting.

The finite map $\overline{\gamma}_f$ can be considered as family over $\mathbb{P}^{n-1}$ of complete intersections (of relative dimension zero). Let $(X_0, 0)$ be a germ of an isolated complete intersection singularity, let $X \to S$ its versal family with discriminant $D \subset S$, the singularity of the discriminat $(D, 0)$ determines the special fiber $(X_0, 0)$ up to isomorphy by a result of Wirthmuller, c.f [17]. If $\dim(X_0) = 0$ the multiplicity of the discriminant fulfills $\mathrm{mult}(D, 0) = \dim_{\mathbb{C}}(\mathcal{O}_{X_0}) - 1 = \dim_{\mathbb{C}}(\mathcal{O}_{Sing(X_0)})$, as a consequnce of [8], for instance. This is globalized straight forward.

**Proposition 4.** *Let $\gamma : X \to S$ be a finite morphism with discriminant $D \subset S$ and each $X_s$ is a complete intersection, then holds:*

$$\mathrm{mult}_s(D) \geqslant \sum_{z_i \in X_s} \mathrm{mult}(Sing(X_s), z_i) = \sum_{z_i \in X_s} (\mathrm{mult}(X_s, z_i) - 1).$$

*Moreover, equality holds at $s \in S$, if $\gamma$ induces a versal deformation of $X_s$.*

*Proof.* The local branches of $D$ at $s$ are corresponding to the discriminants $D_i$ of the germs $(X, z_i) \to (D, s)$, hence the multiplicities of $D_i$ add up to the multiplicity of $D$. Any family is locally induced from a versal one, hence the discriminant is induced by base chance from the discriminant of the versal family and its multiplicity cannot become smaller. □

Note, versality is an open property and corresponds to some kind of *stability* in the sence of Mather. It is not clear for us, whether (or under which additional assumtions) the logarithmic Gauss map $\gamma_f$ for a generic function $f(z)$ with fixed Newton polyhedron $\mathcal{N}$ has this stability property. It holds in all computed examples. But, an answer needs further investigation.

Inspecting the classification of hypersurface singularities we get the types of possible critical points for small multiplicities of the discriminant, which are listed in the following Corollary.

**Corollary 3.** *Given a Laurent polynomial $f(z)$, non-singular with respect to its Newton polyhedron, and let $\gamma$ be the corresponding logarithmic Gauss map with discriminant $D \subset \mathbb{P}^{n-1}$. Let $m = m(q) := \mathrm{mult}(D, q)$, then the following configurations are met for the fiber $F_q := \gamma_f^{-1}(q)$, respectively for the collection of critical points of the phase function $\varphi_q(z)$:*

- *$m = 1$: $F_q$ has exactly one point $z_*$ of multiplicity 2, $\varphi_q$ has non-degenerated critical point and one $A_2$-singularity at $z_*$.*
- *$m = 2$: $F_q$ either one point of multiplicity 3 or at most two points of multiplicity 2, $\varphi_q$ has at most one $A_3$ or two $A_2$-points.*
- *$m = 3$: Besides $A_1$ can occur the following collections of critical points of $\varphi_q$: one $D_4$ or one $A_4$ or a combination $k_2A_2 + k_3A_3$ with $k_2 + 2k_3 \leqslant 3$.*
- *$m \leqslant 6$: Type of critical set of $\varphi_q$: Only (simple) ADE-critical points can occur*

$$\sum_{i \geqslant 1} k_i A_i + \sum_{i \geqslant 4} l_i D_i + \sum_{i=6}^{8} n_i E_i,$$

*such that*

$$\sum i(k_i + l_i + n_i) \leqslant n!\, vol(\mathcal{N})$$

*and*

$$\sum (i-1)(k_i + l_i + n_i) \leqslant m.$$

- *$m \leqslant 6$: all critical points are quasihomogeneous (and simple or unimodal).*
- *$m \leqslant 14$: all critical points are almost quasihomogeneous (and simple or unimodal).*

The first critical point, which is not almost quasihomogeneous are the bimodal exceptional singularities with smallest Milnor number $\mu = 16$ of type $Q_{16}$ or $U_{16}$, cf. [1]. They can occur only at multiplicity $m \geqslant 15$.

# 4. Representation of Diagonal Coefficient by Oscillating Integrals and its Phase Function

In this section we return to Laurent series (3) converging in $\mathrm{Log}^{-1}(E_\nu)$. We explane the residue asymptotics formula for its diagonal coefficient in the direction $q \in \mathbb{Z}^n \cap K_\nu$.

Recall that the Laurent series coefficient can be represented in the form

$$c_\alpha^\nu = \frac{1}{(2\pi\imath)^n} \int_{\Gamma_\nu} \frac{\omega}{z^{\alpha+\mathbf{1}}},$$

where $\omega := F(z)dz$ and the cycle $\Gamma_\nu$ is $n$-dimensional real torus $\mathrm{Log}^{-1}(x_\nu)$, $x_\nu \in E_\nu$. The direction $q$ induces series of diagonal coefficients

$$c^\nu_{q\cdot k} = \frac{1}{(2\pi\imath)^n}\int_{\Gamma_\nu}\frac{\omega}{z^{q\cdot k+\mathbf{1}}}. \tag{6}$$

We may assume that the point $x_\nu$ generates a line $L := \mathbb{R}x_\nu \subset \mathbb{R}^n$ which is transversal to the boundary $\partial E_\nu$ and intersects it at a point $p$, and the normal vector at $p$ to $\partial E_\nu$ coincides with the vector $q$. In other words, $p$ is the Log-image of points $w^{(1)}(q), \ldots, w^{(N)}(q)$ from the fiber $\gamma^{-1}(q)$ of the logarithmic Gauss mapping. The torus $\mathrm{Log}^{-1}(p) \subset \mathrm{Log}^{-1}(L)$ intersects the hypersurface $V^*$ at most in $N \leqslant n!\cdot \mathrm{Vol}(\mathcal{N}_f)$ points.

Consider a heighbourhood $U_i$ in $\mathbb{C}^n$ of the point $w^{(i)}(q)$, then $\mathrm{Log}^{-1}(L)$ intersects the hypersurface $V^*$ in $U_i$ along an $(n-1)$-dimensional chain $h_i \subset V^*$. It can be shown, cf. [15] for the case $n = 2$, that integral (6) is asymptotically equivalent for $k \to +\infty$ to the sum

$$c^\nu_{q\cdot k} = \frac{1}{(2\pi\imath)^{n-1}}\sum_{i=1}^{N}\int_{h_i}\mathrm{res}\left(\frac{\omega}{z}\right)\cdot e^{-\langle q,\log z\rangle\cdot k}, \tag{7}$$

where $\log z = (\log z_1, \ldots, \log z_n)$ and $\mathrm{res}\,(\omega/z)$ is the residue form. In local coordinates $z' = (z_1, \ldots, z_{n-1})$ of $V^*$ (assuming $f_{z_n} \neq 0$) we have $\mathrm{res}\,(\frac{\omega}{z}) = \frac{gdz'}{z'\, f_{z_n}|_{V^*}}$. Therefore, the diagonal coefficient can be represented as the sum of oscillating integrals with the *phase function* $\varphi_q(z') = \langle q, \log z\rangle|_{V^*}$. The critical points of this phase function give the main contribution to the asymptotic of such integrals. From Proposition 2 follows that the support of $h_i$ contains only one critical point of $\varphi_q$. It is a point $w^{(i)}(q) \in \gamma^{-1}(q)$.

The asymptotics of an oscillating integral is the most simple for Morse critical points. In this case it is given by stationary phase method (also called saddle-point method, see [18]). The Corollary 1 of Proposition 3 states that for directions $y = q$ outside the ramification locus of the logarithmic Gauss map $\gamma$ the phase function $\varphi_q$ has only Morse critical points.

The situation of a degenerated critical point is much more complicated. First of all we are looking only for rational critical points! By a result of Varchenko some information about asymptotics of oscillating integral can be read from the distance of the Newton diagramm of the phase function at the corresponding point in case of a Newton non-degenerated phase function (and then it depends only of the $\mathcal{K}$-equivanence class of the hypersurface singularity). Otherwise, the distance is only a lower bound. So called *adapted coordinates* exist always in dimension 2 such that the phase function is Newton non-degenerated. Adapted coordinates can be computed algorithmically, for more details cf. [5] and [16].

## 5. Discussion of examples

**Example 1.** Consider the smooth hypersurface $V^*(f)$ defined as a zero set of the polynomial

$$f = z_1^2z_2 + z_1z_2^2 - z_1z_2 + a,\ a \in \mathbb{R},\ a \neq 0, \frac{1}{27},$$

which is non-degenerated for its Newton polyhedron. The cubic $V^*(f)$ is a two-dimensional real torus with three removed points.

The solutions $z(y) = (z_1(y), z_2(y))$ of

$$\begin{cases} z_1^2z_2 + z_1z_2^2 - z_1z_2 + a = 0, \\ h := (2y_2 - y_1)z_1^2z_2 + (y_2 - 2y_1)z_1z_2^2 + (y_1 - y_2)z_1z_2 = 0 \end{cases} \tag{8}$$

for fixed parameter $(y_1 : y_2) \in \mathbb{P}^1$ are zeroes of ideal (5), and for real parameter $y$, they are projected to the contour $\mathcal{C}_{V^*}$ by Log-map. We are interested in the real ramification locus of $\gamma_f$.

We compute the resultant of $f, h$ with respect to the variable $z_2$

$$\begin{aligned} Res(f,h) := \quad & (-y_1^2 + y_1y_2 + 2y_2^2)z_1^3 + (2y_1^2 - 2y_1y_2 - y_2^2)z_1^2 \\ & +(-y_1^2 + y_1y_2)z_1 + 4ay_1^2 - 4ay_1y_2 + ay_2^2. \end{aligned}$$

The multiplicity of an isolated zero $z(y)$ of system (8) coincides with the multiplicity of the zero $z_1(y)$ in $Res(f,h)$. The discriminant of the polynomial $Res(f,h)$ with respect to $z_1$ is the homogeneous polynomial

$$\begin{aligned} D(y_1,y_2) \quad & = (1-27a)(-2y_1+y_2)^2(4ay_1^6 - 12ay_1^5y_2 + (-3a+1)y_1^4y_2^2 - \\ & -2(1-13a)y_1^3y_2^3 + (-3a+1)y_1^2y_2^4 - 12ay_1y_2^5 + 4ay_2^6) \end{aligned}$$

in variables $y_1, y_2$.

Interested in roots of (8) in $\mathbb{T}^2$ we can omit the factor $(-2y_1 + y_2)^2$ in the last expression. Substituting in $D(y)$ an affine parameter $\lambda = y_1/y_2$, we get the polynomial

$$D(\lambda) = 4a\lambda^6 - 12a\lambda^5 + (-3a+1)\lambda^4 - 2(1-13a)\lambda^3 + (-3a+1)\lambda^2 - 12a\lambda + 4a,$$

whose real zeroes $\lambda_i$ give the points $(\lambda_i : 1) \in \mathbb{P}^1_{\mathbb{R}}$ of the real ramification locus of $\gamma_f$.

We have three real intervals of the parameter line $\mathbb{R}_a$: for $a < 0$ the polynomial $D(\lambda)$ has six real roots, for $0 < a < \dfrac{1}{27}$ and $\dfrac{1}{27} < a$ the polynomial $D(\lambda)$ has no real roots.

Choosing values of $a$ from the different intervals of $\mathbb{R}_a$, we obtain different configurations of the contour $\mathcal{C}_{V^*}$ and the amoeba $\mathcal{A}_{V^*}$. Because the volume $2! \cdot \mathrm{Vol}(\mathcal{N}_f) = 3$ does not depend on $a$, all these configurations have a following common property: The number of preimages $\mathrm{Log}^{-1}(p)$ of a point $p \in \mathcal{C}_{V^*}$ with normal vector $(y_1, y_2) \in \mathbb{R}^2$ is equal to three. We count such preimages, which are solutions to (8) for corresponding $(y_1 : y_2) \in \mathbb{P}^1_{\mathbb{R}}$, with their multiplicity in (8). Hence, one can find for every $\lambda \in \mathbb{R}$ three points on $\mathcal{C}_{V^*}$ with the normal vector $(\lambda, 1)$. Moreover, each one lies on its own colored or black part of the contour (see Fig. 1).

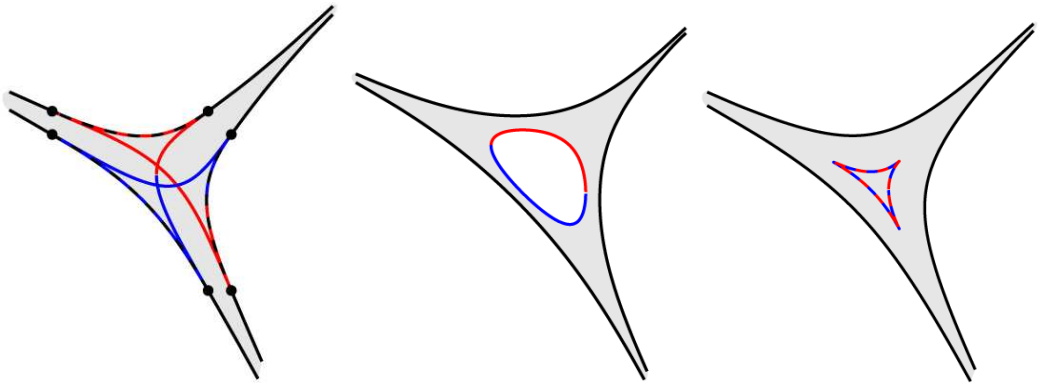

Fig. 1. The contour and the amoeba (shaded) for the polynomial $f = z_1^2z_2 + z_1z_2^2 - z_1z_2 + a$: on the left $a < 0$, in the middle $0 < a < 1/27$,on the right $a > 1/27$

On the left Fig. 1 six black points on $\mathcal{C}_{V^*}$ are images of pleat singularities of the mapping $\mathrm{Log}|_{V^*}$; they correspond to values $\lambda_i$ that belong to the real ramification locus of $\gamma_f$.

Although for $a > 0$ the real ramification locus of $\gamma_f$ is empty, we can distinguish two situations. If $0 < a < 1/27$ the hypersurface $V^*(f)$ is a complexification of the so-called Harnack curve and the complement of its amoeba has the maximal number of components. In this case $\mathrm{Log}|_{V^*}$ has

only fold singularities, which coincide with $V^*(f) \cap \mathbb{R}^2$. For $a > 1/27$ the complement of the amoeba $\mathcal{A}_{V^*}$ has no bounded, component and the mapping $\mathrm{Log}|_{V^*}$ has three pleat singularities; other singularities are folds.

Therefore, for $a > 0$ $\gamma_f$-fiber of any rational $\lambda$ contains only the Morse critical points of the phase function. For example, set the parameter $a = 3/100$ then the $\gamma_f$-fiber of $\lambda = 1/3$ consists of the Morse points $(3/10, 1/2)$, $(7/40+\sqrt{57}/40, 9/8-\sqrt{57}/8)$ and $(7/40-\sqrt{57}/40, 9/8+\sqrt{57}/8)$. For $a < 0$ we can get degenerated rational points in a real ramification locus, e.g. there are six rational points $-2, -1/2, 2/3, 3/2, 1/3, 3$ in the real ramification locus of $\gamma_f$, $a = -9/10$. The $\gamma_f$-fiber of such points has a simple point and an $A_2$-point of the phase function.

**Example 2.** We consider the polynomial $f$ in $n = 3$ variables, which is non-degenerated for its Newton polyhedron;

$$f = 1 + z_1 + z_2 + z_3 + 3z_1z_2 + 3z_1z_3 + 3z_2z_3 + 11z_1z_2z_3.$$

As in Example 1 the real ramification locus of $\gamma_f$ is determined by the following system

$$\begin{cases} 1 + z_1 + z_2 + z_3 + 3z_1z_2 + 3z_1z_3 + 3z_2z_3 + 11z_1z_2z_3 = 0, \\ y_3z_1 - y_1z_3 + 3y_3z_1z_2 + (3y_3 - 3y_1)z_1z_3 - 3y_1z_2z_3 \\ \qquad +(11y_3 - 11y_1)z_1z_2z_3 = 0, \\ y_3z_2 - y_2z_3 + 3y_3z_1z_2 - 3y_2z_1z_3 + (-3y_2 + 3y_3)z_2z_3 \\ \qquad +(11y_3 - 11y_2)z_1z_2z_3 = 0. \end{cases} \tag{9}$$

With similar computations we obtain the discriminant $D(y)$ of the logarithmic Gauss map:

$$D(y) := y_1^4 \cdot (y_2 - y_3)^2 \cdot (4y_1 + 5y_2 + 5y_3)^2 \cdot d(y), \tag{10}$$

where $d(y)$ is a homogeneous polynomial of degree 12, it consists of 91 terms. Its Newton's polyhedron is a triangle with vertices $(12, 0, 0)$, $(0, 12, 0)$ and $(0, 0, 12)$.

We do not consider zeroes of the first three factors in (10) because they do not give us multiple roots of (9) in the torus. The ramification locus of $\gamma_f$ is given by zero set of $d(y)$. Let $\lambda_1 = y_1/y_3$, $\lambda_2 = y_2/y_3$ be coordinates in affine part of $\mathbb{P}^2_{\mathbb{R}}$, where $y_3 \neq 1$. Fig. 2 depicts the zero set of $d(\lambda_1, \lambda_2, 1) = d(y)/y_3^{12}$, which coincides with the affine part of the real ramification locus of $\gamma_f$.

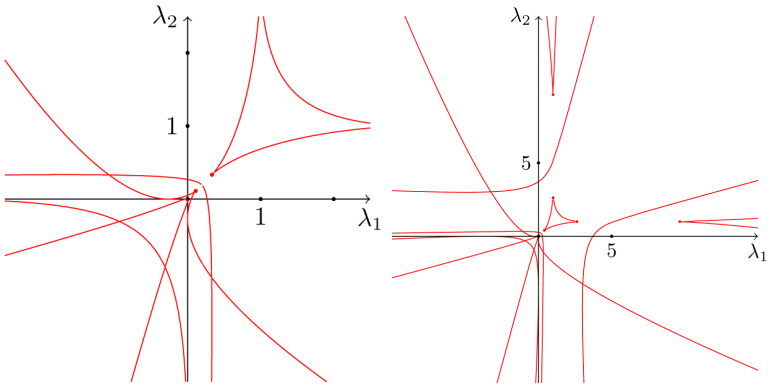


Fig. 2. The real ramification locus of $\gamma_f$

The red points $(1/9, 1/9)$, $(1/3, 1/3)$, $(1, 3)$, $(3, 1)$, $(1, 9)$, $(9, 1)$ on Fig. 2 are degenerated rational critical points of the discriminant with Milnor number $\mu = 2$. This example of the polynomial $f$ is a special one because the existence of rational degenerated points in a real

ramification locus is not a generic property. We are interested in such points because they lead to degenerated critical points of a phase function. In this example the $\gamma_f$-fiber of any $A_2$-point contains excatly one $A_3$-critical point of the phase function (see Appendix for details).

# Appendix. Computation with SINGULAR (Some Experiences)

The computer algebra system SINGULAR, cf. [2], was used for the computation of examples. We tried several strategies for computing the discriminant of the Log-Gauss map with different success, i.e. to get a result for non-trivial examples without overflow and in reasonable time. Here we give a small introduction how proceed in SINGULAR, demonstrated with the equation of Example 2.

Start with a base ring that contains the ideal $I$ of the graph of the log Gauss map $\gamma_f$ of a polynomial $f = f(z)$ and compute $I$, (here $n = 3$):

```
ring R=0,(y1,y2,y3,z1,z2,z3),dp;
poly f=1+z1+z2+z3+3*z1*z2 +3*z1*z3+3*z2*z3+11*z1*z2*z3;
matrix A[2][3] = z1*diff(f,z1),z2*diff(f,z2),z3*diff(f,z3),y1,y2,y3;
ideal I = f,minor(A,2);
```

Next we project the graph restricted to some affine chart $U_3 := \{y_3 \neq 0\}$ into $\mathbb{A}^3 := U_3 \times \mathbb{A}^1$ ($\mathbb{A}^1_1$ — a coordinate axes of $\mathbb{A}^3_z$). The image is a hypersurface defined by the next polynomial $h(y_1, y_2, z_1)$, which we could closure in $\mathbb{P}^2_y$ by homogenizing in the $y's$. Using the elimination of variable, the multiplicities of multiple factors may be lost, but it does not effect the result.

```
I = subst(I,y3,1); ideal J = eliminate(I,z2*z3);
poly h1 = J[1];
```

The choice of the projection direction was good, if $\deg_{z1}(h_1) = n! \cdot \mathrm{Vol}(\mathcal{N}) = 6$. The discriminant variety of $\gamma_f$ is contained in the discriminant hypersurface of the projection $V(h_1) \subset U_3 \times \mathbb{A}^1_1 \longrightarrow U_3$, computed in the next step.

```
poly d1 = resultant(h1,diff(h1,z1),z1);
d1 = homog(d1,y3);
list Ld = factorize(d1);
```

The plane curve $V(d_1) \subset \mathbb{P}^2$ has several components, it may have components with certain multiplicities, some of them induced from the closure $V^*(f)$ or not belonging to the discriminant. If our polynomial is generic, then we expect the discriminant of $\gamma_f$ (i.e. restricted to the torus) being irreducible. We should test which factor is correct. Some components of $V(d_1)$ have empty fiber with respect to $\gamma_f$ or no multiple points in its $\gamma_f$-fibers. We can reduce sometimes the number of factor as follows: Compute for any coordinate $z_i$ (as above for $i = 1$) polynomials $h_i$ and $d_i$ and factorize only $d := gcd(d_1, \ldots, d_n)$.

Having found the equation of the discriminant polynomial $d_0(y)$, we can compute its (discrete) singular locus.

```
poly d0 = Ld[1][2];            (choose the right factor in this example)
d0 = subst(d0,y3,1);
ring S = 0,(y1,y2,y3),dp;
poly d0 = imap(R,d0);
ideal sl = slocus(d0);
list Lsl = primdecGTZ(sl);
```

Here, the singular locus has six rational double points $Q_1 = (1, 3)$, $Q_2 = (\frac{1}{9}, \frac{1}{9})$, $Q_3 = (1, 9)$, $Q_4 = (\frac{1}{3}, \frac{1}{3})$, $Q_5 = (9, 1)$, $Q_6 = (3, 1)$ and more irrational singular points. We choose $Q_1$ and check, that it is an $A_2$-singularity of $D$.

```
show(Lsl[2]);                 (choose one of the singular points of D)
ring S' = 0,(y1,y2),ds;
poly d0 = imap(R,d0);
d0 = subst(d0,y1,y1+1);       (translate that singularity to zero)
d0 = subst(d0,y2,y2+3);
"mu =",milnor(d0);             (Milnor number of the singularity)
```

Compute the $\gamma_f$-fiber of $Q_1$. It has 3 simple points and exactly one point $P_* = (-1, -\frac{1}{3}, -1)$ of multiplicity 3, being an $A_3$-point of the phase function.

```
setring R;
I = subst(I,y1,1); I = subst(I,y2,3);
ring R0 = 0,(z1,z2,z3),dp;
ideal I = imap(R,I);
list Lfib = primdecGTZ(i);            (list contains the points of the fiber).
option(redSB);
show(std(Lfib[1][2]));
"mult =",vdim(std(Lfib[1][1]));
```

Similar computations lead to similar results at the other 5 rational singularities of the discriminant.

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# References

[1] V.I.Arnold, S.M.Gusein-Zade, A.N.Varchenko, Singularities of differentiable maps, Vol. I, Boston, Bikhäuser, 1985.

[2] W.Decker, G.-M.Greuel, G.Pfister, H.Schönemann, SINGULAR 3-1-3 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2011).

[3] M.Forsberg, M.Passare, A.Tsikh, Laurent determinants and arrangements of hyperplane amoebas, *Adv. in Math.*, **151**(2000), 45–70.

[4] I.Gelfand, M.Kapranov, A.Zelevinsky, Discriminants, Resultants and Multidimentional Determinants, Boston, Bikhäuser, 1994.

[5] I.A.Ikromov, D.Müller, On adapted coordinate system, *Trans. Amer. Math. Soc.*, **363**(2011), 2821–2848.

[6] A.G.Khovanskii, Newton polyhedra and toroidal varieties, *Funct. Anal. Appl.*, **11**(1977), no. 4, 289–296.

[7] A.G.Kouchnirenko, Polyèdres de Newton et nombres de Milnor, *Invent. Math.*, **32**(1976), 1–31.

[8] D.T.Le, Calculation of Milnor number of isolated singularity of complete intersection, *Funk. Anal. and its Appl.*, **8**(1972), no. 2, 45–49.

[9] J.N.Mather, Classification of Isolated Hypersurface Singularities by Their Moduli Algebras, *Invet. math.*, **69**(1982), 243–251.

[10] B.Martin, Singularities are determined by their cotangent complexe, *Ann. Global Anal. Geom.*, **3**(1985), no. 2, 197–217.

[11] G.Mikhalkin, Real algebraic curves, moment map and amoebas, *Ann. of Math.*, **151**(2000), no. 1, 309–326.

[12] M.Passare, A.K.Tsikh, Amoebas: their spines and their contours, *Contemporary Math.*, **377**(2005), 275–288.

[13] H.Rullgård, Topics in geometry, analysis and inverse problems, Ph. D diss., Stockholm University, 2003.

[14] B.Teissier, The hunting of invariants in the geometry of discriminants. Real and complexe singularities, Oslo 1076, 565–678.

[15] A.K.Tsikh, Conditions for absolute convergence of the Taylor coefficient series of a meromorphic fucntion of two variables, *Math. USSR Sb.*, **74**(1993), no. 2, 337–360.

[16] A.N.Varchenko, Newton polyhedra and estimates of oscillating integrals, *Funct. Anal. Appl.*, **10**(1976), no. 3, 175–196.

[17] K.Wirthmüller, Singularities Determined by Their Discriminant, *Math. Ann.*, **252**(1980), 237–245.

[18] R.Wong, Asymptotic approximations of integrals, SIAM, 2001.

## Дискриминант и особенности логарифмического отображения Гаусса, примеры и приложение

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*Изучение гиперповерхностей, заданных в торе, приводит к прекрасному зоопарку амеб и их контуров, возможные конфигурации которых читаются из комбинаторных данных. Существует глубокая связь между теорией амеб и логарифмическим отображением Гаусса, а также его критическими точками, изучение которых находит приложения в различных областях.*

*В статье мы напоминаем основные понятия и результаты из теории амеб, раскрываем некоторые ее связи с алгебраической теорией сингулярностей. Более того, мы приводим вычисления критических точек логарифмического отображения Гаусса в системе компьютерной алгебры SINGULAR, а также обсуждаем различные варианты и их эффективность. Здесь мы приходим к существенному наблюдению: содержательные примеры требуют наличия вещественных или даже рациональных решений соответствующей системы алгебраических уравнений.*

*Ключевые слова: логарифмическое отображение Гаусса, особенности, дискриминант, амеба гиперповерхности, асимптотика.*