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An Equilibrium Problem for the Timoshenko-type Plate Containing a Crack on the Boundary of a Rigid Inclusion

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An equilibrium problem for an elastic Timoshenko type plate containing a rigid inclusion is considered. On the interface between the elastic plate and the rigid inclusion, there is a vertical crack. It is assumed that at both crack faces, boundary conditions of inequality type are considered describing a mutual non-penetration of the faces. A solvability of the problem is proved, and a complete system of boundary conditions is found. It is also shown that the problem is the limit one for a family of other problems posed for a wider domain and describing an equilibrium of elastic plates with a vertical crack as the rigidity parameter goes to infinity.

Keywords: crack, Timoshenko-type plate, rigid inclusion, energy functional, mutual non-penetration condition.

Introduction

The interest in studying mathematical models of bodies containing rigid inclusions is due to the wide application of composites. In [1–5], some problems of the theory of elasticity for bodies with cracks and rigid inclusions are considered. It is known that different problems in regard to bodies containing rigid inclusions may be successfully formulated and studied by using variational methods [6–13]. In particular, the theory of two-dimensional problems of the theory of elasticity with thin rigid inclusions and possible delamination is given in [6]. The three-dimensional case is considered in [7].

In studying the deformation of plates and shells, the Kirchhof-Love and Timoshenko-type models [14, 15] are successfully used. A great number of papers [8–13] is devoted to investigating the models of the Kirchhof-Love plates containing rigid inclusions. In the [8–12] assumed that the rigid inclusion in a plate volumetric. In [13] the thin rigid inclusion is modeled by a plane two-dimensional curve.

In the present work, the equilibrium problem of elastic transversally isotropic Timoshenko-type plate containing a through crack on the boundary of a rigid inclusion is investigated. The unique solvability of the variational problem on the equilibrium of the plate with a crack is proved. The system of boundary-value conditions, which defines the equivalent boundary-value problem, is obtained from the variational formulation of the problem. For the equilibrium problem of the elastic Timoshenko-type plate with a crack, it is shown out that the equilibrium problem of the plate with a rigid inclusion is obtained as the rigidity parameter tends to the infinity at some fixed part of the plate.

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1. Formulation of the Problem

Consider the bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary Γ. Let the subdomain ω be strictly inside Ω, i.e., $\overline{\omega} \cap \Gamma = \varnothing$ and let its boundary Ξ be sufficiently smooth. Let us consider that Ξconsists of two parts γ and $\Xi \backslash \overline{\gamma}$, moreover, $\partial\gamma \notin \gamma$ (Fig. 1).

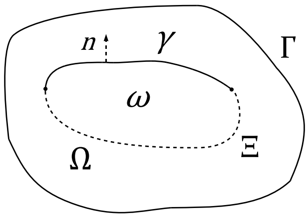

Fig. 1.

The thickness of the plate is considered to be constant and equal to $2h$. We define a three-dimensional Cartesian space $\{x_1, x_2, z\}$ such that the set $\{\Omega_\gamma\} \times \{0\} \subset \mathbb{R}^3$ corresponds to the middle plane of the plate, where $\Omega_\gamma = \Omega \backslash \overline{\gamma}$. Here, the curve γ defines a crack (a cut) in the plate. This means that the cylindrical surface of the through crack may be given by the relations $x = (x_1, x_2) \in \gamma, -h \leqslant z \leqslant h$, where $|z|$ is the distance to the middle plane. According to our arguments, the rigid inclusion is specified by the set $\omega \times [-h, h]$, i.e. the boundary of the rigid inclusion is given by the cylindrical surface $\Xi \times [-h, h]$. Elastic part of the plate corresponds to the domain $\Omega \backslash \overline{\omega}$.

Denote by $\chi = \chi(x) = (U, u)$ the vector of displacements of points of the mid-surface ($x \in \Omega_\gamma$, $U = (u^1, u^2)$ are the displacements in the plane $\{x_1, x_2\}$, and u are the displacement along the axis z. We denote the angles of rotation of a normal fiber by $\phi = \phi(x) = (\phi^1, \phi^2)$ ($x \in \Omega_\gamma$). Let us consider that the values χ and ϕ are infinitesimal by the classical theory of elasticity. In accordance with the direction of the outer (with respect to ω) normal $n = (n_1, n_2)$ to Ξ, there is a positive face Ξ^+ and a negative face Ξ^- of the curve Ξ. If the trace of a function v is chosen on the positive (from the side of the domain $\Omega \backslash \overline{\omega}$) face Ξ^+, we use the notation $v^+ = v|_{\Xi^+}$ and if it is chosen on the negative face, then $v^- = v|_{\Xi^-}$. The jump $[v]$ of the function v on the curve Ξ is found by the formula $[v] = v|_{\Xi^+} - v|_{\Xi^-}$. Analogous notation is used for γ^+ and γ^-.

Introduce the tensors describing the deformation of the plate

$$\varepsilon_{ij}(\phi) = \frac{1}{2}(\phi,^i_j + \phi,^j_i), \quad \varepsilon_{ij}(U) = \frac{1}{2}(u,^i_j + u,^j_i), \ i, j = 1, 2, \ (v,_i = \frac{\partial v}{\partial x_i}).$$

The tensors of moments $m(\boldsymbol{\phi}) = \{m_{ij}(\boldsymbol{\phi})\}$ and stresses $\sigma(\boldsymbol{U}) = \{\sigma_{ij}(\boldsymbol{U})\}$, are expressed by the formulas (summation is performed over repeated indices)

$$m_{ij}(\phi) = a_{ijrl}\varepsilon_{rl}(\phi), \quad \sigma_{ij}(U) = 3h^{-2}a_{ijrl}\varepsilon_{rl}(U), \quad i, j, r, l = 1, 2,$$

where the nonzero components of elasticity tensor $A = \{a_{ijrl}\}$ are expressed by the relations

$$a_{iiii} = D, \, a_{iijj} = D\text{æ}, \, a_{ijij} = a_{ijji} = D(1 - \text{æ})/2, \ i \neq j, \ i, j = 1, 2,$$

where D and æ are the constants: D is a cylindrical rigidity of the plate, æ is the Poisson ratio, $0 < \text{æ} < 1/2$. The transverse forces in the Timoshenko-type model are specified by the expressions

$$q_i(u, \phi) = J(u,_i + \phi^i), \quad i = 1, 2, \tag{1}$$

where $J = 2k^2Gh$, k^2 is the shear coefficient, G is the shear modulus in areas perpendicular to the middle plane of the plate, and J, k^2, G are the constants. Let $B_M(\cdot\,,\cdot)$ be a bilinear form defined by the equality

$$B_M(\xi,\eta) = \langle m_{ij}(\phi), \varepsilon_{ij}(\psi)\rangle_M + J\langle(u_{,i} + \phi^i), (v_{,i} + \psi^i)\rangle_M + \langle\sigma_{ij}(U), \varepsilon_{ij}(V)\rangle_M,$$

where $\langle\cdot,\cdot\rangle_M$ is the scalar product in $L_2(M)$, for the subdomain $M \subset \Omega$, $\xi = (U, u, \phi)$, $\eta = (V, v, \psi)$. The potential energy functional of the deformed plate occupying the region Ω_γ has the form

$$\Pi(\xi) = \frac{1}{2}B_{\Omega_\gamma}(\xi,\xi) - \langle f, \xi\rangle_{\Omega_\gamma}, \quad \xi = (U, u, \phi),$$

where $f = (f_1, f_2, f_3, \mu_1, \mu_2) \in L^2(\Omega_\gamma)^5$ is the vector specifying the external loads [15].

Introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \left\{u \in H^1(\Omega_\gamma) \;\middle|\; u = 0 \text{ a.e. on } \Gamma\right\}, \quad H = H^{1,0}(\Omega_\gamma)^5, \quad \|\cdot\| = \|\cdot\|_H.$$

Let $\xi \in H$. We write the relations for ξ conditioned by the presence of rigid inclusion and crack in the plate. It is known that deformations equal zero for a rigid body. With respect to the considered rigid inclusion in the plate of the Timoshenko-type model, this means that in the domain ω tangential strains $\varepsilon_{ij}(U)$, $i, j = 1, 2$, bending strains $\varepsilon_{ij}(\phi)$, $i, j = 1, 2$, and transverse strains $u_{,i} + \phi^i$, $i = 1, 2$ equal zero. Consequently, the restriction of the function ξ to the domain ω has the given structure (for example, see [16]):

$$U = \rho, \quad \phi = d, \quad \phi + \nabla u = 0 \quad \text{on } \omega, \quad \rho, d \in R(\omega), \tag{2}$$

where the space $R(\omega)$ is defined as follows

$$R(\omega) = \{\rho = (\rho_1, \rho_2) \,|\, \rho(x) = Bx + C, \; x \in \omega\},$$

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad C = (c^1, c^2); \quad b, c^1, c^2 = \text{const}.$$

In component-wise notation, two last equalities in (2) take the form

$$\begin{cases} u_{,i} + \phi^i = 0, \quad i = 1, 2, \quad \text{on } \omega, \\ \phi^1 = d_1 = bx_2 + c_1 \quad \text{on } \omega, \\ \phi^2 = d_2 = -bx_1 + c_2 \quad \text{on } \omega. \end{cases}$$

This implies that $u_{,12} = b$ and $u_{,21} = -b$. This means that $b = 0$ and the restriction of components of the function ξ satisfies the relations

$$U = \rho, \quad u = l, \quad \phi + \nabla u = 0 \quad \text{on } \omega, \quad \rho \in R(\omega), \, l \in L(\omega), \tag{3}$$

where

$$L(\omega) = \{l \,|\, l(x) = a_0 + a_1x_1 + a_2x_2, \; a_i \in \mathbb{R}, \; i = 0, 1, 2, \; x = (x_1, x_2) \in \omega\}.$$

In turn, it is possible to define that equalities (3) follow from (2).

Derive the nonpenetration conditions of the crack faces. Since the dependence of horizontal displacements $U(z)$ on z for the Timoshenko-type model is expressed by the following formulas [15]:

$$u^i(z) = u^i + z\phi^i, \quad i = 1, 2, \quad |z| \leqslant h, \quad U(z) = (u^1(z), u^2(z)),$$

after scalar product of the jump of the displacement vector $[\chi(z)] = ([u^1(z)], [u^2(z)], [u])$ by the normal to the vertical crack surface with coordinates $(n_1, n_2, 0)$, we have

$$([u^1(z)], [u^2(z)], [u]) \cdot (n_1, n_2, 0) = [U(z)] \cdot n = [U] \cdot n + z[\phi] \cdot n \geqslant 0 \quad |z| \leqslant h, \quad \text{on } \gamma.$$

Hereinafter, by " $\cdot$ " denote the operation of scalar product. Substitution of $z = h$ and $z = -h$ into this inequality gives the condition of the mutual non-penetration.

$$[U] \cdot n \geqslant h|[\phi] \cdot n| \quad \text{on } \gamma.$$

Note that the vertical displacements u are absent in this inequality. It is due to the fact that the generatrix of the cylindrical surface, which bounds the rigid inclusion, are perpendicular to the mid-surface of the plate. Next, taking into consideration the fact that the traces of functions U and ϕ on γ^- are defined by (3), from the last inequality, we derive

$$(U - \rho) \cdot n \geqslant h\Big| (\phi + \nabla l) \cdot n \Big| \quad \text{on } \gamma^+, \tag{4}$$

where $U = \rho$, $u = l$ on ω, $\rho \in R(\omega)$, $l \in L(\omega)$.

The equilibrium problem of the plate with a crack on the boundary of the rigid inclusion may be formulated in the form of the minimization problem

$$\inf_{\xi \in K_\omega} \Pi(\xi), \tag{5}$$

where $K_\omega = \Big\{ \xi = (U, u, \phi) \in H \;\Big|\; \xi \quad \text{satisfies} \quad (3)\ ,\ (4) \Big\}$ is the set of admissible functions. Note that the inclusion $\xi \in H$ assumes that the homogeneous boundary-value conditions hold:

$$u = 0, \quad \phi = U = \mathbf{0} = (0, 0) \text{ on } \Gamma. \tag{6}$$

By virtue of the equivalence of (2) and (3), for the functions $\xi \in K_\omega$, the following equalities hold: $\varepsilon_{ij}(U) = 0$, $\varepsilon_{ij}(\phi) = 0$, $\phi^i + u_{,i} = 0$ a.e. in ω, $i, j = 1, 2$. Hence, the energy functional may be represented in the form

$$\Pi(\xi) = \frac{1}{2} B_{\Omega\backslash\overline{\omega}}(\xi, \xi) - \big\langle f, \xi \big\rangle_{\Omega_\gamma}, \quad \xi = (U, u, \phi) \in K_\omega.$$

One can easily show that the set K_ω is convex and closed in the Hilbert space H. Due to the estimate

$$B_{\Omega_\gamma}(\xi, \eta) \leqslant c_1 \|\xi\| \|\eta\|,$$

where the constant $c_1 > 0$ is independent of $\xi \in H$ and $\eta \in H$, the symmetric bilinear form of $B_{\Omega_\gamma}(\xi, \eta)$ is continuous with respect to H. The coercivity of the functional $\Pi(\xi)$ follows from the inequality

$$B_{\Omega_\gamma}(\xi, \xi) \geqslant c_2 \|\xi\|^2, \quad \xi \in H \tag{7}$$

where the constant $c_2 > 0$ is independent of ξ (see [17]). The above properties of energy functional $\Pi(\xi)$, form $B_{\Omega_\gamma}(\cdot, \cdot)$, and set K_ω allow one to state on the unique solvability of problem (5) (see [18]).

In what follows, by $\xi = (U, u, \phi)$ we denote the solution of problem (5). Let the functions $\rho_0 \in R(\omega)$ and $l_0 \in L(\omega)$ be defined by equalities $\rho_0 = U$ and $l_0 = u$ on ω, where U and u are the components of the solution ξ.

Symmetry and continuity of the bilinear form $B_{\Omega_\gamma}(\cdot, \cdot)$ and the properties of the set K_ω provide (see [18]) the equivalence of problem (5) to the variational inequality

$$\xi \in K_\omega, \quad B_{\Omega\backslash\overline{\omega}}(\xi, \eta - \xi) \geqslant \langle f, \eta - \xi \rangle_{\Omega_\gamma} \quad \text{for any} \quad \eta = (V, v, \psi) \in K_\omega. \tag{8}$$

Compare two inequalities obtained by substituting of the test functions $\eta = \xi + \tilde{\eta}$ and $\eta = \xi - \tilde{\eta}$, where $\tilde{\eta} = (\tilde{V}, \tilde{v}, \tilde{\psi}) \in C_0^\infty(\Omega\backslash\overline{\omega})^5$. As a result, we get an equality, which may be written as follows:

$$\big\langle m_{ij}, \varepsilon_{ij}(\tilde{\psi}) \big\rangle_{\Omega\backslash\overline{\omega}} + \big\langle \sigma_{ij}, \varepsilon_{ij}(\tilde{V}) \big\rangle_{\Omega\backslash\overline{\omega}} + \big\langle q_i, (\tilde{v}_{,i} + \tilde{\psi}^i) \big\rangle_{\Omega\backslash\overline{\omega}} = \langle f, \tilde{\eta} \rangle_{\Omega\backslash\overline{\omega}}, \tag{9}$$

where $m_{ij} = m_{ij}(\phi)$, $\sigma_{ij} = \sigma_{ij}(U)$, $q_i = q_i(u, \phi)$, $i, j = 1, 2$; U, u, ϕ are the components of the solution ξ. From (9), with allowance for the independence of $\tilde{v}^1$, $\tilde{v}^2$, $\tilde{v}$, $\tilde{\psi}^1$, $\tilde{\psi}^2$, we have

$$\langle \sigma_{ij}, \varepsilon_{ij}(\tilde{V}) \rangle_{\Omega\setminus\overline{\omega}} = \langle f_i, \tilde{v}^i \rangle_{\Omega\setminus\overline{\omega}} \quad \text{for any} \quad \tilde{V} \in C_0^\infty(\Omega\setminus\overline{\omega})^2,$$

$$\langle q_i, (\tilde{v}_{,i}) \rangle_{\Omega\setminus\overline{\omega}} = \langle f_3, \tilde{v} \rangle_{\Omega\setminus\overline{\omega}} \quad \text{for any} \quad \tilde{v} \in C_0^\infty(\Omega\setminus\overline{\omega}),$$

$$\langle m_{ij}, \varepsilon_{ij}(\tilde{\psi}) \rangle_{\Omega\setminus\overline{\omega}} + \langle q_i, \tilde{\psi}^i \rangle_{\Omega\setminus\overline{\omega}} = \langle \mu_i, \tilde{\psi}^i \rangle_{\Omega\setminus\overline{\omega}} \quad \text{for any} \quad \tilde{\psi} \in C_0^\infty(\Omega\setminus\overline{\omega})^2.$$

Taking into account that the representations $\langle m_{ij}, \varepsilon_{ij}(\tilde{\psi}) \rangle_{\Omega\setminus\overline{\omega}} = \langle m_{ij}, (\tilde{\psi}_{,j}^{i}) \rangle_{\Omega\setminus\overline{\omega}}$ and $\langle \sigma_{ij}, \varepsilon_{ij}(\tilde{V}) \rangle_{\Omega\setminus\overline{\omega}} = \langle \sigma_{ij}, (\tilde{v}_{,j}^{i}) \rangle_{\Omega\setminus\overline{\omega}}$ take place, three previous equalities imply that the following equilibrium equations hold in the sense of distributions:

$$\sigma_{ij,j} = -f_i, \quad i = 1, 2, \quad \text{in } \Omega\setminus\overline{\omega}, \tag{10}$$

$$q_{i,i} = -f_3 \quad \text{in } \Omega\setminus\overline{\omega}, \tag{11}$$

$$m_{ij,j} - q_i = -\mu_i, \quad i = 1, 2, \quad \text{in } \Omega\setminus\overline{\omega}. \tag{12}$$

2. Differential Statement of the Problem

Below, we get the equivalent differential statement of problem (5). Namely, by the variational inequality (8), using an appropriate choice of test functions, we derive a complete set of boundary-value conditions on the curve γ. In order to find the relations on the inner boundary γ from inequality (8), we use Green's formulas. Since the derivatives of functions in the space $H^1(\Omega_\gamma)$ have generally no traces, we assume a sufficient smoothness ξ in this paragraph.

For the domain $\Omega\setminus\overline{\omega}$, Green's formula is valid (see [19], [20])

$$\int_{\Omega\setminus\overline{\omega}} \sigma_{ij}\, \varepsilon_{ij}(V) = -\int_{\Omega\setminus\overline{\omega}} \sigma_{ij,j} v^i - \int_{\Xi} \sigma_{ij} n_j v^i, \quad V = (v^1, v^2) \in H^{1,0}(\Omega_\gamma)^2, \tag{13}$$

where $n = (n_1, n_2)$ is the normal to Ξ. Similarly, for the arbitraries $\psi \in H^{1,0}(\Omega_\gamma)^2$ and $v \in H^{1,0}(\Omega_\gamma)$, Green's formulas take place

$$\langle m_{ij}, \varepsilon_{ij}(\psi) \rangle_{\Omega\setminus\overline{\omega}} = -\langle m_{ij,j}, \psi^i \rangle_{\Omega\setminus\overline{\omega}} - \int_{\Xi} m_{ij} n_j \psi^i, \tag{14}$$

$$\langle \nabla u, \nabla v \rangle_{\Omega\setminus\overline{\omega}} = -\langle \triangle u, v \rangle_{\Omega\setminus\overline{\omega}} - \int_{\Xi} \frac{\partial u}{\partial n}\, v, \tag{15}$$

$$\langle \phi, \nabla v \rangle_{\Omega\setminus\overline{\omega}} = -\langle (\phi_{,i}^{i}), v \rangle_{\Omega\setminus\overline{\omega}} - \int_{\Xi} (\phi \cdot n) v. \tag{16}$$

Let the function $\tilde{\eta} = (\tilde{V}, \tilde{v}, \tilde{\psi}) \in K_\omega$ such that $\tilde{\eta} \in H_0^1(\Omega)^5$. Substitution of the test functions of the form $\eta = \xi \pm \tilde{\eta}$ into the variational inequality (8) yields the following integral identity valid for all $\tilde{\eta} \in H_0^1(\Omega)^5$:

$$\langle m_{ij}, \varepsilon_{ij}(\tilde{\psi}) \rangle_{\Omega_\gamma} + \langle \sigma_{ij}, \varepsilon_{ij}(\tilde{V}) \rangle_{\Omega_\gamma} + \langle q_i, (\tilde{v}_{,i} + \tilde{\psi}^i) \rangle_{\Omega_\gamma} = \langle f, \tilde{\eta} \rangle_{\Omega_\gamma}. \tag{17}$$

Next, integrating by parts, with allowance for (3) in (17), we derive

$$-\int_{\Xi} \sigma_{ij} n_j \rho_i + \int_{\Xi} m_{ij} n_j l_{,i} - J\int_{\Xi} (\phi \cdot n + \frac{\partial u}{\partial n}) l = \int_{\omega} (F \cdot \rho + f_3 l - \mu_i l_{,i}), \tag{18}$$

where $F = (f_1, f_2)$. Taking into consideration the independence of ρ and l in (18), we have

$$-\int_{\Xi}\sigma_{ij}n_j\rho_i = \int_{\omega} F\cdot\rho \quad \text{for any } \rho\in R(\omega), \tag{19}$$

$$\int_{\Xi}m_{ij}n_jl_{,i} - J\int_{\Xi}(\phi\cdot n + \frac{\partial u}{\partial n})l = \int_{\omega}(f_3l - \mu_il_{,i}) \quad \text{for any } l\in L(\omega). \tag{20}$$

Comparing two inequalities obtained by substituting the test functions $\eta = 0$ and $\eta = 2\xi$ into (8), we derive the identity

$$-\int_{\Xi^+}\sigma_{ij}n_ju^i - \int_{\Xi^+}m_{ij}n_j\phi^i - J\int_{\Xi^+}(\phi\cdot n + \frac{\partial u}{\partial n})u = \int_{\omega}(F\cdot\rho_0 + f_3l_0 - \mu_il_{0,i}). \tag{21}$$

Here Ξ^+ denotes that the values of functions on Ξ are taken on the positive face. Consider the function $\eta = \xi\pm\tilde\eta$, where $\tilde\eta = (\tilde V, \tilde v, \tilde\psi)$ such that $\tilde V\equiv 0$, $\tilde\psi\equiv 0$ in Ω_γ, the function $\tilde v\in H^{1,0}(\Omega_\gamma)$, $\tilde v = 0$ in ω. By construction, it is obvious that $\eta\in K_\omega$. Note that the values of function $\tilde v$ are equal to zero on the curve $\Xi\backslash\overline{\gamma}$ and arbitrary on γ^+. Substitute the function η into (8) and transform the obtained relation by using formulas (15), (16). As a result, we get

$$\int_{\gamma^+}(\phi\cdot n + \frac{\partial u}{\partial n})\tilde v \geqslant 0.$$

Hence, with allowance for the arbitrariness of values $\tilde v$ on the boundary γ^+, we find

$$\phi\cdot n + \frac{\partial u}{\partial n} = 0 \quad \text{on } \gamma^+. \tag{22}$$

The integral over the boundary in (13) may be represented as follows (see [20]):

$$\int_{\Xi}\sigma_{ij}n_jv^i = \int_{\Xi}\big(\sigma_n(V\cdot n) + \sigma_{\tau i}V_{\tau i}\big), \tag{23}$$

where $\sigma_n n$ and $\sigma_\tau = (\sigma_{\tau 1}, \sigma_{\tau 2})$ are normal and tangential components of the vector $\sigma n = \{\sigma_{ij}n_j\}$; here the following relations are valid:

$$\sigma_{ij}n_j = \sigma_n n_i + \sigma_{\tau i},\ \sigma_n = \sigma_{ij}n_jn_i, \quad \tau = (-n_2, n_1),$$

$$V = (v^1, v^2), \quad v^i = (V\cdot n)n_i + V_{\tau i}, \quad i = 1,2, \quad V_\tau = (V_{\tau 1}, V_{\tau 2}).$$

The representation of the form (23) is valid for the integral over the boundary in (14) as well. . Substitute the test functions of the form $\eta = \xi\pm\tilde\eta$, where $\tilde\eta = (\tilde V, \tilde v, \tilde\psi)\in H$ such that $\tilde v\equiv 0$, $\tilde\psi\equiv 0$ in Ω_γ, $\tilde V\equiv 0$ in ω, and $\tilde V\cdot n = 0$ on γ^+ into the variational inequality (8). By using Green's formulas, the transformation of the obtained integrals yields

$$\int_{\Xi^+}\sigma_{ij}n_j\tilde v^i = \int_{\Xi^+}\big(\sigma_n(\tilde V\cdot n) + \sigma_{\tau i}\tilde V_{\tau i}\big) = \int_{\Xi^+}\sigma_{\tau i}\tilde V_{\tau i} = 0.$$

Hence, by virtue of the arbitrariness of the values $\tilde V$ on γ^+, we infer that $\sigma_\tau = \mathbf{0}$ on γ^+. By using analogous calculations and choosing the test function of special form (considering ψ, which satisfy $\psi\cdot n = 0$), we can get that $m_\tau = \mathbf{0}$ on γ^+. Now let $\eta = \xi + \tilde\eta$, where $\tilde\eta = (\tilde V, \tilde v, \tilde\psi)\in H$ such that $\tilde v\equiv 0$ in Ω_γ, $\tilde\psi\equiv 0$, $\tilde V\equiv 0$ in ω, $\tilde V\cdot n = h|\tilde\psi\cdot n|$, $\tilde V\cdot n\geq 0$ on γ^+. It is easy to note that $\eta\in K_\omega$. Substitute it into (8). By using Green's formulas, we find

$$\int_{\Xi^+}\big(-\sigma_n(\tilde V\cdot n)\pm hm_n(\tilde\psi\cdot n)\big)\geqslant 0.$$

The choice of functions $\tilde{V}$, $\tilde{\psi}$ allows one to obtain the following relation from the last inequality:

$$-h\sigma_n \geqslant |m_n| \quad \text{on } \gamma^+. \tag{24}$$

From identities (18) and (21), we have

$$-\int_{\Xi^+} \sigma_{ij} n_j (u^i - \rho_{0i}) - \int_{\Xi^+} m_{ij} n_j \big(\phi^i + l_{0,i}\big) - J\int_{\Xi^+} (\phi \cdot n + \frac{\partial u}{\partial n})(u - l_0) = 0. \tag{25}$$

From (25), taking into account the equalities $\sigma_\tau = m_\tau = \mathbf{0}$ on γ^+, (3), (22) and (24), we get

$$\sigma_n (U - \rho_0) \cdot n + m_n \big(\phi + \nabla l_0\big) \cdot n = 0 \quad \text{on } \gamma^+.$$

As a result, from the variational inequality (8), we derive equilibrium equations (10)–(12), identities (19), (20), and the following boundary-value conditions on γ^+:

$$\begin{cases} \phi \cdot n + \dfrac{\partial u}{\partial n} = 0, \quad \sigma_\tau = \mathbf{0}, \quad m_\tau = \mathbf{0}, \quad -h\sigma_n \geqslant |m_n|, \\ \sigma_n (U - \rho_0) \cdot n + m_n \big(\phi + \nabla l_0\big) \cdot n = 0. \end{cases} \tag{26}$$

We can also establish the converse; the solution to boundary-value problem (10)–(12), (19), (20) with conditions (6), (26) is the solution to the variational problem (8) as well.

Thus, the following statement is valid.

Theorem 1. *The smooth function* $\xi = (U, u, \phi)$ *is the solution to the variational problem* (5) *if and only if it is the solution to the boundary-value problem consisting of equilibrium equations* (10)–(12), *relations* (19), (20) *and conditions* (6), (26).

3. Passage to the Limit

Problem (5) turns out to be considered as limit one for the family of problems describing the equilibrium of plates, each of which occupies the domain Ω_γ. The family is characterized by the parameter $\lambda > 0$ and the limit case corresponds to $\lambda = 0$. In other words, for each $\lambda > 0$, the subdomain ω corresponds to the elastic inclusion in the plate, and in the limit, each point $x \in \omega$ has the displacement $(\rho_0(x), l_0(x))$, where $\rho_0 \in R(\omega)$ and $l_0 \in L(\omega)$.

Let the tensor of elastic moduluses $A^\lambda = \{a^\lambda_{ijrl}\}$ and coefficient J^λ depend on λ as follows:

$$a^\lambda_{ijrl} = \begin{cases} a_{ijrl} & \text{in } \Omega \backslash \overline{\omega}, \\ \lambda^{-1} a_{ijrl} & \text{in } \omega \end{cases}; \quad J^\lambda = \begin{cases} J & \text{in } \Omega \backslash \overline{\omega}, \\ \lambda^{-1} J & \text{in } \omega. \end{cases} \tag{27}$$

For $i, j = 1, 2$, define the functions

$$m^\lambda_{ij}(\phi) = a^\lambda_{ijrl} \varepsilon_{rl}(\phi), \quad \sigma^\lambda_{ij}(U) = 3a^\lambda_{ijrl} h^{-2} \varepsilon_{rl}(U), \quad q^\lambda_i(u, \phi) = J^\lambda(u_{,i} + \phi^i).$$

Let the bilinear form $B^\lambda_{\Omega_\gamma}$ be defined by formula

$$B^\lambda_{\Omega_\gamma}(\xi, \eta) = \langle m^\lambda_{ij}(\phi), \varepsilon_{ij}(\psi) \rangle_{\Omega_\gamma} + J^\lambda \langle (u_{,i} + \phi^i), (v_{,i} + \psi^i) \rangle_{\Omega_\gamma} + \langle \sigma^\lambda_{ij}(U), \varepsilon_{ij}(V) \rangle_{\Omega_\gamma}.$$

Formulate the equilibrium problem of the elastic plate containing a crack in the variational form

$$\inf_{\xi \in K} \Pi^\lambda(\xi), \tag{28}$$

where $\Pi^\lambda(\xi) = \dfrac{1}{2}B^\lambda_{\Omega_\gamma}(\xi,\xi) - \langle f,\xi\rangle_{\Omega_\gamma}$ and

$$K = \{\xi \in H \mid [U]\cdot n \geqslant h[\phi]\cdot n \quad \text{a.e. on } \gamma\}.$$

Here $[g] = g^+ - g^-$ the jump of the function g on γ. Problem (28) has the unique solution ξ^λ for each $\lambda \in (0,\lambda_0)$, which satisfies the variational inequality (see [17])

$$\xi^\lambda \in K, \quad B^\lambda_{\Omega_\gamma}(\xi^\lambda, \eta - \xi^\lambda) \geqslant \langle f, \eta - \xi^\lambda\rangle_{\Omega_\gamma} \quad \text{for any } \eta \in K. \tag{29}$$

Here, if the solution ξ^λ is sufficiently smooth, then one can show that it is also the solution of the boundary-value problem (see [17])

$$\sigma^\lambda_{ij,j} = -f_i, \quad i = 1,2, \quad \text{a.e. in } \Omega_\gamma,$$

$$q^\lambda_{i,i} = -f_3 \quad \text{a.e. in } \Omega_\gamma,$$

$$m^\lambda_{ij,j} - q^\lambda_i = -\mu_i, \quad i = 1,2, \quad \text{a.e. in } \Omega_\gamma,$$

$$U^\lambda = \phi^\lambda = \mathbf{0}, \quad u^\lambda = 0 \quad \text{on } \Gamma,$$

$$[U^\lambda]\cdot n \geqslant h|[\phi^\lambda]\cdot n|, \quad [\sigma^\lambda_n] = [m^\lambda_n] = 0, \quad \sigma^\lambda_\tau = m^\lambda_\tau = \mathbf{0} \quad \text{on } \gamma,$$

$$\frac{\partial u^\lambda}{\partial n} + \phi^\lambda\cdot n = 0, \quad \sigma^\lambda_n[U^\lambda]\cdot n + m^\lambda_n[\phi^\lambda]\cdot n = 0, \quad -h\sigma^\lambda_n \geq |m^\lambda_n| \quad \text{on } \gamma.$$

Here $\sigma^\lambda_{ij} = \sigma^\lambda_{ij}(U^\lambda)$, $m^\lambda_{ij} = m^\lambda_{ij}(\phi^\lambda)$, $q^\lambda_i = q^\lambda_i(u^\lambda,\phi^\lambda)$, the values σ^λ_τ, $\sigma^\lambda_n n$, and m^λ_τ, $m^\lambda_n n$ are the components of vectors $\{\sigma^\lambda_{ij}n_j\}$, $\{m^\lambda_{ij}n_j\}$ respectively (for them, the formulas, which are analogous to those in (23), are valid). Further, we substantiate the passage to the limit as $\lambda \to 0$ in (29). It turns out that the limit problem completely coincides with (8). The comparison of two inequalities obtained by substituting $\eta = 2\xi^\lambda$, $\eta = 0$ into (29) yields

$$\langle m^\lambda_{ij}, \varepsilon_{ij}(\phi^\lambda)\rangle_{\Omega_\gamma} + \langle \sigma^\lambda_{ij}, \varepsilon_{ij}(U^\lambda)\rangle_{\Omega_\gamma} + \langle q^\lambda_i, (u^\lambda{}_{,i} + (\phi^i)^\lambda)\rangle_{\Omega_\gamma} = \langle f, \xi^\lambda\rangle_{\Omega_\gamma}.$$

Hence, by using (7), we get two following estimates, which are uniform in λ:

$$\|\xi^\lambda\| \leqslant c_3,$$

$$\frac{1}{\lambda}B_\omega(\xi^\lambda,\xi^\lambda) \leqslant c_4. \tag{30}$$

Choosing subsequences if necessary, one can consider that as $\lambda \to 0$

$$\xi^\lambda \to \xi_\star \quad \text{weakly in } H. \tag{31}$$

Then, by (30), we have

$$\varepsilon_{ij}(U_\star) = 0, \quad \phi_\star + \nabla u_\star = 0, \quad \varepsilon_{ij}(\phi_\star) = 0 \quad \text{in } \omega, \quad i,j = 1,2.$$

According to the arguments coming about from deriving (2) and (3), it follows from the last equalities that there exist functions $\rho_\star$, $l_\star$ such that

$$U_\star = \rho_\star, \quad u_\star = l_\star, \quad \phi_\star + \nabla u_\star = 0 \quad \text{on } \omega, \quad \rho_\star \in R(\omega), \quad l_\star \in L(\omega).$$

Weak convergence $\xi^\lambda \to \xi_\star$ in H implies strong convergence $\xi^\lambda \to \xi_\star$ in $L_2(\gamma)^5$. Choosing subsequences if necessary, one can consider that $\xi^\lambda \to \xi_\star$ a.e. on γ as $\lambda \to 0$. Passing to the limit, as $\lambda \to 0$, in the relation

$$[U^\lambda]\cdot n \geqslant h|[\phi^\lambda]\cdot n| \quad \text{a.e. on } \gamma^+,$$

as $\lambda \to 0$ we find

$$(U_\star - \rho_\star) \cdot n \geqslant h|(\phi_\star + \nabla l_\star) \cdot n| \quad \text{a.e. on } \gamma^+.$$

This means that the limit function $\xi_\star$ belongs to the set K_ω. Fix the arbitrary function $\eta \in K_\omega$. It is obvious that one can substitute it into (29) as a test one. We obtain

$$B^\lambda_{\Omega_\gamma}(\xi^\lambda, \eta - \xi^\lambda) \geqslant \langle f, \eta - \xi^\lambda \rangle_{\Omega_\gamma}. \tag{32}$$

With allowance for weak convergence (31) and relations (27), it is possible to pass to the lower limit in (32) as $\lambda \to 0$ that gives

$$\xi_\star \in K_\omega, \quad B_{\Omega\setminus\overline{\omega}}(\xi_\star, \eta - \xi_\star) \geqslant \langle f, \eta - \xi_\star \rangle_{\Omega_\gamma} \quad \text{for all} \quad \eta = (V, v, \psi) \in K_\omega.$$

Due to the uniqueness of the solution to problem (5), we get that $\xi_\star = \xi$. Thus, for the family of problems (28) describing the equilibrium of elastic plates, as $\lambda \to 0$, we have the equilibrium problem (5) of the plate with the rigid inclusion as a limit one.

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Задача о равновесии пластины Тимошенко, содержащей трещину на границе жесткого включения

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Исследуется нелинейная задача о равновесии пластины, содержащей жесткое включение. Предполагается, что пластина имеет вертикальную трещину вдоль некоторой части поверхности, ограничивающей жесткое включение. Деформирование пластины описывается моделью Тимошенко. На кривой, задающей трещину, налагаются нелинейные условия вида неравенств, описывающие взаимное непроникание противоположных берегов трещины. В работе установлена однозначная разрешимость задачи о равновесии пластины. Получены соотношения, описывающие контакт противоположных берегов трещины. Показано, что задача является предельной для семейства задач, моделирующих равновесие упругих пластин при стремлении параметра жесткости к бесконечности в той области, которая соответствует жесткому включению.

Ключевые слова: трещина, пластина Тимошенко, жесткое включение, функционал энергии, условие непроникания.