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Control Strategies for Multi-Controller Multi-Objective Systems

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CONTROL STRATEGIES FOR MULTI-CONTROLLER MULTI-OBJECTIVE SYSTEMS

by

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ABSTRACT

This dissertation's focus is control systems controlled by multiple controllers, each having its own objective function. The control of such systems is important in many practical applications such as economic systems, the smart grid, military systems, robotic systems, and others. To reap the benefits of feedback, we consider and discuss the advantages of implementing both the Nash and the Leader-Follower Stackelberg controls in a closed-loop form. However, closed-loop controls require continuous measurements of the system's state vector, which may be expensive or even impossible in many cases. As an alternative, we consider a sampled closed-loop implementation. Such an implementation requires only the state vector measurements at pre-specified instants of time and hence is much more practical and cost-effective compared to the continuous closed-loop implementation. The necessary conditions for existence of such controls are derived for the general linear-quadratic system, and the solutions developed for the Nash and Stackelberg controls in detail for the scalar case.

To illustrate the results, an example of a control system with two controllers and state measurements available at integer multiples of 10% of the total control interval is presented. While both Nash and Stackelberg are important approaches to develop the controls, we then considered the advantages of the Leader-Follower Stackelberg strategy. This strategy is appropriate for control systems controlled by two independent controllers whose roles and objectives in terms of the system's performance and implementation of the controls are generally different. In such systems, one controller has an advantage over the other in that it has the capability of designing and implementing its control first, before the other controller. With such a control hierarchy, this controller is designated as the leader while the other is the follower. To take advantage of its primary role, the leader's control is designed by anticipating and considering the follower’s control. The follower becomes the sole controller in the system after the leader’s control has been implemented. In this study, we describe such systems and derive in detail the controls of both the leader and follower. In systems where the roles of leader and follower are negotiated, it is important to consider each controller's leadership property. This property considers the question for each controller as to whether it is preferable to be a leader and let the other controller be a follower or be a follower and let the other controller be the leader. In this dissertation, we try to answer this question by considering two models, one static and the other dynamic, and illustrating the results with an example in each case. The final chapter of the dissertation considers an application in microeconomics. We consider a dynamic duopoly problem, and we derive the necessary conditions for the Stackelberg solution with one firm as a leader controlling the price in the market.

I dedicate this dissertation is to my mother and my father with eternal love.

My words cannot express my appreciation.

*To my lovely family, my amazing wife, **Shatha**, and my kids **Baqer, Ridha**, and **Lujain,***

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CHAPTER ONE
INTRODUCTION

This dissertation considers the important properties and applications of multi-controller multi-objective systems. These systems fall in the general framework of differential non-cooperative non-zero-sum systems. In this chapter, we review these systems' general background, which will be important for the remaining chapters of this dissertation.

Many control engineering, economic, biological, and social science applications can be represented and modeled by simple ordinary differential equations. These control systems have been extensively studied for the past several decades [1-10]. In these systems, state variables are represented by functions of time and affected by the input parameters and if any external signals, such as disturbance signals. One of the main demands of such systems is to be stable because the output changes are bounded as the inputs are bounded. From another view, for stable or stabilized systems, some systems need to have better behaviors in the time domain. Examples of such enhancing systems characteristics include fast-reaching to steady-state and decreasing or eliminating the overshooting. The classical control theory tries to solve such a problem for single-input-single-output. However, for more general systems with higher-order, the problem is more complex. The use of the optimization theory to solve such systems with constructing the desired characteristics with efficient control signal by mathematically represent what is known as performance index or objective function. This construct problem is called the optimal control problem.

Arising from the systems discussed above are have a single input, and single output and system are controlled by only one controller. Classical and advanced control approaches can design this controller; the controller is designed by applying optimal control theory tools. However, the systems with single input are well defined for such the principle of optimality, and it is clear and straight forward for many applications. For example, tools are applied for designing a closed-loop controller for linear systems and single quadratic objective function, which is well-known as LQ systems. On the other hand, many applications cannot be controlled by one controller. Thus, no one of the controllers can govern the system's behavior by itself, and neither can be the only minimizer for its objective. In other words, the controller must consider the other controllers' syntheses over the time horizon, finite or infinite. The latter makes the problem more complex for such systems, i.e., the multiple controllers with multiple objectives, even for the simple open-loop systems. The application of the principle of optimality is not clear in these systems. However, such systems' design is strategic and based on the method in which all controllers have committed. From focusing on the work done in this research dissertation, these strategies (or solutions) could be done simultaneously or in a hierarchical approach. The next sections will consider the optimal control problem framework and the multiple-controller framework as well.

1.1 Single-Controller with Single objective problems (Optimal control problems):

This problem is based on the fact that there is one controller that governs the system's dynamics, and thus this controller will be the optimizer (usually the minimizer) for its objective function. Thus, choosing a controller should result in the best value for its objective, in the sense its value is the lowest value among all the other possible controller choices. In mathematical representation, the optimal control problems for the ordinary differential equation, such as the system's states , evolving over time-horizon , t_f can be finite or infinite.

The state differential equation is

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad x(0) = x_o,\ t \in [t_o, t_f] \tag{1.1}$$

If the controller is chosen from the admissible control strategy, i.e. $u(t) \in \upsilon$. In the open-loop design, the controller synthesis provided the initial state values x_o. For a closed-loop design approach, continuous information $x(t)$ has to be available at each instance of time. For both open/ closed-loop approaches, the designed controller role as the minimizer for its objective functions:

$$J = S(t_f, x(t_f)) + \int_{t_o}^{t_f} L(\tau, x(\tau), u(\tau)) d\tau \tag{1.2}$$

Where the $S(t_f, x(t_f))$ is the terminal cost and the $L \in \mathbb{R}$ is the local (or called running) cost. Many approaches have been invested in finding solutions, control designs, and optimal control problems. The next section includes a background introduction for the multi-controller multi-objective systems framework based on the two main approaches: Nash and leader-follower Stackelberg approaches.

1.2 Control Systems with Multi-Controller with Multi-Objectives

Control systems whose state variables are controlled by two or more independent controllers, each trying to optimize the system performance based on its own criterion, occur in many applications such as in transportation systems [11-17], robotics [18-26], biomedical systems

[27-36], economic systems [37-47], power systems [48-59], smart energy buildings [60-65], military systems [66-71], and many other applications for in networks. The applications example of such network systems is unmanned underwater vehicles and satellites[72-76] in different context. In work in [72] focuses on team cooperation, namely consensus, for both leaderless (LL) and modified leader-follower (MLF) architectures. The recent papers are application of multiple-controller on cyber-physical systems [77-82], to mention a few.

Unlike the classical one controller control systems where the sequence of control implementation is not an issue, control systems with more than one controller have the additional complexity of designating whether the control actions are implemented simultaneously at the same time or whether there is a sequence by which the controls are implemented. The Nash approach [83], first introduced to the control literature in [84, 85], describes a situation where the control actions are implemented simultaneously and exactly at the same time. Thus, no one controller has an advantage over the other in knowing ahead of time how the other controller reacts to its control actions. Such an approach may result in an equilibrium that prohibits each controller from deviating from its control actions; simply because if such a deviation is taken, the outcome will be unfavorable to the controller exercising such an action. Another approach, known as the Stackelberg approach [86], first introduced into the control literature in [87, 88], describes a situation where one controller is more powerful than the other resulting in the control actions implemented according to a specific hierarchical sequence. The more powerful controller implements its control first and is assigned as the leader, and the other controller is then set as the follower. Such a hierarchy of decision making was first introduced in the 1950's by Von Stackelberg[86] in the context of two firms making decisions about supplying a product into a common market. The more powerful firm, which was labeled as the “leader”, decided on its production level first and the less powerful firm, which was labeled as the “follower”, followed by making its decision after knowing the production level of the leader firm. Since then, there has been considerable interest in the control literature in this hierarchical control structure for control systems with two controllers. [89, 90] Assuming that each controller has its own objective function that it wants to optimize, the question is how would the leader controller design its control so that when the follower controller follows, the final outcome will be favorable to the leader? This question is very important to address whenever such a hierarchy in control design exists.

1.3 Nash Control Solution

The Nash rational in control has been studied by many researchers over the past fifty years or so [84, 85]. This type of multiple controller strategy has the property that no one controller will benefit by deviating from its agreed Nash control. The optimality is defined by assuming that both controllers know each other objective functions during their control design determination, and they are designed and implemented simultaneously.

Suppose the objective function of controller 1 is $J_1\left(u_1, u_2\right)$, and the objective function of controller 2 is $J_2\left(u_1, u_2\right)$ where u1 and u2 are the respective control functions of the two controllers and U_1 U_2 are the domains of these controls. The symbolical representation for deriving the Nash controls are as follows:

$$u_1^* = \min_{u_1 \in U_1} J_1\left(u_1, u_2^*\right) \tag{1.1}$$

$$u_2^* = \min_{u_2 \in U_2} J_2\left(u_1^*, u_2\right) \tag{1.2}$$

Thus, both above equations yield to the Nash strategy inequality

$$J_1\left(u_1^*,u_2^*\right) \le J_1\left(u_1,u_2^*\right) \tag{1.3}$$

$$J_2\left(u_1^*,u_2^*\right) \le J_2\left(u_1^*,u_2\right) \tag{1.4}$$

These inequalities assure each controller that the other controller has no incentive to deviate from its Nah control because if it does, it will only be hurting its own objective function. In this sense, the Nash controls provide an equilibrium situation for both controllers.

The necessary condition for the Nash open-loop solutions for both controllers is illustrated in the following

1.3.1 Nash Open-Loop Solution in Dynamic Systems

Consider two controllers C_1 and C_2, responsible for finding the controls u_1 and u_2, respectively, in which these control variables are continuous functions on the interval $\left[t_0,t_f\right]$. The two-controller differential system with state equation

$$\dot{x} = f\left(x,u_1,u_2,t\right), \qquad x\left(t_0\right) = x_0 \tag{1.5}$$

and controller 1 has its own objective function

$$J_1\left(u_1,u_2\right) = K_1\left(x\left(t_f\right)\right) + \int_{t_0}^{t_f} L_1\left(x,u_1,u_2,t\right)dt \tag{1.6}$$

and for controller 2 the objective function is

$$J_2\left(u_1,u_2\right) = K_2\left(x\left(t_f\right)\right) + \int_{t_0}^{t_f} L_2\left(x,u_1,u_2,t\right)dt \tag{1.7}$$

Where x_0 is the initial state known by both controllers, and, $u_1 \in U_1$ and $u_2 \in U_2 \quad \forall t \in \left[t_0,t_f\right]$, where $\left[t_0,t_f\right]$ is the fixed time-horizon.

Depending on the structure of the information for both controllers, such as open-loop structure. Hence, the controls depend on the time and initial state x_0, $u_1 = u_1\left(t,x_0\right)$ and $u_2 = u_2\left(t,x_0\right)$

To derive the necessary condition for such systems implementing their controls using the Nash strategy, both act simultaneously. In other words, C_1 picks u_1 and C_2 picks u_2 , at the same time. Where the pair $(u_1,u_2)\in U_1\times U_2$.

The Hamiltonian function for C_1 and C_2 are as follow:

$$H_1(x,u_1,u_2,\lambda_1,t)=L_1(x,u_1,u_2,t)+\lambda_1 f(x,u_1,u_2) \tag{1.8}$$

$$H_2(x,u_1,u_2,\lambda_2,t)=L_2(x,u_1,u_2,t)+\lambda_2 f(x,u_1,u_2) \tag{1.9}$$

Where $\lambda_1=\lambda_1(t)$ and $\lambda_2=\lambda_2(t)$ are costate variables for C_1 and .Thus, the necessary conditions for C_1 and C_2 are addressed below:

$$\dot{x}=\frac{\partial H_1}{\partial \lambda_1}=\frac{\partial H_2}{\partial \lambda_2}=f(x,u_1,u_2,t), \qquad x(t_0)=x_0 \tag{1.10}$$

$$\dot{\lambda}_1=-\frac{\partial H_1}{\partial x}=-\frac{\partial L_1(x,u_1,u_2,t)}{\partial x}-\lambda_1\frac{\partial f(x,u_1,u_2)}{\partial x}, \qquad \lambda_1(T)=\frac{\partial K_1(x(T))}{\partial x(T)} \tag{1.11}$$

$$\dot{\lambda}_2=-\frac{\partial H_2}{\partial x}=-\frac{\partial L_2(x,u_1,u_2,t)}{\partial x}-\lambda_2\frac{\partial f(x,u_1,u_2)}{\partial x}, \qquad \lambda_2(T)=\frac{\partial K_2(x(T))}{\partial x(T)} \tag{1.12}$$

$$u_1^*=\min_{u_1\in U_1} H_1(x,u_1,\hat{u}_2,t)\Rightarrow\frac{\partial H_1}{\partial u_1}=0 \tag{1.13}$$

$$u_2^*=\min_{u_2\in U_2} H_2(x,\hat{u}_1,u_2,t)\Rightarrow\frac{\partial H_2}{\partial u_2}=0 \tag{1.14}$$

These are the necessary conditions for the open-loop Nash strategy nonzero-sum differential system. Notice that these equations have mixed boundary where the state equation has a known initial state, while the costate equations have the terminal costate values with solving these three differential equations with two algebraic equations (stationary conditions 4 and 5) yield to determine the open-loop Nash equilibrium. However, solving such an equation in general not easy to solve. Though for some application of linear state variable and quadratic objective function can be solved analytically, most of them must are solved numerically.

1.4 Stackelberg Control Startegy

The Stackelberg control option is based on one controller having the capability to design its controller first, due to its size or faster means of information processing and refers to it as a "leader" [87, 88]. Thus, the Stackelberg solution of the two-controller nonzero-sum solution is based on assuming these two-controller systems are different in their roles in such the control systems are designed hierarchically. Therefore, a controller is called a leader, and the other controller is referred to as a follower. The follower follows the leader's strategy in which the leader announces his strategy first, and the follower determines his controller according to the leader's announced strategy. The leader foresees and effectively dominate the entire controls determination process

Let U_1 and U_2 be the controller 1and controller 2 admissible sets, respectively, and in turn, if their corresponding objective functions are $J_1(u_1,u_2)$ and $J_2(u_1,u_2)$, where the $u_1 \in U_1$. and $u_2 \in U_2$ Now if controller 2 is assigned to be the leader and supposed there is exist a mapping such that $T: U_2 \to U_1$ such that

$$J_1\left(Tu_2,u_2\right) \le J_1\left(u_1,u_2\right) \qquad \forall u_1 \in U_1 \tag{1.15}$$

For every $u_2 \in U_2$, then as a rational reaction from controller 1, the following set

$$D_1 = \left\{\left(u_1,u_2\right) \in U_1 \times U_2 : u_1 = Tu_2, \forall u_2 \in U_2\right\} \tag{1.16}$$

is called the rational reaction set for controller 1 when controller 2 is the leader. Moreover, if there is a pair $(u_{1S2}, u_{2S2}) \in D_1$ such that[1]

$$J_2(u_{1S2}, u_{2S2}) \le J_2(u_1, u_2) \qquad \forall (u_1, u_2) \in D_1 \tag{1.17}$$

Where (u_{1S2}, u_{2S2}) is called a Stackelberg strategy pair when the controller is the leader. Hence, the same process is done when controller 1 is the leader with a change of the indices. The rational reaction is set for controller 2, which is now the follower, denoted by D_2 and the pair $(u_{1S1}, u_{2S1}) \in D_2$. For the above approach, it is clear that the intersection for both the rational reaction sets $D_l \cap D_2$, then the common pair (u_{1N}, u_{2N}) is the Nash solution of the two-controller system. In this case: if controller 2 is the leader, $J_2(u_{1S2}, u_{2S2}) \le J_2(u_{1N}, u_{2N})$ similarly if controller 1is the leader $J_1(u_{1S1}, u_{2S1}) \le J_1(u_{1N}, u_{2N})$.

1.4.1 Stackelberg Open-Loop Solution in Dynamic Systems

The necessary conditions for the existence of the Stackelberg open-loop control can be obtained based on the Stackelberg solution is obtained in the hierarchical scheme. For instance, assume the leader C_2 and announce his strategy first, which leaves controller 1, the follower, with no choice just to solve his problem by considering the leader's announcement.

1.4.2 For the Following controller

The Hamiltonian function for the follower, Controller, C_1

1 where the subscription beside the controls index;S_2 , the letter S=Stackelberg and number 2 is when the leader is controller

$$H_1\left(x,u_1,u_2,\lambda_1,t\right)=L_1\left(x,u_1,u_2,t\right)+\lambda_1 f\left(x,u_1,u_2\right)$$

The necessary condition for the follower is as follow:

$$\dot{x}=\frac{\partial H_1}{\partial \lambda_1}=f\left(x,u_1,u_2,t\right), \qquad x\left(t_0\right)=x_0 \tag{1.18}$$

$$\dot{\lambda}_1=-\frac{\partial H_1}{\partial x}=-\frac{\partial L_1\left(x,u_1,u_2,t\right)}{\partial x}-\lambda_1\frac{\partial f\left(x,u_1,u_2,t\right)}{\partial x}, \qquad \lambda_1\left(T\right)=\frac{\partial K_1\left(x\left(T\right)\right)}{\partial x\left(T\right)} \tag{1.19}$$

$$\frac{\partial H_1}{\partial u_1}=0 \Rightarrow u_1^*=\min_{u_1\in U_1} H_1\left(x,u_1,\hat{u}_2,t\right) \tag{1.20}$$

And from the latter condition, there is a reaction for the follower, and for each possible action from the leader, there is a reaction from the follower then $u_1^*=u_1\left(\hat{u}_2\left(t\right)\right)$.

1.4.3 For the Leading controller

Now, the leader must face two constrains to optimize his objective function, the state dynamic equation and the co-state dynamic condition for the follower

The constructed Hamiltonian function for the leader C_2 becomes as shown in equation (1.21):

$$\begin{aligned}&H_2\left(x,u_1^*\left(u_2\right),u_2,\lambda_2,\zeta,t\right)=\\&L_2\left(x,u_1^*\left(u_2\right),u_2,t\right)+\lambda_2 f\left(x,u_1^*\left(u_2\right),u_2\right)+\zeta\left(-\frac{\partial L_1\left(x,u_1^*\left(u_2\right),u_2,t\right)}{\partial x}-\lambda_1\frac{\partial f\left(x,u_1^*\left(u_2\right),u_2,t\right)}{\partial x}\right)\end{aligned} \tag{1.21}$$

Where λ_2,ζ are the costate variables of the leading controller. Thus, the necessary conditions for the leading controller are

$$\dot{x}=\frac{\partial H_2}{\partial \lambda_2}=f\left(x,u_1^*\left(u_2\right),u_2,t\right), \qquad x\left(t_0\right)=x_0 \tag{1.22}$$

$$\dot{\lambda}_1 = \frac{\partial H_2}{\partial \zeta} = -\frac{\partial L_1\left(x, u_1^*(u_2), u_2, t\right)}{\partial x} - \lambda_1 \frac{\partial f\left(x, u_1^*(u_2), u_2, t\right)}{\partial x}, \qquad \lambda_1(T) = \frac{\partial K_1(x(T))}{\partial x(T)} \tag{1.22}$$

And the costate differential equations are derived as follow:

$$\dot{\lambda}_2 = -\frac{\partial H_2}{\partial x} = -\frac{\partial L_2\left(x, u_1^*(u_2), u_2, t\right)}{\partial x} - \lambda_2 \frac{\partial f\left(x, u_1^*(u_2), u_2\right)}{\partial x} + \zeta\left(\frac{\partial^2 L_1\left(x, u_1^*(u_2), u_2, t\right)}{\partial x^2} + \lambda_1 \frac{\partial f\left(x, u_1^*(u_2), u_2, t\right)}{\partial x^2}\right)$$

With terminal value $\lambda_2(T) = \frac{\partial K_2(x(T))}{\partial x(T)} - \zeta(T)\left[\frac{\partial^2 K_1(x(T))}{\partial x^2(T)}\right]$ (1.23)

$$\dot{\zeta} = -\frac{\partial H_2}{\partial \lambda_1} = \zeta\left(\frac{\partial^2 f\left(x, u_1^*(u_2), u_2, t\right)}{\partial \lambda_1 \partial x}\right), \qquad \zeta(0) = 0 \tag{1.24}$$

And the stationary condition for controller 2, the leader, is as follow:

$$\frac{\partial H_2}{\partial u_2} = 0 \Rightarrow u_2^* = \min_{u_2 \in U_2} H_2\left(x, u_1^*(u_2), u_2, t\right) \tag{1.25}$$

Again, the resulting conditions consider as a two-boundary point problem, and solving such a problem, in general, is not easy in general. In this dissertation, the solution is obtained for some illustrative examples, depending on the system's type, numerically, and analytically.

The optimal control of systems governed by multiple controllers rather than one controller leads to many questions that need to be addressed. This dissertation addresses the Nash and Stackelberg strategies to construct solutions for such problems. More specifically, the implementation of the feedback controls in sample data form is addressed.

In the open-loop control design for multiple controller systems, the design control needs only for the state vector's initial state and will be a function of time as well. The other option is the closed-loop control option, in which the measurements of the state vector in each instance of time must be available. This requirement for designing a closed-loop may be expensive, and in some cases, not possible. The dissertation's proposed method is that the sampled closed loop and the samples are pre-specified and equally distributed. In each sample of this proposed approach, the control is designed in its open-loop scheme, and thus the whole interval is represented by these open-loop controls. This type of control is sampled closed-loop, which is a trade-off between the simplicity of designing a single open-loop and reap the advantage of the feedback closed-loop. Remarkably, the more samples, the sampled closed loop is approaching to continuous closed loop on the Nash control design. For the Stackelberg, increasing the samples leads to the sampled closed-loop approaching the continuous Nash closed loop.

The Stackelberg option is a Leader-Follower approach in which there is a leader controller who announces his strategy first, and the other controller is a follower. However, if these roles are predetermined, this option can be in mutual agreement: when both realize that they are better off with the agreed leader's selection, the other is selected as a leader. The other possible situation is when both prefer to be leaders or both followers. In this dissertation, we suggest when the parameter uncertainty, the role selection option availability is dependent on the parameters space over which the system is defined. In this dissertation, we define the partitioning of the parameter space to classify the ability and availability of the design of the Stackelberg implementation or not. The resulting region of the parameter space is used to determine the probability of leadership determination.

1.5 Organization of the Dissertation

This dissertation is organized into four main chapters. Chapter one is an introductory chapter with a general review of the optimal control problem and its natural extension to the multiple controllers' problem, each with its designated own objective functions. Chapter two proposes a new method to reap the advantage of the closed-loop properties by designing a less-costly and straightforward controller for multi-controller systems, which needs fewer amounts of state vector measurements. In chapter three, we discuss a significant property for Leader-Follower Stackelberg systems, which is so little explored, and we try to give attention to the importance of the leadership role with possible changing of the system parameters. Chapter four applies multi-controller multi-objective systems in dynamic microeconomic systems where two firms are maximizing profits by controlling their production outputs. Finally, we conclude this dissertation and suggest new paths for future research in chapter five.

CHAPTER TWO
SAMPLED CLOSED-LOOP

2.1 Introduction

In this chapter, we consider control systems controlled by multiple controllers, each having its own objective function. The Nash and leader-follower Stackelberg options for designing the controls are considered. These control options are heavily used as decision options in the non-zero-sum differential solution theory. To reap many of the benefits of feedback, the resulting designs are best implemented as closed-loop controls. However, closed-loop controls require continuous measurements of the system's state vector, which may be expensive or even impossible in many cases. As an alternative, in this chapter, we consider a sampled closed-loop implementation. Such an implementation requires the state-vector measurements only at pre-specified instants of time and hence is much more practical and cost-effective compared to the continuous closed-loop implementation. We derive the necessary conditions for the general linear-quadratic problem, develop the Nash's solutions, and Stackelberg controls in detail for the scalar case. An example of a control system with two controllers and state measurements available at integer multiples of 10% of the total control interval is presented to illustrate the results.

2.2 Motivation

The multi-controller multi-objective control systems theory deals with control systems whose state variables are controlled by two or more independent controllers, each trying to minimize its own objective function. Systems of this type occur in many applications in smart energy buildings [60], load frequency control, and automatic voltage regulation in power systems

[48, 91], biomedical systems [27], to mention a few. Unlike single controller problems where optimality is easily defined in minimizing one objective function, defining what optimality means in multi-controller multi-objective problems is much more complicated. Since the state variables, and consequently the objective functions, depending on all controllers' control action, defining optimality must be done in terms of the choice of controls as implemented by all controllers simultaneously and collectively. The principle of optimality in these systems is defined in terms of the rationale assumed by each controller in determining its control variables. Several different rationales, leading to different definitions of optimality, have been explored in the past several decades, mainly within differential solution theory.

The Nash rationale [83], first introduced to the control literature in [84], describes a non-cooperative situation in which each controller's control rationale is to safeguard itself against attempts by any other controller from further improving its objective by deviating from its agreed Nash control. This concept of optimality assumes that all controllers know each other's objective functions and that when the controls have been determined, they are all determined and implemented simultaneously at the same time. Another concept, which has proven to be very useful in two-controller systems, is the Stackelberg rationale [86]. This concept, first introduced to the control literature in [27], also describes a non-cooperative situation except that due to size, importance, or faster means of information processing, one controller can arrive and implement its control actions before the other. The controller that can implement its control first is referred to as the "leader" and the other as the "follower." Thus, the Stackelberg solution is based on a hierarchy of control decision-making and is very powerful in deriving optimal controls for the leader controller that would benefit it due to the timing advantage over the follower controller.

Whether the controllers in a multi-control system are applying Nash or Stackelberg controls, they always must make an additional decision on how their controls are to be implemented in practice. As is well known in single controller problems, control variables can be implemented in either open-loop or closed-loop form. The open-loop form is simpler to implement in that it only requires knowledge by all controllers of the state variables' initial conditions. It represents control functions of time that do not depend on the evolution of the state of the system and hence cannot be adjusted if system parameters drift from their nominal values or unknown nonlinear distortions occur at any time during the implementation of the control. On the other hand, the closed-loop form is more complex to implement in that it requires knowledge of the state vector at every instant of time during implementation, thus necessitating the placement of sensors or filters at critical locations of the system to provide measurements of the state variables. This form has a clear advantage [92] over the open-loop form in that should any small perturbations occur is the system's parameters, or should any unknown distortions occur, the state variables would change accordingly, causing an adjustment in the control variables to keep the state variables as close as possible to their prescribed optimal trajectories. Such an adjustment would not occur in the open-loop implementation because the control variables are completely unaware of the system's state once the system is past its initial state.

An added complication in multi-controller systems, which does not exist in single controller systems, is that the open-loop and closed-loop controls are different and produce different state trajectories even under ideal conditions [27, 48]. Because the state variable may or may not be available to the other controllers for implementation, making the design of the control variables for each controller completely different and dependent on the information structure available to the controllers; thus, the controllers must decide whether they would be implementing open-loop or closed-loop controls prior to determining their control variables. This decision is very important and must be communicated or known to all the controllers in the system. Furthermore, the controllers must accept and implement the same control structure used in the controls' design process. Some controllers' option using open-loop and others using closed-loop is also possible; however, simplicity will not be considered in this chapter.

One of the main issues that may deter the controllers from implementing closed-loop controls is cost. Clearly, the placement of sensors, or measurement devices, to measure the state variables continuously over time and transmitting that information to the controllers at every instant of time is a very costly process and in some cases may even be impossible due to the environment in which the state variables are measured. For example, in a metal forming process, the high temperatures of the environment around the state variables (could be greater than 800^{o} F) may prevent the possibility of permanently placing sensors in that environment. Instead, a less costly and more practical option could be to measure the state variables at pre-specified instants of time [93], which may or may not necessarily be uniformly distributed over the interval of optimization. The controllers would then implement closed-loop controls only at the instants when measurements are obtained and implement open-loop controls following those instants until the next instant when measurement becomes available again. The design of such controllers would be an intermediate option between the open-loop and continuous closed-loop options. The fewer the measurement samples would produce controls that are closer to the open-loop option, and the larger the number of measurement samples would produce controls that are closer to the continuous closed-loop option. We will refer to this control structure as a sampled-closed-loop. In this chapter, we will investigate the implementation of this new control structure in the case of linear quadratic multi-controller systems. Without loss of generality and for simplicity of notation, we will consider only the case of two controllers. Almost all the results derived in this chapter can be easily extended to the case of M-controller systems where M > 2. Furthermore, in the case of the Stackelberg controls, without loss of generality, we will only consider the case where controller 2 is the leader controller. The results can be easily duplicated for the case where controller 1 is the leader.

2.3 Linear Quadratic Multi-Controller Systems.

Linear quadratic systems are a very important class of control systems for which optimal controllers can be easily derived analytically [1]. A linear-quadratic two-controller system is a control system described by the linear differential equation:

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(t_0) = \mathrm{x}_0 \tag{2.1}$$

where x is the state vector, u_1 is the control vector of controller 1, and u_2 is the control vector of controller 2. The objective functions for the two controllers are quadratic in the form:

$$J_1(u_1, u_2) = \frac{1}{2} x_f' C_1 x_f + \frac{1}{2} \int_{t_o}^{t_f} (x' Q_1 x + u_1' R_1 u_1) dt \tag{2.2}$$

and $$J_2(u_1, u_2) = \frac{1}{2} x_f' C_2 x_f + \frac{1}{2} \int_{t_o}^{t_f} (x' Q_2 x + u_2' R_2 u_2) dt \tag{2.3}$$

respectively, where all matrices are symmetric and of proper dimensions and R_1 and R_2 are positive definite matrices. Controller 1 wants to minimize J_1, while controller 2 wants to minimize J_2. Defining the following matrices $E_1 = B_1 R_1^{-1} B_1'$ and $E_2 = B_2 R_2^{-1} B_2'$, it is known [84] that the open-loop Nash controls for this problem are of the form:

$$u_1^N(t) = -R_1^{-1}B_1'K_1\varphi(t,t_o)x_o \tag{2.4}$$

and
$$u_2^N(t) = -R_2^{-1}B_2'K_2\varphi(t,t_o)x_o \tag{2.5}$$

where $\varphi(t,t_o)$, $K_1(t)$, and $K_2(t)$ satisfy the coupled differential equations:

$$\dot{\varphi}(t,t_o) = (A - E_1K_1 - E_2K_2)\varphi(t,t_o), \qquad \varphi(t_o,t_o) = I \tag{2.6}$$

$$\dot{K}_1 = -A'K_1 - K_1A - Q_1 + K_1E_1K_1 + K_1E_2K_2, \qquad K_1(t_f) = C_1 \tag{2.7}$$

$$\dot{K}_2 = -A'K_2 - K_2A - Q_2 + K_2E_2K_2 + K_2E_1K_1, \qquad K_2(t_f) = C_2 \tag{2.8}$$

It is also known [88] that the open-loop Stackelberg controls with controller 2 as the leader is of the form:

$$u_1^{S2}(t) = -R_1^{-1}B_1'S_1\phi(t,t_o)x_o \tag{2.9}$$

and
$$u_2^{S2}(t) = -R_2^{-1}B_2'S_2\phi(t,t_o)x_o \tag{2.10}$$

Where $\phi(t,t_0)$, $S_1(t)$, and $S_2(t)$ satisfy the coupled differential equations:

$$\dot{\phi}(t,t_o) = (A - E_1S_1 - E_2S_2)\phi(t,t_o) \qquad \phi(t_o,t_o) = I \tag{2.11}$$

$$\dot{S}_1 = -A'S_1 - S_1A - Q_1 + S_1E_1S_1 + S_1E_2S_2, \qquad S_1(t_f) = C_1 \tag{2.12}$$

$$\dot{S}_2 = -A'S_2 - S_2A - Q_2 + Q_1P + S_2E_2S_2 + S_2E_1S_1, \qquad S_2(t_f) = C_2 - C_1P(t_f) \tag{2.13}$$

$$\dot{P} = AP - PA + PE_1S_1 + PE_2S_2 + E_1S_2 \qquad P(t_o) = 0 \tag{2.14}$$

Note that equations (2.9)-(2.11) are very similar to equations (2.5)-(2.7) except that equation (3.11) has the extra term $Q_1 P$ in it that depends on the solution of the linear equation in (2.12).

In both cases, the values of the objective functions when either the Nash or Stackelberg controls are used are given by:

$$J_1 = \frac{1}{2} x_o' M_1(t_o) x_o \tag{2.15}$$

$$J_2 = \frac{1}{2} x_o' M_2(t_o) x_o \tag{2.16}$$

where $M_1(t)$ and $M_2(t)$ satisfy the linear differential equations:

$$\dot{M}_1 = -(A - E_1 L_1 - E_2 L_2)' M_1 - M_1 (A - E_1 L_1 - E_2 L_2) - Q_1 - L_1' E_1 L_1, \; M_1(t_f) = C_1 \tag{2.17}$$

$$\dot{M}_2 = -(A - E_1 L_1 - E_2 L_2)' M_2 - M_2 (A - E_1 L_1 - E_2 L_2) - Q_2 - L_2' E_2 L_2, \quad M_2(t_f) = C_2 \tag{2.18}$$

Where L_1 and L_2 are replaced by K_1 and K_2 from equations (2.6) and (2.7) in the case of the Nash controls and are replaced by S_1 and S_2 from equations (2.10) and (2.11) in the case of the Stackelberg controls.

2.4 Implementation of the Sampled Closed-Loop Controls

To simplify the notation, we will now illustrate these controls' implementation on a scalar linear-quadratic system. The extension to higher dimensionality systems can be easily done in a very similar way. Consider the two-controller scalar system:

$$\dot{x} = ax + b_1 u_1 + b_2 u_2, \quad t \in [t_o, t_f), \qquad x(t_o) \text{ is given} \tag{2.19}$$

$$J_1(t_o) = \frac{1}{2} c_1 x^2(t_f) + \frac{1}{2} \int_{t_o}^{t_f} (q_1 x^2 + r_1 u_1^2) dt \tag{2.20}$$

$$J_2(t_o) = \frac{1}{2} c_2 x^2(t_f) + \frac{1}{2} \int_{t_o}^{t_f} (q_2 x^2 + r_2 u_2^2) dt \tag{2.21}$$

Let us now assume that measurements of the state vector can be obtained at discrete pre-specified instants of time $t_o, t_1, \ldots, t_{n-1}$ such $t_o < t_1 < \ldots < t_{n-1} < t_f$ and which may or may not be uniformly distributed over the interval $[t_o, t_f)$. Starting at the interval $[t_{n-1}, t_f)$ and proceeding backward in time until the first interval $[t_o, t_1)$ is reached, we will solve for the controls successively as described below. Assume that at instant t_i the state vector $x(t_i)$ can is available for measurement by both controllers who will then design and implement sampled closed-loop controls of the form $u_1(t, x(t_i))$ and $u_2(t, x(t_i))$ over the interval $t \in [t_i, t_{i+1})$. The system equation over this interval $[t_i, t_{i+1})$ is:

$$\dot{x} = ax + b_1 u_1 + b_2 u_2, \quad t \in [t_i, t_{i+1}), \qquad x(t_i) \text{ available} \tag{2.22}$$

$$\text{and } J_1(t_i) = \frac{1}{2} m_1(t_{i+1}) x^2(t_{i+1}) + \frac{1}{2} \int_{t_i}^{t_{i+1}} (q_1 x^2 + r_1 u_1^2) dt \tag{2.23}$$

$$J_2(t_i) = \frac{1}{2} m_2(t_{i+1}) x^2(t_{i+1}) + \frac{1}{2} \int_{t_i}^{t_{i+1}} (q_2 x^2 + r_2 u_2^2) dt \tag{2.24}$$

The Nash controls over the interval $t \in [t_i, t_{i+1})$ are obtained from (2.4) and (2.5)-(2.7) and are of the form:

$$\left. \begin{aligned} u_1^N(t, x(t_i)) &= -\frac{b_1}{r_1} k_1 \varphi x(t_i) \\ u_2^N(t, x(t_i)) &= -\frac{b_2}{r_2} k_2 \varphi x(t_i) \end{aligned} \right\} \tag{2.25}$$

Where, assuming that $e_1 = \frac{b_1^2}{r_1}$ and $e_2 = \frac{b_2^2}{r_2}$, we have

$$\dot{\varphi} = (a - e_1 k_1 - e_2 k_2)\varphi, \qquad \varphi(t_i) = 1 \tag{2.26}$$

$$\dot{k}_1 = -2ak_1 - q_1 + e_1 k_1^2 + e_2 k_1 k_2, \qquad k_1(t_{i+1}) = m_1(t_{i+1}) \tag{2.27}$$

$$\dot{k}_2 = -2ak_2 - q_2 + e_2 k_2^2 + e_1 k_1 k_2, \qquad k_2(t_{i+1}) = m_2(t_{i+1}) \tag{2.28}$$

Once $k_1(t)$ and $k_2(t)$ are determined from the above equations, the values of the objective functions are obtained as:

$$J_1(t_i) = \frac{1}{2} m_1(t_i) x^2(t_i) \tag{2.29}$$

$$J_2(t_i) = \frac{1}{2} m_2(t_i) x^2(t_i) \tag{2.30}$$

Where $m_1(t)$ and $m_2(t)$ satisfy the linear differential equations obtained (2.17) and(2.18):

$$\dot{m}_1 = -2(a - e_1k_1 - e_2k_2)m_1 - q_1 - e_1k_1^2, \qquad m_1^i(t_{i+1}) = m_1^{i+1}(t_i) \tag{2.31}$$

$$\dot{m}_2 = -2(a - e_1k_1 - e_2k_2)m_2 - q_2 - e_2k_2^2, \qquad m_2^i(t_{i+1}) = m_2^{i+1}(t_i) \tag{2.32}$$

We can follow a similar procedure for the Stackelberg controls. For the interval $t \in [t_i, t_{i+1})$ the controls are obtained from (2.4) and (2.9)-(2.10) and are of the form:

$$\left.\begin{aligned} u_1^S(t, x(t_i)) &= -\frac{b_1}{r_1} s_1 \phi x(t_i) \\ u_2^S(t, x(t_i)) &= -\frac{b_2}{r_2} s_2 \phi x(t_i) \end{aligned}\right\} \tag{2.33}$$

Where:

$$\dot{\phi} = (a - e_1s_1 - e_2s_2)\phi, \qquad \phi(0) = 1 \tag{2.34}$$

$$\dot{s}_1 = -2as_1 - q_1 + e_1s_1^2 + e_2s_1s_2, \qquad s_1(t_{i+1}) = m_1(t_{i+1}) \tag{2.35a}$$

$$\dot{s}_2 = -2as_2 - q_2 + q_1p + e_2s_2^2 + e_1s_1s_2, \qquad s_2(t_{i+1}) = m_2(t_{i+1}) - m_1(t_{i+1})p(t_{i+1}) \tag{2.35b}$$

$$\dot{p} = (e_1s_1 + e_2s_2)p + e_1s_2, \quad p(t_i) = 0 \tag{2.36}$$

Once $s_1(t)$ and $s_2(t)$ are determined from the above equations, the values of the objective functions are obtained as:

$$\left.\begin{aligned} J_1(t_i) &= \frac{1}{2} m_1(t_i) x^2(t_i) \\ J_2(t_i) &= \frac{1}{2} m_2(t_i) x^2(t_i) \end{aligned}\right\} \tag{2.37}$$

where $m_1(t)$ and $m_2(t)$ satisfy the linear differential equations obtained from (2.17) and (2.8):

$$\dot{m}_1 = -2(a - e_1 s_1 - e_2 s_2) m_1 - q_1 - e_1 s_1^2, \qquad m_1^i(t_{i+1}) = m_1^{i+1}(t_i) \tag{2.38}$$

$$\dot{m}_2 = -2(a - e_1 s_1 - e_2 s_2) m_2 - q_2 - e_2 s_2^2, \qquad m_2^i(t_{i+1}) = m_2^{i+1}(t_i) \tag{2.39}$$

As mentioned earlier, for both the Nash and Stackelberg controls, the process starts at the last interval of time $[t_{n-1}, t_f)$ where the boundary conditions at $t = t_f$ are given as c_1 and c_2 proceeds backward in time until the interval $[t_o, t_1)$ is reached. We will now illustrate the derivation of the Nash and Stackelberg controls for the following two-controller control system.

2.5 An Illustrative Example

Consider the following two-controller system:

$$\dot{x} = x + u_1 - u_2, \quad t \in [0,1), \ \ x(0) = 1 \tag{2.40}$$

Let the objective function of controller 1 be:

$$J_1 = x^2(1) + \frac{1}{2}\int_0^1 (2x^2(t) + u_1^2(t))dt \tag{2.41}$$

And the objective function of controller 2 be:

$$J_2 = -2x^2(1) + \frac{1}{2}\int_0^1 (-4x^2(t) + 3u_2^2(t))dt \tag{2.42}$$

With the coefficients multiplying x^2 being positive in J_1 and negative in J_2, clearly minimizing these objective functions represents a situation where controller 1 is trying to drive the state variable *x* towards the origin (i.e., regulate the system) while controller 2 is trying to drive it away from the origin (i.e., destabilize the system). The interval of control is $[0,1]$. We will consider the case of sampled-closed-loop control, where only ten state vector measurements are available $t_i = 0.1(n-1)$, for $n = 1, \ldots, 10$. Comparing (2.40)-(2.42) with (2.19)-(2.21) the system parameters are $a = 1, b_1 = 1$ and $b_2 = -1$, the parameters for controller 1 are $c_1 = q_1 = 2$ and $r_1 = 1$, the parameters for controller 2 are $c_2 = q_2 = -4$, and $r_2 = 3$. Following (2.25), the Nash controls for this system are $u_1^N(t, x(t_i)) = -k_1\varphi x(t_i)$ and $u_2^N(t, x(t_i)) = 1/3k_2\varphi x(t_i)$. Similarly, following (2.33) the Stackelberg controls with controller 2 as a leader are $u_1^S(t, x(t_i)) = -s_1\phi x(t_i)$ and $u_2^S(t, x(t_i)) = 1/3s_2\phi x(t_i)$. Plots of all solution variables for this problem are shown in Figures (2.1) through (2.8). Figures (2.1) and (2.2) show plots of the feedback gains for both the Nash and Stackelberg controls. These are plotted as a pair on the same graph to illustrate the difference between the two solutions. While k_1 and s_1 having the same boundary condition at t=1, k_2 and s_2 have different boundary conditions due to the *p(t)* variable in (3.33), which is plotted versus time in figure (2.3).

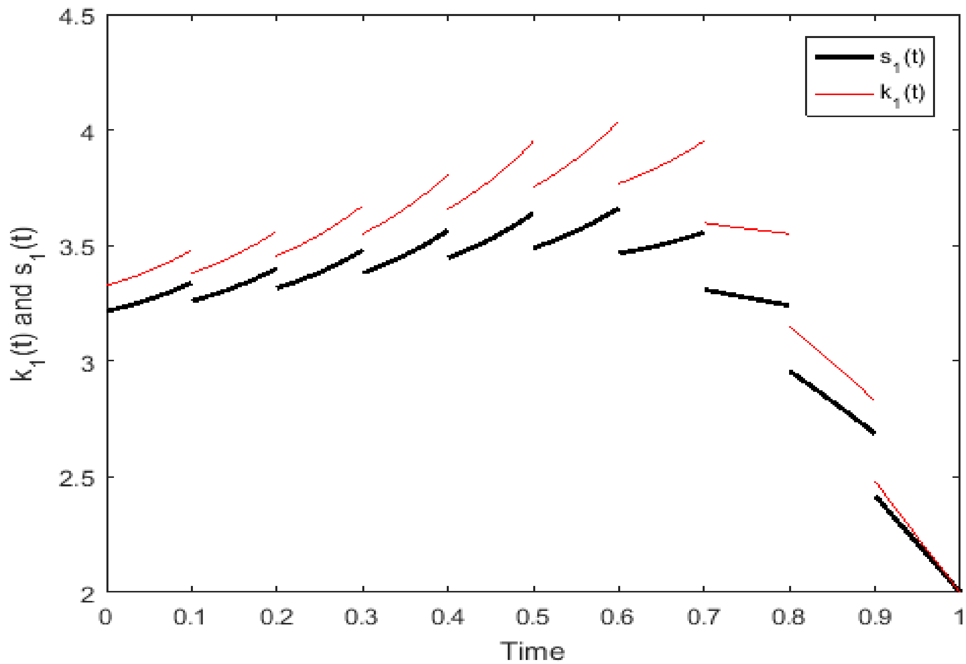

Figure (2.1): Plots of $k_1(t)$ *and* $s_1(t)$ *vs. time*

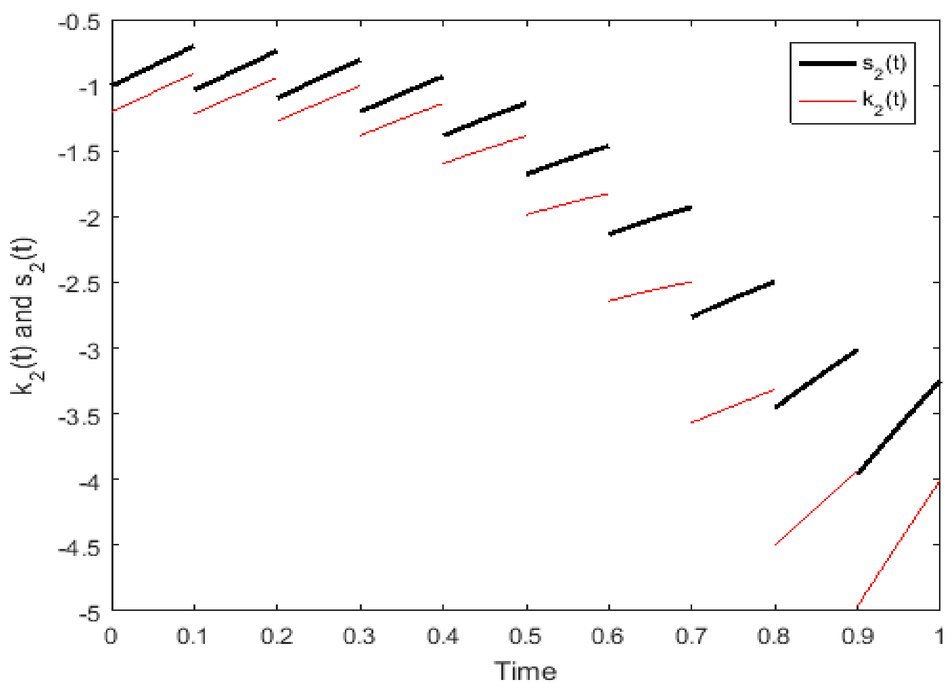

Figure (2.2): Plots of $k_2(t)$ *and* $s_2(t)$ *vs. time*

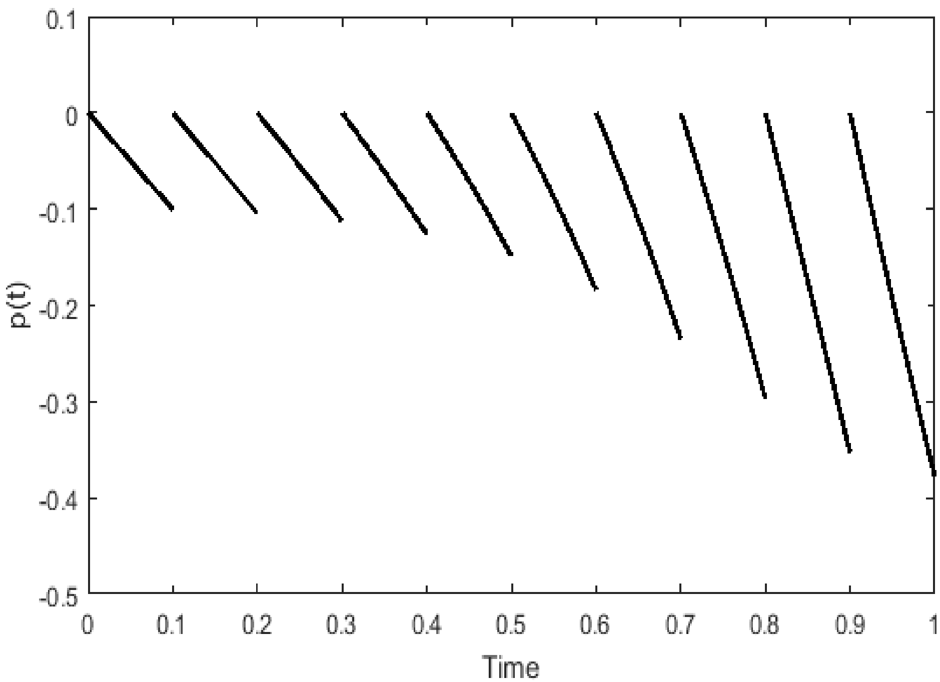

Figure (2.3): Plot of $p(t)$ *for the Stackelberg control vs. time*

Plots of the parameters $m_1(t)$ and $m_2(t)$ which characterize the objective functions are shown in Figures (2.4) and (2.5), and plots of the controls and state trajectories are shown in Figures (2.6)-(2.8), respectively. Clearly, the state variable's trajectory is approaching the origin, which means it has been regulated. This result indicates that controller1 has been able to accomplish its objective in spite of the fact that controller 2 was trying to drive the state away from the origin. Plots of the objective functions' values J_1 and J_2 for both the Nash and Stackelberg controls are shown in Figures (2.9) and (2.10). It is clear from Figure (2.10) that $J_2^S < J_2^N$ at every $t \in [0,1]$ which means, as expected, that the leader in the Stackelberg solution (controller 2 in this case) achieves better performance using a Stackelberg control rather than a Nash control.

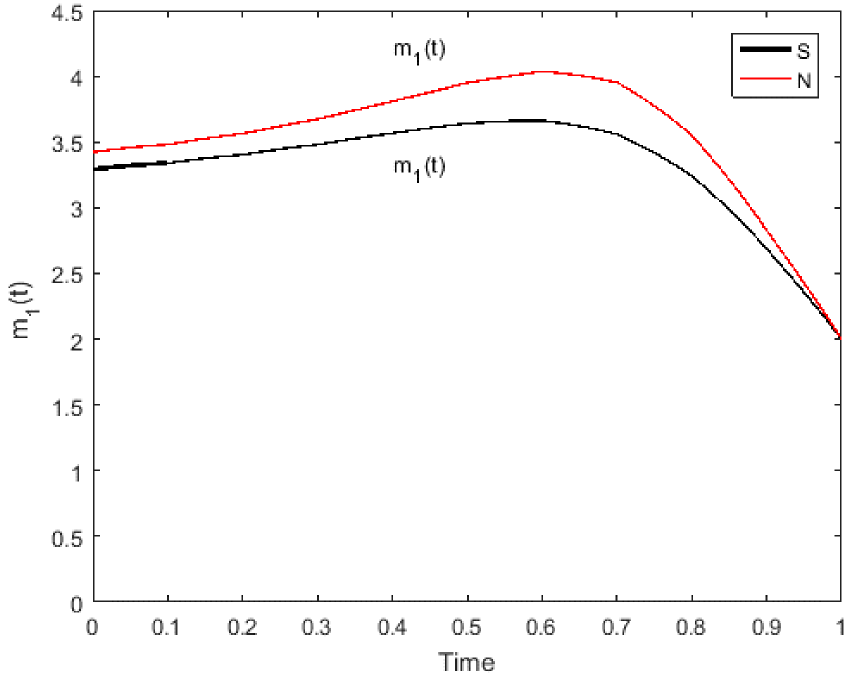

Figure (2.4): plots of $m_1(t)$ *vs. time*

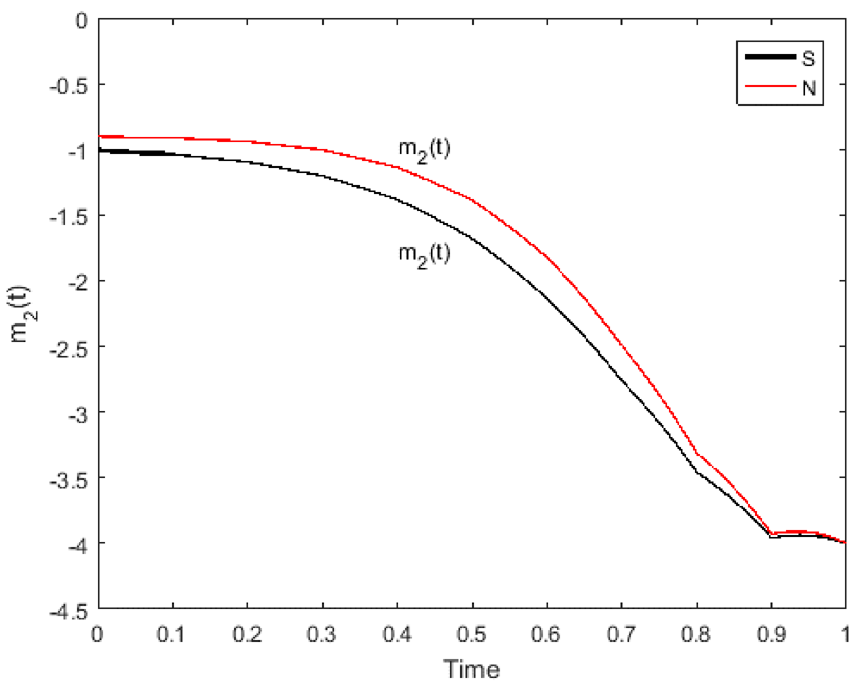

Figure (2.5): Plots of $m_2(t)$ *vs. time*

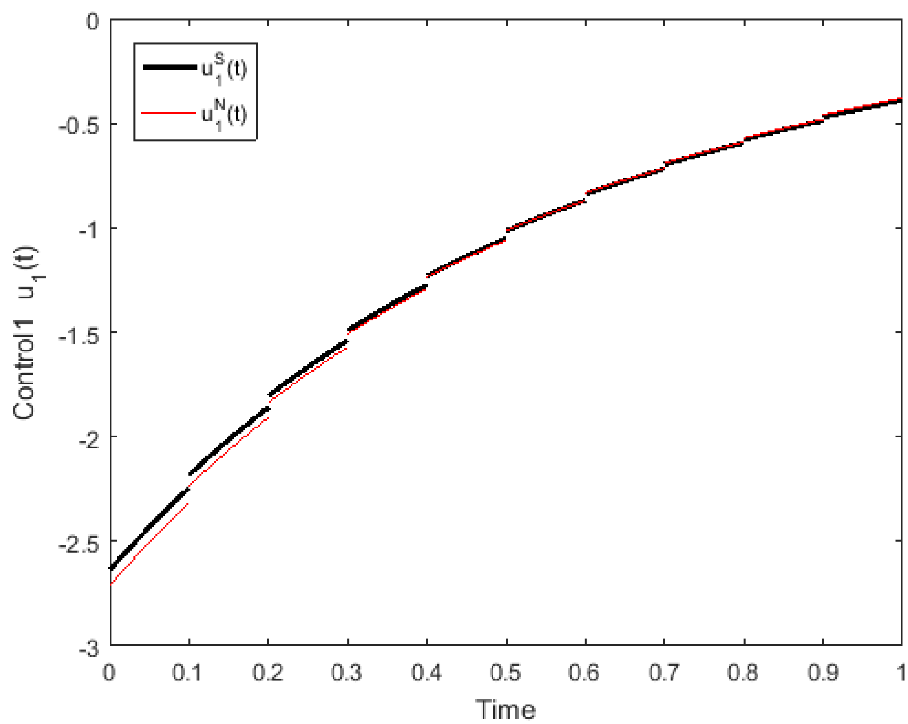

Figure (2.6): Plots of the Nash and Stackelberg controls for controller 1 vs. time.

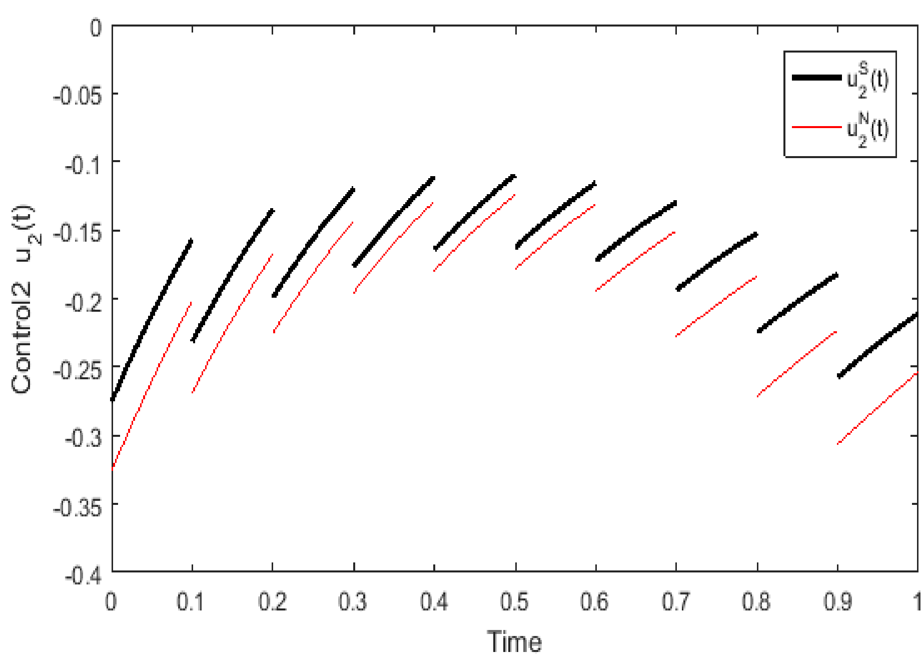

Figure (2.7): Plots of the Nash and Stackelberg controls for controller 2 vs. time.

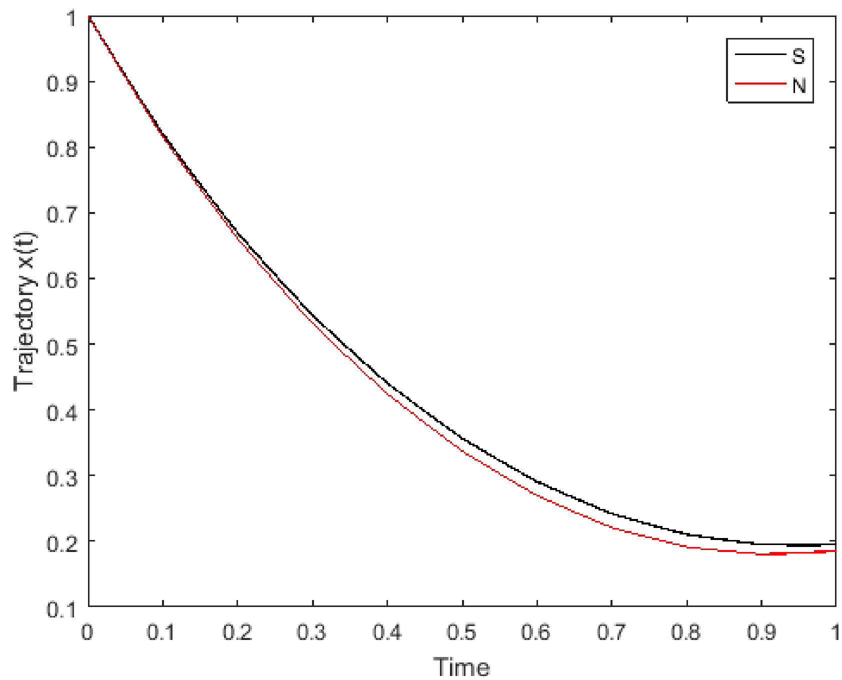

Figure (2.8): Plots $x(t)$ *for both the Nash and the Stackelberg controls vs. time*

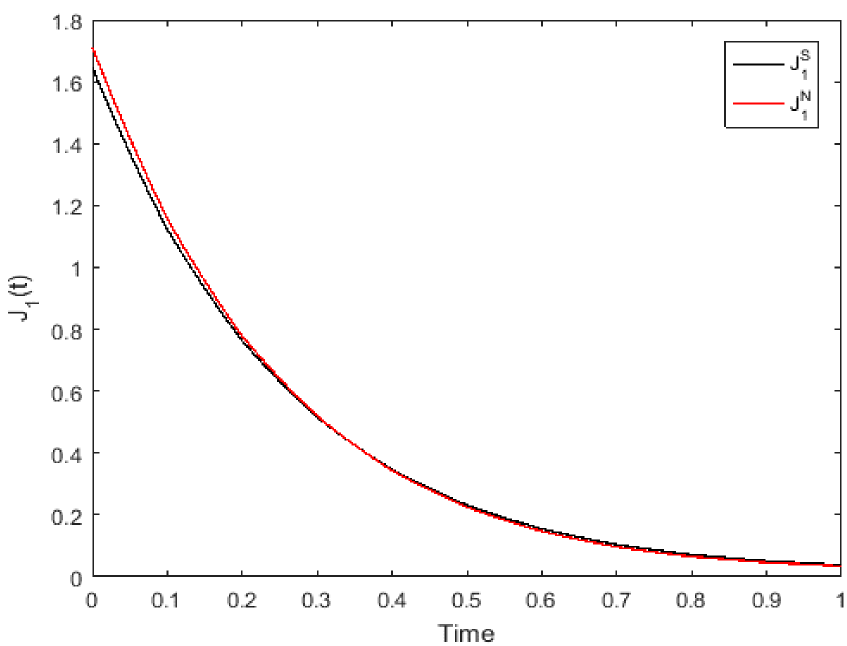

Figure (2.9): Plots of controller 1 cost-to-go (Nash and the Stackelberg) vs. time.

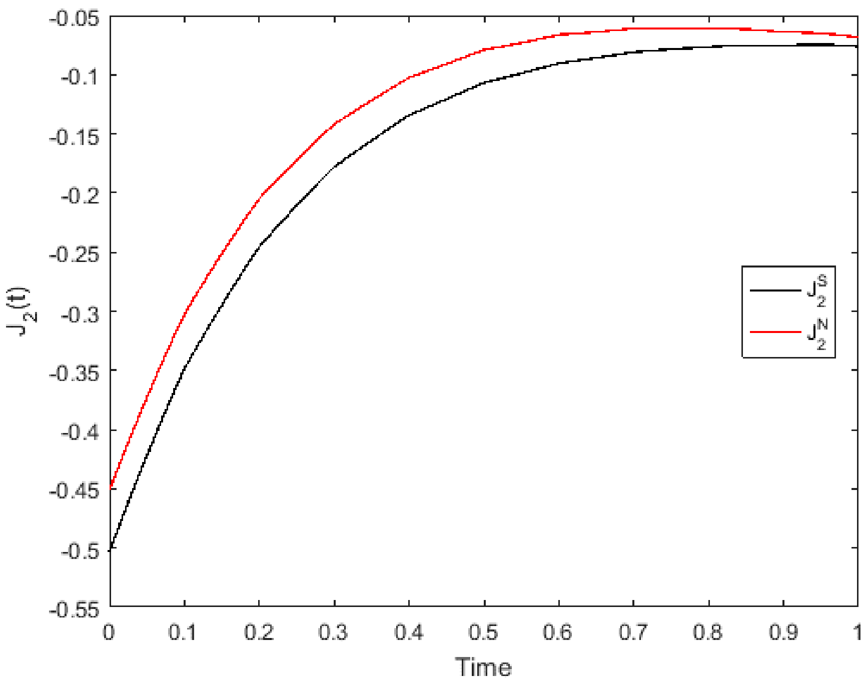

Figure (2.10): Plots of controller 2 cost-to-go (Nash and the Stackelberg) vs. time.

2.6 Conclusion

In this chapter, we have considered systems that are controlled by more than one controller, each having its own objective function. The optimal control of such systems does not simply involve minimizing the objective functions, but it also involves how the various controllers interact with each other and how they take the controls of the other controllers into account. In this chapter, we have considered the Nash and Stackelberg control rationales. These solution concepts are very popular in the context of dynamic solutions. We have considered the special case of linear quadratic systems with two controllers and derived and solved in detail all the accompanying necessary differential equations for the scalar case. We then considered the implementation of sampled closed-loop controls. These controls are closed-loop types expect that the feedback loops are closed only at specific instants of time when the state-vector is available for measurement. We have included a two-controller example to illustrate the results.

CHAPTER THREE
LEADERSHIP SELECTION WITH PARAMETER UNCERTAINTY

In leader-follower Stackelberg games, the leader determines and announces its strategy first by anticipating the follower's reaction function, and the follower determines its strategy as the best response to the leader's strategy. Thus, there is a perceived advantage in assuming the role of a leader in a Stackelberg game. When the controllers' roles are not determined a propri, both controllers must mutually agree on the selection of the leader. Such an agreement is possible only if the controllers realize that they are both better off with the agreed selection of leader than when the other controller is selected as leader. In games with parameter uncertainty, this option's availability depends on the parameter space over which the game is defined. This chapter describes the partitioning of the parameter space to characterize when a Stackelberg solution based on an agreed leader selection exists and when it does not. The resulting partition can then be used to determine the probability of all possible games where agreement can and cannot be reached. We illustrate the results using two examples.

3.1 Introduction

The Stackelberg solution [86-88] in two-player nonzero-sum static and dynamic games provides an alternative to the Nash solution [83-85] when the two players' roles can be defined as leader and follower. The leader in a Stackelberg game decides on its strategy first, and the follower determines its strategy as the best response to the leader's strategy. The Nash and Stackelberg solutions have received considerable attention in both the multiple controller's literature over the past 50 years or so. The recent books [94] by Basar and Olsder, [95] Yong, [90]Ungureanu , and [96] Lambertini provide a very good summary and include most of the relevant references related to these two solutions. One of the main advantages of the Stackelberg solution, as was demonstrated in [88] and [97], is that the Stackelberg solution is advantageous over the Nash solution for the leader. A controller is always better off being a leader in a Stackelberg game than an equal controller in the Nash game. Hence, as expected, both controllers will compete for the leadership role either by acquiring faster means of decision-making or by trying to become dominant in size. In doing so, however, each controller has ignored considering the possibility that being a follower might be a more beneficial option to it than being a leader. While this would not be possible in a duopoly with two profit-maximizing firms, Hou et al [98], it is a highly probable outcome in two-player games in general. To illustrate this point let us first consider the simple 2- controller matrix game is shown in Figure (3.1), where each controller has three decision choices. controller C_1 decides on the x variables and wants to minimize its payoff consisting of the first entries in the matrix and controller C_2 decides on the y variables and wants to minimize its payoff consisting of the second entries in the matrix. The Stackelberg Solution with C_1 as leader is $\{x_2, y_3\}$ yielding payoffs of (3, 4) and the Stackelberg Solution with P2 as leader is $\{x_1, y_1\}$ yielding payoffs of (4, 7). Clearly, in this case, both controllers will do better when P1 is the leader and would therefore would readily agree that P1 should be the leader.

	y_1	y_2	y_3
x_1	4, 7	6,16	7, 5
x_2	9, 14	11,13	3, 4
x_3	10,15	5,10	2,11

Figure (3.1): A Matrix Example

In games where the leader's selection is negotiable, each of the two controllers will need to compare the outcome of the games when it plays the role of leader and when it agrees for the other controller to play the role of leader. Thus, there are four possible options for the controllers to consider. Two options occur when each controller determines that it is preferable for it to be a leader while simultaneously, the other determines that it is preferable for it to be a follower. These two latter options are implementable since, in each case, the controllers can reach a mutual agreement on the selection of the leader. A third option occurs when each controller determines that it is preferable for it to be a leader and for the other controller to be a follower, and a fourth option occurs when each controller determines that it is preferable for the other controller to be a leader and for it to be a follower. Clearly, neither of these last two options is implementable since the controllers cannot mutually agree on the leader's selection, and a stalemate will prevail. The deadlock can be resolved either by the controllers adopting a Nash approach or by one controller considering making side payments to the other controller as an incentive to agree to be a follower. The possibility of distributing the roles in a Stackelberg game was first mentioned in [99] Basar in the context of a scalar differential game example. It was also considered in [100] Boyer & Moreaux, and [101] Dowrick in the context of static duopoly problems where it is agreed that even in the competitive framework between two firms in a duopoly, it is not unreasonable to expect that they may end up coordinating the distribution of their roles as leader and follower in a mutually advantageous way. Later on, [102]Van Damme and Hurkens argued that committing to the role of leadership is less risky for the low-cost firm so that such a firm will emerge as a leader is a Stackelberg duopoly. In a more recent paper, [103] Liu questioned whether a leader firm in a duopoly really has a strategic advantage in practice under demand uncertainty. The paper cites several examples of market leaders in the dotcom era that ended up not sustaining the business due to uncertainty in demand. Another recent paper [104] Nie, Wand, and Cui argue that in repeated games, controllers acting as leaders, in turn, improves cooperation and consequently enhances social welfare.

While most of the early applications of the Stackelberg strategy were in duopoly type economic problems, in recent years, there has been an emergence of interest in the Stackelberg solution as an effective mechanism for analyzing many of today's complex engineering systems. These include the smart electric grids [105-108], wireless communication systems [109, 110], cyber-physical systems [111, 112], and others. The Stackelberg solution has also been of interest in problems related to security resource allocation [113], artificial intelligence [114, 115], economics, management, and marketing systems [116-118], and others. Many of these complex systems do not have a naturally designated leader leaving the leadership position open for negotiation. The selection of a leader becomes a very important issue that will affect the entire system's performance. Selecting or negotiating who should be the leader in these systems becomes very important, especially when it is not obvious that being a leader is always advantageous.

In this chapter, we explore the partitioning of the parameter space to characterize the various regions when the controllers can mutually agree, and when they do not, on the selection of the game leader. If the parameters probability distribution is known, the resulting partition of the parameter space can then be used to determine the probability of all possible options of leader selection. We illustrate the results with two examples where the parameters are uniformly distributed over a bounded space and show how the probabilities of the existence of either controller's mutually acceptable selection as a leader can be determined.

3.2 Stackelberg Solutions with Uncertain Parameters:

In Stackelberg games defined over a space of uncertain parameters where the leader is to be selected by mutual agreement, the challenge is to determine probabilities of occurrence of a game where it is advantageous for each of the two controllers to be select as the leader. To accomplish this, the regions in the parameter space that delineate when it is advantageous for each controller to be selected as a leader need to be determined. Let Ω be the space of uncertain game parameters, and let $\Omega_i^L \subset \Omega,\ i=1,2$ be the set of parameters such that the controller i prefers[2] [119] to be the leader and $\Omega_i^F = \Omega - \Omega_i^L$ be the set of parameters such that the Controller i prefers to be a follower. Then Ω can be divided into two regions: (1) A region of Agreement, Ω_A, representing parameters that characterize Stackelberg solutions where both controllers agree on the leader selection, and a region of disagreement, Ω_D, representing parameters that characterize Stackelberg solutions where both controllers cannot agree on the leader selection. The region of agreement Ω_A

[2] The preference can be either based on a cardinal ranking of the choices available to each player according to an objective function or on an ordinal raking of the choices based on each players' subjective preference

consists of two feasible sub-regions $\Omega_{A1} = \Omega_1^L \cap \Omega_2^F$ representing the region where both controllers agree that controller 1 should be selected as the leader and $\Omega_{A2} = \Omega_1^F \cap \Omega_2^L$ represent the region where both controllers agree that controller 2 should be selected. Similarly, Ω_D can be divided into two sub-regions: $\Omega_{DL} = \Omega_1^L \cap \Omega_2^L$ representing solutions where both controllers disagree in that both want to be selected as leaders and $\Omega_{DF} = \Omega_1^F \cap \Omega_2^F$ where both controllers disagree in that both want the other controller to be selected as leader. Characterizing these regions in the parameter space will not only provide information about solutions where the leader selections by mutual agreement are possible but also will allow for a determination of the probabilities of occurrence of each of these solutions. Such probabilities will help the controllers decide a priori on the most advantageous selection of the leader between them.

To clarify these concepts, let us first consider the simple 2-controller matrix solution shown in Figure (3.1), where each controller has three decision choices. The controller C_1 controls the x variables and wants to minimize the first entry in the matrix, and the controller C_2 controls the y variables and wants to minimize the second entry in the matrix. The solution has two uncertain parameters α and β in the matrix entries that are uniformly distributed over the bounded region $\Omega = \{(\alpha, \beta) \text{ such that } 0 \le \alpha \le 8 \text{ and } 0 \le \beta \le 12\}$. Following [99], it can be easily shown that the Stackelberg solution with controller 1, C_1 as a leader, always occurs at the location $\{x_2, y_3\}$, and the Stackelberg solution with controller 2, C_2, as a leader, always occurs at the location $\{x_1, y_1\}$ for all $(\alpha, \beta) \in \Omega$. Furthermore, it can be easily shown, as illustrated in Figure (3.2), that:

1) $\Omega_{A1} = \{(\alpha, \beta) \in \Omega \text{ such that } 3 < \alpha \le 4 \text{ and } 0 \le \beta < 7\}$

2) $\Omega_{A2} = \{(\alpha, \beta) \in \Omega \text{ such that } 0 \le \alpha < 3 \text{ and} 7 < \beta \le 12\}$

3) $\Omega_{D1} = \{(\alpha, \beta) \in \Omega \text{ such that } 3 < \alpha \le 8 \text{ and } 7 < \beta \le 12\}$

4) $\Omega_{D2} = \{(\alpha, \beta) \in \Omega \text{ such that } 0 \le \alpha < 3 \text{ and } 0 \le \beta < 7\}$

The four regions and their probabilities of occurrence of corresponding solutions are indicated in Figure (3.2). Clearly, the largest region in the parameter space is the region of agreement Ω_A implying that there is a 36% probability that a solution will occur where both controllers agree that should be selected as leader and 16% probability that a solution will occur where both agree that C_2 be selected as to be the leader. These are both feasible Stackelberg solutions by mutual agreement. Figure (3.2) also indicates that there is a 48% probability that a solution will occur where an agreement is not possible with a 26% probability due to both controllers want to be selected as leaders and 22% probability due to neither controller wants to be selected as leader. This simple example illustrates the importance of determining the regions in the parameter space where agreement can be reached on selecting the solution leader.

	y_1	y_2	y_3
x_1	$\alpha, 7$	6,16	7,5
x_2	9, 14	11,13	$3, \beta$
x_3	10,15	5,10	2,11

Figure (3.2): Matrix example with variables of α, β

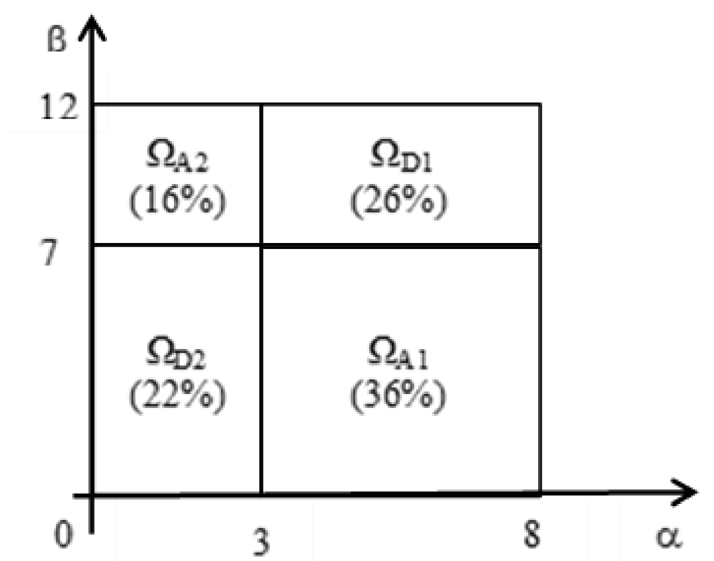

Figure (3.3): Characterization of the Parameter Space Ω for the solution in figure (3.2)

3.3 Differential Solution example

Consider the first-order linear-quadratic differential game example described by the linear differential equation:

$$\dot{x} = u_1 - u_2 \ , \ t \in [0,1] \ , \text{ and } x(0) = x_0 \tag{3.1}$$

and quadratic cost functions

$$J_1(u_1, u_2) = \frac{1}{2} c_1 x^2(1) + \frac{1}{2c_p} \int_0^1 u_1^2(t) dt \tag{3.2}$$

$$J_2(u_1, u_2) = \frac{1}{2} c_2 x^2(1) + \frac{1}{2c_e} \int_0^1 u_2^2(t) dt \tag{3.3}$$

where x is the state variable, u_1 and u_2 are the control variables of Controllers C_1 and C_2 respectively and J_1 J_2 are their respective cost functions. This classic simple example was first considered in [120] Ho, Bryson, and Baron (1965) and has been used since then as a benchmark example to illustrate numerous solution concepts in differential game theory. Note that J_1 and J_2 include the scalars parameters c_p, c_e, c_1, and c_2, which for the problem to be well defined, must satisfy the conditions:

$$\{c_p > 0, c_e > 0, c_1 \neq 0, c_2 \neq 0\} \tag{3.4}$$

If the value of these parameters were known a priori, then it will be possible to determine whether or not both controllers can agree on who should be selected as leader. Using the notation u_{isj} and $J_i(u_{1sj}, u_{2sj})$, or simply J_i^j, to denote the Stackelberg strategy and corresponding cost for controller *i* when controller *j* is leader, then both controller s will agree for controller 1 to be selected leader if $J_1^1 < J_1^2$ and $J_2^1 < J_2^2$ both will agree for controller 2 to be selected leader if $J_1^2 < J_1^1$ and $J_2^2 < J_2^1$. In this chapter, we will assume that the parameters c_p, c_e, c_1, and c_2 are uncertain, and as a result, it is not possible to determine a priori whether a leader selection is feasible or not. To the simplicity of notation, let us define the parameters

$$c_p, c_e, c_1, \text{ and } c_2 \tag{3.5}$$

To determine the regions of agreement Ω_A and disagreement Ω_D, we first need to determine the set of parameters $\{a_1, a_2\} \in R^2$ over which the two Stackelberg solutions with either controller as leader exist. It follows (from Eqs. (48)-(53) in [88]) that the open-loop Stackelberg solution with P1 as leader exists provided $\{a_1, a_2\} \in R^2$ satisfy:

$$\left.\begin{aligned} (1+a_2) > 0 \\ (1+a_2)^2 + a_1 > 0 \end{aligned}\right\} \tag{3.6}$$

The control variables for this solution are

$$u_{1s1} = -\frac{a_1}{(1+a_2)^2 + a_1} x_o \quad \text{and} \quad u_{2s1} = \frac{a_2(1+a_2)}{(1+a_2)^2 + a_1} x_o \tag{3.7}$$

and the corresponding cost functions for the two controllers are:

$$J_1^1 = \frac{1}{2} \frac{c_1}{(1+a_2)^2 + a_1} x_o^2 \quad \text{and} \quad J_2^1 = \frac{1}{2} \frac{c_2(1+a_1)^3}{[(1+a_2)^2 + a_1]^2} x_o^2 \tag{3.8}$$

Similarly, the open-loop Stackelberg solution with P2 as the leader exists provide $\{a_1, a_2\} \in R^2$ satisfy:

$$\left.\begin{array}{l} 1 + a_1 > 0 \\ (1+a_1)^2 + a_2 > 0 \end{array}\right\} \tag{3.9}$$

The control variables for this solution are:

$$u_{1s2} = -\frac{a_1(1+a_1)}{(1+a_1)^2 + a_2} x_o \quad \text{and} \quad u_{2s2} = \frac{a_2}{(1+a_1)^2 + a_2} x_o \tag{3.10}$$

and the corresponding cost functions for the two controllers are:

$$J_1^2 = \frac{1}{2} \frac{c_1(1+a_1)^3}{[(1+a_1)^2 + a_2]^2} x_o^2 \quad \text{and} \quad J_2^2 = \frac{1}{2} \frac{c_2}{(1+a_1)^2 + a_2} x_o^2 \tag{3.11}$$

Thus, the parameter space $\Omega \in R^2$ over which both Stackelberg solutions exist is:

$$\Omega = \left\{ \{a_1, a_2\} \in R^2 \ni a_1 \neq 0, a_2 \neq 0, (1+a_1) > 0, (1+a_2) > 0, (1+a_1)^2 + a_2 > 0, \text{and } (1+a_2)^2 + a_1 > 0 \right\}$$

Figure (3.3) shows the region Ω in R^2. Note that the reason the coordinate axes correspond to $a_1 = 0$ and $a_2 = 0$ are shown as dotted lines is that these lines are not included as a part of Ω.

Next, we need to determine the region of agreement $\Omega_A \subset \Omega$. As mentioned earlier, Ω_A consists of two sub-regions $\Omega_{A1} = \Omega_1^L \cap \Omega_2^F$ where both controllers agree that C_1 should be selected as the leader and $\Omega_{A2} = \Omega_1^F \cap \Omega_2^L$ where both controllers agree that C_2 it should be selected as leader. The region Ω_1^L is determined by the set of parameters that yield $J_1^1 < J_1^2$. Using (3.8) and (3.11), this means that:

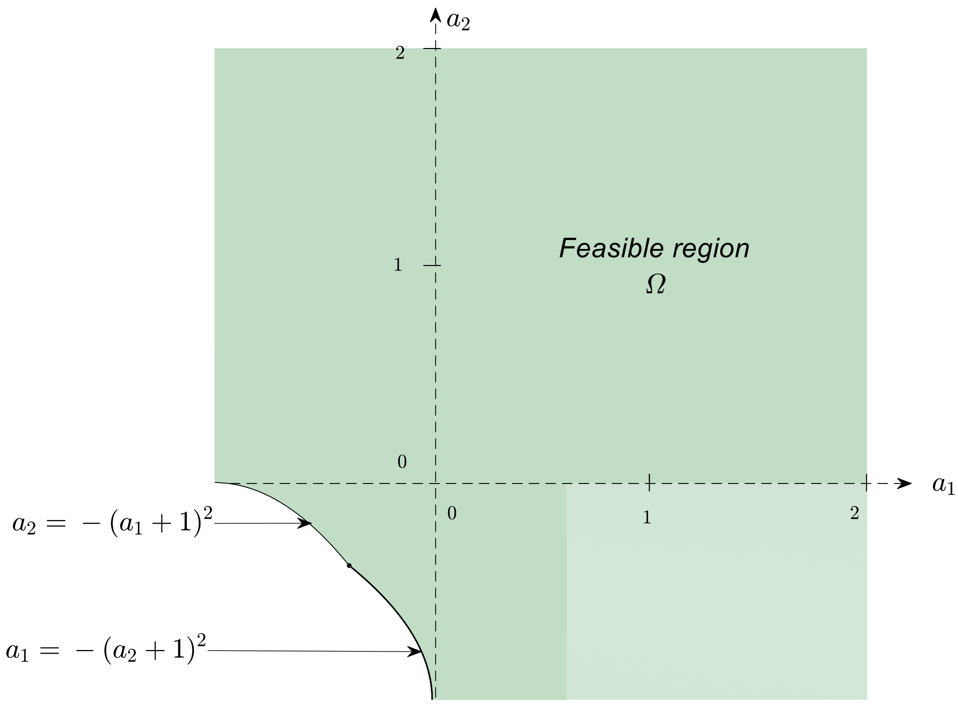

Figure (3.4): Feasible Region Ω (shown in color)

$$\frac{c_1}{[(1+a_2)^2+a_1]} < \frac{c_1(1+a_1)^3}{[(1+a_1)^2+a_2]^2} \tag{3.12}$$

After numerous algebraic manipulations, the above inequality reduces to:

$$c_1\left\{(1+a_1)^4+2a_2(1+a_1)^2+a_2^2-((1+a_1)^3[(1+a_1)+2a_2+a_2^2]\right\}<0 \tag{3.13}$$

and after additional manipulations, it reduces further to:

$$c_1\left\{2a_2(1+a_1)^2+a_2^2-a_2(2+a_2)(1+a_1)^3\right\}<0 \tag{3.14}$$

and finally, after more manipulations, it simplifies to

$$-2c_1a_1a_2(1+a_1)^2-c_1a_1a_2^2((2+a_1)+(1+a_1)^2)<0 \tag{3.15}$$

Dividing the above inequality by the expression $c_1 a_1 a_2^2 (1+a_1)^2$ which is > 0 since $c_1 a_1 = c_1^2 c_p$ yields the condition:

$$\frac{2}{a_2} + 1 + \frac{(2+a_1)}{(1+a_1)^2} > 0 \tag{3.16}$$

Now, if $a_2 > 0$ (i.e., if $c_2 > 0$) then the above inequality (4.16) will always be satisfied when $\{a_1, a_2\} \in \Omega$. If $a_2 < 0$ then (4.16) must also be satisfied to characterize Ω_1^L. Thus, in summary, the set of parameters for which C_1 prefers to be the leader is:

$$\Omega_1^L = \left\{ \{a_1, a_2\} \in \Omega \ni (a_2 > 0) \text{ or } \left(a_2 < 0 \text{ and } \frac{2}{a_2} + 1 + \frac{(2+a_1)}{(1+a_1)^2} > 0 \right) \right\} \tag{3.17}$$

and the set of parameters for which C1 prefers to be a follower is $\Omega_1^F = \Omega - \Omega_1^L$. Figure (3.4) illustrates the division of Ω into Ω_1^L and Ω_1^F.

Now following a similar derivation, the set Ω_2^L such that $J_2^2 < J_2^1$ can be determined as:

$$\Omega_2^L = \left\{ \{a_1, a_2\} \in \Omega \ni (a_1 > 0) \text{ or } \left(\frac{2}{a_1} + 1 + \frac{(2+a_2)}{(1+a_2)^2} > 0 \text{ if } a_1 < 0 \right) \right\} \tag{3.18}$$

and $\Omega_2^F = \Omega - \Omega_2^L$. Figure (3.5) illustrates the division of Ω into Ω_2^L and Ω_2^F. The superposition of Figures (3.4) and (3.5) when Ω it is bounded by $a_1 \le A$ and $a_2 \le A$ and $A = 2$ is shown in Figure (3.6). This figure shows the two regions of agreement (Ω_{A1} and Ω_{A2}) and the two regions of disagreement (Ω_{D1} and Ω_{D2}). It is interesting to note that the feasible region is now divided into eight separate regions (labeled I through VIII for ease of referencing) that are related to the allocation of roles between the two controllers as follows:

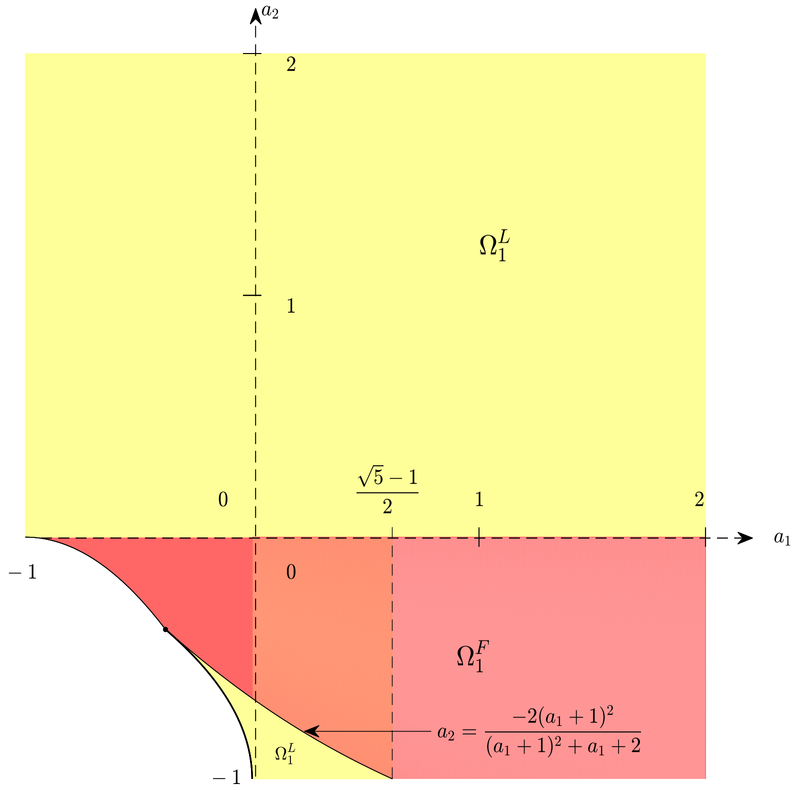

Figure (3.5): Regions Ω_1^L and Ω_1^F in Ω

1) Ω_{A1} It consists of sub-regions I and II, where both agree that C_1 should be selected as a leader.

2) Ω_{A2} It consists of sub-regions III and IV, where both agree that C_2 should be selected as a leader.

3) Ω_{D1} It consists of sub-regions V, VI, and VII, where both disagree, each preferring itself to be selected as a leader.

4) Ω_{D2} It consists of sub-region VIII, where both disagree, each preferring the other controller to be elected as leader.

Note that Ω_{A1} and Ω_{A2} each consists of a large and a small region that is disconnected, while Ω_{D1} consists of three disconnected regions with VII being dominant in size, indicating that most solutions would result in each controller wanting itself to be selected as leader.

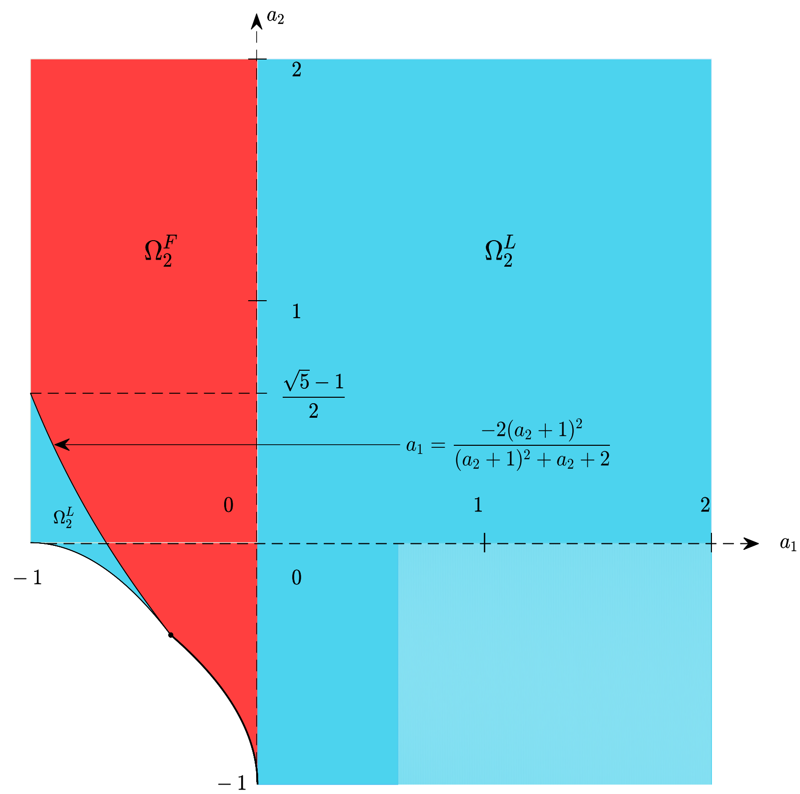

Figure (3.6): Regions Ω_2^L and Ω_2^F in Ω

When the upper bound *A* of the parameters a_1 and a_2 changes, some of the above observations will change accordingly. For example, when *A=0* (i.e., when Ω is bounded by $a_1 < 0$ and $a_2 < 0$), it will follow that Ω_{D1} will no longer exist, leaving only three possible options with Ω_{D2} being the most dominant indicating a strong preference for solutions where both controllers prefer the other controller be selected as leader.

Finally, solutions that correspond to parameters in the small regions (e.g., II and IV) are very sensitive to small perturbations in the parameters causing a wrong potential distribution of roles for the controllers.

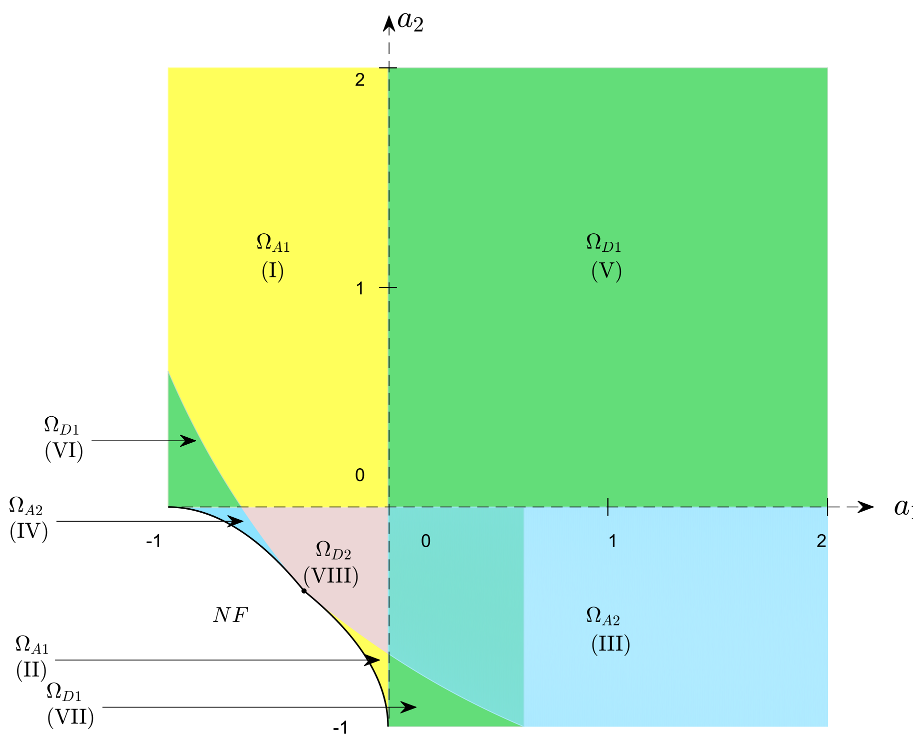

Figure (3.7): Characterization of Ω_{A1} , Ω_{A2} , Ω_{D1} *and* Ω_{D2} *in Ω*

3.4 Probabilities of Occurrence

Assuming that the parameters a_1 and a_2 are uniformly distributed over, Ω as shown in Figure (3.3) with A as upper bounds a_1 and a_2, we can calculate the probabilities of occurrence of solutions where both agreement and disagreement occur. The total area of the region Ω in Figure (3.4) can be determined analytically as a function of A by direct integration to be equal to $(A+1)^2 - \frac{5\sqrt{5}-7}{6} = A^2 + 2A + 0.30328$. Tables (3.1) and (3.2) list the areas calculated by direct integration for each of the various regions in figure (3.6) for three different values of *A*. These tables also show the probability of occurrence of solutions corresponding to all sub-regions individually as well as to the cumulative regions representing agreement and disagreement.

Table (3.1): Probabilities of Occurrence of all individual Sub-regions within Ω

Type of Solution	Region	Area of Region				Probability of Occurrence			
		A=0-	$A=\frac{\sqrt{5}-1}{2}$	*A=1*	*A=2*	*A=0-*	$A=\frac{\sqrt{5}-1}{2}$	*A=1*	*A=2*
Ω_{A1}	I	0.0	0.5227	0.9047	1.9047	0.0%	27.20%	27.39%	22.94%
	II	0.0526	0.0526	0.0526	0.0526	17.34%	2.74%	1.59%	0.63%
Ω_{A2}	III	0.0	0.5227	0.9047	1.9047	0.0%	27.20%	27.39%	22.94%
	IV	0.0526	0.0526	0.0526	0.0526	17.34%	2.74%	1.59%	0.63%
Ω_{D1}	V	0.0	0.3820	1.0000	4.00000	0.0%	19.88%	30.27%	48.17%
	VI	0.0	0.0953	0.0953	0.0953	0.0%	4.96%	2.89%	1.15%
	VII	0.0	0.0953	0.0953	0.0953	0.0%	4.96%	2.89%	1.15%
Ω_{D2}	VIII	0.1981	0.1981	0.1981	0.1981	65.32%	10.32%	5.99%	2.39%
Total Area		0.3033	1.9213	3.3033	8.3033				

Table (3.2): Cumulative Probabilities of Occurrence of Agreement/Disagreement Regions

Region	Area of Region				Probability of Occurrence			
	A=0-	$A=\frac{\sqrt{5}-1}{2}$	*A=1*	*A=2*	*A=0-*	$A=\frac{\sqrt{5}-1}{2}$	*A=1*	*A=2*
Ω_{A1}	0.0526	0.5753	0.9573	1.9573	17.34%	29.94%	28.98%	23.57%
Ω_{A2}	0.0526	0.5753	0.9573	1.9573	17.34%	29.94%	28.98%	23.57%
Ω_{D1}	0.0	0.5726	1.1906	4.1906	0.0%	29.80%	36.05%	50.47%
Ω_{D2}	0.1981	0.1981	0.1981	0.1981	65.32%	10.32%	5.99%	2.39%
Area	0.3033	1.9213	3.3033	8.3033				

Clearly, for a large parameters space (*A=2*), the dominant region with the highest overall probability of occurrence is Ω_{D1}. This means that there is a 50.47% probability that both controllers will end up disagreeing that C_1 should be selected as a leader. This probability, however, decreases rapidly to 36.05% as the parameter space becomes smaller (*A=1*) and to 0% as space becomes even smaller (A=0$^-$). The probability of occurrence of an agreement-solution Ω_{A1} or Ω_{A2} is 0% when $A=0^-$, becomes equal to the probability of disagreement Ω_{D1} at about 30% for each Ω_{A1}, Ω_{A2} and Ω_{D1} when $A=\frac{\sqrt{5}-1}{2}$ then increases to 28.98% when *A=1* and then dropping substantially to 23.57% when *A=2*. Clearly, the highest probability of agreement (at 59.88%) occurs when $A=\frac{\sqrt{5}-1}{2}$ and the highest probability of disagreement (at 65.32%) occurs when A=0$^-$. The size of the parameter space is clearly the determining factor of whether a mutual agreement can be reached $\Omega_{D1} \to \infty$ between the two controllers. Note that if $A \to \infty$ then and the probability of disagreement with each controller wanting to be the leader will approach 100%.

3.5 Conclusions

In this chapter, we have explored the problem of selecting the leader in Stackelberg games with uncertain parameters. We have shown that the parameter space can be divided into four regions, two of which representing games in which mutual agreement can be reached on the selection of leader and the other two representing games in which disagreement between the controllers takes place with both controllers either wanting to be leaders or both preferring to be followers. A leader's selection can be easily accomplished if the game parameters fall within the region of agreement. However, if they do not, then a stalemate condition may prevail, and the selection of a leader becomes more complicated. The Nash solution or the possibility of side payments may become options to break the stalemate. The probabilities of occurrence of agreement and disagreement games are very useful information for the controllers to use in the process of negotiations. We have illustrated these concepts with two examples and showed how the probabilities of occurrence of all games are determined when the parameters are uniformly distributed over a bounded space. This chapter provides an illustration of the type of analysis that needs to be performed on all Stackelberg games defined over uncertain parameters where the leader is to be selected by mutual agreement or negotiation and between the two controllers.

CHAPTER FOUR:
ECONOMIC APPLICATION: DUOPOLIST CASE

This chapter is devoted to the important application of multiple controllers with multiple objective systems in economics. We consider the real case when there are two-firm that control the dynamics of a demand function. We derive the necessary condition for Leader-Follower-Firm economic Stackelberg control systems. The general results are not easy to solve. In this work we proposed the novel demand linear differential state function with quadratic cost functions. However, the presence of cross term in the integrating profits function. The derivation of the necessary conditions for such systems are presented in this chapter. To demonstrate the important of the proposed model, the results, simulation results are presented for the numerical example.

4.1 Problem Description

A control system with two controllers, one labeled as a leader whose control is $u_L(t)$, and the other labeled as follower whose control is $u_F(t)$, which is typically described by the differential equation over an interval of time $[0,T]$ of the form:

$$\frac{dx(t)}{dt} = \dot{x} = f(x(t), u_L(t), u_F(t)) \tag{4.1}$$

where $x(t)$ is the state variable. For simplicity of notation and without loss of generality, we will consider only scalar systems, although most of the derivations can easily be extended to higher dimensions. We will assume that the two controllers have two different discounted objective functions (profits) that they wish to maximize:

$$J_L(u_L, u_F) = \int_0^T e^{-rt} L_L(x, u_L, u_F)\,dt \tag{4.2}$$

$$J_F(u_L, u_F) = \int_0^T e^{-rt} L_F(x, u_L, u_F) dt \tag{4.3}$$

Where e^{-rt} is the discounted factor, and r is the discounted rate for both firms.

To best describe the Stackelberg Leader-Follower control design process, we will consider a specific model of a dynamic economic system of two controlling firms. This model describes two firms controlling a common dynamically evolving market through their product supply functions. The model that we will consider assumes that the product price x(t) (i.e., state variable) depends on the total product supply $u(t) = u_L(t) + u_F(t)$ where $u_L(t)$ is the leader supply control and $u_F(t)$ is the follower supply control according to the differential equation:

$$\frac{dx(t)}{dt} = \dot{x} = f(x(t), u_L(t) + u_F(t)) \tag{4.4}$$

This relationship essentially implies that at any instant of time, the product price depends on the total supply according to:

$$x(t) = x_0 + \int_0^t f(x(\tau), u_L(\tau) + u_F(\tau)) d\tau \tag{4.5}$$

Where x_0 is the initial price at some arbitrary initial time $t_0 = 0$. Furthermore, we shall assume that the objective of each firm is to maximize its profits over the time horizon $[0, T]$, which are now described as:

$$J_L(u_L, u_F) = \int_0^T e^{-rt} [x u_L - C_L(u_L)].dt \tag{4.6}$$

and

$$J_F(u_L, u_F) = \int_0^T e^{-rt} [x u_F - C_F(u_F)].dt \tag{4.7}$$

where the term $xu_I - C_I(u_I)$ represents the difference between the revenues xu_I and production costs $C_I(u_I)$ for firms $I = \{L \text{ and } F\}$. In this dynamic control model, the supply control functions are determined continuously as a function of time to maximize the profits over the specified time horizon.

The Stackelberg control of the leader is first determined as a function of the control of the follower. To do this, the leader controller must anticipate the follower's maximization problem (4.7) for every possible leader control to arrive at its control function. This is accomplished by the follower controller using a standard optimal control methodology [1, 121-124]. The Hamiltonian for the follower is defined (consider the discount factor for both firms is $r = 0$) as:

$$H_F = xu_F - C_F(u_F) + \lambda_F f(x, u_L + u_F) \tag{4.8}$$

and the necessary conditions for the follower's control function are [27]:

$$\dot{\lambda}_F = -u_F - \lambda_F \frac{\partial f(x, u_L + u_F)}{\partial x}, \quad \lambda_F(T) = 0 \tag{4.9}$$

$$0 = x - \frac{\partial C(u_F)}{\partial u_F} + \lambda_F \frac{\partial f(x, u_L + u_F)}{\partial u_F} \tag{4.10}$$

Where λ_F is the follower's Lagrange-multiplier. Thus, for every possible control u_L that the leader can implement, the follower will determine its control u_F by solving (4.9) and (4.10) where x and H_F satisfy (4.4) and (4.8). These expressions, therefore, define how the follower reacts to every possible control choice by the leader. Now the problem faced by the leader is a little more complex. The leader must determine its control u_L that maximizes (4.6) subject to the constraint that u_F satisfies the differential equation (4.9), and the algebraic equation (4.10) can be solved implicitly for the follower's control as a function of x and the leader's control u_L ; that is,

$u_F = u_F(u_L, x)$. Then, the leader's control can be determined using a different leader's Hamiltonian defined as:

$$H_L = xu_L - C_L(u_L) + \lambda_L f(x, u_L + u_F(u_L, x)) + \beta_L \left(-u_F(u_L, x) - \lambda_F \frac{\partial f(x, u_L + u_F(u_L, x))}{\partial u_F} \right) \qquad (4.11)$$

And the necessary conditions for the leader's control function become:

$$\dot{\lambda}_L = -\lambda_L \left(\frac{\partial f(x, u_L + u_F)}{\partial x} + \frac{\partial f(x, u_L + u_F)}{\partial u_F} \frac{\partial u_F}{\partial x} \right) + \beta_L \left(\frac{\partial u_F}{\partial x} + \lambda_F \frac{\partial^2 f(x, u_L + u_F)}{\partial u_F \partial x} + \frac{\partial^2 f(x, u_L + u_F)}{\partial u_F^2} \frac{\partial u_F}{\partial x} \right) - u_L, \qquad \lambda_L(T) = 0 \qquad (4.12)$$

$$\dot{\beta}_L = -\frac{\partial H_L}{\partial \lambda_F} = \beta_L \frac{\partial f(x, u_L + u_F)}{\partial u_F}, \quad \beta_L(0) = 0 \qquad (4.13)$$

$$0 = \frac{\partial H_L}{\partial u_L} = x - \frac{\partial C_L(u_L)}{\partial u_L} + \lambda_L \left(\frac{\partial f(x, u_L + u_F)}{\partial u_L} + \frac{\partial f(x, u_L + u_F)}{\partial u_F} \frac{\partial u_F}{\partial u_L} \right) - \beta_L \left(\frac{\partial u_F}{\partial u_L} + \frac{\partial^2 f(x, u_L + u_F)}{\partial u_F \partial u_L} + \frac{\partial^2 f(x, u_L + u_F)}{\partial u_F{}^2} \frac{\partial u_F}{\partial u_L} \right) \qquad (4.14)$$

These expressions are, in general, very difficult to solve. However, like many of the optimal control problem, a solution can be determined analytically in the case where the system is linear and the cost functions are quadratic. This will be discussed in detail in the next section.

4.2 The Case of Linear Demand and Quadratic Cost Function

The special case of one controller linear system and quadratic cost functions has received considerable interest in the control literature since the 1970s [122, 123]. As in the one controller case, considerable insight can be obtained about the system behavior by analyzing this special linear quadratic two-controller case. Let the system dynamics in (4.4) be linear and described as follows:

$$\dot{x} = ax - b(u_L + u_F) \qquad x(0) = x_0 \tag{4.15}$$

Where a and b are positive constants and representing the rate of growth of the state variable when no control is applied. Similarly, let the cost functions in (4.6) and (4.7) be quadratic of the form:

$$C_L(u_L) = \frac{1}{2}c_L u_L^2 \quad \text{and} \quad C_F(u_F) = \frac{1}{2}c_F u_F^2 \tag{4.16}$$

Where c_L and c_F are positive constants, and the factor ½ is introduced for mathematical convenience. It is clear from (1) that the two controllers can keep the state variable constant x_0 throughout the entire time horizon [87] if they both reach a consensus to simultaneously adjust their controls so that $u_L + u_F = (a/b)x$. However, this is unlikely to happen since if one controller increases its control supply to increase its profits, the other controller will have to reduce its control to keep the consensus resulting in a reduction in its profits, which may not be acceptable. In the dynamic model described in (4.15), both controllers will continuously adjust their controls in order to maximize their objective functions. Thus, the follower's maximization problem as described in (4.4), (4.8)-(4.10) will have a Hamiltonian in the form:

$$H_F = xu_F - \frac{1}{2}c_F u_F^2 + \lambda_F\left(ax - b\left(u_L + u_F\right)\right) \tag{4.17}$$

As well as the following necessary conditions:

$$\dot{x} = ax - b\left(u_L + u_F\right) \quad x(0) = x_0 \tag{4.18}$$

$$\dot{\lambda}_F = -u_F - a\lambda_F, \quad \lambda_F(T) = 0 \tag{4.19}$$

$$0 = x - c_F u_F - b\lambda_F \tag{4.20}$$

Thus, from(4.20), we have

$$u_F = \frac{1}{c_F}(x - b\lambda_F) \tag{4.21}$$

As a result, the problem faced by the leader is to maximize (4.7)subject to the two constraints:

$$\dot{x} = (a - \frac{b}{c_F})x - bu_L + \frac{b^2}{c_F}\lambda_F, \quad x(0) = x_0 \tag{4.22}$$

And

$$\dot{\lambda}_F = -(a - \frac{b}{c_F})\lambda_F - \frac{1}{c_F}x, \quad \lambda_F(T) = 0 \tag{4.23}$$

Derived from (4.18) and (4.19) by replacing u_F as described in (4.21). The corresponding Hamiltonian (4.11) becomes:

$$H_L = xu_L - \frac{1}{2}c_L u_L^2 + \lambda_L\left((a - \frac{b}{c_F})x - bu_L + \frac{b^2}{c_F}\lambda_F\right) + \beta_L\left(-(a - \frac{b}{c_F})\lambda_F - \frac{1}{c_F}x\right) \tag{4.24}$$

And the necessary conditions (4.12)-(4.14) reduce to:

$$\dot{\lambda}_L = -u_L - (a - \frac{b}{c_F})\lambda_L + \frac{1}{c_F}\beta_L, \quad \lambda_L(T) = 0 \tag{4.25}$$

$$\dot{\beta}_L = -\frac{b^2}{c_F}\lambda_L + (a - \frac{b}{c_F})\beta_L, \quad \beta_L(0) = 0 \tag{4.26}$$

$$0 = x - c_L u_L - b\lambda_L \tag{4.27}$$

From (4.27) we have:

$$u_L = \frac{1}{c_L}(x - b\lambda_L) \tag{4.28}$$

Now combining equations,(4.22)-(4.23) and (4.28) and replacing the controls u_F and u_L with their expressions in (4.21) and (4.28), we get:

$$\dot{x} = (a - \frac{b}{c_L} - \frac{b}{c_F})x + \frac{b^2}{c_L}\lambda_L + \frac{b^2}{c_F}\lambda_F \quad x(0) = x_0 \tag{4.29}$$

$$\dot{\lambda}_L = -(a - \frac{b}{c_L} - \frac{b}{c_F})\lambda_L - \frac{1}{c_L}x + \frac{1}{c_F}\beta_L, \quad \lambda_L(T) = 0 \tag{4.30}$$

$$\dot{\lambda}_F = -(a - \frac{b}{c_F})\lambda_F - \frac{1}{c_F}x, \quad \lambda_F(T) = 0 \tag{4.31}$$

with β_L satisfying equations(4.26). Now, introducing the transformations $\lambda_L = k_L x$, $\lambda_F = k_F x$, and $\beta_L = kx$; and after considerable mathematical manipulations, the solution of these equations yields the following control functions for the leader and follower respectively:

$$u_L(t) = \frac{1}{c_L}\left(1 - bk_L(t)\right)\varphi(t)x_0 \tag{4.32}$$

$$u_F(t) = \frac{1}{c_F}\left(1 - bk_F(t)\right)\varphi(t)x_0 \tag{4.33}$$

where $k_L(t)$, $k_F(t)$, and $\varphi(t)$ are functions of time that satisfy the following differential equations:

$$\dot{k}_L(t) = \left(-2a + \frac{2b}{c_L} + \frac{2b}{c_F}\right)k_L(t) - \frac{b^2}{c_L}k_L^2(t) - \frac{b^2}{c_F}k_L(t)k_F(t) + \frac{1}{c_F}k(t) - \frac{1}{c_L}, \qquad k_L(T) = 0 \tag{4.34}$$

$$\dot{k}_F(t) = \left(-2a + \frac{b}{c_L} + \frac{2b}{c_F}\right)k_F(t) - \frac{b^2}{c_L}k_L(t)k_F(t) - \frac{b^2}{c_F}k_F^2(t) - \frac{1}{c_F}, \quad k_F(T) = 0 \tag{4.35}$$

and where $k(t)$ satisfies

$$\dot{k}(t) = \frac{b}{c_L}k(t) - \left(\frac{b^2}{c_L}k_L(t) + \frac{b^2}{c_F}k_F(t)\right)k(t) - \frac{b^2}{c_F}k_L(t) \qquad k(0) = 0 \tag{4.36}$$

and $\varphi(t) = e^{\int_0^t \sigma(\tau)d\tau}$

where:

$$\sigma(t) = \left(a - \frac{b}{c_L} + \frac{b^2}{c_L}k_L(t) - \frac{b}{c_F} + \frac{b^2}{c_F}k_F(t)\right) \tag{4.37}$$

At this point, we should mention that equations (4.34)-(4.36) are a two-point boundary value problem consisting of coupled nonlinear differential equations. Equations (4.34) and (4.35) have boundary conditions at the terminal time t=T, while equation (4.36) has a boundary condition at the initial time t=0. Once this system of equations is solved for $k_L(t)$, $k_F(t)$ and $k(t)$, only $k_L(t)$ and $k_F(t)$ are used to generate the function $\sigma(t)$ in (4.37), which in turn is used to calculate the function $\varphi(t)$.

4.3 Illustrative Example

As an illustrative example, we consider a control system model of a dynamic market with two firms producing and selling the same product. The model would follow the differential equation (4.15) and profit functions as described in (4.6) and (4.7) with production costs as described in (4.16). Let the problem parameters be defined as follows: $a = 0.03$, $b = 0.012$, $c_L = 0.60$ and $c_F = 0.70$, and let the time horizon be such that $T = 30$. Assuming that the product has an initial unit price $x_0 = 10$, plots of the functions $k_L(t)$ $k_F(t)$, and $k(t)$ that satisfy (4.34)-(4.36) are shown in Figures (4.1) and (4.2), respectively. Plots of the control functions $u_L(t)$ and $u_F(t)$ are shown in figure (4.3), and a plot of the state variable $x(t)$, which represents the product price, is shown in Figure (4.4). The total profits accumulated by the firms in this case over the entire time horizon are $J_L = 2,843.70$ and $J_F = 2,471.31$ indicating that the firm that has lower production costs has achieved higher profits

Clearly, figure (4.3) shows that in the case of both firms, to maximize their profits, they must continuously increase production over the entire time horizon. Also, it appears that the leader firm whose production cost is lower seems to be producing at a higher rate than the follower firm whose production cost is higher. A close examination of the figure (4.4) reveals that the product price increases rapidly at the beginning but reaches a peak of almost 12 around t=20 before tapering down to 11.5 at t=30., the end of the profit maximization horizon.

One interesting aspect of this analysis is to examine the profits of both firms and the product price as a function of the follower's production cost. Table (4.1) and Figures (4.4-4.5) show the variations of the profits of both firms and the price behavior overtime when the Leader's cost parameter is fixed $c_L = 0.60$, and the follower cost parameter c_F is increased from 0.50 to 1.2. Clearly, as the follower's production costs increase, the product price increases and the follower's profits decrease, but the leader's profits increase. Thus the leader has an incentive to ensure that the follower's production costs remain as high as possible. In fact, when the follower's production costs are double those of the leader, i.e. $c_F = 1.20$, the leader's profits will be double the profits of the follower. Table 1 also shows that, when $c_F = 0.58128$, both firms accomplish the same profits $J_L = J_F = 2,761.20$. It is also interesting to note that the product price behavior as a function of time changes markedly as a function of c_F. The lower c_F, the more the price tends to reach a peak value. This peak value shifts to later in time as c_F increases and vanishes when $c_F = 1.00$. Beyond this value, the price becomes monotonically increasing in time. This type of price behavior is interesting from the consumer point of view. A price that exhibits a peak followed by a drop after a certain time is more favorable to the consumer leading to the conclusion that the consumer prefers that the follower firm's production costs be more on par with the costs of the leading firm. In the next section, some results will be presented for the sensitivity of the parameters variation in the proposed model and their corresponding effects on the profits for the two firms.

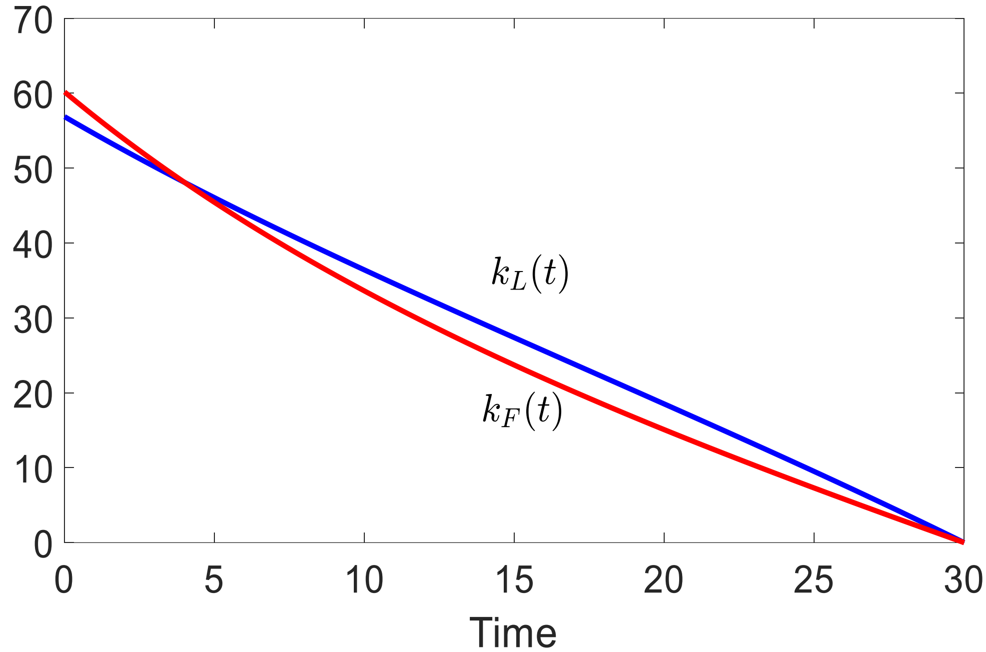

Figure (4.1): Plots of functions $k_L(t)$ *and* $k_F(t)$ *of equations(4.34) and (4.35)*

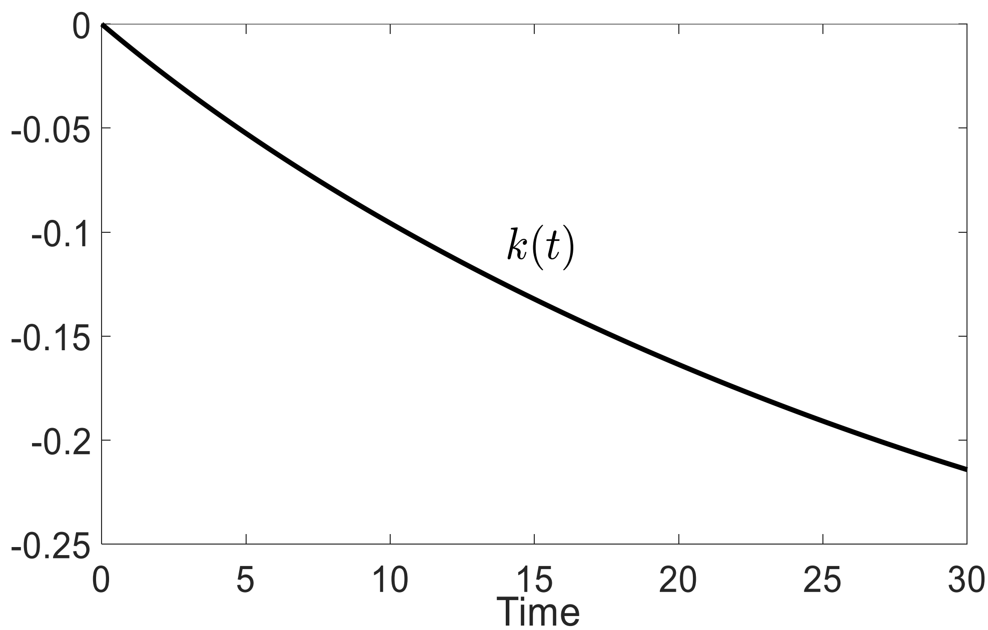

Figure (4.2): Plots of function $k(t)$*, equation (4.36)*

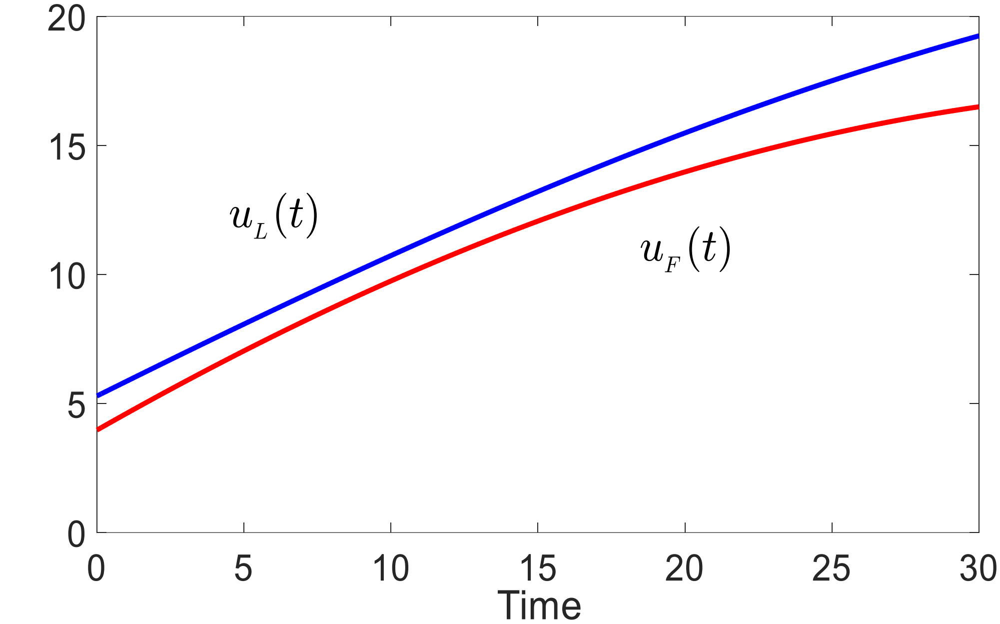

Figure (4.3): plots of $u_L(t)$ *and* $u_F(t)$ *production rates*

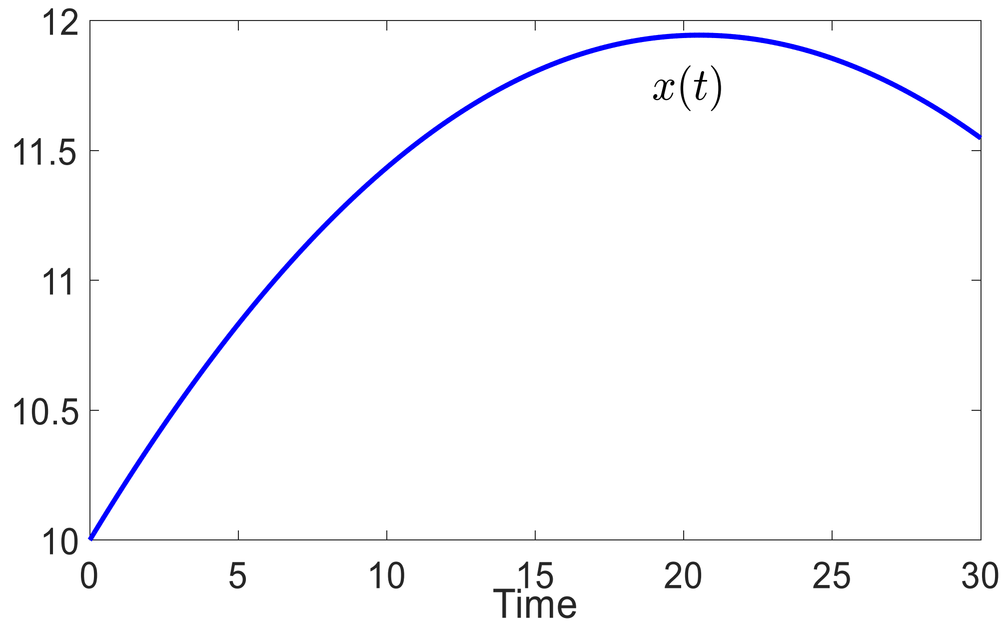

Figure (4.4): plot of $x(t)$ *production price*

Table (4.1): Profits of both firms as a function of c_F *when* $c_L = 0.60$

c_F	J_L	J_F
0.50	2708.14	3008.48
0.52	2720.50	2943.00
0.54	2733.40	2880.72
0.56	2746.72	2821.38
0.58	2760.32	2764.74
0.58128	2761.20	2761.20
0.6	2774.11	2710.58
0.62	2788.02	2658.73
0.64	2801.97	2609.01
0.66	2815.93	2561.29
0.68	2829.85	2515.43
0.70	2843.70	2471.31
1.00	3032.24	1963.00
1.20	3134.67	1729.25

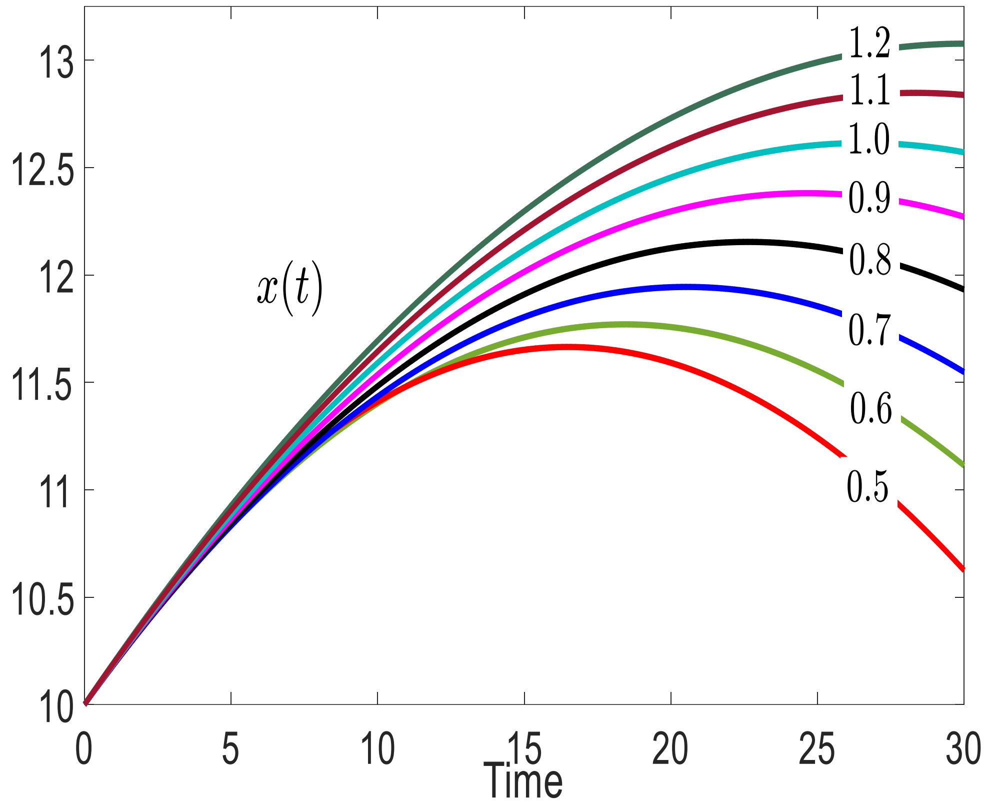

Figure (4.5): plot of $x(t)$ *when* $c_L = 0.6$ *and* c_F *is increased from 0.50 to 1.20.*

4.4 Sensitivity with parameter variation of the demand function

Following the results, we have gotten in chapter three and applied to the proposed model and results of the duopolistic. If we suppose that uncertainty happens in the demand function parameters, i.e., changing in a and b.

First, supposing there is an uncertain value of a around its nominal value, with a fixed value of b, the plot in figure (4.6) shows proportional relations between the a and both firms' profits. Whereas $a = 0.024$ which is 80% from its nominal value $a = 0.03$, the leader firm will have profit $J_L = 2298$ which is around 80% from its profit when a is nominal, and on another side, follower firm $J_F = 1582$ which is 64.7% almost the same percentage of losses. Thus, as the value of a going up to +20%, $a = 0.036$, the profits for leader and follower increased by +29% and+30% , respectively.

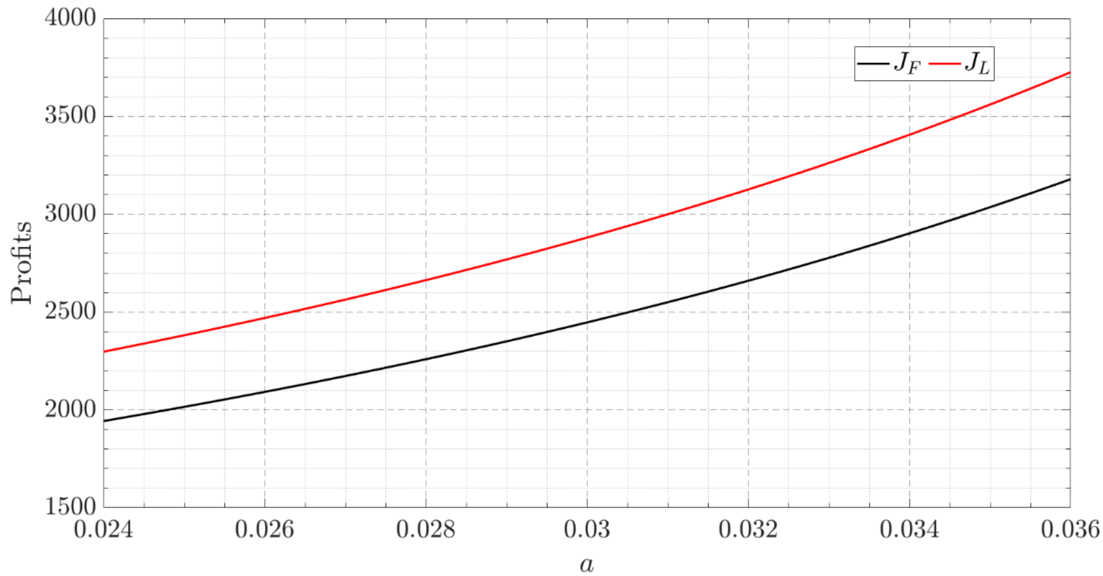

Figure (4.6): Firms' profits vs. a *fixed* $b = 0.012$.

The other possibility is that change in b 's value with fixed a . If the variation b is ranging as follow: $\{-20\% \rightarrow +20\%\}$ from its nominal value, both firms' profits decrease as follows: Leader $\{+13 \rightarrow -10\%\}$ while the follower will change $\{+13.5\% \rightarrow -11\%\}$. Figure (4.7) shows the full range for the above b 's variation and the firms' profits' corresponding effects.

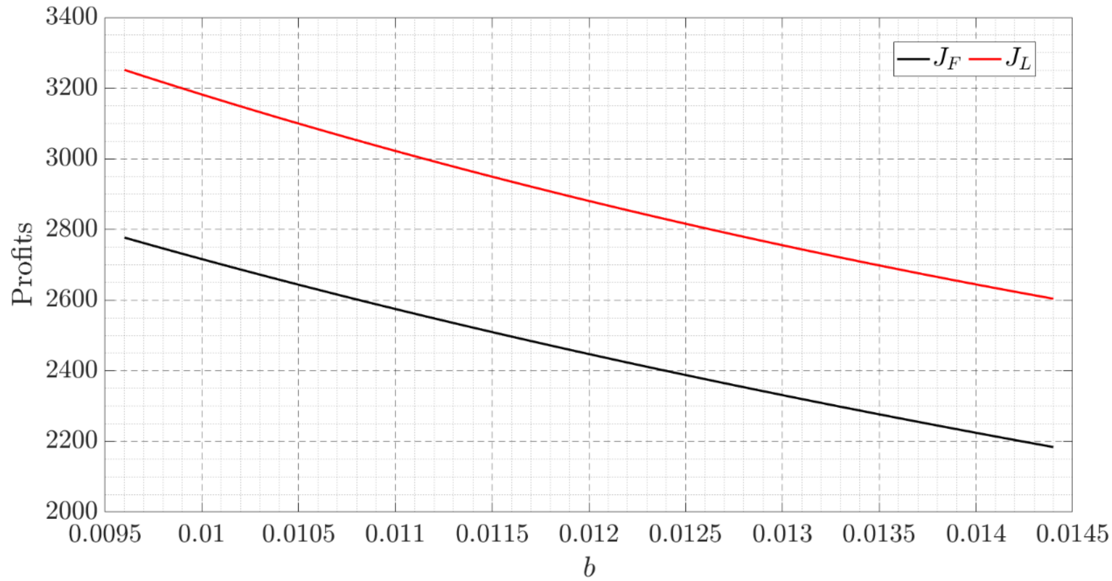

Figure (4.7): Firms' profits vs. b *and fixed* $a = 0.03$.

From the above two possibilities, both firms' profits will increase as the value a increases and decreases. As the final possible if both parameters are varying $\pm 20\%$, the 3-dimensional plot in figure (4.8) shows the relationship as (a,b) pair are change and outcome leader firm profit J_L. As expected from the previous results, the possible upper value of a combined with the lowest possible of b the best profit of leader firm within the range of variations and vice-versa. Numerically speaking, as shown from the figure (4.8) the lowest possible value for the leader profit $J_L(a = 0.024, b = 0.0216) = 2631$ and best possible value is $J_L(a = 0.036, b = 0.0144) = 5554$

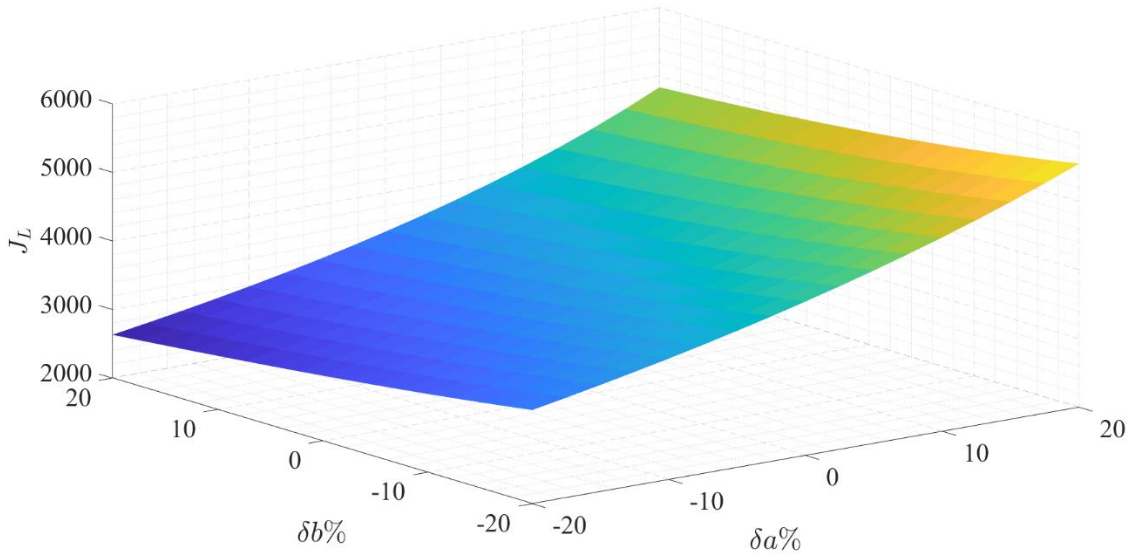

Figure (4.8) Leader firm profits versus a and b variation

For the same range of variations for the demand function parameters, the profits changes are represented by the surface, as shown in figure (4.9). The results can be shown for the lowest and best values of the follower profit as follow: $J_F(a = 0.024, b = 0.0216) = 1507$ and best possible value is $J_F(a = 0.036, b = 0.0144) = 3384$

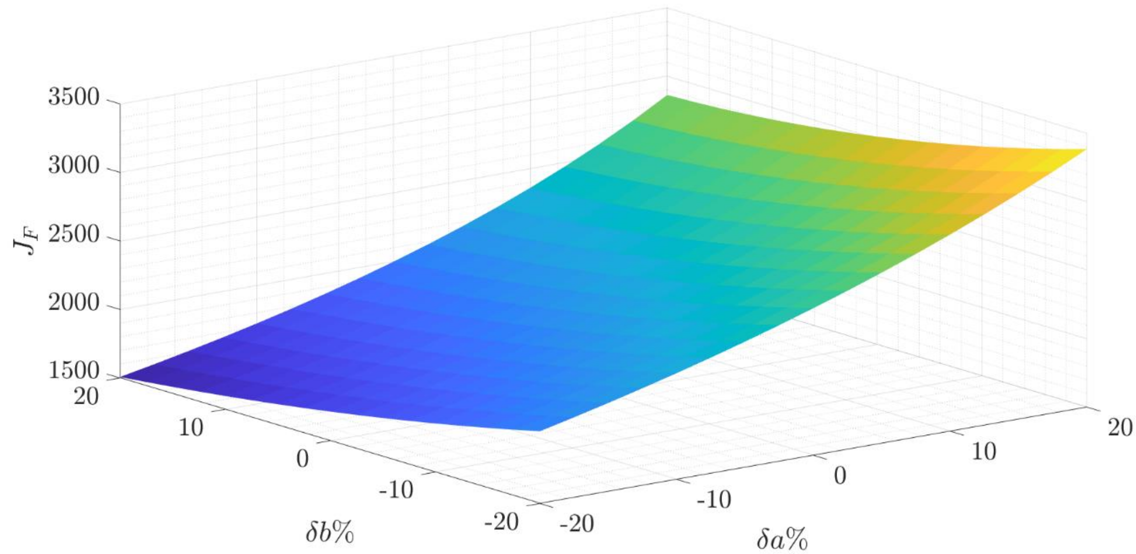

Figure (4.9): Follower firm profits versus a and b variation

4.4 Conclusions

In this chapter, we considered control systems controlled by two independent controllers. Motivated by real problems in dynamic economics where more powerful big firms can implement their production strategies before the less powerful small firms, the leader-follower structure as a variation of the traditional control systems has attracted considerable attention in recent years. In such systems, due to either size or power, one controller has an advantage over the other in that it is capable of designing and implementing its control actions before the other. This controller is referred to as the leader controller and the other as the follower controller. To take advantage of the leadership role, the leader controller anticipates the follower controller's reaction and designs its control actions taking this reaction into account as a constraint that needs to be satisfied. This makes the design process of the leader control much more complicated than the follower control design.

Once the leader's control is designed and implemented, the follower control is traditionally designed as an optimal one controller control. In this chapter, we have examined the design process of both controllers in detail, and we have shown that the leader implicitly determines the best follower's control that optimizes its performance and designs its control taking into account that the follower's optimal choice is that specific control. These types of leader-follower control systems can be used to describe many practical control systems. An illustrative example and simulation results are presented to demonstrate the our proposed differential demand function with using Stackelberg framework in such applications.

CHAPTER FIVE: CONCLUSIONS

The control of multi-controller multi-objective systems presented in this dissertation is considered a natural extension of the standard optimal control theory. In this dissertation, we have discussed properties and applications related to these types of systems. The focus of the dissertation is on dynamic systems with two controllers, each having its own objective function to minimize over a finite-horizon. The contributions of the dissertation are summarized below.

5.1 Contributions

5.1.1 Sampled closed-loop

Chapter two proposed a sampled, instead of continuous, closed-loop schemes for two-controller multi-objective systems. The necessary conditions for the proposed approach are derived in detail for Linear Quadratic (LQ) systems. The theoretical results and implementation of the sampled closed loop controls are applied for both Nash, and Stackelberg approaches. The main consideration of this approach is in designing the controls, which is a trade-off between the simplicity of implementation of the open-loop framework and the robustness property of the closed-loop framework. The proposed scheme can be a special type of feedback loops that are closed only at specific instants of time when the state-vector is available for measurement. As an application for the derived results, we have illustrated a two-controller example for both the Nash and Stackelberg solutions where the time horizon is divided into several number of samples. Several observations can be made as a result of this example. For the Nash controllers it was observed that as the number of samples increased, the system's behaviors for both controllers and state trajectory resemble the behavior of the continuous closed loop. However, the sampled-closed-loop Stackelberg implemented controller with a high rate of samples approached the Nash continuous closed-loop controls, and state trajectory.

5.1.2 Chosen Roles with Parameters uncertainty

Chapter four is devoted to an application of two controller systems in economics, the duopoly model. This model is a linear differential price equation while the profit functions include quadratic costs. In this chapter, we presented an illustrative example and derived the necessary conditions for the leader-follower Stackelberg approach. One can conclude that the solution does not exist for all possible ranges of cost parameters.

5.1.3 Economic Application: Duopolist Case

Chapter four is devoted to one important application in economic, the duopoly case. The literature for two-firm shares the same market and produces the same good and control the demand function is mostly static. However, this dissertation tries to consider the price as controlled by a differential equation from a control system view. The proposed model is new and has not been considered in previous literature. This model is a linear differential price equation while the profit functions are quadratic cost and have a cross term. However, due to the fact that the Nash controller's derivation is more straightforward than the Stackelberg controller, the Nash case is not considered in this work. An illustrative example is presented to apply the necessary derived conditions for such systems using the leader-follower Stackelberg approach. One can conclude that the solution does not exist for all possible ranges of cost parameters.

5.2 Suggestions for Future Research

There are several interesting problems that have not been explored in this dissertation and remain open for future research. An important problem is to extend the sampled closed loop approach to multiple controllers in nonlinear systems. Another problem is generalizing the leader-follower role assignment over system parameters distribution for more general models including linear quadratic systems. Exploring the possible solutions when both controllers are in disagreement would also be interesting.

APPENDIX LIST OF PUBLICATIONS

Journal

Raaed S. Al-Azzawi and Marwan A. Simaan, "On the selection of leader in Stackelberg games with parameter uncertainty," *International Journal of Systems Science*, pp. 1-9, 2020, doi: 10.1080/00207721.2020.1820097.

Conferences

Raaed S. Al-Azzawi and Marwan. A. Simaan, "Sampled Closed-Loop Control in Multi-Controller Multi-Objective Control Systems," in *SoutheastCon 2018*, 19-22 April 2018 2018, pp. 1-7, doi: 10.1109/SECON.2018.8478971.

Raaed S. Al-Azzawi and Marwan A. Simaan, "Leader-Follower Controls in Systems with Two Controllers," in *2019 SoutheastCon*, 11-14 April 2019 2019, pp. 1-6, doi: 10.1109/SoutheastCon42311.2019.9020435.

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