December 2020

Uncertainty Principles in Framed Hilbert Spaces

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A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Science

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by
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December 2020

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Introduction

The uncertainty principle is one of the most fundamental concepts in harmonic analysis. It has many facets and appears in many different (non-equivalent) forms. However, a common theme of all uncertainty principles can be vaguely summarized into two points of view. From Fourier analysis perspective: In 1925, Wiener first formulate the uncertainty principle [19] in his Göttingen lecture. It says a function and its Fourier transform cannot both be sharply localized. From operator theoretical perspective: In 1927, two years after the Wiener's lecture, Heisenberg formulated his famous uncertainty principle [7], saying that the position and the momentum of a quantum particle cannot be measure simultaneously. Mathematically, this can be expressed as the multiplication and differentiation operators X and D satisfy the canonical commutation relation:

$$[D, X] := DX - XD = \frac{1}{2\pi i} I.$$

Which shows a strong non-commutativity of X and D. These two points of view are related with each other through the fact that the Fourier transform $\mathcal{F}$ conjugates the operators X and D. Namely, $X = \mathcal{F} D \mathcal{F}^*$.

The thesis mainly consists of three projects whose common theme is the uncertainty principle. Two of them concern interpolation and sampling problems, while the third one is about the so-called generalized Balian-Low theorem. We now briefly describe each of them, and give an outline of how the thesis is organized.

The main results of the thesis are included in the last three chapters. The main purpose of Chapter 1 is to introduce the well-known concepts that will be used throughout the thesis, and set up the notations. Besides this, we state the classical uncertainty principle and the classical Balian-Low theorem that will be generalized in the later chapters. We also include proofs of these results which are based on the similar ideas with our proofs of more general results. The reason for including them is to illustrate those ideas in the simplest possible setting.

In Chapter 2, we introduce the concept of framed Hilbert spaces as a general setting for studying problems of uncertainty principle type. These spaces could be viewed as a special type of reproducing kernel Hilbert spaces which include many function spaces that play an important role in harmonic analysis. In the chapter, we prove some basic results about framed Hilbert spaces that will be used in the rest of the thesis. Many of those are new results in this level of generality.

Sampling and interpolation problems could be viewed as a manifestation of the uncertainty principle in the following way. Take the classical Paley-Wiener space as an example. This space consists of functions (band-limited functions) in $L^2(\mathbb{R})$ whose Fourier transform is localized in a fixed interval. As a consequence of the classical uncertainty principle, these functions cannot be localized on any finite intervals. However, we have more than that. It turns out that the oscillation of the functions get controlled by their band limit. Therefore, it is difficult to solve interpolation problems for such functions. More precisely, to be able to solve interpolation problems, the density of interpolating points must be sparse enough relative to the band limit. However, at the same time, it is easy to solve sampling problems. In other words, such functions could be completely determined on the set (sampling set) which is sufficiently dense relative to the band limit.

In Chapter 3, we define a very general interpolation set called d-approximate (weak) interpolation set in framed Hilbert spaces. This type of sets were relatively recently introduced by Olevskii and Ulanovskii in the classical Paley-Wiener space. And we will prove a necessary density condition for those sets similar to the one for usual interpolation sets.

Like Chapter 3, sampling problems could be also studied in the general framework of framed Hilbert spaces. However, we will concentrate on a more precise problem of estimating a sampling constant in the space of multiband-limited functions. A sharp estimate of this constant in classical Paley-Wiener spaces is given by Kovrijkine, who also provides a quantitative estimate for this constant in multiband cases. In both of these cases, the dependence of this sampling constant on the length of the (multi)band is exponential. In Chapter 4, we will impose additional (suboptimal) conditions on the sampling set which allow us to obtain a much-improved sampling constant which depends linearly on the length of the multiband.

The classical uncertainty principle gives a bound on how close a function could become a joint eigenvector of X and D. This classical version of uncertainty principles also gives us the optimal approximate joint eigenvectors, which are the time-frequency shifts of the Gaussian. However, it turns out that the time-frequency shifts of the Gaussian, unfortunately, cannot form an orthonormal basis (even a Riesz basis). Moreover, if we use an integer lattice of time-frequency shifts of some generating function to form an orthonormal basis, it turns out this generating function must have a very poor time-frequency localization. This result of uncertainty principle type is the so-called Balian-Low theorem.

In Chapter 5, we will give a very general version of Balian-Low theorem, which only requires the operator pair to satisfy a weak non-commutativity property and does not necessarily require the orthonormal (or Riesz) basis to be the time-frequency shifts of a single function. Furthermore, we examine the relationship between diagonalization and possibility of joint orthonormal (or Riesz) eigenbasis of the operator pair both in a very weak sense.

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Chapter 1

Preliminaries

1.1 Multiplication and Differentiation Operators

Arguably, the two most important operators in analysis are the multiplication and differentiation operators:

The **multiplication** $X : \mathcal{D}(X) \to L^2(\mathbb{R})$ is defined by

$$Xf(x) := xf(x), \quad \forall x \in \mathbb{R}.$$

And the **differentiation** $D : \mathcal{D}(D) \to L^2(\mathbb{R})$ is defined by

$$Df(x) := \frac{1}{2\pi i} f'(x), \quad \forall x \in \mathbb{R}.$$

Where the domains of X and D are defined in the usual way:

$$\mathcal{D}(X) := \{f \in L^2(\mathbb{R}) : Xf \in L^2(\mathbb{R})\}; \quad \mathcal{D}(D) := \{f \in L^2(\mathbb{R}) : Df \in L^2(\mathbb{R})\}.$$

By definitions, we could show X and D are both self-adjoint operators, i.e.,

$$X^* = X; \quad D^* = D.$$

Using X and D, we could also define the following two groups of operators:

The group of **translations** $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by

$$T_a := e^{-2\pi i a D}, \quad \forall a \in \mathbb{R}.$$

And the group of **modulations** $M_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by

$$M_b := e^{2\pi i b X}, \quad \forall b \in \mathbb{R}.$$

By Taylor expansions, we have

$$T_a f(x) = f(x-a); \quad M_b f(x) = e^{2\pi i b x} f(x).$$

That's why we call T_a translation and M_b modulation.

Moreover, we could show $a \to T_a$ and $b \to M_b$ are two unitary representations from $(\mathbb{R}, +)$ to $\mathcal{U}(\mathcal{H})$, which means T_a and M_b are the unitary operators such that

$$T_a T_b = T_{a+b}, \quad T_a^* = T_{-a};$$
$$M_a M_b = M_{a+b}, \quad M_b^* = M_{-a}.$$

Since M_b is generated by X and T_a is generated by D, we have two pairs of commuting operators:

$$X M_b = M_b X; \quad D T_a = T_a D.$$

However, simple calculation shows that X and D do not commute but satisfy the so-called **canonical commutation relation**:

$$[D, X] := DX - XD = \frac{1}{2\pi i} I, \tag{1.1}$$

where $I : \mathcal{D}(X) \cap \mathcal{D}(D) \to \mathcal{D}(X) \cap \mathcal{D}(D)$ is the identity map. Rewrite (1.1) in terms of T_a and M_b, it is equivalent to

$$T_a M_b = e^{-2\pi i b a} M_b T_a.$$

As a consequence of the canonical commutation relation, we obtain

$$[X, T_a] := XT_a - T_aX = aT_a;$$
$$[D, M_b] := DM_b - M_bD = bM_b. \tag{1.2}$$

In next section, we will see the canonical commutation relation plays an important role in the classical uncertainty principle.

1.2 Classical Uncertainty Principle

Before we state the classical uncertainty principle, we need to introduce a very important concept in analysis which is the Fourier transform:

For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the **Fourier transform** $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}dx, \quad \forall \xi \in \mathbb{R}.$$

And we will also denote $\widehat{f}$ by $\mathcal{F}f$. This notation indicates that we would consider the Fourier transform $\mathcal{F}$ as an operator as well. By *Plancherel theorem*, $\mathcal{F}$ could be extended as an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. Throughout the thesis, we would view the Fourier transform $\mathcal{F}$ as an unitary operator on $L^2(\mathbb{R})$. Since $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$, its **inverse Fourier transform** $\mathcal{F}^{-1}$ equals its adjoint $\mathcal{F}^*$ given by

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi, \quad \forall x \in \mathbb{R}.$$

For us, the importance of the Fourier transform $\mathcal{F}$ mainly comes from the fact that it connects multiplication X and differentiation D in the following way:

$$\mathcal{F}X = -D\mathcal{F}; \quad \mathcal{F}D = X\mathcal{F}.$$

As a consequence, $\mathcal{F}$ could also transform T_a and M_b as the following:

$$\mathcal{F}M_b = T_b\mathcal{F}; \quad \mathcal{F}T_a = M_{-a}\mathcal{F}.$$

We now could state the classical uncertainty principle (which is often referred to as the *Heisenberg-Pauli-Weyl inequality*) as the following:

Theorem 1.2.1 ([5], Theorem 1.1)**.** *For any $f \in L^2(\mathbb{R})$ and any $a, b \in \mathbb{R}$,*

$$\frac{1}{16\pi^2} \|f\|_2^4 \leq \int_{\mathbb{R}} |(x-a)f(x)|^2 dx \int_{\mathbb{R}} |(\xi - b)\widehat{f}(\xi)|^2 d\xi. \tag{1.3}$$

Equality holds if and only if $f(x) = Ce^{2\pi i b x} e^{-c(x-a)^2}$ for some $C \in \mathbb{C}$ and $c > 0$.

Note the inequality (1.3) can be written as the following equivalent form in terms of X and D. Namely, for any $f \in \mathcal{D}(X) \cap \mathcal{D}(D)$ and any $a, b \in \mathbb{R}$,

$$\frac{1}{16\pi^2} \|f\|_2^4 \leq \|(X-a)f\|_2^2 \|(D-b)f\|_2^2. \tag{1.4}$$

Proof. Here is a simple proof for Schwartz class $\mathcal{S}(\mathbb{R})$ using the canonical commutation relation. Assume $f \in \mathcal{S}(\mathbb{R})$. For every $a, b \in \mathbb{R}$, by (1.1) we have

$$\begin{aligned}
\frac{1}{2\pi i} \|f\|_2^2 &= \left\langle \frac{1}{2\pi i} f, f \right\rangle \\
&= \langle [D, X] f, f \rangle \\
&= \langle (DX - XD) f, f \rangle \\
&= \langle ((D-b)(X-a) - (X-a)(D-b)) f, f \rangle \\
&= \langle (X-a) f, (D-b) f \rangle - \langle (D-b) f, (X-a) f \rangle \\
&= 2i \mathbf{Im} \langle (X-a) f, (D-b) f \rangle.
\end{aligned}$$

Apply Cauchy-Schwartz, we get the desired inequality,

$$\begin{aligned}
\frac{1}{16\pi^2} \|f\|_2^4 &= |\mathbf{Im} \langle (X-a) f, (D-b) f \rangle|^2 \\
&\leq |\langle (X-a) f, (D-b) f \rangle|^2 \\
&\leq \|(X-a)f\|_2^2 \|(D-b)f\|_2^2. \qquad (1.5)
\end{aligned}$$

Notice equality in (1.5) holds if and only if $(D-b)f = ic(X-a)f$ for some $c \in \mathbb{R}$. This is the differential equation:

$$f' - 2\pi i b f = -2\pi c(x-a)f.$$

Its solutions are $Ce^{2\pi i b x}e^{-c(x-a)^2}$ for every $C \in \mathbb{C}$. Finally, $f \in L^2(\mathbb{R})$ requires $c > 0$. $\square$

The proof for $f \in L^2(\mathbb{R})$ is similar. Instead of using (1.1), we need the **generalized canonical commutation relation**. That is, for any $f \in \mathcal{D}(X) \cap \mathcal{D}(D)$,

$$\frac{1}{2\pi i}\|f\|_2^2 = \langle Xf, Df\rangle - \langle Df, Xf\rangle. \tag{1.6}$$

Since (1.6), X and D cannot have a common eigenvector in $L^2(\mathbb{R})$ (otherwise we would get a contradiction). And (1.4) could be viewed as a quantitive version of this argument. It tell us there is no sequence of $L^2(\mathbb{R})$ functions which approximates to and performs like a common eigenvector for X and D. However, we could find some functions in $L^2(\mathbb{R})$ which minimize the right hand side in (1.4). Naturally, we might call these functions the "best approximate common eigenvectors". And it turns out these approximants are the multiples of M_bT_ag, where $g(x) = e^{-cx^2}$ is the Gaussian function for some $c > 0$.

1.3 Classical Balian–Low Theorem

Recall the spectral theorem in functional analysis, if there are two compact self-adjoint operators on a Hilbert space commuting with each other, then we could find a sequence of common eigenvectors which forms an orthonormal basis for the Hilbert space. Go back to operators X and D, the classical uncertainty principle shows that instead of common eigenvectors, we could find some "best approximate common eigenvectors" for X and D. The natural question is: can we use these "best approximants" to form an orthonormal basis just like in the spectral theorem. Unfortunately, the answer is No!

If we set $g(x) := 1_{[0,1]}(x)$, the sequence $\{M_mT_ng\}_{m,n\in\mathbb{Z}}$ which does not consist of the "best approximants" does form an orthonormal basis of $L^2(\mathbb{R})$. However, in this case, the Fourier transform of g has a very poor localization property. The question is: can we find some $g \in L^2(\mathbb{R})$ such that $\{M_mT_ng\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis, in addition, g and $\widehat{g}$ are both well localized. Again, the answer is No! It is given by the so-called Balian-Low theorem. In order to state the theorem, we need to introduce the Gabor systems.

Definition 1.3.1. *A* ***Gabor system*** *is a sequence in* $L^2(\mathbb{R})$ *of the form* $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$*, where* $a, b > 0$ *and* $g \in L^2(\mathbb{R})$ *is a fixed non-zero function.*

The function g is called the window function or the generator. Explicitly,

$$M_{mb}T_{na}g(x) = e^{2\pi imbx}g(x - na), \quad x \in \mathbb{R}.$$

The set $\Lambda = \{(na, mb)\}_{m,n\in\mathbb{Z}}$ is called the lattice with volume ab. Actually, we could generalize the definition for high dimensional cases, i.e., we could define the Gabor system $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}^n}$ for $L^2(\mathbb{R}^n)$ in a similar way.

Now we state the classical Balian-Low theorem:

Theorem 1.3.2 ([2], Theorem 1.1)**.** *Let* $g \in L^2(\mathbb{R})$ *and let* $a, b > 0$ *satisfy* $ab = 1$*. If the Gabor system* $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is an orthonormal basis for* $L^2(\mathbb{R})$*, then*

$$\int_{\mathbb{R}} |xg(x)|^2 dx \int_{\mathbb{R}} |\xi\widehat{g}(\xi)|^2 d\xi = \infty. \tag{1.7}$$

Note the expression (1.7) can be replaced by: for any $x_0, \xi_0 \in \mathbb{R}$

$$\int_{\mathbb{R}} |(x - x_0)g(x)|^2 dx \int_{\mathbb{R}} |(\xi - \xi_0)\widehat{g}(\xi)|^2 d\xi = \infty.$$

Proof. We prove the theorem by contradiction. Suppose

$$\int_{\mathbb{R}} |xg(x)|^2 dx \int_{\mathbb{R}} |\xi\widehat{g}(\xi)|^2 d\xi < \infty.$$

Which implies $Xg, Dg \in L^2(\mathbb{R})$, then by (1.2) we have

$$\begin{aligned}
&\langle Xg, Dg\rangle \\
=&\sum_{m,n} \langle Xg, M_{mb}T_{na}g\rangle \langle M_{mb}T_{na}g, Dg\rangle \\
=&\sum_{m,n} \langle T_{-na}M_{-mb}Xg, g\rangle \langle g, T_{-na}M_{-mb}Dg\rangle \\
=&\sum_{m,n} \langle (XM_{-mb}T_{-na} + naM_{-mb}T_{-na})g, g\rangle \langle g, (DM_{-mb}T_{-na} + mbM_{-mb}T_{-na})g\rangle \\
=&\sum_{m,n} \langle XM_{-mb}T_{-na}g, g\rangle \langle g, DM_{-mb}T_{-na}g\rangle \\
=&\sum_{m,n} \langle M_{-mb}T_{-na}g, Xg\rangle \langle Dg, M_{-mb}T_{-na}g\rangle \\
=&\sum_{m,n} \langle Dg, M_{-mb}T_{-na}g\rangle \langle M_{-mb}T_{-na}g, Xg\rangle \\
=&\langle Dg, Xg\rangle .
\end{aligned}$$

Combining (1.6), we have

$$\frac{1}{2\pi i} \|g\|_2^2 = \langle Xg, Dg\rangle - \langle Dg, Xg\rangle = 0.$$

It contradicts with the fact that $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis. □

The classical Balian-Low theorem tells us that there is no well-localized window function g in both time and frequency, which generates an orthonormal basis. Furthermore, in some sense (1.7) is sharp. Actually, Benedetto et al. [1] shows that for any $\varepsilon > 0$, there exists an orthonormal basis $\{M_m T_n g\}_{m,n\in\mathbb{Z}}$ such that

$$\int_{\mathbb{R}} |g(x)|^2 \frac{1+|x|^2}{\log^{1+\varepsilon}(2+|x|)} dx \int_{\mathbb{R}} |\widehat{g}(\xi)|^2 \frac{1+|\xi|^2}{\log^{2+\varepsilon}(2+|\xi|)} d\xi < \infty.$$

1.4 Frames and Riesz Sequences

If we ask for the Gabor system to be a weaker type of basis such that the Balian-Low theorem still holds. The answer would be the Riesz basis which is slightly weaker than the orthonormal basis. Here is the definition of such basis:

Definition 1.4.1. *A sequence $\{f_k\}_{k=1}^{\infty}$ in a complex and separable Hilbert space $\mathcal{H}$ is called a* ***Riesz***

basis, *if there exists an invertible operator* $U : \mathcal{H} \to \mathcal{H}$ *such that*

$$f_k = Ue_k, \quad \forall k \in \mathbb{N},$$

where $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis of* $\mathcal{H}$.

Throughout the thesis, we always view $\mathcal{H}$ as a complex and separable Hilbert space. Just as any basis, the Riesz basis has two dual features: spanning and independence. The sequence in $\mathcal{H}$ which is usually viewed as a representative of spanning property is the so-called frame. And the sequence in $\mathcal{H}$ which is usually viewed as a representative of independence property is the so-called Riesz sequence. In this section, for completeness we will give their definitions and state their main properties. And we will not include proofs which can be found in [4, 20].

Definition 1.4.2. *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}$ *is a (discrete)* ***frame*** *of* $\mathcal{H}$*, if there exist constants* $A, B > 0$ *such that*

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.8}$$

The (optimal: maximal for A and minimal for B) constants are called the (optimal) lower and upper frame bounds. A frame is said to be **tight** if we can pick $A = B$ as frame bounds. And a frame is called a **Parseval frame** if $A = B = 1$, i.e.,

$$\|f\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2, \quad \forall f \in \mathcal{H}.$$

Definition 1.4.3. *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}$ *is said to be* ***complete*** *if*

$$\overline{span}\{f_k\}_{k=1}^{\infty} = \mathcal{H}.$$

It is not hard to see that if a sequence satisfies the inequality to the left in (1.8), then it has to be complete. As a consequence, any frame is a complete sequence. If a sequence satisfies the inequality to the right in (1.8), such sequence is said to be Bessel:

Definition 1.4.4. *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}$ *is called a* ***Bessel sequence*** *if there exists a constant* $B > 0$ *such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

The (optimal) constant B is called the (optimal) Bessel bound. By the definition, we can prove every Bessel sequence is a bounded sequence. Another equivalent definition of Bessel sequences is as the following.

Proposition 1.4.5. *A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a Bessel sequence if and only if there exist constant $B > 0$ such that*

$$\left\| \sum_{k=1}^{\infty} c_k f_k \right\|^2 \leq B \sum_{k=1}^{\infty} |c_k|^2 ,$$

for any finite complex sequence $\{c_k\}$.

Given a Bessel sequence, we could well define the following operators:

Definition 1.4.6. *Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence of $\mathcal{H}$.*

*1. The **synthesis operator** T is defined by*

$$T : l^2(\mathbb{N}) \to \mathcal{H}, \quad T\{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k f_k.$$

*2. Its adjoint operator T^* called the **analysis operator** is defined by*

$$T^* : \mathcal{H} \to l^2(\mathbb{N}), \quad T^* f := \{\langle f, f_k \rangle\}_{k=1}^{\infty}.$$

*3. The **frame operator** S is the composition of T and T^*,*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf := TT^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

Actually, $\{f_k\}_{k=1}^{\infty}$ being a Bessel sequence not only guarantees the operators are well-defined, but also make them being bounded. A very interesting fact is that if the operator T or T^* is well defined, then the sequence $\{f_k\}_{k=1}^{\infty}$ has to be Bessel.

From the definition, it is easy to see that the frame operator S is positive and self-adjoint. In addition, if $\{f_k\}_{k=1}^{\infty}$ is a frame of $\mathcal{H}$, then the frame operator S is invertible, i.e., S^{-1} exists. It follows from the definition that

$$f = \sum_{k=1}^{\infty} \left\langle f, S^{-1} f_k \right\rangle f_k, \quad \forall f \in \mathcal{H};$$
$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \quad \forall f \in \mathcal{H}.$$

Notice there are some elements $\widetilde{f}_k := S^{-1} f_k$ (for any $k \in \mathbb{N}$) appearing in the last two series. The sequence $\{\widetilde{f}_k\}_{k=1}^{\infty}$ is called the **canonical dual frame** of $\{f_k\}_{k=1}^{\infty}$. And one could show the canonical dual frame is also a frame of $\mathcal{H}$.

Furthermore, if $\{f_k\}_{k=1}^{\infty}$ is a Parseval frame, then $S = I$. So the canonical dual frame is itself, i.e., $\widetilde{f}_k = f_k$ for any $k \in \mathbb{N}$. In this case

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

Definition 1.4.7. *A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a **Riesz sequence**, if there exist constants $A, B > 0$ such that*

$$A \sum_{k=1}^{\infty} |c_k|^2 \leq \left\| \sum_{k=1}^{\infty} c_k f_k \right\|^2 \leq B \sum_{k=1}^{\infty} |c_k|^2, \tag{1.9}$$

for any finite complex sequence $\{c_k\}$.

The (optimal) constants A and B are called the (optimal) lower and upper Riesz bounds.

By Proposition 1.4.5, we have already known a sequence satisfies the inequality to the right in (1.9) is a Bessel sequence. Now we introduce another important sequence which satisfies the inequality to the left in (1.9), such sequence is the so-called Riesz-Fischer sequence.

Definition 1.4.8. *A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a **Riesz-Fischer sequence** if there exists a constant $A > 0$ such that*

$$A \sum_{k=1}^{\infty} |c_k|^2 \leq \left\| \sum_{k=1}^{\infty} c_k f_k \right\|^2, \tag{1.10}$$

for any finite complex sequence $\{c_k\}$.

By previous discussions, we have an equivalent definition of Riesz sequences.

Proposition 1.4.9. *$\{f_k\}_{k=1}^{\infty}$ is a Riesz sequence of $\mathcal{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is not only a Bessel sequence but also a Riesz-Fischer sequence of $\mathcal{H}$.*

About Riesz-Fischer sequences, we also have the following two useful equivalent definitions. And they will be applied in Chapter 3.

Proposition 1.4.10. *$\{f_k\}_{k=1}^{\infty}$ is a Riesz-Fischer sequence of $\mathcal{H}$ if and only if the moment problem of sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is solvable, i.e., there exists an element $f \in \mathcal{H}$ such that*

$$\langle f, f_k \rangle = c_k, \quad \forall k \in \mathbb{N},$$

for any sequence $\{c_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$.

Proposition 1.4.11. *$\{f_k\}_{k=1}^{\infty}$ is a Riesz-Fischer sequence of $\mathcal{H}$ if and only if there exists a Bessel sequence $\{g_k\}_{k=1}^{\infty}$ of $\mathcal{H}$ such that*

$$\langle f_k, g_j \rangle = \delta_{kj}, \quad \forall k, j \in \mathbb{N},$$

where δ_{kj} is the Kronecker delta symbol.

Definition 1.4.12. *Two sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ are said to be* ***biorthogonal*** *if*

$$\langle f_k, g_j \rangle = \delta_{kj}, \quad \forall k, j \in \mathbb{N}.$$

And a sequence that admits a biorthogonal sequence will be called ***minimal****.*

By Proposition 1.4.11, every Riesz-Fischer sequence is a biorghogonal sequence (or a minimal sequence). Go back to the Riesz basis, we have the following two important propositions:

Proposition 1.4.13. *A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a Riesz basis if and only if it is a complete Riesz sequence.*

Proposition 1.4.14. *A Riesz basis $\{f_k\}_{k=1}^{\infty}$ of $\mathcal{H}$ is a frame. And its canonical dual frame $\{\widetilde{f_k}\}_{k=1}^{\infty}$ is also a Riesz basis. In addition, $\{f_k\}_{k=1}^{\infty}$ and $\{\widetilde{f_k}\}_{k=1}^{\infty}$ are biorthogonal sequences.*

Chapter 2

Continuous Frames and Toeplitz Operators in Framed Hilbert Spaces

2.1 Continuous Frames and Examples

In Chapter 2, we will introduce a very important concept for the thesis which is the so-called "continuous frame", and make some preparations for the rest of the chapters.

2.1.1 Continuous Frames

Let $\mathcal{H}$ be a Hilbert space.

Definition 2.1.1. *A* ***continuous frame*** *of $\mathcal{H}$ is a family of elements $\{k_x\}_{x\in(X,\sigma)}$ indexed on a measure space (X,σ), such that*

1. σ is a σ-finite positive measure;

2. $x \to \langle f, k_x\rangle$ is a measurable function on X for any $f \in \mathcal{H}$;

3. there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \int_X |\langle f, k_x\rangle|^2 \, d\sigma(x) \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Throughout the thesis, we may omit the measure σ for some continuous frame $\{k_x\}_{x\in X}$, when doing this will not cause any doubt.

A continuous frame $\{k_x\}_{x\in(X,\sigma)}$ is said to be a **continuous Parseval frame** if $A = B = 1$. Namely,

$$\|f\|^2 = \int_X |\langle f, k_x\rangle|^2 \, \sigma(x), \quad \forall f \in \mathcal{H}.$$

And a continuous frame $\{k_x\}_{x\in X}$ is said to be a **normalized continuous frame** if

$$\|k_x\| = 1, \quad \forall x \in X.$$

Note that if the measure space $X = \mathbb{N}$ with σ being the counting measure, then the continuous frame $\{k_x\}_{x\in\mathbb{N}}$ becomes a discrete frame (see 1.4.2). So continuous frames are the generalization of discrete frames, and some mathematicians use the phrase the generalized frame instead of the continuous frame.

Proposition 2.1.2. *Let $\{k_x\}_{x\in(X,\sigma)}$ be a continuous frame of $\mathcal{H}$, then $\{k_x\}_{x\in(X,\sigma)}$ is complete, i.e.,*

$$\overline{span}\{k_x\}_{x\in X} = \mathcal{H}.$$

Definition 2.1.3. *Let $\{k_x\}_{x\in(X,\sigma)}$ be a continuous frame of $\mathcal{H}$, the **frame operator** $S : \mathcal{H} \to \mathcal{H}$ is given by*

$$Sf := \int_X \langle f, k_x\rangle \, k_x d\sigma(x),$$

where the right-hand side is defined in the weak sense, i.e., as the unique element in $\mathcal{H}$ such that

$$\langle Sf, g\rangle = \int_X \langle f, k_x\rangle \langle k_x, g\rangle \, d\sigma(x), \quad \forall g \in \mathcal{H}.$$

Just like the discrete case, the frame operator is invertible for continuous frames. Let $\widetilde{k}_x := S^{-1}k_x$ for any $x \in X$. The family of elements $\{\widetilde{k}_x\}_{x\in X}$ is called the **canonical dual frame** of $\{k_x\}_{x\in X}$. In addition, the canonical dual frame is also a continuous frame of $\mathcal{H}$.

Again, if $\{k_x\}_{x\in X}$ is a continuous Parseval frame, the corresponding frame operator is the identity map $I : \mathcal{H} \to \mathcal{H}$. Then the canonical dual frame is itself, i.e., $\widetilde{k}_x = k_x$ for any $x \in X$. In this case

$$f = \int_X \langle f, k_x\rangle\, k_x d\sigma(x), \quad \forall f \in \mathcal{H}.$$

2.1.2 Continuous Gabor Frames

One example of continuous frames is the continuous Gabor frame.

Definition 2.1.4. *A* ***continuous Gabor frame*** *is a family in $L^2(\mathbb{R}^n)$ of the form $\{M_bT_ag\}_{(a,b)\in\mathbb{R}^{2n}}$, where $g \in L^2(\mathbb{R}^n)$ is a fixed non-zero function.*

Proposition 2.1.5. *The continuous Gabor frame $\{M_bT_ag\}_{(a,b)\in(\mathbb{R}^{2n},m)}$ is a continuous frame of $L^2(\mathbb{R}^n)$, where m is the Lebesgue measure on $\mathbb{R}^{2n}$. If $\|g\|_2 = 1$, then $\{M_bT_ag\}_{(a,b)\in\mathbb{R}^{2n}}$ is a normalized continuous Parseval frame of $L^2(\mathbb{R}^n)$.*

Proof. We give the following proof for $n = 1$. In high dimensional case, the proof is similar. For any $f \in L^2(\mathbb{R})$,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle f, M_bT_ag\rangle|^2\, dbda \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\int_{\mathbb{R}} f(x)\overline{e^{2\pi ibx}g(x-a)}dx\right|^2 dbda \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\int_{\mathbb{R}} f(x)\overline{g(x-a)}e^{-2\pi ibx}dx\right|^2 dbda \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{F}\left(f(\cdot)\overline{g(\cdot-a)}\right)(b)\right|^2 dbda \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(b)\overline{g(b-a)}\right|^2 dbda \\
=&\int_{\mathbb{R}}\int_{\mathbb{R}} |f(b)g(b-a)|^2\, dadb \\
=&\int_{\mathbb{R}} |f(b)|^2 \int_{\mathbb{R}} |g(b-a)|^2\, dadb \\
=&\int_{\mathbb{R}} |f(b)|^2 \int_{\mathbb{R}} |g(a)|^2\, dadb \\
=&\|g\|_2^2\, \|f\|_2^2\,.
\end{aligned} \tag{2.1}$$

Denote $\|g\|_2^2$ by A, then (2.1) shows that

$$A\,\|f\|_2^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |\left\langle f, M_b T_a g \right\rangle|^2 dbda, \quad \forall f \in L^2(\mathbb{R}).$$

So $\{M_b T_a g\}_{(a,b)\in(\mathbb{R}^2,m)}$ is a continuous frame of $L^2(\mathbb{R})$.

Note if $\|g\|_2 = 1$, we have $\|M_b T_a g\|_2 = 1$ for every $(a,b) \in \mathbb{R}^2$. In addition,

$$\|f\|_2^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle f, M_b T_a g\rangle|^2\, dbda, \quad \forall f \in L^2(\mathbb{R}).$$

In this case, $\{M_b T_a g\}_{(a,b)\in\mathbb{R}^2}$ is a normalized continuous Parseval frame of $L^2(\mathbb{R})$. □

2.1.3 Continuous Exponential Frames

Another example of continuous frames is the exponential functions, which is very familiar to us but easy to ignore as a continuous frame.

Proposition 2.1.6. *Let E be a subset of $\mathbb{R}$ with finite Lebesgue measure, then the exponential functions $\{e^{2\pi i \xi x}\}_{\xi\in(\mathbb{R},m)}$ is a continuous Parseval frame of $L^2(E)$. If $m(E) = 1$, then $\{e^{2\pi i \xi x}\}_{\xi\in(\mathbb{R},m)}$ is a normalized continuous Parseval frame.*

Proof. Let f be any function in $L^2(E)$, and we view f as a $L^2(\mathbb{R})$ function supported on E. Notice

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \left\langle f, e^{2\pi i \xi (\cdot)} \right\rangle_{L^2(E)} \right|^2 d\xi \\
&= \int_{\mathbb{R}} \left| \int_E f(x) e^{-2\pi i \xi x} dx \right|^2 d\xi \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx \right|^2 d\xi \\
&= \int_{\mathbb{R}} \left| \widehat{f}(\xi) \right|^2 d\xi \\
&= \int_{\mathbb{R}} |f(x)|^2\, dx \\
&= \int_E |f(x)|^2\, dx \\
&= \|f\|_{L^2(E)}^2 .
\end{aligned}$$

It follows that $\{e^{2\pi i\xi x}\}_{\xi\in(\mathbb{R},m)}$ is a continuous Parseval frame of $L^2(E)$. Furthermore, if $m(E)=1$,

$$\left\|e^{2\pi i\xi(\cdot)}\right\|^2_{L^2(E)} = \int_E \left|e^{2\pi i\xi x}\right|^2 dx = m(E) = 1.$$

Then, $\{e^{2\pi i\xi x}\}_{\xi\in(\mathbb{R},m)}$ is a normalized continuous Parseval frame. □

2.2 Framed Hilbert Spaces and Examples

2.2.1 Framed Hilbert Spaces

Definition 2.2.1. *A* ***framed Hilbert space*** *(FHS) is a pair* $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$*, where* $\mathcal{H}$ *is a complex Hilbert space, and* $\{k_x\}_{x\in(X,\sigma)}$ *is a normalized continuous Parsevel frame of* $\mathcal{H}$*.*

Actually, one can view a *FHS* as another more familiar space, which is the so-called reproducing kernel Hilbert space.

Definition 2.2.2. *Let* X *be an arbitrary set. A Hilbert space* $\mathcal{H}$ *consisting of functions* $f : X \to \mathbb{C}$ *is said to be a* ***reproducing kernel Hilbert space*** *(RKHS) if for any* $x \in X$*, the evaluation functional* δ_x *given by*

$$\delta_x : \mathcal{H} \to \mathbb{C}, \quad \delta_x(f) := f(x),$$

is bounded.

By Riesz representation theorem, for any $x \in X$, there exists an unique element $K_x \in \mathcal{H}$ such that

$$f(x) = \langle f, K_x\rangle\,, \quad \forall f \in \mathcal{H}.$$

The collection of elements $\{K_x\}_{x\in X}$ is called the **reproducing kernel** of $\mathcal{H}$; the element K_x is called the reproducing kernel of $\mathcal{H}$ at point x. And the collection $\{k_x := K_x/\left\|K_x\right\|\}_{x\in X}$ is called the **normalized reproducing kernel** of $\mathcal{H}$; the element k_x is called the normalized reproducing kernel of $\mathcal{H}$ at point x.

The following two propositions show that every *RKHS* which is embedded in a $L^2(X,\gamma)$ for some positive measure γ is a *FHS* and vice versa.

Proposition 2.2.3. *Let* $\mathcal{H} \subseteq L^2(X,\gamma)$ *be a RKHS, and* $\{K_x\}_{x\in X}$ *be its reproducing kernel. Then*

$(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ forms a FHS, where $d\sigma(x) := \|K_x\|^2 \, d\gamma(x)$, and $\{k_x\}_{x\in X}$ is the normalized reproducing kernel.

Proof. Obviously, $\|k_x\| = 1$ for any $x \in X$. In addition,

$$\begin{aligned}
\|f\|^2 &= \int_X |f(x)|^2 \, d\gamma(x) \\
&= \int_X |\langle f, K_x\rangle|^2 \, d\gamma(x) \\
&= \int_X |\langle f, k_x\rangle|^2 \|K_x\|^2 \, d\gamma(x) \\
&= \int_X |\langle f, k_x\rangle|^2 \, d\sigma(x),
\end{aligned}$$

for any $f \in \mathcal{H}$. It implies $\{k_x\}_{x\in(X,\sigma)}$ is a normalized continuous Parsevel frame, and $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ forms a *FHS*. □

Proposition 2.2.4. *Let $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ be a framed Hilbert space. Then $\mathcal{H}$ can be viewed as a RKHS embedded in $L^2(X, \sigma)$.*

Proof. Define the space $\widetilde{\mathcal{H}}$ by

$$\widetilde{\mathcal{H}} := \{\widetilde{f} : X \to \mathbb{C} | \ \widetilde{f}(x) := \langle f, k_x\rangle \, , f \in \mathcal{H}\}.$$

Notice for any $f \in \mathcal{H}$,

$$\begin{aligned}
\|\widetilde{f}\|_2^2 :&= \int_X \left|\widetilde{f}(x)\right|^2 d\sigma(x) \\
&= \int_X |\langle f, k_x\rangle|^2 \, d\sigma(x) \\
&= \|f\|^2 .
\end{aligned}$$

So the linear map $f \to \widetilde{f}$ is an isometry from $\mathcal{H}$ onto $\widetilde{\mathcal{H}} \subseteq L^2(X, \sigma)$. Then the map $f \to \widetilde{f}$ is an unitary operator from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$. Notice for any $x \in X$, the evaluation functional δ_x on $\widetilde{\mathcal{H}}$ satisfies the boundedness property:

$$\begin{aligned}\left|\delta_x(\widetilde{f})\right| &:= \left|\widetilde{f}(x)\right| \\ &= |\langle f, k_x\rangle| \\ &\le \|f\|\,\|k_x\| \\ &= \|f\| \\ &= \|\widetilde{f}\|_2,\end{aligned}$$

for any $\widetilde{f} \in \widetilde{\mathcal{H}}$. By δ_x is bounded, $\widetilde{\mathcal{H}}$ is a RKHS. Since $f \to \widetilde{f}$ is unitary, for any $x \in X$ we have

$$\widetilde{f}(x) = \langle f, k_x\rangle = \left\langle \widetilde{f}, \widetilde{k}_x \right\rangle_2, \quad \forall \widetilde{f} \in \widetilde{\mathcal{H}}.$$

So $\{\widetilde{k}_x\}_{x\in X}$ is the reproducing kernel of $\widetilde{\mathcal{H}}$. □

Now we are going to list some classical examples of *FHS*.

2.2.2 $L^2(\mathbb{R}^n)$ Space with Continuous Gabor Frame

Define $L^2(\mathbb{R}^n)$ as the Hilbert space as usual. And let $\{M_bT_ag\}_{(a,b)\in\mathbb{R}^{2n}}$ be the continuous Gabor frame with $\|g\|_2 = 1$, i.e., for any $(a,b) \in \mathbb{R}^{2n}$

$$M_bT_ag(x) = e^{2\pi i b\cdot x}g(x-a), \quad \forall x \in \mathbb{R}^n.$$

By Proposition 2.1.5, $(L^2(\mathbb{R}^n), \{M_bT_ag\}_{(a,b)\in(\mathbb{R}^{2n},m)})$ is a *FHS*.

2.2.3 Classical Bargmann-Fock Spaces

The **classical Bargmann-Fock space** $\mathcal{F}_\alpha(\mathbb{C}^n)$ with the parameter $\alpha > 0$ is the space of all entire functions $f : \mathbb{C}^n \to \mathbb{C}$ satisfying the following integrability condition

$$\|f\|_\alpha^2 := \frac{\alpha^n}{\pi^n}\int_{\mathbb{C}^n} |f(z)|^2\, e^{-\alpha|z|^2}dm(z) < \infty,$$

where m is the Lebesgue measure on $\mathbb{C}^n$. It is known that $\mathcal{F}_\alpha(\mathbb{C}^n)$ is a reproducing kernel Hilbert space. Its reproducing kernel at point $z \in \mathbb{C}^n$ equals

$$K_z^\alpha(w) = e^{\alpha\langle w,z\rangle}, \quad \forall w \in \mathbb{C}^n;$$

and the normalized reproducing kernel at point z equals

$$k_z^\alpha(w) = e^{\alpha\langle w,z\rangle - \frac{\alpha}{2}|z|^2}, \quad \forall w \in \mathbb{C}^n.$$

Define measure m_α by

$$dm_\alpha(z) := \|K_z\|_\alpha^2 \frac{\alpha^n}{\pi^n} e^{-\alpha|z|^2} dm(z) = \frac{\alpha^n}{\pi^n} dm(z),$$

then the classical Bargmann-Fock space $(\mathcal{F}_\alpha, \{k_z^\alpha\}_{z\in(\mathbb{C}^n,m_\alpha)})$ forms a framed Hilbert space.

When $\alpha = \pi$, by convention, we will drop the sub(super)scripts α in the above notations. We will use $\|\cdot\|$, $K_z(w)$, and $k_z(w)$ to denote the norm, the reproducing kernel, and the normalized reproducing kernel of $\mathcal{F}(\mathbb{C}^n)$ at z in this case.

Because there is an unitary operator $\mathcal{B}$ (the *Bargmann transform*) from $L^2(\mathbb{R}^n)$ onto $\mathcal{F}(\mathbb{C}^n)$, which maps the continuous Gabor frame $\{M_bT_ag\}_{(a,b)\in\mathbb{R}^{2n}}$ generated by the normalized Gaussian window function g onto the normalized reproducing kernel $\{k_z\}_{z\in\mathbb{C}^n}$ of $\mathcal{F}(\mathbb{C}^n)$ (up to some complex numbers with modulus 1). One could view the classical Bargmann-Fock space $(\mathcal{F}_\alpha, \{k_z^\alpha\}_{z\in(\mathbb{C}^n,m_\alpha)})$ as a special case of the *FHS* $(L^2(\mathbb{R}^n), \{M_bT_ag\}_{(a,b)\in(\mathbb{R}^{2n},m)})$, that is when g is the normalized Gaussian function.

2.2.4 General Paley-Wiener Spaces

Let E be a subset of $\mathbb{R}$ with finite Lebesgue measure, we define the **general Paley-Wiener space** as the following

$$L^2(E) := \{f \in L^2(\mathbb{R}) : \ supp f \subseteq E\}.$$

let $\{\frac{e^{2\pi i\xi x}}{\sqrt{m(E)}}\}_{\xi\in\mathbb{R}}$ be the $\frac{1}{\sqrt{m(E)}}$ multiple of exponential functions. As shown in Proposition 2.1.6, we obtain $\{\frac{e^{2\pi i\xi x}}{\sqrt{m(E)}}\}_{\xi\in(\mathbb{R},m_E)}$ is a normalized continuous Parseval frame of $L^2(E)$, where $dm_E := m(E)dm$ is the $m(E)$ multiple of the Lebesgue measure.

The general Paley-Wiener space $(L^2(E), \{\frac{e^{2\pi i\xi x}}{\sqrt{m(E)}}\}_{\xi\in(\mathbb{R},m_E)})$ forms a *FHS.*

2.2.5 Classical Paley-Wiener Spaces

The **classical Paley-Wiener space** $\mathcal{PW}_\alpha$ with the parameter $\alpha > 0$ is given by

$$\mathcal{PW}_\alpha := \{f \in L^2(\mathbb{R}, \|\cdot\|_2) : \ supp\widehat{f} \subseteq [-\alpha, \alpha]\}.$$

$\mathcal{PW}_\alpha$ is a *RKHS* as well. Its reproducing kernel at point $x \in \mathbb{R}$ equals

$$K_x^\alpha(y) = \frac{\sin\alpha(y-x)}{\pi(y-x)}, \quad \forall y \in \mathbb{R};$$

and the normalized reproducing kernel at x equals

$$k_x^\alpha(y) = \frac{\sqrt{\alpha}}{\sqrt{\pi}}\frac{\sin\alpha(y-x)}{\alpha(y-x)}, \quad \forall y \in \mathbb{R}.$$

Define measure m_α by

$$dm_\alpha(x) := \|K_x^\alpha\|_2^2 \, dm(x) = \frac{\alpha}{\pi}dm(x),$$

Then the classical Paley-Wiener space $(\mathcal{PW}_\alpha, \{k_x^\alpha\}_{x\in(\mathbb{R},m_\alpha)})$ forms a *FHS.*

Because the Fourier transform $\mathcal{F}$ is an unitary operator from $\mathcal{PW}_\alpha$ onto $L^2[-\alpha,\alpha]$, which maps the normalized reproducing kernel $\{k_x^\alpha\}_{x\in\mathbb{R}}$ of $\mathcal{PW}_\alpha$ onto the continuous frame $\{\frac{e^{2\pi i\xi x}}{\sqrt{2\alpha}}\}_{\xi\in\mathbb{R}}$ of $L^2[-\alpha,\alpha]$. One could view the space $(L^2[-\alpha,\alpha], \{\frac{e^{2\pi i\xi x}}{\sqrt{2\alpha}}\}_{\xi\in(\mathbb{R},m_{[-\alpha,\alpha]})})$ as the classical Paley-Wiener space $(\mathcal{PW}_\alpha, \{k_x^\alpha\}_{x\in(\mathbb{R},m_\alpha)})$. In addition, for any finite interval $I \subseteq \mathbb{R}$ with length 2α, there exists a translation operator mapping from $L^2[-\alpha,\alpha]$ onto $L^2(I)$, which also maps $\{\frac{e^{2\pi i\xi x}}{\sqrt{2\alpha}}\}_{\xi\in\mathbb{R}}$ onto $\{\frac{e^{2\pi i\xi x}}{\sqrt{m(I)}}\}_{\xi\in\mathbb{R}}$ up to some constants with modulus 1. Based on these arguments, we could also view the space $(L^2(I), \{\frac{e^{2\pi i\xi x}}{\sqrt{m(I)}}\}_{\xi\in(\mathbb{R},|\cdot|,m_I)})$ as a classical Paley-Wiener space.

2.3 Toeplitz Operators on Framed Hilbert Spaces

2.3.1 Toeplitz Operators

Let $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ be a *FHS.*

Definition 2.3.1. *Let μ be a σ-finite positive measure on the same measurable sets with σ, such* *that*

$$\int_X |\langle k_x, k_y \rangle| \, d\mu(y) < \infty, \quad \forall x \in X.$$

We densely define the ***Toeplitz operator*** T_μ *of* $\{k_x\}_{x\in X}$ *with symbol* μ *by*

$$T_\mu : span\{k_x\}_{x\in X} \to \mathcal{H}, \quad T_\mu f := \int_X \langle f, k_x \rangle \, k_x d\mu(x),$$

where the right-hand side is defined in the weak sense.

Proposition 2.3.2. *The Toeplitz operator* T_μ *with symbol* μ *is a positive symmetric linear operator.*

Proof. Obviously, T_μ is linear.

1. T_μ is positive:

$$\begin{aligned} \langle T_\mu f, f \rangle &= \int_X \langle f, k_x \rangle \langle k_x, f \rangle \, d\mu(x) \\ &= \int_X |\langle f, k_x \rangle|^2 \, d\mu(x) \\ &\geq 0, \end{aligned}$$

   for any $f \in span\{k_x\}_{x\in X}$.

2. T_μ is symmetric:

$$\begin{aligned} \langle T_\mu f, g \rangle &= \int_X \langle f, k_x \rangle \langle k_x, g \rangle \, d\mu(x) \\ &= \overline{\int_X \langle g, k_x \rangle \langle k_x, f \rangle \, d\mu(x)} \\ &= \overline{\langle T_\mu g, f \rangle} \\ &= \langle f, T_\mu g \rangle, \end{aligned}$$

   for any $f, g \in span\{k_x\}_{x\in X}$.

□

Proposition 2.3.3. *Let* T_μ *be a Toeplitz operator with a finite measure* μ. *Then* T_μ *can be extend to a positive bounded self-adjoint linear operator on* $\mathcal{H}$.

Proof. By Proposition 2.3.2, we only need to show the boundedness of T_μ on $span\{k_x\}_{x\in X}$. For any $f \in span\{k_x\}_{x\in X}$, we have

$$\begin{aligned} \|T_\mu f\|^2 &= \sup_{\|g\|=1} |\langle T_\mu, g\rangle|^2 \\ &= \sup_{\|g\|=1} \left| \int_X \langle f, k_x\rangle \langle k_x, g\rangle \, d\mu(x) \right|^2 \\ &\leq \sup_{\|g\|=1} \int_X |\langle f, k_x\rangle|^2 \, d\mu(x) \int_X |\langle k_x, g\rangle|^2 \, d\mu(x) \\ &\leq \sup_{\|g\|=1} \int_X \|f\|^2 \|k_x\|^2 \, d\mu(x) \int_X \|k_x\|^2 \|g\|^2 \, d\mu(x) \\ &= \sup_{\|g\|=1} \mu(X)^2 \|f\|^2 \|g\|^2 \\ &= \mu(X)^2 \|f\|^2 . \end{aligned}$$

□

Now we could view the Toeplitz operator T_μ as a bounded operator on the whole $\mathcal{H}$, if we impose the assumption that μ is a finite measure.

Proposition 2.3.4. *Let T_μ be a Toeplitz operator with a finite measure μ. Then T_μ is in the trace class, and its trace and Hilbert-Schmidt norm satisfy the following identities:*

$$\|T_\mu\|_{Tr} = \mu(X) = \int_X \int_X |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\mu(x); \tag{2.2}$$

$$\|T_\mu\|_{HS}^2 = \int_X \int_X |\langle k_x, k_y\rangle|^2 \, d\mu(y) d\mu(x). \tag{2.3}$$

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. On one hand,

$$\begin{aligned}
\|T_\mu\|_{Tr} &= Tr(|T_\mu|) \\
&= Tr(T_\mu) \\
&= \sum_{n=1}^{\infty} \langle T_\mu e_n, e_n \rangle \\
&= \sum_{n=1}^{\infty} \int_X |\langle e_n, k_x \rangle|^2 \, d\mu(x) \\
&= \int_X \sum_{n=1}^{\infty} |\langle e_n, k_x \rangle|^2 \, d\mu(x) \\
&= \int_X \|k_x\|^2 \, d\mu(x) \\
&= \mu(X) < \infty.
\end{aligned}$$

Which implies T_μ is in the trace class, and $\|T_\mu\|_{Tr} = \mu(X)$. Then (2.2) comes from the following

$$\begin{aligned}
&\int_X \int_X |\langle k_x, k_y \rangle|^2 \, d\sigma(y) d\mu(x) \\
=&\int_X \|k_x\|^2 \, d\mu(x) \\
=&\mu(X).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\|T_\mu\|_{HS}^2 &= Tr(T_\mu^* T_\mu) \\
&= \sum_{n=1}^{\infty} \left\langle T_\mu^* T_\mu e_n, e_n \right\rangle \\
&= \sum_{n=1}^{\infty} \langle T_\mu e_n, T_\mu e_n \rangle \\
&= \sum_{n=1}^{\infty} \left\langle \int_X \langle e_n, k_x \rangle k_x d\mu(x), \int_X \langle e_n, k_y \rangle k_y d\mu(y) \right\rangle \\
&= \sum_{n=1}^{\infty} \int_X \langle e_n, k_x \rangle \left\langle k_x, \int_X \langle e_n, k_y \rangle k_y d\mu(y) \right\rangle d\mu(x) \\
&= \sum_{n=1}^{\infty} \int_X \langle e_n, k_x \rangle \int_X \langle k_y, e_n \rangle \langle k_x, k_y \rangle d\mu(y) d\sigma(x) \\
&= \sum_{n=1}^{\infty} \int_X \int_X \langle k_y, e_n \rangle \langle e_n, k_x \rangle \langle k_x, k_y \rangle d\mu(y) d\mu(x) \\
&= \int_X \int_X \sum_{n=1}^{\infty} \langle k_y, e_n \rangle \langle e_n, k_x \rangle \langle k_x, k_y \rangle d\mu(y) d\mu(x) \\
&= \int_X \int_X \langle k_y, k_x \rangle \langle k_x, k_y \rangle d\mu(y) d\mu(x) \\
&= \int_X \int_X |\langle k_x, k_y \rangle|^2 d\mu(y) d\mu(x).
\end{aligned}
$$

□

In general, it is not that easy to characterize the boundedness, compactness and the trace class membership for Toeplitz operators without the assumption $\mu(X) < \infty$. We will continue to do the characterizations later.

2.3.2 Concentration Operators

Let $(\mathcal{H}, \{k_x\}_{x \in (X,\sigma)})$ be a *FHS*, we now introduce a very special Toeplitz operator by letting the measure $d\mu := 1_E d\sigma$ for some measurable subset E of X, such operator is called the concentration operator.

Definition 2.3.5. *Let E be a measurable subset of X. Define the* ***concentration operator*** T_E *over E by*

$$
T_E : \mathcal{H} \to \mathcal{H}, \quad T_E f := \int_E \langle f, k_x \rangle k_x d\sigma(x),
$$

where the right-hand side is defined in the weak sense.

Proposition 2.3.6. *Let T_E be a concentration operator over the set E, then T_E is bounded with $\|T_E\| \leq 1$.*

Proof. For any $f \in \mathcal{H}$, we have

$$\begin{aligned}
\|T_E f\|^2 &= \sup_{\|g\|=1} |\langle T_E, g\rangle|^2 \\
&= \sup_{\|g\|=1} \left| \int_E \langle f, k_x\rangle \langle k_x, g\rangle \, d\sigma(x) \right|^2 \\
&\leq \sup_{\|g\|=1} \int_E |\langle f, k_x\rangle|^2 \, d\sigma(x) \int_E |\langle k_x, g\rangle|^2 \, d\sigma(x) \\
&\leq \sup_{\|g\|=1} \int_X |\langle f, k_x\rangle|^2 \, d\sigma(x) \int_X |\langle k_x, g\rangle|^2 \, d\sigma(x) \\
&= \sup_{\|g\|=1} \|f\|^2 \|g\|^2 \\
&= \|f\|^2 .
\end{aligned}$$

So $\|T_E\| \leq 1$. □

Proposition 2.3.6 shows that as a special Toeplitz operator, the concentration operator T_E (equals T_μ, $d\mu := 1_E d\sigma$) will always being bounded. In addition, if we assume $\sigma(E)$ is finite, we have $\mu(X) = \sigma(E) < \infty$. By Proposition 2.3.3 and 2.3.4, we obtain the following corollary.

Corollary 2.3.7. *Let T_E be the concentration operator over the set E with $\sigma(E) < \infty$. Then T_E is a positive compact self-adjoint operator, and its trace and Hilbert-Schmidt norm satisfy the following identities:*

$$\|T_E\|_{Tr} = \sigma(E) = \int_E \int_X |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\sigma(x); \tag{2.4}$$

$$\|T_E\|_{HS}^2 = \int_E \int_E |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\sigma(x). \tag{2.5}$$

Let E be a finite measure set, Corollary 2.3.7 tell us the concentration operator T_E is a positive compact self-adjoint operator. Combine Proposition 2.3.6, we know all the eigenvalues of T_E are belonged to the interval $[0, 1]$.

Definition 2.3.8. *Let E be a measurable subset of X with $\sigma(E) < \infty$. Given a number $0 < c < 1$,* *we say that a subspace $\mathcal{G}$ of $\mathcal{H}$ is c-***concentrated** *on set E if*

$$c\|f\|^2 \leq \int_E |\langle f, k_x\rangle|^2 \, d\sigma(x),$$

for every $f \in \mathcal{G}$.

Lemma 2.3.9. *Let E be a Borel subset of X with $\sigma(E) < \infty$. Given a number $0 < c < 1$, if $\mathcal{G}$ is a subspace of $\mathcal{H}$ which is c-concentrated on E, then*

$$dim\mathcal{G} \leq \frac{\sigma(E)}{c}.$$

Proof. Let T_E be the concentration operator over E. Denote all the eigenvalues of T_E in the decreasing order by $1 \geq l_1 \geq l_2 \geq \cdots \geq 0$, where entries are repeated with multiplicity.

Let $\mathcal{G}'$ be an arbitrary finite-dimensional subspace of $\mathcal{G}$ and $k = dim\mathcal{G}'$. By the min-max principle,

$$\begin{aligned}
l_k &= \max_{dim\mathcal{H}=k} \min_{f\in\mathcal{H}, \|f\|=1} \langle T_E f, f\rangle \\
&\geq \min_{f\in\mathcal{G}', \|f\|=1} \langle T_E f, f\rangle \\
&= \min_{f\in\mathcal{G}', \|f\|=1} \int_E |\langle f, k_x\rangle|^2 \, d\sigma(x) \\
&\geq \min_{f\in\mathcal{G}', \|f\|=1} c\|f\|^2 \\
&= c.
\end{aligned}$$

Combine (2.4) we have

$$\sigma(E) = \|T_E\|_{Tr} = Tr(T_E) = \sum_{j=1}^{\infty} l_j \geq kl_k \geq kc.$$

So we obtain

$$dim\mathcal{G}' = k \leq \frac{\sigma(E)}{c}.$$

Since $\mathcal{G}'$ was an arbitrary finite-dimensional subspace of $\mathcal{G}$, we obtain that $\mathcal{G}$ is finite-dimensional and the same estimate of the dimension holds for $\mathcal{G}$. □

Lemma 2.3.9 shows that, given a finite measure set E, we could find a "inverse" relation between the dimension of a concentrated subspace $\mathcal{G}$ on E and its concentration level c. Notice, from the proof, if we knew the distribution of eigenvalues of T_E, we could get a more precise relation between $dim\mathcal{G}$ and c. We will see it later.

2.4 Additional Conditions on Framed Hilbert Spaces

In this section, we will impose additional conditions on *FHS* to be able to obtain additional properties. All these conditions are natural, and almost all of them are satisfied in the previous examples.

2.4.1 Framed Hilbert Spaces with Metric

Definition 2.4.1. *We say* $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ *is a* ***framed Hilbert space with metric*** d*, if* $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ *is a FHS and* σ *is a Borel measure with respect to the imposed metric* d *on* X*.*

Recall the examples of *FHS* in section 2.2,

1. The space $(L^2(\mathbb{R}^n), \{M_bT_ag\}_{(a,b)\in(\mathbb{R}^{2n},m)})$,
2. the classical Bargmann-Fock space $(\mathcal{F}_\alpha, \{k_z^\alpha\}_{z\in(\mathbb{C}^n,m_\alpha)})$,
3. the general Paley-Wiener space $(L^2(E), \{\frac{e^{2\pi i\xi x}}{\sqrt{m(E)}}\}_{\xi\in(\mathbb{R},m_E)})$,
4. the classical Paley-Wiener space $(\mathcal{PW}_\alpha, \{k_x^\alpha\}_{x\in(\mathbb{R},m_\alpha)})$

Because all of the measures in these *FHS* are the Borel measure with respect to the Euclidean metric $|\cdot|$. If we impose the Euclidean metric $|\cdot|$ on their index space respectively, then all of them are the framed Hilbert space with the Euclidean metric $|\cdot|$.

For convenience, we will continue using the phrase "framed Hilbert space" or "*FHS*" to call a framed Hilbert space $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ with the metric d.

2.4.2 Additional Conditions

Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a *FHS*, and we continue to impose conditions on it.

We say the metric measure space (X, d, σ) satisfies the **doubling measure property** (DMP), if there exists a constant $C > 0$ such that

$$0 < \mu(B(a, 2r)) \leq C\mu(B(a, r)) < \infty,$$

for all $a \in X$ and $r > 0$.

Simple calculation shows that the Euclidean metric space with the Lebesgue measure $(\mathbb{R}^n, |\cdot|, m)$ satisfies DMP with $C = 2^n$.

The metric measure space (X, d, σ) is said to satisfy the **annular decay property** (ADP), if for any $\rho > 0$ we have

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a, r+\rho))}{\sigma(B(a, r))} = 1. \tag{2.6}$$

Notice (2.6) is equivalent to

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a, r+\rho)) \setminus \sigma(B(a, r))}{\sigma(B(a, r))} = 0.$$

Easily see $(\mathbb{R}^n, |\cdot|, m)$ satisfies ADP as well. For other metric measure spaces, we provide a method to verify ADP by the following proposition.

Proposition 2.4.2 ([3]). *Let (X, d, σ) be a metric measure space. If (X, d, σ) satisfies DMP and (X, d) is also a length space, then (X, d, σ) satisfies ADP.*

We say (X, d, σ) satisfies the **uniform measure distribution** (UMD), if for every $r > 0$ there exist constants $D(r) \geq c(r) > 0$ such that

$$c(r) \leq \sigma(B(x, r)) \leq D(r),$$

for any ball $B(x, r)$ in X with the radius r, where $c(r) \to \infty$ as $r \to \infty$.

By translation invariance of the Lebesgue measure, $(\mathbb{R}^n, |\cdot|, m)$ satisfies UMD.

We say $\mathcal{H}$ satisfies the **mean value property** (MVP), if for every $r > 0$ there exists $C_r > 0$ such that

$$|\langle f, k_a\rangle|^2 \leq C_r \int_{B(a,r)} |\langle f, k_x\rangle|^2 \, d\sigma(x),$$

for any $f \in \mathcal{H}$ and any $a \in X$.

Combining *MVP* with *UMD*, we have

$$|\langle f, k_a\rangle|^2 \lesssim \frac{1}{\sigma(B(a,r))}\int_{B(a,r)} |\langle f, k_x\rangle|^2 \, d\sigma(x).$$

It tell us that the averaging value of the function $x \to \langle f, k_x\rangle$ over a ball is greater or equal to the value of the center. That's the reason we borrow the terminology "mean value property" from complex analysis here. Actually, the *MVP* usually holds when the Hilbert space is an analytic function space.

We say $\{k_x\}_{x\in(X,d,\sigma)}$ satisfies the **approximate orthogonality property** (*AOP*), if the followings hold:

1. there exists $\delta > 0$ and $c > 0$ such that for any $x, y \in X$ with $d(x,y) < \delta$,

$$c \le |\langle k_x, k_y\rangle|;$$

2.

$$|\langle k_x, k_y\rangle| \to 0, \text{ as } d(x,y) \to \infty.$$

Notice *AOP* tell us that the angle between continuous frame k_x and k_y is small when x is closed to y, and the angle is big when x is far from y.

The continuous frame $\{k_x\}_{x\in(X,d,\sigma)}$ is said to satisfy the **uniform localization property** (*ULP*), if

$$\lim_{r\to\infty} \sup_{a\in X} \int_{B(a,r)^c} |\langle k_a, k_x\rangle|^2 \, d\sigma(x) = 0.$$

In the rest part of the thesis, for convenience we just say $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ satisfies *DMP*, *ADP*, *UMD*, *MVP*, *AOP* and *ULP*, instead of mentioning (X, d, σ), $\mathcal{H}$ or $\{k_x\}_{x\in(X,d,\sigma)}$ respectively.

In the previous examples of *FHS*, we have the following properties hold:

1. The space $(L^2(\mathbb{R}^n), \{M_bT_ag\}_{(a,b)\in(\mathbb{R}^{2n},|\cdot|,m)})$ satisfies *DMP*, *ADP*, *UMD* and *AOP*.

2. The classical Bargmann-Fock space $(\mathcal{F}_\alpha, \{k_z^\alpha\}_{z\in(\mathbb{C}^n,|\cdot|,m_\alpha)})$ satisfies *DMP*, *ADP*, *UMD*, *MVP*, *AOP* and *ULP*.

3. The general Paley-Wiener space $(L^2(E), \{\frac{e^{2\pi i\xi x}}{\sqrt{m(E)}}\}_{\xi\in(\mathbb{R},|\cdot|,m_E)})$ satisfies *DMP*, *ADP*, *UMD* and *AOP*.

4. The classical Paley-Wiener space $(\mathcal{PW}_\alpha, \{k_x^\alpha\}_{x\in(\mathbb{R},|\cdot|,m_\alpha)})$ satisfies *DMP*, *ADP*, *UMD*, *MVP*, *AOP* and *ULP*.

2.4.3 Additional Properties

Proposition 2.4.3. *Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a FHS, which satisfies ADP, UMD and ULP. Then*

$$\lim_{r\to\infty} \sup_{a\in X} \frac{1}{\sigma(B(a,r))} \int_{B(a,r)} \int_{B(a,r)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\sigma(x) = 0. \tag{2.7}$$

Proof. Let $\rho > 0$. We break the double integral above as follows

$$\int_{B(a,r)} \int_{B(a,r+\rho)^c} + \int_{B(a,r)} \int_{B(a,r+\rho)\setminus B(a,r)} .$$

Estimate each term separately, and in both estimates we will divide by $\sigma(B(a,r))$ ($\sigma(B(a,r)) < \infty$, by *UMD*). About the first term, we have for any $a \in X$ and $r > 0$,

$$\begin{aligned}
&\frac{1}{\sigma(B(a,r))} \int_{B(a,r)} \int_{B(a,r+\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\sigma(x) \\
\leq &\frac{1}{\sigma(B(a,r))} \int_{B(a,r)} \int_{B(x,\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\sigma(x) \\
\leq &\frac{1}{\sigma(B(a,r))} \int_{B(a,r)} \left(\sup_{x\in X} \int_{B(x,\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) \right) d\sigma(x) \\
= &\frac{1}{\sigma(B(a,r))} \sigma(B(a,r)) \sup_{x\in X} \int_{B(x,\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) \\
= &\sup_{x\in X} \int_{B(x,\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y)
\end{aligned}$$

For any $\varepsilon > 0$, by *ULP* we can find a positive ρ such that

$$\sup_{x\in X} \int_{B(x,\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) < \varepsilon.$$

It follows that for such ρ

$$\begin{aligned}
&\limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\sigma(B(a,r))} \int_{B(a,r)} \int_{B(a,r+\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) d\sigma(x) \\
\leq &\sup_{x\in X} \int_{B(x,\rho)^c} |\langle k_x, k_y\rangle|^2 \, d\sigma(y) < \varepsilon.
\end{aligned} \tag{2.8}$$

Now we estimate the second term, using the positive $\rho > 0$ from above,

$$\begin{aligned}
&\frac{1}{\sigma(B(a,r))}\int_{B(a,r)}\int_{B(a,r+\rho)\setminus B(a,r)}|\langle k_x,k_y\rangle|^2\,d\sigma(y)d\sigma(x)\\
=&\frac{1}{\sigma(B(a,r))}\int_{B(a,r+\rho)\setminus B(a,r)}\int_{B(a,r)}|\langle k_x,k_y\rangle|^2\,d\sigma(x)d\sigma(y)\\
\leq&\frac{1}{\sigma(B(a,r))}\int_{B(a,r+\rho)\setminus B(a,r)}\int_{X}|\langle k_x,k_y\rangle|^2\,d\sigma(x)d\sigma(y)\\
=&\frac{1}{\sigma(B(a,r))}\int_{B(a,r+\rho)\setminus B(a,r)}\|k_y\|^2\,d\sigma(y)\\
=&\frac{\sigma(B(a,r+\rho)\setminus B(a,r))}{\sigma(B(a,r))}.
\end{aligned}$$

By *ADP*, we obtain for such ρ

$$\begin{aligned}
&\limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\sigma(B(a,r))}\int_{B(a,r)}\int_{B(a,r+\rho)\setminus B(a,r)}|\langle k_x,k_y\rangle|^2\,d\sigma(y)d\sigma(x)\\
\leq&\limsup_{r\to\infty}\sup_{a\in X}\frac{\sigma(B(a,r+\rho)\setminus B(a,r))}{\sigma(B(a,r))}=0.
\end{aligned}\tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\limsup_{r\to\infty}\sup_{a\in X}\frac{1}{\sigma(B(a,r))}\int_{B(a,r)}\int_{B(a,r)^c}|\langle k_x,k_y\rangle|^2\,d\sigma(y)d\sigma(x)<\varepsilon.$$

Since ε is arbitrary, we get the desired equality. □

Notice (2.7) is very similar with *ULP*, for some particular spaces, one implies another. Out of the thesis, we might sometimes name (2.7) as *ULP*. As a direct consequence of Proposition 2.4.3, we have the following corollary.

Corollary 2.4.4. *Let $(\mathcal{H},\{k_x\}_{x\in(X,d,\sigma)})$ be a FHS satisfying ADP, UMD and ULP, and $T_{B(a,r)}$ be the concentration operator over the ball $B(a,r)$. Denote all its eigenvalues in the decreasing order by $1\geq l_1(T_{B(a,r)})\geq l_2(T_{B(a,r)})\geq\cdots\geq 0$, where entries are repeated with multiplicity. Then for any $\epsilon>0$, there exists $R>0$ such that*

$$(1-\epsilon)\sum_{i=1}^{\infty}l_i(T_{B(a,r)})\leq\sum_{i=1}^{\infty}l_i(T_{B(a,r)})^2,$$

for any $a\in X$ and all $r>R$.

Proof. By *UMD*, we have $\sigma(B(a,r)) < \infty$ for any $a \in X$ and all $r > 0$. So we can apply Corollary 2.3.7 and obtain

$$\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) = \left\|T_{B(a,r)}\right\|_{Tr} = \sigma(B(a,r)).$$

In order to prove the corollary, it is sufficient to show

$$\limsup_{r\to\infty} \sup_{a\in X} \frac{1}{\sigma(B(a,r))} \left(\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) - \sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2 \right) = 0.$$

Again by Corollary 2.3.7

$$\begin{aligned} &\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) - \sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2 \\ &= \left\|T_{B(a,r)}\right\|_{Tr} - \left\|T_{B(a,r)}\right\|_{HS}^2 \\ &= \int_{B(a,r)} \int_X \left|\langle k_x, k_y\rangle\right|^2 d\sigma(y) d\sigma(x) - \int_{B(a,r)} \int_{B(a,r)} \left|\langle k_x, k_y\rangle\right|^2 d\sigma(y) d\sigma(x) \\ &= \int_{B(a,r)} \int_{B(a,r)^c} \left|\langle k_x, k_y\rangle\right|^2 d\sigma(y) d\sigma(x). \end{aligned}$$

Then by Proposition 2.4.3, we get the desired equality. □

Because *ADP* and *UMD* are common properties, we usually view Corollary 2.4.4 as a consequence of *ULP*. Study Corollary 2.4.4, it tell us every eigenvalue $l_i(T_{B(a,r)})$ of $T_{B(a,r)}$ should be closed to either 1 or 0, and the closeness depends on how large of r. It somehow tell us the information about the distribution of eigenvalues of $T_{B(a,r)}$ when r is large. Based on this concern, we have the following Lemma which improves the inequality in Lemma 2.3.9 when r is large.

Lemma 2.4.5. *Given a number $0 < c < 1$, then for every $0 < \theta < 1$, there exists $R > 0$ such that for every ball $B(a,r) \subseteq X$ with $r > R$,*

$$dim\mathcal{G} < \frac{\sigma(B(a,r))}{\theta},$$

where $\mathcal{G}$ denote any subspace of $\mathcal{H}$ which is c-concentrated on $B(a,r)$.

Proof. Let $T_{B(a,r)}$ be the concentration operator over an arbitrary ball $B(a,r)$. Denote all the eigenvalues of $T_{B(a,r)}$ by $\{l_i(T_{B(a,r)})\}_{i=1}^{\infty}$ indexed in decreasing order. And Let $\mathcal{G}$ be any c-concentrated subspace of $\mathcal{H}$ on $B(a,r)$ and denote $dim\mathcal{G}$ by k ($k < \infty$, by Lemma 2.3.9). By the min-max principle,

$$\begin{aligned} l_k(T_{B(a,r)}) &= \max_{dim\mathcal{F}=k} \min_{g\in\mathcal{F},\|g\|=1} \left\langle T_{B(a,r+\frac{\delta}{2})} g, g \right\rangle \\ &\geq \min_{g\in\mathcal{G},\|g\|=1} \left\langle T_{B(a,r+\frac{\delta}{2})} g, g \right\rangle \\ &\geq c. \end{aligned}$$

For any $0 < \varepsilon < 1 - c$, we have

$$\begin{aligned} dim\mathcal{G} &= k \\ &\leq \# \left\{ i : l_i(T_{B(a,r)} \geq c \right\} \\ &= \# \left\{ i : l_i(T_{B(a,r)}) > 1 - \varepsilon \right\} + \# \left\{ i : c \leq l_i(T_{B(a,r)}) \leq 1 - \varepsilon \right\} \\ &\leq \sum_{l_i > 1-\varepsilon} \frac{l_i(T_{B(a,r)})}{1-\varepsilon} + \sum_{c \leq l_i \leq 1-\varepsilon} \frac{l_i(T_{B(a,r)})}{c} \\ &\leq \frac{1}{1-\varepsilon} \sum_{i=1}^{\infty} l_i(T_{B(a,r)}) + \frac{1}{c} \sum_{l_i \leq 1-\varepsilon} l_i(T_{B(a,r)}). \end{aligned} \tag{2.10}$$

Let $\epsilon = c\varepsilon^2$. By Corollary 2.4.4, there exists $R > 0$ such that for any $a \in X$ and all $r > R$, we have

$$\begin{aligned} &(1-\epsilon) \sum_{i=1}^{\infty} l_i(T_{B(a,r)}) \\ \leq &\sum_{i=1}^{\infty} l_i(T_{B(a,r)})^2 \\ = &\sum_{l_i \leq 1-\varepsilon} l_i(T_{B(a,r)})^2 + \sum_{l_i > 1-\varepsilon} l_i(T_{B(a,r)})^2 \\ \leq &(1-\varepsilon) \sum_{l_i \leq 1-\varepsilon} l_i(T_{B(a,r)}) + \sum_{l_i > 1-\varepsilon} l_i(T_{B(a,r)}) \\ = &\sum_{i=1}^{\infty} l_i(T_{B(a,r)}) - \varepsilon \sum_{l_i \leq 1-\varepsilon} l_i(T_{B(a,r)}). \end{aligned}$$

It follows that for any $a \in X$ and all $r > R$,

$$\frac{1}{c} \sum_{l_i \leq 1-\varepsilon} l_i(T_{B(a,r)}) \leq \varepsilon \sum_{i=1}^{\infty} l_i(T_{B(a,r)}). \tag{2.11}$$

Combining (2.10) and (2.11), we obtain that for any $a \in X$ and all $r > R$,

$$\frac{1-\varepsilon}{1+\varepsilon-\varepsilon^2} dim\mathcal{G} \le \sum_{i=1}^{\infty} l_i(T_{B(a,r)}). \tag{2.12}$$

Notice as $\varepsilon \to 0^+$,

$$\frac{1-\varepsilon}{1+\varepsilon-\varepsilon^2} \uparrow 1.$$

So for any $0 < \theta < 1$, there exists $\varepsilon > 0$ such that

$$\frac{1-\varepsilon}{1+\varepsilon-\varepsilon^2} > \theta.$$

Then by (2.12), we have

$$dim\mathcal{G} \le \frac{\sum_{i=1}^{\infty} l_i(T_{B(a,r)})}{\frac{1-\varepsilon}{1+\varepsilon-\varepsilon^2}} < \frac{\sum_{i=1}^{\infty} l_i(T_{B(a,r)})}{\theta} = \frac{\sigma(B(a,r))}{\theta},$$

for any $a \in X$ and all $r > R$. Where R only dependents on c and θ. □

2.5 Boundedness, Compactness and Trace Class Membership of Toeplitz Operators

In this section, based on the additional conditions we impose on *FHS*, we could give some criteria to characterize the boundedness, compactness and trace class membership of Toeplitz operators without assuming $\mu(X)$ is finite. All of the results in this section are based on my master's thesis [11].

Definition 2.5.1. *Let $\{k_x\}_{x\in(X,\sigma)}$ be a continuous frame of $\mathcal{H}$, and μ be another positive measure for the same measurable sets with measure σ. We define the* ***Berezin transform*** *$\widetilde{\mu}$ of measure μ by*

$$\widetilde{\mu} : X \to \mathbb{R}, \quad \widetilde{\mu}(x) := \int_X |\langle k_x, k_y\rangle|^2 \, d\mu(y).$$

It is easy to check that if $\int_X |\langle k_x, k_y\rangle| \, d\mu(y) < \infty$ for every $x \in X$, then

$$\widetilde{\mu}(x) = \langle T_\mu k_x, k_x\rangle, \quad \forall x \in X.$$

Therefore, the Berezin transform $\widetilde{\mu}(x)$ of measure μ can be viewed as the "continuous diagonal terms" of the matrix of the Toeplitz operator T_μ.

Definition 2.5.2. *Let μ be a positive Borel measure on a metric space (X, d). For any $r > 0$, we define the* ***volume function*** *$\widehat{\mu}_r$ of measure μ by*

$$\widehat{\mu}_r : X \to \mathbb{R}, \quad \widehat{\mu}_r(x) := \mu(B(x, r)).$$

We now list the following theorems to characterize the boundedness, compactness and trace class membership of Toeplitz operators T_μ.

Theorem 2.5.3 ([11], Theorem 5)**.** *Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a FHS which satisfies DMP, UMD, MVP and AOP. And Let μ be a σ-finite positive Borel measure on X that satisfies*

$$\int_X |\langle k_x, k_y\rangle|\, d\mu(y) < \infty, \quad \forall x \in X.$$

Then the followings are equivalent.

a. $T_\mu : span\{k_x : x \in X\} \to \mathcal{H}$ can be extended to a bounded operator on $\mathcal{H}$.

b. The Berezin transform $\tilde{\mu} : X \to \mathbb{R}$ is bounded.

c. The volume function $\hat{\mu}_r : X \to \mathbb{R}$ is bounded for some $r > 0$.

Theorem 2.5.4 ([11], Theorem 6)**.** *Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a FHS which satisfies DMP, UMD, MVP and AOP. And Let μ be a σ-finite positive Borel measure on X that satisfies*

$$\int_X |\langle k_x, k_y\rangle|\, d\mu(y) < \infty, \quad \forall x \in X.$$

Then the followings are equivalent.

a. $T_\mu : \mathcal{H} \to \mathcal{H}$ is a compact operator.

b. $\tilde{\mu}(x) \to 0$ as $x \to \infty$.

c. $\hat{\mu}_r(x) \to 0$ as $x \to \infty$ for some $r > 0$.

Theorem 2.5.5 ([11], Theorem 8)**.** *Given* $1 \leq p < \infty$*, Let* $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ *be a FHS which satisfies DMP, UMD, MVP and AOP. And Let* μ *be a* σ*-finite positive Borel measure on* X *that satisfies*

$$\int_X |\langle k_x, k_y\rangle| \, d\mu(y) < \infty, \quad \forall x \in X.$$

Then the followings are equivalent.

a. T_μ *belongs to the Schatten class* $\mathcal{S}_p$*.*

b. $\tilde{\mu}$ *belongs to* $L^p(X, d\sigma)$*.*

c. $\hat{\mu}_r$ *belongs to* $L^p(X, d\sigma)$ *for some* $r > 0$*.*

2.6 Beurling Densities of Sampling and Interpolation Sets

2.6.1 Sampling and Interpolation Sets

For *RKHS*, one could define the sampling and interpolation set. Similarly, we could give the definitions of those for *FHS*.

Definition 2.6.1. *Let* $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ *be a FHS. A countable subset* $\Lambda := \{\lambda\}$ *of* X *is called a* ***sampling set*** *for* $\mathcal{H}$*, if there exist constants* $A, B > 0$ *such that*

$$A \|f\|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, k_\lambda\rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

From the definition, Λ is a sampling set for $\mathcal{H}$ if and only if the corresponding sequence of continuous frame $\{k_\lambda\}_{\lambda\in\Lambda}$ is a frame of $\mathcal{H}$.

Definition 2.6.2. *Let* $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ *be a FHS. A countable subset* $\Lambda := \{\lambda\}$ *of* X *is called an* ***interpolation set*** *for* $\mathcal{H}$*, if for any complex sequence* $(c_\lambda) \in l^2(\Lambda)$*, there exists* $f \in \mathcal{H}$ *such that*

$$\langle f, k_\lambda\rangle = c_\lambda, \quad \forall \lambda \in \Lambda.$$

By Propositions 1.4.10, Λ is an interpolation set if and only if the corresponding sequence of continuous frame $\{k_\lambda\}_{\lambda\in\Lambda}$ is a Riesz-Fischer sequence of $\mathcal{H}$.

2.6.2 Density Theorems in Particular Framed Hilbert Spaces

Now we are going to list some important results about sampling and interpolation sets which will be applied in the rest of the chapters. All of them are related to the so-called Beurling densities.

Definition 2.6.3. *Let Λ be a countable subset of a metric measure space (X, d, σ). The **lower Beurling density** of Λ is defined as*

$$D_\sigma^-(\Lambda) := \liminf_{r\to\infty} \inf_{a\in X} \frac{\#\{\Lambda \cap B(a,r)\}}{\sigma(B(a,r))}.$$

*Similarly, the **upper Beurling density** of Λ is defined as*

$$D_\sigma^+(\Lambda) := \limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda \cap B(a,r)\}}{\sigma(B(a,r))}.$$

If σ is the Lebesgue measure, by convention, we will drop the subscript σ in the above notations. We will use $D^-(\Lambda)$ and $D^+(\Lambda)$ to denote the lower and upper Beurling densities in this case.

For example, all the integers $\mathbb{Z}$ is a countable subset of $(\mathbb{R}, |\cdot|, m)$. It is not hard to see $D^-(\mathbb{Z}) = D^+(\mathbb{Z}) = 1$. If we pick all the natural numbers $\mathbb{N}$ as the countable subset, then $D^-(\mathbb{N}) = 0$ and $D^+(\mathbb{N}) = 1$.

The following density theorem is based on the general Paley-Wiener space which is proved by Landua in 1967.

Theorem 2.6.4 ([10])**.** *Let E be a bounded measurable set on $\mathbb{R}$, and Λ be a countable subset of $\mathbb{R}$. For the general Paley-Wiener space $(L^2(E), \{\frac{e^{2\pi i\xi x}}{\sqrt{m(E)}}\}_{\xi\in(\mathbb{R},|\cdot|,m_E)})$,*

1. if Λ is a sampling set for $L^2(E)$, then

$$D_{m_E}^-(\Lambda) \geq 1;$$

2. if Λ is an interpolation set for $L^2(E)$, then

$$D_{m_E}^+(\Lambda) \leq 1.$$

Note Theorem 2.6.4 gives us a necessary density condition for sampling and interpolation sets in the general Paley-Wiener space. Besides this, Beurling and Kahane proved a sufficient density condition for sampling and interpolation sets in the classical Paley-Wiener space by the following theorem:

Theorem 2.6.5 ([8])**.** *Let I be a finite interval of $\mathbb{R}$, and Λ be countable subset of $\mathbb{R}$. For the classical Paley-Wiener space* $(L^2(I), \{\frac{e^{2\pi i \xi x}}{\sqrt{m(I)}}\}_{\xi \in (\mathbb{R}, |\cdot|, m_I)})$,

1. *if Λ is relatively separated with $D^-_{m_I}(\Lambda) > 1$, then Λ is a sampling set for $L^2(I)$, i.e. there exist constants $A, B > 0$ such that*

$$A \|f\|^2_{L^2(I)} \leq \sum_{\lambda \in \Lambda} \left| \left\langle f, e^{2\pi i \lambda(\cdot)} \right\rangle_{L^2(I)} \right|^2 \leq B \|f\|^2_{L^2(I)}, \quad \forall f \in L^2(I);$$

2. *if Λ is separated with $D^+_{m_I}(\Lambda) < 1$, then Λ is an interpolation set for $L^2(I)$, i.e. there exists a constant $C > 0$ such that*

$$C \sum_{\lambda \in \Lambda} |c_\lambda|^2 \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda e^{2\pi i \lambda(\cdot)} \right\|^2_{L^2(I)},$$

   for any finite sequence $\{c_\lambda\}_{\lambda \in \Lambda}$.

Similar results for the one-dimensional classical Bargmann-Fock space are proved by Seip and Wallstén as the following two theorems:

Theorem 2.6.6 ([17, 18])**.** *Let Λ be a countable subset in $\mathbb{C}$. For the one-dimensional classical Bargmann-Fock space $(\mathcal{F}_\alpha(\mathbb{C}), \{k^\alpha_z\}_{z \in (\mathbb{C}, |\cdot|, m_\alpha)})$, Λ is a sampling set for $\mathcal{F}_\alpha(\mathbb{C})$ if and only if Λ is relatively separated and contains a separated subset Λ' for which $D^+_{m_\alpha}(\Lambda') > 1$.*

Theorem 2.6.7 ([17, 18])**.** *Let Λ be a countable subset in $\mathbb{C}$. For the one-dimensional classical Bargmann-Fock space $(\mathcal{F}_\alpha(\mathbb{C}), \{k^\alpha_z\}_{z \in (\mathbb{C}, |\cdot|, m_\alpha)})$, Λ is an interpolation set for $\mathcal{F}_\alpha(\mathbb{C})$ if and only if Λ is separated and $D^+_{m_\alpha}(\Lambda) < 1$.*

For high dimensional Bargmann-Fock spaces, we don't have the perfect if and only if density conditions for sampling and interpolation sets. But still, Lindholm gives a very similar necessary density condition for high dimensional weighted Bargmann-Fock spaces which implies the following theorem:

Theorem 2.6.8 ([12])**.** *Let Λ be a countable subset of $\mathbb{C}^n$. For the classical Bargmann-Fock space* $(\mathcal{F}_\alpha(\mathbb{C}^n), \{k^\alpha_z\}_{z \in (\mathbb{C}^n, |\cdot|, m_\alpha)})$, *1. if Λ is a sampling set for $\mathcal{F}_\alpha(\mathbb{C}^n)$, then Λ is relatively separated and contains a separated subset Λ' which is also sampling and satisfies*

$$D^-_{m_\alpha}(\Lambda') \geq 1;$$

2. if Λ is an interpolation set for $\mathcal{F}_\alpha(\mathbb{C}^n)$, then Λ is separated with

$$D^+_{m_\alpha}(\Lambda) \leq 1.$$

In general cases, we have the following theorem proved by Mitkovski and Ramirez which gives a necessary density condition for the general *FHS*.

Theorem 2.6.9 ([13], Theorem 4.5)**.** *Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a FHS which satisfies the ADP, UMD, MVP and ULP, and Λ be a separated subset of X,*

1. If Λ is a sampling set for $\mathcal{H}$, then

$$D^+_\sigma(\Lambda) \geq 1;$$

2. If Λ is an interpolation set for $\mathcal{H}$, then

$$D^+_\sigma(\Lambda) \leq 1.$$

Chapter 3

Approximate Interpolation in Framed Hilbert Spaces

3.1 Introduction

3.1.1 (Weak) Interpolation Sets

Let $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ be a *FHS*. A countable subset $\Lambda := \{\lambda\}$ of X is called an interpolation set for $\mathcal{H}$, if the corresponding sequence of continuous frame $\{k_\lambda\}_{\lambda\in\Lambda}$ is a Riesz-Fischer sequence of $\mathcal{H}$ (see 2.6.2 and 1.4.10).

By Proposition 1.4.11, Λ is an interpolation set if and only if there exists a Bessel sequence $\{f_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{H}$ such that

$$\langle f_\lambda, k_\nu\rangle = \delta_{\lambda\nu}, \quad \forall \lambda, \nu \in \Lambda.$$

If we replace the Bessell sequence condition by the condition of bounded sequence, we would get a weak type of interpolation sets.

Definition 3.1.1. *Let $(\mathcal{H}, \{k_x\}_{x\in(X,\sigma)})$ be a FHS. A countable subset $\Lambda := \{\lambda\}$ of X is called a* ***weak interpolation set*** *for $\mathcal{H}$, if there exists a bounded sequence $\{f_\lambda\}_{\lambda\in\Lambda}$ such that*

$$\langle f_\lambda, k_\nu\rangle = \delta_{\lambda\nu}, \quad \forall \lambda, \nu \in \Lambda.$$

Because any Bessel sequence is bounded, we have every interpolation set is a weak interpolation set. And for particular *FHS*, the converse is also true. For example, in the classical one-dimensional Bargmann-Fock space $\mathcal{F}(\mathbb{C})$, these two classes of sets coincide.

3.1.2 d-Approximate (Weak) Interpolation Sets

Recall Theorem 2.6.9: let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a *FHS* which satisfies the *ADP*, *UMD*, *MVP* and *ULP* , if a separated set Λ (will be defined in next section) is an interpolation set for $\mathcal{H}$, then its upper Beurling density satisfies

$$D_\sigma^+(\Lambda) \leq 1.$$

The goal of this chapter is to provide a similar type necessary density condition for an even larger class of "interpolation sets". This type of sets (to be defined momentarily) were relatively recently introduced by Olevskii and Ulanovskii (see [15]) in the classical Paley-Wiener space, where it was shown that all such sets have to satisfy a Beurling density condition similar to the one for usual interpolation sets. Our results can be viewed as the generalization of their results.

Definition 3.1.2. *Given $0 \leq d < 1$, we will say that a countable subset $\Lambda := \{\lambda\}$ of X is a **d-approximate interpolation set** for $\mathcal{H}$, if there exists a Bessel sequence $\{h_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{H}$ such that*

$$\sum_{\nu\in\Lambda} |\langle h_\lambda, k_\nu\rangle - \delta_{\lambda\nu}|^2 \leq d^2, \quad \forall\lambda \in \Lambda.$$

Namely, the $l^2(\Lambda)$ distance between the sequences $(\langle h_\lambda, k_\nu\rangle)$ and $(\delta_{\lambda\nu})$ is no greater than d for any λ.

Note that 0-approximate interpolation sets coincide with interpolation sets. Again, using a bounded sequence instead of a Bessel sequence in Definition 3.1.2, we could define a weak type of d-approximate interpolation sets.

Definition 3.1.3. *Given $0 \leq d < 1$, we will say that a countable subset $\Lambda = \{\lambda\}$ of X is a **d-approximate weak interpolation set** for $\mathcal{H}$, if there exists a bounded sequence $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ such that*

$$\sum_{\nu\in\Lambda} |\langle f_\lambda, k_\nu\rangle - \delta_{\lambda\nu}|^2 \leq d^2, \quad \forall\lambda \in \Lambda.$$

Again, 0-approximate weak interpolation sets coincide with weak interpolation sets in $\mathcal{H}$.

And every d-approximate interpolation set is a d-approximate weak interpolation set. But we don't know whether the converse is true in general.

3.2 d-Approximate Interpolation in Framed Hilbert Spaces

3.2.1 Main Theorem

Our first result is based on *FHS* which gives a necessary density condition for d-approximate interpolation sets.

Theorem 3.2.1. *Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a FHS which satisfy ADP, UMD, MVP and ULP. Given $0 \le d < 1$, suppose a relatively separated subset Λ of X is a d-approximate interpolation set for $\mathcal{H}$. Then*

$$D^+_\sigma(\Lambda) \le \frac{1}{1-d^2}.$$

Before we give a proof of the theorem, we need to collect some basics.

3.2.2 Relatively Separated Sets

Definition 3.2.2. *A subset Λ in a metric space (X, d) is said to be* ***separated*** *or uniformly discrete if*

$$\delta := \inf\{d(\lambda, \nu) : \lambda \ne \nu \in \Lambda\} > 0.$$

And the constant $\delta > 0$ is called a separation constant of Λ.

Definition 3.2.3. *A subset Λ in a metric space (X, d) is said to be* ***relatively separated*** *if it is a finite union of separated sets. Namely,*

$$\Lambda = \cup_{k=1}^K \Lambda_k,$$

where Λ_k is a separated set for any $k = 1, \cdots, K$.

The following two propositions will show some simple but important properties for Λ being relatively separated.

Proposition 3.2.4. *Let (X, d, σ) be a metric measure space satisfying UMD, and Λ be a relatively separated set in X. Then for any ball $B(a, r)$ in X with center $a \in X$ and radius $r > 0$,*

$$\#\{\Lambda \cap B(a, r)\} < \infty.$$

If (X, d, σ) also has ADP, then

$$D^+_\sigma(\Lambda) < \infty.$$

Proof. Since Λ is relatively separated, we have

$$\Lambda = \cup_{k=1}^{K} \Lambda_k,$$

where Λ_k is a separated set with the separation constant δ_k for any $k = 1, \cdots, K$. Let $\delta := \min_{1 \le k \le K} \delta_k$, and $\{\lambda_i^k\}_{i \in I_k} := \{\Lambda_k \cap B(a, r)\}$ for any $k = 1, \cdots, K$. For every $1 \le k \le K$, by Λ_k is separated,

$$B(\lambda_i^k, \frac{\delta}{2}) \cap B(\lambda_j^k, \frac{\delta}{2}) = \emptyset, \quad \forall i \neq j.$$

In addition, for every $1 \le k \le K$,

$$\cup_{i \in I_k} B(\lambda_i^k, \frac{\delta}{2}) \subseteq B(a, r + \frac{\delta}{2}).$$

It follows that for every $1 \le k \le K$,

$$\sum_{i \in I_k} \sigma(B(\lambda_i^k, \frac{\delta}{2})) \le \sigma(B(a, r + \frac{\delta}{2})).$$

By *UMD*, there exists $c(\frac{\delta}{2}) > 0$ such that

$$\#\{\Lambda_k \cap B(a, r)\} c(\frac{\delta}{2}) \le \sum_{i \in I_k} \sigma(B(\lambda_i^k, \frac{\delta}{2})) \le \sigma(B(a, r + \frac{\delta}{2})).$$

Again by *UMD*, there exists $D(r + \frac{\delta}{2}) > 0$ such that

$$\#\{\Lambda \cap B(a, r)\} \le \sum_{k=1}^{K} \#\{\Lambda_k \cap B(a, r)\} \le \sum_{k=1}^{K} \frac{\sigma(B(a, r + \frac{\delta}{2}))}{c(\frac{\delta}{2})} \le \frac{mD(r + \frac{\delta}{2})}{c(\frac{\delta}{2})} < \infty.$$

Combine *ADP*,

$$\begin{aligned}
D^+_\sigma(\Lambda) :&= \limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda\cap B(a,r)\}}{\sigma(B(a,r))} \\
&\le \limsup_{r\to\infty} \sup_{a\in X} \frac{\sum_{k=1}^K \#\{\Lambda_k\cap B(a,r)\}}{\sigma(B(a,r))} \\
&\le \limsup_{r\to\infty} \sup_{a\in X} \sum_{k=1}^K \frac{\sigma(B(a,r+\frac{\delta}{2}))}{c(\frac{\delta}{2})\sigma(B(a,r))} \\
&= \frac{K}{c(\frac{\delta}{2})} \limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a,r+\frac{\delta}{2}))}{\sigma(B(a,r))} \\
&= \frac{K}{c(\frac{\delta}{2})} < \infty. \qquad (3.1)
\end{aligned}$$

□

Actually, Theorem 3.2.1 tell us under additional interpolation assumptions on Λ, this trivial density upper bound (3.1) could be significantly improved (especially when δ is close to 0).

The second consequence of Λ being relatively separated is that it will generate a Bessel sequence of continuous frames $\{k_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{H}$.

Proposition 3.2.5. *Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a FHS satisfying ADP, UMD, MVP and ULP. If Λ is a relatively separated set in X, then there exists a constant $C>0$ such that*

$$\sum_{\lambda\in\Lambda} |\langle f, k_\lambda\rangle|^2 \le C\,\|f\|^2, \quad \forall f\in\mathcal{H}.$$

Proof. Since Λ is relatively separated, we have

$$\Lambda = \cup_{k=1}^K \Lambda_k,$$

where Λ_k is a separated set with the separation constant δ_k for any $k=1,\cdots,K$. Let $\delta := \min_{1\le k\le K}\delta_k$, and $\{\lambda_i^k\}_{i\in I_k} := \{\Lambda_k\cap B(a,r)\}$ for any $k=1,\cdots,K$. Then by *MVP*, there exists

$C_\delta > 0$ such that

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle f, k_\lambda\rangle|^2 &\le \sum_{k=1}^{K}\sum_{\lambda\in\Lambda_k} |\langle f, k_\lambda\rangle|^2 \\
&\le \sum_{k=1}^{K}\sum_{\lambda\in\Lambda_k} C_\delta \int_{B(\lambda,\frac{\delta}{2})} |\langle f, k_x\rangle|^2 \, d\sigma(x) \\
&= C_\delta \sum_{k=1}^{K}\sum_{\lambda\in\Lambda_k} \int_{B(\lambda,\frac{\delta}{2})} |\langle f, k_x\rangle|^2 \, d\sigma(x) \\
&\le C_\delta \sum_{k=1}^{K} \int_X |\langle f, k_x\rangle|^2 \, d\sigma(x) \\
&= K C_\delta \|f\|^2 .
\end{aligned}$$

□

3.2.3 Some Lemmas

In order to prove Theorem 3.2.1, we will adopt the proof strategy of Olevskii and Ulanovskii [15]. The argument has two crucial ingredients. The first one (essentially going back to Landau [10]) says that any subspace of $\mathcal{H}$ which is c-concentrated on a fixed finite measure set cannot have arbitrarily large dimension. We have these results as Lemma 2.3.9 and Lemma 2.4.5 in Chapter 2, for convenience we restate them here.

Lemma. *2.3.9 Let E be a Borel subset of X with $\sigma(E) < \infty$. Given a number $0 < c < 1$, if $\mathcal{G}$ is a subspace of $\mathcal{H}$ which is c-concentrated on E, then*

$$dim\mathcal{G} \le \frac{\sigma(E)}{c}.$$

Lemma. *2.4.5 Given a number $0 < c < 1$, then for every $0 < \theta < 1$, there exists $R > 0$ such that for every ball $B(a, r) \subseteq X$ with $r > R$,*

$$dim\mathcal{G} < \frac{\sigma(B(a, r))}{\theta},$$

where $\mathcal{G}$ denote any subspace of $\mathcal{H}$ which is c-concentrated on $B(a, r)$.

The following lemma is the second important ingredient in our proofs. It allows us to generate a fairly high-dimensional subspace which is concentrated on some ball so that we can apply the above two lemmas. Note the following result is a finite-dimensional result, which can be applied in our proofs only when we restrict the set Λ to a ball.

Lemma 3.2.6 ([15], Lemma 2)**.** *Let $\{\mathbf{u}_j\}_{1\leq j\leq N}$ be an orthonormal basis in an N-dimensional complex Euclidean space $\mathcal{U}$. Given $0 < d < 1$, suppose that $\{\mathbf{v}_j\}_{1\leq j\leq N}$ is a set of vectors in $\mathcal{U}$ satisfying*

$$\|\mathbf{v}_j - \mathbf{u}_j\|^2 \leq d^2,\ 1 \leq j \leq N.$$

Then for any b with $1 < b < 1/d$, there is a subspace X of $\mathbb{C}^N$, such that

1. $(1 - b^2 d^2)N - 1 < dim X$;

2. *the estimate*

$$(1 - \frac{1}{b})^2 \sum_{j=1}^{N} |c_j|^2 \leq \| \sum_{j=1}^{N} c_j \mathbf{v}_j \|^2,$$

   holds for any vector $\mathbf{c} = (c_1, c_2, ..., c_N) \in X$.

3.2.4 Proof of Theorem 3.2.1

Proof. By $\Lambda = \{\lambda\} \subseteq X$ is a d-approximate interpolation set for $\mathcal{H}$, there exists a Bessel sequence $\{h_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{H}$ such that

$$\sum_{\nu\in\Lambda} |\langle h_\lambda, k_\nu\rangle - \delta_{\lambda\nu}|^2 \leq d^2, \quad \forall\lambda \in \Lambda.$$

Let $B(a, r)$ be an arbitrary open ball in X. Since Λ is relatively separated, $\Lambda \cap B(a, r)$ is finite set. Let $\Lambda \cap B(a, r) = \{\lambda_1, \lambda_2, ..., \lambda_N\}$. Consider the following vectors in $\mathbb{C}^N$,

$$\mathbf{v}_j := (\langle h_{\lambda_j}, k_{\lambda_1}\rangle, \cdots, \langle h_{\lambda_j}, k_{\lambda_N}\rangle), \quad 1 \leq j \leq N,$$

and the standard basis of $\mathbb{C}^N$,

$$\mathbf{u}_j := (\delta_{\lambda_j\lambda_1}, \cdots, \delta_{\lambda_j\lambda_N}), \quad 1 \leq j \leq N.$$

Notice that

$$
\begin{aligned}
\|\mathbf{v}_j - \mathbf{u}_j\|^2 &= \sum_{i=1}^{N} \left|\left\langle h_{\lambda_j}, k_{\lambda_i}\right\rangle - \delta_{\lambda_j\lambda_i}\right|^2 \\
&\le \sum_{\nu\in\Lambda} \left|\left\langle h_{\lambda_j}, k_{\nu}\right\rangle - \delta_{\lambda_j\nu}\right|^2 \\
&\le d^2, \quad 1 \le j \le N.
\end{aligned}
$$

By Lemma 3.2.6, for any $1 < b < 1/d$, there exists a subspace U of $\mathbb{C}^N$, such that

1.
$$(1 - b^2d^2)N - 1 < dimU; \tag{3.2}$$

2. the inequality
$$(1 - \frac{1}{b})^2 \sum_{j=1}^{N} |c_j|^2 \le \|\sum_{j=1}^{N} c_j\mathbf{v}_j\|^2, \tag{3.3}$$

   holds for any vector $\mathbf{c} = (c_1, ..., c_N) \in U$.

Let $Y := \{\sum_{j=1}^{N} c_j\mathbf{v}_j | \mathbf{c} = (c_1, ..., c_N) \in U\}$ be the subspace of $\mathbb{C}^N$. we want to show $dimU \le dimY$. The idea is that using a basis of U to create some linearly independent vectors of Y.

Let $M = dimU$ and let $\{\mathbf{c}^i := (c_1^i, ..., c_N^i)\}_{i=1}^{M}$ be a basis of $U \subseteq \mathbb{C}^N$. And let $\{\mathbf{w}_i := \sum_{j=1}^{N} c_j^i\mathbf{v}_j\}_{i=1}^{M}$ be the vectors of Y. Assume $a_1, a_2, \cdots, a_M$ are the scalars such that the linear combination $\sum_{i=1}^{M} a_i\mathbf{w}_i = \mathbf{0}$, then

$$
\begin{aligned}
\mathbf{0} &= \sum_{i=1}^{M} a_i\mathbf{w}_i \\
&= \sum_{i=1}^{M} a_i \sum_{j=1}^{N} c_j^i\mathbf{v}_j \\
&= \sum_{j=1}^{N} \left(\sum_{i=1}^{M} a_i c_j^i\right)\mathbf{v}_j
\end{aligned}
$$

Notice the scalars of coefficients $\left(\sum_{i=1}^{M} a_i c_j^i\right)_{1\leq j\leq N} = \sum_{i=1}^{M} a_i \mathbf{c}^i \in U$, by (3.3) we have

$$0 = \left\| \sum_{j=1}^{N} \left(\sum_{i=1}^{M} a_i c_j^i \right) \mathbf{v}_j \right\|^2$$
$$\geq (1 - \frac{1}{b})^2 \sum_{j=1}^{N} \left| \sum_{i=1}^{M} a_i c_j^i \right|^2 ,$$

which implies the linear combination $\sum_{i=1}^{M} a_i \mathbf{c}^i = \mathbf{0}$. By $\{\mathbf{c}^i\}_{i=1}^{M}$ is a basis, $a_i = 0,\ 1 \leq i \leq M$. So $\{\mathbf{w}_i\}_{i=1}^{M}$ are the linearly independent vectors of Y. Then

$$dim U \leq dim Y. \tag{3.4}$$

Let $\mathcal{G} := \{\sum_{j=1}^{N} c_j h_{\lambda_j} | \mathbf{c} = (c_1, ..., c_N) \in U\}$ be the subspace of $\mathcal{H}$, now we want to prove $dim Y \leq dim \mathcal{G}$.

Let T^* be the analysis operator of the finite sequence $\{k_{\lambda_1}, k_{\lambda_2}, ..., k_{\lambda_N}\}$ on $\mathcal{H}$. By $T^* h_{\lambda_j} = \mathbf{v}_j,\ 1 \leq j \leq N$, we have $T^*(\mathcal{G}) = Y$. It follows that

$$dim Y \leq dim \mathcal{G}. \tag{3.5}$$

Combine (3.2), (3.4) and (3.5), we obtain

$$(1 - b^2 d^2) N - 1 < dim \mathcal{G}. \tag{3.6}$$

Let $g = \sum_{j=1}^{N} c_j h_{\lambda_j} \in \mathcal{G}$ for some $(c_1, ..., c_N) \in U$. Using that $\{h_\lambda\}_{\lambda \in \Lambda}$ is a Bessel sequence and

(3.3), we get

$$\begin{aligned}\sum_{i=1}^{N}|\langle g,k_{\lambda_i}\rangle|^2 &= \sum_{i=1}^{N}\left|\left\langle\sum_{j=1}^{N}c_jh_{\lambda_j},k_{\lambda_i}\right\rangle\right|^2\\ &= \|\sum_{j=1}^{N}c_j\mathbf{v}_j\|^2\\ &\geq (1-\frac{1}{b})^2\sum_{j=1}^{N}|c_j|^2\\ &\geq (1-\frac{1}{b})^2C\|\sum_{j=1}^{N}c_jh_{\lambda_j}\|^2\\ &= C_1\|g\|^2, \qquad (3.7)\end{aligned}$$

for any $g\in\mathcal{G}$.

Let δ be the separation constant of Λ. Then $B(\lambda,\delta/2)\cap B(\nu,\delta/2)=\emptyset$ for any $\lambda\neq\nu\in\Lambda$. It follows from the *MVP* that

$$\begin{aligned}\sum_{i=1}^{N}|\langle g,k_{\lambda_i}\rangle|^2 &\leq \sum_{i=1}^{N}C_\delta\int_{B(\lambda_i,\frac{\delta}{2})}|\langle g,k_x\rangle|^2\,d\sigma(x)\\ &\leq C_\delta\int_{B(a,r+\frac{\delta}{2})}|\langle g,k_x\rangle|^2\,d\sigma(x), \qquad (3.8)\end{aligned}$$

for any $g\in\mathcal{G}$. Combining (3.7) and (3.8), we obtain

$$c\|g\|^2\leq\int_{B(a,r+\frac{\delta}{2})}|\langle g,k_x\rangle|^2\,d\sigma(x)=\left\langle T_{B(a,r+\frac{\delta}{2})}g,g\right\rangle, \qquad (3.9)$$

for any $g\in\mathcal{G}$, where $0<c:=C_1/C_\delta<1$ is independent of a and r. It follows that $\mathcal{G}$ is a c-concentrated subspace of $\mathcal{H}$ on $B(a,r+\frac{\delta}{2})$.

For every $0<\theta<1$, apply Lemma 2.4.5, there exists $R>0$ such that for any ball $B(a,r+\frac{\delta}{2})$ with $r>R$,

$$dim\mathcal{G}<\frac{\sigma(B(a,r+\frac{\delta}{2}))}{\theta}. \qquad (3.10)$$

Combine (3.6) and (3.10), we have for every $a\in X$ and $r>R$,

$$(1-b^2d^2)\#\{\Lambda\cap B(a,r)\}-1<\frac{\sigma(B(a,r+\frac{\delta}{2}))}{\theta}.$$

Then for every $a \in X$ and $r > R$,

$$\#\{\Lambda \cap B(a,r)\} < \frac{\sigma(B(a, r + \frac{\delta}{2}))}{\theta(1 - b^2 d^2)} + \frac{1}{(1 - b^2 d^2)}.$$

Finally, using ADP and UMD, we obtain

$$\begin{aligned}
D^+_\sigma(\Lambda) &= \limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda \cap B(a,r)\}}{\sigma(B(a,r))} \\
&\le \limsup_{r\to\infty} \sup_{a\in X} \left(\frac{\sigma(B(a, r + \frac{\delta}{2}))}{\theta(1 - b^2 d^2)\sigma(B(a,r))} + \frac{1}{(1 - b^2 d^2)\sigma(B(a,r))} \right) \\
&\le \limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a, r + \frac{\delta}{2}))}{\theta(1 - b^2 d^2)\sigma(B(a,r))} + \limsup_{r\to\infty} \sup_{a\in X} \frac{1}{(1 - b^2 d^2)\sigma(B(a,r))} \\
&= \limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a, r + \frac{\delta}{2}))}{\theta(1 - b^2 d^2)\sigma(B(a,r))} \\
&= \frac{1}{\theta(1 - b^2 d^2)} \limsup_{r\to\infty} \sup_{a\in X} \frac{\sigma(B(a, r + \frac{\delta}{2}))}{\sigma(B(a,r))} \\
&= \frac{1}{\theta(1 - b^2 d^2)}.
\end{aligned}$$

Since $\theta < 1$ and $b > 1$ are arbitrary,

$$D^+(\Lambda) \le \frac{1}{1 - d^2}.$$

□

3.3 d-Approximate Weak Interpolation in Classical Bargmann-Fock Spaces

Our next result is based on the classical Bargmann-Fock space $(\mathcal{F}(\mathbb{C}^n), |\cdot|, m)$. In the classical one-dimensional Bargmann-Fock space $\mathcal{F}(\mathbb{C})$, it was shown by Seip and Schuster [16] that the class of weak interpolation sets coincides with the class of interpolation sets.

Another important result (Theorem 2.6.7) of Seip and Wallstén shows that, in the classical one-dimensional Bargmann-Fock space $\mathcal{F}(\mathbb{C})$, interpolation sets Λ can be completely characterized in terms of the upper Beurling density $D^+(\Lambda)$. Namely, Λ is an interpolation set (or equivalently weak interpolation set) if and only if Λ is separated, and $D^+(\Lambda) < 1$.

3.3.1 Main Theorem

For the classical Bargmann-Fock space $(\mathcal{F}(\mathbb{C}^n), |\cdot|, m)$, as a special *FHS*, we could improve Theorem 3.2.1 by applying a weaker condition that Λ is a d-approximate weak interpolation set.

Theorem 3.3.1. *Let $(\mathcal{F}(\mathbb{C}^n), |\cdot|, m)$ be the classical Bargmann-Fock space. Given $0 \leq d < 1$, suppose a relatively separated subset Λ of $\mathbb{C}^n$ is a d-approximate weak interpolation set for $\mathcal{F}(\mathbb{C}^n)$. Then*

$$D^+(\Lambda) \leq \frac{1}{1-d^2}.$$

This result can be easily extended to all of the classical Bargmann-Fock spaces $\mathcal{F}_\alpha(\mathbb{C}^n)$.

3.3.2 Basics in Classical Bargmann-Fock Spaces

Recall in Chapter 2, the classical Bargmann-Fock space $(\mathcal{F}_\alpha(\mathbb{C}^n), \{k_z\}_{z\in(\mathbb{C}^n,|\cdot|,m_\alpha)})$ is a *FHS* satisfying *DMP*, *ADP*, *UMD*, *MVP*, *AOP* and *ULP*. In addition, $\mathcal{F}_\alpha(\mathbb{C}^n)$ is a *RKHS*. Its reproducing kernel at point $z \in \mathbb{C}^n$ equals

$$K_z^\alpha(w) = e^{\alpha\langle w,z\rangle}, \quad \forall w \in \mathbb{C}^n,$$

and its normalized reproducing kernel at point $z \in \mathbb{C}^n$ is

$$k_z^\alpha(w) = e^{\alpha\langle w,z\rangle - \frac{\alpha}{2}|z|^2}, \quad \forall w \in \mathbb{C}^n.$$

About the reproducing kernels, we have the following important equalities hold:

$$\|K_z^\alpha\|_\alpha^2 = \langle K_z^\alpha, K_z^\alpha\rangle_\alpha = e^{\alpha|z|^2}, \quad \forall z \in \mathbb{C}^n,$$

$$|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha| = e^{-\frac{\alpha}{2}|z-w|^2}, \quad \forall z, w \in \mathbb{C}^n.$$

3.3.3 Proof of Theorem 3.3.1

Proof. By $\Lambda = \{\lambda\} \subseteq \mathbb{C}^n$ is a d-approximate weak interpolation set for $\mathcal{F}(\mathbb{C}^n)$, there exists a bounded sequence $\{f_\lambda\}_{\lambda\in\Lambda}$ such that

$$\sum_{\nu\in\Lambda} |\langle f_\lambda, k_\nu\rangle - \delta_{\lambda\nu}|^2 \leq d^2, \quad \forall \lambda \in \Lambda.$$

Let $\varepsilon > 0$. For any $\lambda \in \Lambda$ define $g_\lambda(z) := f_\lambda(z) k_\lambda^\varepsilon(z)$. Clearly $g_\lambda : \mathbb{C}^n \to \mathbb{C}$ is entire as a product of two entire functions. Also, since

$$\begin{aligned}
|k_\lambda^\varepsilon(z)|^2 &= |\langle k_\lambda^\varepsilon, K_z^\varepsilon \rangle|^2 \\
&\leq \|k_\lambda^\varepsilon\|_\varepsilon^2 \, \|K_z^\varepsilon\|_\varepsilon^2 \\
&= e^{\varepsilon |z|^2},
\end{aligned}$$

for all $z, \lambda \in \mathbb{C}^n$, we have

$$\begin{aligned}
&\int_{\mathbb{C}^n} |g_\lambda(z)|^2 e^{-(\pi+\varepsilon)|z|^2} dm(z) \\
&= \int_{\mathbb{C}^n} |f_\lambda(z) k_\lambda^\varepsilon(z)|^2 e^{-(\pi+\varepsilon)|z|^2} dm(z) \\
&\leq \int_{\mathbb{C}^n} |f_\lambda(z)|^2 e^{-\pi |z|^2} dm(z) < \infty.
\end{aligned}$$

Therefore, $g_\lambda \in \mathcal{F}_{\pi+\varepsilon}(\mathbb{C}^n)$ for all $\lambda \in \Lambda$. Moreover,

$$\begin{aligned}
&\sum_{\nu \in \Lambda} \left| \left\langle g_\lambda, k_\nu^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu \in \Lambda} \left| \left\langle g_\lambda, K_\nu^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu \in \Lambda} \left| g_\lambda(\nu) \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu \in \Lambda} \left| f_\lambda(\nu) k_\lambda^\varepsilon(\nu) \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu \in \Lambda} \left| \langle f_\lambda, K_\nu \rangle \langle k_\lambda^\varepsilon, K_\nu^\varepsilon \rangle_\varepsilon \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu \in \Lambda} \left| \langle f_\lambda, k_\nu \rangle \|K_\nu\| \langle k_\lambda^\varepsilon, k_\nu^\varepsilon \rangle_\varepsilon \|K_\nu^\varepsilon\|_\varepsilon \left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-1} - \delta_{\lambda\nu} \right|^2 \\
&= \sum_{\nu \in \Lambda} \left| \langle f_\lambda, k_\nu \rangle \langle k_\lambda^\varepsilon, k_\nu^\varepsilon \rangle_\varepsilon - \delta_{\lambda\nu} \right|^2 \\
&\leq \sum_{\nu \in \Lambda} \left| \langle f_\lambda, k_\nu \rangle - \delta_{\lambda\nu} \right|^2 \leq d^2,
\end{aligned} \tag{3.11}$$

for any $\lambda \in \Lambda$. Note that in this simple computation we used that

$$\left\| K_\nu^{\pi+\varepsilon} \right\|_{\pi+\varepsilon} = \|K_\nu\| \, \|K_\nu^\varepsilon\|_\varepsilon .$$

Let $B(a,r)$ be any ball in $\mathbb{C}^n$. Since Λ is relatively separated, $\Lambda \cap B(a,r)$ is a finite set. Let $\Lambda \cap B(a,r) = \{\lambda_1, \lambda_2, ..., \lambda_N\}$. Consider the following vectors in $\mathbb{C}^N$

$$\mathbf{v}_j := (\langle g_{\lambda_j}, k_{\lambda_1}^{\pi+\varepsilon}\rangle_{\pi+\varepsilon}, ..., \langle g_{\lambda_j}, k_{\lambda_N}^{\pi+\varepsilon}\rangle_{\pi+\varepsilon}), \quad 1 \leq j \leq N,$$

and the standard basis

$$\mathbf{u}_j := (\delta_{\lambda_j\lambda_1}, ..., \delta_{\lambda_j\lambda_N}), \quad 1 \leq j \leq N.$$

By the above inequality (3.11), we have

$$\begin{aligned}
\|\mathbf{v}_j - \mathbf{u}_j\|^2 &= \sum_{i=1}^{N} \left| \langle g_{\lambda_j}, k_{\lambda_i}^{\pi+\varepsilon}\rangle_{\pi+\varepsilon} - \delta_{\lambda_j\lambda_i} \right|^2 \\
&\leq \sum_{\nu\in\Lambda} \left| \langle g_{\lambda_j}, k_{\nu}^{\pi+\varepsilon}\rangle_{\pi+\varepsilon} - \delta_{\lambda_j\nu} \right|^2 \\
&\leq d^2, \quad 1 \leq j \leq N.
\end{aligned}$$

By Lemma 3.2.6, for any $1 < b < 1/d$, there exists a subspace U of $\mathbb{C}^N$, such that

1. $(1 - b^2d^2)N - 1 < dimU$;

2. the inequality
$$(1 - \frac{1}{b})^2 \sum_{j=1}^{N} |c_j|^2 \leq \| \sum_{j=1}^{N} c_j\mathbf{v}_j\|^2, \tag{3.12}$$
holds for any vector $\mathbf{c} = (c_1, ..., c_N) \in U$.

Let $\mathcal{G} := \{\sum_{j=1}^{N} c_j g_{\lambda_j} | \mathbf{c} = (c_1, ..., c_N) \in U\}$. By the same argument of (3.6) in the proof of Theorem 3.2.1,

$$(1 - b^2d^2)N - 1 \leq dim\mathcal{G}. \tag{3.13}$$

Let $g = \sum_{j=1}^{N} c_j g_{\lambda_j} \in \mathcal{G}$ for some $(c_1, ..., c_N) \in U$. Using that $\{k_\lambda^{\pi+\varepsilon}\}_{\lambda\in\Lambda}$ is a Bessel sequence (due to the relative separateness of Λ) and (3.12), we have

$$\begin{aligned}
\|g\|_{\pi+\varepsilon}^2 &\geq C \sum_{\lambda\in\Lambda} \left| \left\langle g, k_\lambda^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \right|^2 \\
&\geq C \sum_{i=1}^{N} \left| \left\langle \sum_{j=1}^{N} c_j g_{\lambda_j}, k_{\lambda_i}^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \right|^2 \\
&\geq C \sum_{i=1}^{N} \left| \sum_{j=1}^{N} c_j \left\langle g_{\lambda_j}, k_{\lambda_i}^{\pi+\varepsilon} \right\rangle_{\pi+\varepsilon} \right|^2 \\
&= C \left\| \sum_{j=1}^{N} c_j \mathbf{v}_j \right\|^2 \\
&\geq C(1-1/b)^2 \sum_{j=1}^{N} |c_j|^2 \\
&= C_1 \sum_{j=1}^{N} |c_j|^2, \qquad (3.14)
\end{aligned}$$

for any $g \in \mathcal{G}$, where C_1 is independent of the radius r.

Fix a small $\rho > 0$. A simple application of the Cauchy-Schwarz inequality and the already mentioned identity $\|K_z^{\pi+\varepsilon}\|_{\pi+\varepsilon} = \|K_z\| \, \|K_z^{\varepsilon}\|_{\varepsilon}$ yields

$$\begin{aligned}
&\int_{B(a,r+\rho r)^c} |g(z)|^2 \, e^{-(\pi+\varepsilon)|z|^2 - n\log(\frac{\pi}{\pi+\varepsilon})} dm(z) \\
&= \int_{B(a,r+\rho r)^c} |g(z)|^2 \, e^{-(\pi+\varepsilon)|z|^2} dm_{\pi+\varepsilon}(z) \\
&= \int_{B(a,r+\rho r)^c} \left| \sum_{j=1}^{N} c_j f_{\lambda_j}(z) k_{\lambda_j}^{\varepsilon}(z) \right|^2 e^{-(\pi+\varepsilon)|z|^2} dm_{\pi+\varepsilon}(z) \\
&\leq \sum_{j=1}^{N} |c_j|^2 \int_{B(a,r+\rho r)^c} \sum_{j=1}^{N} \left| f_{\lambda_j}(z) k_{\lambda_j}^{\varepsilon}(z) \right|^2 e^{-(\pi+\varepsilon)|z|^2} dm_{\pi+\varepsilon}(z) \\
&= \sum_{j=1}^{N} |c_j|^2 \int_{B(a,r+\rho r)^c} \sum_{j=1}^{N} \left| \left\langle f_{\lambda_j}, K_z \right\rangle \left\langle k_{\lambda_j}^{\varepsilon}, K_z^{\varepsilon} \right\rangle_{\varepsilon} \right|^2 \left\| K_z^{\pi+\varepsilon} \right\|_{\pi+\varepsilon}^{-2} dm_{\pi+\varepsilon}(z) \\
&= \sum_{j=1}^{N} |c_j|^2 \int_{B(a,r+\rho r)^c} \sum_{j=1}^{N} \left| \left\langle f_{\lambda_j}, k_z \right\rangle \left\langle k_{\lambda_j}^{\varepsilon}, k_z^{\varepsilon} \right\rangle_{\varepsilon} \right|^2 dm_{\pi+\varepsilon}(z), \qquad (3.15)
\end{aligned}$$

for any $g \in \mathcal{G}$. We now estimate the integral term in (3.15). Applying $\{f_\lambda\}_{\lambda\in\Lambda}$ is bounded and doing a simple change of variables, we obtain

$$
\begin{aligned}
&\int_{B(a,r+\rho r)^c} \sum_{j=1}^{N} \left| \left\langle f_{\lambda_j}, k_z \right\rangle \left\langle k^{\varepsilon}_{\lambda_j}, k^{\varepsilon}_z \right\rangle_{\varepsilon} \right|^2 dm_{\pi+\varepsilon}(z) \\
\leq &\int_{B(a,r+\rho r)^c} \sum_{j=1}^{N} \left| \left\| f_{\lambda_j} \right\| \left\| k_z \right\| \left\langle k^{\varepsilon}_{\lambda_j}, k^{\varepsilon}_z \right\rangle_{\varepsilon} \right|^2 dm_{\pi+\varepsilon}(z) \\
\leq &C \int_{B(\lambda_j,\rho r)^c} \sum_{j=1}^{N} \left| \left\langle k^{\varepsilon}_{\lambda_j}, k^{\varepsilon}_z \right\rangle_{\varepsilon} \right|^2 dm_{\pi+\varepsilon}(z) \\
= &C \sum_{j=1}^{N} \int_{B(\lambda_j,\rho r)^c} \left| \left\langle k^{\varepsilon}_{\lambda_j}, k^{\varepsilon}_z \right\rangle_{\varepsilon} \right|^2 dm_{\pi+\varepsilon}(z) \\
= &C \sum_{j=1}^{N} \int_{B(\lambda_j,\rho r)^c} e^{-\varepsilon|z-\lambda_j|^2} dm_{\pi+\varepsilon}(z) \\
= &CN \int_{B(\mathbf{0},\rho r)^c} e^{-\varepsilon|z|^2} dm_{\pi+\varepsilon}(z).
\end{aligned} \tag{3.16}
$$

Since Λ is relatively separated, we have

$$
\Lambda = \cup_{k=1}^{K} \Lambda_k,
$$

where Λ_k is a separated set with the separation constant δ_k for any $k = 1, \cdots, K$. Let $\delta := \min_{1\leq k\leq K} \delta_k$, and $\{\lambda^k_i\}_{i\in I_k} := \{\Lambda_k \cap B(a, r)\}$ for any $k = 1, \cdots, K$. The simple counting argument shows, for $r > 1$

$$
\begin{aligned}
N &= \#\{\Lambda \cap B(a, r)\} \\
&\leq \sum_{k=1}^{K} \#\{\Lambda_k \cap B(a, r)\} \\
&\leq \sum_{k=1}^{K} \frac{m(B(a, r + \frac{\delta}{2}))}{m(B(\mathbf{0}, \frac{\delta}{2}))} \\
&= K \frac{\frac{\pi^n}{n!}(r + \frac{\delta}{2})^{2n}}{\frac{\pi^n}{n!}(\frac{\delta}{2})^{2n}} \\
&= K(\frac{1}{r} + \frac{2}{\delta})^{2n} r^{2n} \\
&< K(1 + \frac{2}{\delta})^{2n} r^{2n}.
\end{aligned}
$$

And doing change of variables by surface coordinates, we get (3.16) is bounded by

$$\begin{aligned}
&CN\int_{B(\mathbf{0},\rho r)^c} e^{-\varepsilon|z|^2}dm_{\pi+\varepsilon}(z)\\
<&CK(1+\frac{2}{\delta})^{2n}r^{2n}\int_{B(\mathbf{0},\rho r)^c} e^{-\varepsilon|z|^2}dm_{\pi+\epsilon}(z)\\
=&C'r^{2n}\int_{\rho r}^{\infty} e^{-\varepsilon t^2}t^{2n-1}dt\\
=&C_2(r),
\end{aligned} \tag{3.17}$$

where C' does not depend on r (which depends on n, δ and K). Denote the last expression by $C_2(r)$. Observe that $C_2(r)\to 0$ as $r\to\infty$ (to be used in a moment). Combine (3.15), (3.16) and (3.17), we obtain for every $g\in\mathcal{G}$

$$\int_{B(a,r+\rho r)^c} |g(z)|^2 e^{-(\pi+\varepsilon)|z|^2-n\log(\frac{\pi}{\pi+\varepsilon})}dm(z)\le C_2(r)\sum_{j=1}^{N}|c_j|^2. \tag{3.18}$$

Combining (4.2.4) and (4.2.5), we have for every $g\in\mathcal{G}$

$$\begin{aligned}
(1-\frac{C_2(r)}{C_1})\left\|g\right\|_{\pi+\varepsilon}^2 &\le \int_{B(a,r+\rho r)} |g(z)|^2 e^{-(\pi+\varepsilon)|z|^2-n\log(\frac{\pi}{\pi+\varepsilon})}dm(z)\\
&=\int_{B(a,r+\rho r)}\left|\left\langle g,k_z^{\pi+\varepsilon}\right\rangle_{\pi+\varepsilon}\right|^2 dm_{\pi+\varepsilon}(z)\\
&=\left\langle T_{B(a,r+\rho r)}g,g\right\rangle_{\pi+\varepsilon}.
\end{aligned}$$

Let $0<\epsilon<1$. Since $C_2(r)/C_1\to 0$ as $r\to\infty$, there exists $R>0$, such that

$$(1-\epsilon)\left\|g\right\|_{\pi+\varepsilon}^2\le\left\langle T_{B(a,r+\rho r)}g,g\right\rangle_{\pi+\varepsilon},$$

for every $g\in\mathcal{G}$ when $r>R$. In other words, the subspace $\mathcal{G}$ is $(1-\epsilon)$-concentrated on $B(a,r+\rho r)$ whenever $r>R$. By Lemma 2.3.9, we obtain

$$dim\mathcal{G}\le\frac{m_{\pi+\varepsilon}(B(a,r+\rho r))}{1-\epsilon}. \tag{3.19}$$

Combining (3.13) and (3.19), we obtain for every $a \in \mathbb{C}^n$ and $r > R$,

$$(1 - b^2 d^2)N - 1 < \frac{m_{\pi+\varepsilon}(B(a, r + \rho r))}{1 - \epsilon}.$$

Then for every $a \in \mathbb{C}^n$ and $r > R$,

$$\#\{\Lambda \cap B(a, r)\} < \frac{m_{\pi+\varepsilon}(B(a, r + \rho r))}{(1 - \epsilon)(1 - b^2 d^2)} + \frac{1}{(1 - b^2 d^2)}.$$

Therefore,

$$\begin{aligned} D^+(\Lambda) &= \limsup_{r\to\infty} \sup_{a\in\mathbb{C}^n} \frac{\#\{\Lambda \cap B(a, r)\}}{m(B(a, r))} \\ &\le \limsup_{r\to\infty} \sup_{a\in\mathbb{C}^n} \left(\frac{m_{\pi+\varepsilon}(B(a, r + \rho r))}{(1 - \epsilon)(1 - b^2 d^2)m(B(a, r))} + \frac{1}{(1 - b^2 d^2)m(B(a, r))} \right) \\ &= \frac{(\pi + \varepsilon)^n (1 + \rho)^{2n}}{\pi^n (1 - \epsilon)(1 - b^2 d^2)}. \end{aligned}$$

Since $\varepsilon > 0, \rho > 0, \epsilon > 0, b > 1$ are arbitrary,

$$D^+(\Lambda) \le \frac{1}{1 - d^2}.$$

□

Chapter 4

Frame Bound Estimates for Continuous Frames of Exponentials

4.1 Introduction

Recall Beurling and Kahane's density theorem for sampling sets in the classical Paley-Wiener space.

Theorem. *2.6.5 Let I be a finite interval of $\mathbb{R}$, and Λ be countable subset of $\mathbb{R}$. For the classical Paley-Wiener space $(L^2(I), \{\frac{e^{2\pi i x(\cdot)}}{\sqrt{m(I)}}\}_{x\in(\mathbb{R},|\cdot|,m_I)})$, if Λ is relatively separated with $D^-_{m_I}(\Lambda) > 1$, then Λ is a sampling set for $L^2(I)$. Namely, there exist constants $A, B > 0$ such that*

$$A \|f\|^2_{L^2(I)} \leq \sum_{\lambda\in\Lambda} \left| \left\langle f, e^{2\pi i\lambda(\cdot)} \right\rangle_{L^2(I)} \right|^2 \leq B \|f\|^2_{L^2(I)}, \quad \forall f \in L^2(I).$$

The following theorem could be viewed as an analogue for "continuous sampling sets" in the classical Paley-Wiener space.

Theorem 4.1.1 ([9], Theorem 1 Logvinenko-Sereda)**.** *Let E be an interval with the length b and let S be a measurable subset of $\mathbb{R}$. If there exist $a > 0$ and $\gamma > 0$ such that $|S \cap I| := m(S \cap I) \geq \gamma a$ for every interval I with length a, then*

$$\left(\frac{C}{\gamma}\right)^{-C(ab+1)} \left\|\widehat{f}\right\|^2_{L^2(\mathbb{R})} \leq \int_S \left|\widehat{f}(x)\right|^2 dx, \tag{4.1}$$

for any function $f \in L^2(E)$.

Note that the above inequality implies

$$\left(\frac{C}{\gamma}\right)^{-C(ab+1)} \|f\|^2_{L^2(E)} \le \int_S \left| \left\langle f, e^{2\pi i x(\cdot)} \right\rangle_{L^2(E)} \right|^2 dx \le \|f\|^2_{L^2(E)}.$$

It shows us the exponentials $\{\frac{e^{2\pi ix(\cdot)}}{\sqrt{m(E)}}\}_{x\in(S,m_E)}$ forms a continuous frame of $L^2(E)$ with lower frame bound $\left(\frac{C}{\gamma}\right)^{-C(ab+1)}$ and upper frame bound 1. By this observation, Theorem 4.1.1 tell us that when the subset S is pretty "dense" in $\mathbb{R}$, S could be viewed as a "continuous sampling set" of the classical Paley-Wiener space $(L^2(E), \{\frac{e^{2\pi ix(\cdot)}}{\sqrt{m(E)}}\}_{x\in(\mathbb{R},|\cdot|,m_E)})$.

In 1973, Logvinenko and Sereda first proved inequality (4.1) for some constant of lower frame bound. Then Kovrijkine improved it by giving a formula for that lower frame bound. In addition, Kovrijkine developed and generalized Theorem 4.1.1 even for multiband-limited functions.

Theorem 4.1.2 ([9], Theorem 2 Kovrijkine)**.** *Let $E := \cup_{i=1}^n I_i$ be the union of n intervals with the same length b. And let S be a measurable subset of $\mathbb{R}$. If there exist $a > 0$ and $\gamma > 0$ such that $|S \cap I| \ge \gamma a$ for every interval I of length a, then*

$$\left(\frac{C}{\gamma}\right)^{-2ab\left(\frac{C}{\gamma}\right)^n - 2n+1} \left\|\widehat{f}\right\|^2_{L^2(\mathbb{R})} \le \int_S \left|\widehat{f}(x)\right|^2 dx, \tag{4.2}$$

for any function $f \in L^2(E)$.

Notice the inverses of lower frame bound appearing in (4.1) and (4.2) both grow exponentially with the length of E. The goal of this chapter is to provide a better lower frame bound whose inverse would grow linearly with the length of E under certain conditions.

4.2 Lower Frame Bound Estimates for Continuous Frames of Exponentials

4.2.1 Main Theorems

In order to state our theorems in an easier way, it is better to impose the conditions on set Σ, which plays the role of the complement of set S in Theorem 4.1.1 and Theorem 4.1.2. In those theorems, we impose conditions to make set S "dense" in $\mathbb{R}$. For our theorems, we would like to make set Σ kind of "sparse".

When Σ is a countable union of bounded intervals, we find two different ways to make set Σ "sparse". One way is to require that the length of intervals and the density of centers of intervals are small; another way is to require the density of Σ itself is small. Based on these, we formulate the following two theorems.

Theorem 4.2.1. *Let $E := \cup_{i=1}^{n} I_i$ be the union of n bounded intervals. And let $\Lambda := \{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence, and $\Sigma := \cup_{k\in\mathbb{Z}}(\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2})$ be the countable union of intervals with center λ_k and length b. If $[b]D^+(\Lambda) < \frac{1}{n}$, then there exists a constant C such that*

$$C\left\|\widehat{f}\right\|_{L^2(\mathbb{R})}^2 \le \int_{\Sigma^c} |\widehat{f}(x)|^2 dx,$$

for any function $f \in L^2(E)$. And C^{-1} grows linearly with $|E|$ when $|E|$ is large. Note: $[b]$ denote the biggest integer less or equal to b.

Theorem 4.2.2. *Let $E := \cup_{i=1}^{n} I_i$ be the union of n bounded intervals. And let $\Sigma = \cup_{k\in\mathbb{Z}}(\lambda_k - \frac{b_k}{2}, \lambda_k + \frac{b_k}{2})$ be the countable disjoint union of intervals with the center λ_k and the length b_k. If $\inf_k\{b_k\} \ge b > 0$ for some constant b, and $D^+(\Sigma) < \frac{1}{n}$, then there exists a constant C such that*

$$C\left\|\widehat{f}\right\|_{L^2(\mathbb{R})}^2 \le \int_{\Sigma^c} |\widehat{f}(x)|^2 dx,$$

for any function $f \in L^2(E)$. And C^{-1} grows linearly with $|E|$ when $|E|$ is large. Note: $D^+(\Sigma) = \limsup_{r\to\infty}\sup_{a\in\mathbb{R}} \frac{|\Sigma\cap[a-r,a+r]|}{2r}$ as usual.

4.2.2 Random Periodization

In order to prove Theorem 4.2.1 and Theorem 4.2.2, we will adopt the proof strategy of Nazarov in [14], where the random periodization was introduced.

Definition 4.2.3. *Given any function $f \in L^1(\mathbb{R})$ and any positive number ε. We define the random periodization $g^{\varepsilon\nu}$ of f by*

$$g^{\varepsilon\nu}(t) := \frac{1}{\sqrt{\varepsilon\nu}} \sum_{k\in\mathbb{Z}} f(\frac{k+t}{\varepsilon\nu}), \quad t \in [0,1)$$

where ν is a random variable equidistributed on the interval $(1,2)$.

The series in the definition of $g^{\varepsilon\nu}$ converges in $L^1[0,1)$, and for every ν this series represents a 1-periodic function on $\mathbb{R}$. The random periodization $g^{\varepsilon\nu}$ allows some useful properties which will be stated in the following two propositions.

Proposition 4.2.4. *Given any function $f \in L^1(\mathbb{R})$, let $g^{\varepsilon\nu}$ be the random periodization of f. Then the Fourier coefficients of $g^{\varepsilon\nu}$ satisfy*

$$\widehat{g^{\varepsilon\nu}}(k) = \sqrt{\varepsilon\nu}\widehat{f}(\varepsilon\nu k),$$

for any $k \in \mathbb{Z}$.

Proof. By Fubini's theorem, we have

$$\begin{aligned}
\widehat{g^{\varepsilon\nu}}(k) &= \int_0^1 g^{\varepsilon\nu}(t)e^{-2\pi ikt}dt \\
&= \int_0^1 \frac{1}{\sqrt{\varepsilon\nu}} \sum_{n\in\mathbb{Z}} f(\frac{n+t}{\varepsilon\nu})e^{-2\pi ikt}dt \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 \frac{1}{\sqrt{\varepsilon\nu}} f(\frac{n+t}{\varepsilon\nu})e^{-2\pi ikt}dt \\
&= \sum_{n\in\mathbb{Z}} \int_n^{n+1} \frac{1}{\sqrt{\varepsilon\nu}} f(\frac{t}{\varepsilon\nu})e^{-2\pi ikt}dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\varepsilon\nu}} f(\frac{t}{\varepsilon\nu})e^{-2\pi ikt}dt \\
&= \sqrt{\varepsilon\nu}\widehat{f}(\varepsilon\nu k).
\end{aligned}$$

□

Let Σ be a measurable subset of $\mathbb{R}$ and α be a positive number. Define the following subset of $\mathbb{Z}$

$$\Lambda(\alpha, \Sigma) := \{k \in \mathbb{Z} : k\alpha \in \Sigma\},$$

then we have the following property:

Proposition 4.2.5 ([14], Proposition 2.2)**.** *Let $g^{\varepsilon\nu}$ be the random periodization of the function f, and $\Lambda := \Lambda(\varepsilon\nu, \Sigma)$ be the subset of $\mathbb{Z}$. Then*

$$\mathbf{E} \sum_{0\neq k\notin\Lambda} |\widehat{g^{\varepsilon\nu}}(k)|^2 \leq 2\int_{\Sigma^c} |\widehat{f}(x)|^2 dx.$$

Note: **E** *denote the expectation of random variable.*

To prove Proposition 4.2.5, we need the random lattice averaging lemma:

Lemma 4.2.6 ([14], Lemma 2.1)**.** *Let $\varphi : \mathbb{R} \to [0, \infty)$ be a positive function, and let $\varepsilon > 0$ be a fixed number. Then*

$$\int_1^2 \sum_{k \neq 0} \varphi(k\varepsilon\nu) d\nu \leq \frac{1}{\varepsilon} \int_{\mathbb{R}} \varphi(x) dx.$$

Proof. Split the left hand side into two terms,

$$\int_1^2 \sum_{k \neq 0} \varphi(k\varepsilon\nu) d\nu = \int_1^2 \sum_{k>0} \varphi(k\varepsilon\nu) d\nu + \int_1^2 \sum_{k<0} \varphi(k\varepsilon\nu) d\nu.$$

Estimating the first term, by Tonelli's theorem we have

$$\begin{aligned}
\int_1^2 \sum_{k>0} \varphi(k\varepsilon\nu) d\nu &= \sum_{k>0} \int_1^2 \varphi(k\varepsilon\nu) d\nu \\
&= \sum_{k>0} \frac{1}{\varepsilon k} \int_{\varepsilon k}^{2\varepsilon k} \varphi(x) dx \quad (x = k\varepsilon\nu) \\
&= \frac{1}{\varepsilon} \sum_{k>0} \int_{\varepsilon k}^{2\varepsilon k} \frac{\varphi(x)}{k} dx \\
&= \frac{1}{\varepsilon} \int_{\varepsilon}^{\infty} \sum_{\frac{x}{2\varepsilon} < k < \frac{x}{\varepsilon}} \frac{\varphi(x)}{k} dx \\
&= \frac{1}{\varepsilon} \int_0^{\infty} \varphi(x) \sum_{\frac{x}{2\varepsilon} < k < \frac{x}{\varepsilon}} \frac{1}{k} dx \\
&\leq \frac{1}{\varepsilon} \int_0^{\infty} \varphi(x) dx, \qquad (4.3)
\end{aligned}$$

where the last inequality comes from $\sum_{\frac{x}{2\varepsilon} < k < \frac{x}{\varepsilon}} \frac{1}{k} \leq 1$ for any $x > 0$. Similarly, for the second term, we have

$$\int_1^2 \sum_{k<0} \varphi(k\varepsilon\nu) d\nu \leq \frac{1}{\varepsilon} \int_{-\infty}^0 \varphi(x) dx. \qquad (4.4)$$

Adding (4.3) and (4.4) together, we obtain

$$\int_1^2 \sum_{k \neq 0} \varphi(k\varepsilon\nu) d\nu \leq \frac{1}{\varepsilon} \int_{\mathbb{R}} \varphi(x) dx.$$

□

Proof of Proposition 4.2.5. By Proposition 4.2.4 and Lemma 4.2.6, we have

$$\begin{aligned}
\mathbf{E}\sum_{0\neq k\notin\Lambda}|\widehat{g^{\varepsilon\nu}}(k)|^2 &= \mathbf{E}\sum_{0\neq k\notin\Lambda}\varepsilon\nu|\widehat{f}(k\varepsilon\nu)|^2 \\
&= \mathbf{E}\sum_{k\neq 0}\varepsilon\nu|\widehat{f}(k\varepsilon\nu)|^2 1_{\Sigma^c}(k\varepsilon\nu) \\
&\leq 2\varepsilon\mathbf{E}\sum_{k\neq 0}|\widehat{f}(k\varepsilon\nu)|^2 1_{\Sigma^c}(k\varepsilon\nu) \\
&= 2\varepsilon\int_1^2\sum_{k\neq 0}|\widehat{f}(k\varepsilon\nu)|^2 1_{\Sigma^c}(k\varepsilon\nu)d\nu \\
&\leq 2\int_{\Sigma^c}|\widehat{f}(x)|^2 dx.
\end{aligned}$$

□

4.2.3 Proof of Theorem 4.2.1

To prove Theorem 4.2.1, we need Beurling and Kahane's interpolation theorem (see Theorem 2.6.5). For convenience, we modify the statement and restate as the following:

Theorem. *2.6.5 Let Λ be a sequence of real numbers, and I be an interval of torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. If Λ is separated with $D^+(\Lambda) < |I|$, then there exists a constant C such that*

$$\sum_{\lambda\in\Lambda}|c_\lambda|^2 \leq C\|\sum_{\lambda\in\Lambda}c_\lambda e^{2\pi i\lambda(\cdot)}\|^2_{L^2(I)}$$

for any $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$. Where the constant C only depends on $D^+(\Lambda)$ and $|I|$, i.e., $C = C(D^+(\Lambda), |I|)$.

Besides of Beurling and Kahane's interpolation theorem, we also need the following lemmas. Consider any function $f \in L^2(\mathbb{R})$ supported on the δ-neighbourhood E_δ of E for some $\delta > 0$. Let $f_y := M_{-y}f$ be the modulation of f, and $g_y^{\varepsilon\nu}$ be the random periodization of f_y.

Let $E_y^{\varepsilon\nu} := \{t \in \mathbb{T} : g_y^{\varepsilon\nu}(t) \neq 0\}$ be the support of $g_y^{\varepsilon\nu}$, and $F_y^{\varepsilon\nu} := \{t \in \mathbb{T} : g_y^{\varepsilon\nu}(t) = 0\}$ be the zero set of $g_y^{\varepsilon\nu}$ which is the complement of $E_y^{\varepsilon\nu}$ in $\mathbb{T}$.

And denote $\Lambda_y^{\varepsilon\nu} := \{k \in \mathbb{Z} : k\varepsilon\nu \in \Sigma - y\}$, where $\Sigma - y := \{t - y : t \in \Sigma\}$.

Lemma 4.2.7.

$$D^+(\Lambda_y^{\varepsilon\nu}) \leq ([b] + 2\varepsilon)D^+(\Lambda).$$

Proof. Without loss of generality, consider the case when $y = 0$. Let $\varepsilon\nu\Lambda_0^{\varepsilon\nu} := \{k\varepsilon\nu | k\varepsilon\nu \in \Sigma, k \in \mathbb{Z}\}$ be the $\varepsilon\nu$ multiple of $\Lambda_0^{\varepsilon\nu}$. It can be viewed as the intersection of random lattice $\varepsilon\nu\mathbb{Z}$ and Σ. For any $r > 0$ and $a \in \mathbb{R}$

$$
\begin{aligned}
&\#\{\varepsilon\nu\Lambda_0^{\varepsilon\nu} \cap (a-r, a+r)\} \\
=&\#\{k\varepsilon\nu : k\varepsilon\nu \in \Sigma \cap (a-r, a+r)\} \\
\le&\#\{k\varepsilon\nu : k\varepsilon\nu \in \cup_{a-r-\frac{b}{2}<\lambda_k<a+r+\frac{b}{2}}(\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2})\} \\
\le& \sum_{a-r-\frac{b}{2}<\lambda_k<a+r+\frac{b}{2}} \#\{k\varepsilon\nu : k\varepsilon\nu \in (\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2})\} \\
\le& \sum_{a-r-\frac{b}{2}<\lambda_k<a+r+\frac{b}{2}} \lceil \frac{b}{\varepsilon\nu} \rceil \\
=&\lceil \frac{b}{\varepsilon\nu} \rceil \#\{\Lambda \cap (a-r-\frac{b}{2}, a+r+\frac{b}{2})\},
\end{aligned}
$$

where $\lceil \frac{b}{\varepsilon\nu} \rceil$ denote the smallest integer bigger or equal to $\frac{b}{\varepsilon\nu}$. Then

$$
\begin{aligned}
&D^+(\varepsilon\nu\Lambda_0^{\varepsilon\nu}) \\
=& \lim_{r\to\infty} \sup_a \frac{\#\{\varepsilon\nu\Lambda_0^{\varepsilon\nu} \cap (a-r, a+r)\}}{2r} \\
\le& \lim_{r\to\infty} \sup_a \frac{\lceil \frac{b}{\varepsilon\nu} \rceil \#\{\Lambda \cap (a-r-\frac{b}{2}, a+r+\frac{b}{2})\}}{2r} \\
=&\lceil \frac{b}{\varepsilon\nu} \rceil \lim_{r\to\infty} \sup_a \frac{\#\{\Lambda \cap (a-r-\frac{b}{2}, a+r+\frac{b}{2})\}}{2r+b} \frac{2r+b}{2r} \\
=&\lceil \frac{b}{\varepsilon\nu} \rceil D^+(\Lambda).
\end{aligned}
$$

Simple computation shows that

$$
\begin{aligned}
D^+(\Lambda_0^{\varepsilon\nu}) &= \varepsilon\nu D^+(\varepsilon\nu\Lambda_0^{\varepsilon\nu}) \\
&\le \varepsilon\nu \lceil \frac{b}{\varepsilon\nu} \rceil D^+(\Lambda) \\
&\le ([b] + 2\varepsilon) D^+(\Lambda).
\end{aligned}
$$

□

Lemma 4.2.8. *For* $\varepsilon < \frac{1}{2|E_\delta|}$, $F_y^{\varepsilon\nu}$ *contains an interval* $I^{\varepsilon\nu}$ *with length* $|I^{\varepsilon\nu}| > (1 - 2\varepsilon|E_\delta|)\frac{1}{n}$.

Proof. By $g_y^{\varepsilon\nu}$ is the random periodization of f_y,

$$g_y^{\varepsilon\nu}(t) = \frac{1}{\sqrt{\varepsilon\nu}} \sum_{k\in\mathbb{Z}} f(\frac{k+t}{\varepsilon\nu}) e^{-2\pi i(\frac{k+t}{\varepsilon\nu})y}.$$

By $E_y^{\varepsilon\nu} := \{t \in \mathbb{T} : g_y^{\varepsilon\nu}(t) \neq 0\}$ is the support of $g_y^{\varepsilon\nu}$,

$$\begin{aligned}
\forall t \in E_y^{\varepsilon\nu} &\Rightarrow g_y^{\varepsilon\nu}(t) \neq 0, \\
&\Rightarrow \exists k_0 \in \mathbb{Z}, s.t. f(\frac{k_0+t}{\varepsilon\nu}) \neq 0, \\
&\Rightarrow \frac{k_0+t}{\varepsilon\nu} \in E_\delta, \\
&\Rightarrow t \in \varepsilon\nu E_\delta - k_0, \\
&\Rightarrow t \in \varepsilon\nu E_\delta (mod 1),
\end{aligned}$$

where $\varepsilon\nu E_\delta := \{\varepsilon\nu t : t \in E_\delta\}$, and $\varepsilon\nu E_\delta(mod1) := \{t + \mathbb{Z} : t \in \varepsilon\nu E_\delta\} \subseteq \mathbb{T}$.

So $E_y^{\varepsilon\nu} \subseteq \varepsilon\nu E_\delta(mod1) \subseteq \mathbb{T}$. Since E_δ is at most n union of intervals, so is $\varepsilon\nu E_\delta(mod1)$. Then $\mathbb{T} \setminus \varepsilon\nu E_\delta(mod1)$ is at most n union of intervals as well. Notice

$$\begin{aligned}
|\mathbb{T} \setminus \varepsilon\nu E_\delta(mod1)| &= 1 - |\varepsilon\nu E_\delta(mod1)| \\
&\geqslant 1 - |\varepsilon\nu E_\delta| \\
&> 1 - 2\varepsilon |E_\delta|.
\end{aligned}$$

So $\mathbb{T} \setminus \varepsilon\nu E_\delta(mod1)$ contains an interval I with length $|I| > \frac{1-2\varepsilon|E_\delta|}{n}$. By $F_y^{\varepsilon\nu} = \mathbb{T} \setminus E_y^{\varepsilon\nu} \supset \mathbb{T} \setminus \varepsilon\nu E_\delta(mod1)$, $F_y^{\varepsilon\nu}$ contains interval I as well. □

Lemma 4.2.9. *If* $[b]D^+(\Lambda) < \frac{1}{n}$*, then there exists a constant* C *such that*

$$\sup_{y\in\Sigma} |\widehat{f}(y)|^2 \leq C \int_{\Sigma^c} |\widehat{f}(x)|^2 dx,$$

for any function $f \in L^2(\mathbb{R})$ *supported on* E_δ.

Proof. Let $g_y^{\varepsilon\nu}$ be the random periodization of $f \in L^2(\mathbb{R})$ supported on E_δ. Rewrite $g_y^{\varepsilon\nu}$ as the sum of functions $p_y^{\varepsilon\nu}$ and $q_y^{\varepsilon\nu}$,

$$\begin{aligned} g_y^{\varepsilon\nu}(t) &= \sum_{k\in\mathbb{Z}} \widehat{g_y^{\varepsilon\nu}}(k)e^{2\pi ikt} \\ &= p_y^{\varepsilon\nu}(t) + q_y^{\varepsilon\nu}(t), \end{aligned}$$

where $p_y^{\varepsilon\nu}(t) := \sum_{k\in\Lambda_y^{\varepsilon\nu}} \widehat{g_y^{\varepsilon\nu}}(k)e^{2\pi ikt}$ and $q_y^{\varepsilon\nu}(t) := \sum_{k\notin\Lambda_y^{\varepsilon\nu}} \widehat{g_y^{\varepsilon\nu}}(k)e^{2\pi ikt}$.

When $y \in \Sigma$, it implies $0 \in \Lambda_y^{\varepsilon\nu}$. Then for any $y \in \Sigma$,

$$\begin{aligned} |\widehat{f}(y)|^2 &= |\widehat{f}_y(0)|^2 \\ &= \frac{1}{\varepsilon\nu}|\widehat{g_y^{\varepsilon\nu}}(0)|^2 \\ &\leq \frac{1}{\varepsilon}|\widehat{g_y^{\varepsilon\nu}}(0)|^2 \\ &\leq \frac{1}{\varepsilon}\sum_{k\in\Lambda_y^{\varepsilon\nu}} |\widehat{g_y^{\varepsilon\nu}}(k)|^2. \end{aligned} \tag{4.5}$$

On one hand, by lemma 4.2.7, $D^+(\Lambda_y^{\varepsilon\nu}) \leq ([b]+2\varepsilon)D^+(\Lambda)$. Let $\varepsilon \to 0^+$,

$$D^+(\Lambda_y^{\varepsilon\nu}) \leq ([b]+2\varepsilon)D^+(\Lambda) \downarrow [b]D^+(\Lambda).$$

On the other hand, by lemma 4.2.8, $F_y^{\varepsilon\nu}$ contains an interval $I^{\varepsilon\nu}$ with length $|I^{\varepsilon\nu}| > (1-2\varepsilon|E_\delta|)\frac{1}{n}$. Let $\varepsilon \to 0^+$,

$$|I^{\varepsilon\nu}| > (1-2\varepsilon|E_\delta|)\frac{1}{n} \uparrow \frac{1}{n}.$$

Since $[b]D^+(\Lambda) < \frac{1}{n}$, let $d := \frac{1}{n} - [b]D^+(\Lambda) > 0$, and $\varepsilon_0 := \min\{\frac{d}{6D^+(\Lambda)}, \frac{nd}{6|E_\delta|}\}$. Then

$$D^+(\Lambda_y^{\varepsilon_0\nu}) \leq [b]D^+(\Lambda) + \frac{d}{3} < \frac{1}{n} - \frac{d}{3} < |I^{\varepsilon_0\nu}|.$$

By Theorem 2.6.5, there exists a constant $C_0 = C_0([b]D^+(\Lambda)+\frac{d}{3}, \frac{1}{n}-\frac{d}{3})$ such that

$$\sum_{k\in\Lambda_y^{\varepsilon_0\nu}} \left|\widehat{g_y^{\varepsilon_0\nu}}(k)\right|^2 \leq C_0 \left\|\sum_{k\in\Lambda_y^{\varepsilon_0\nu}} \widehat{g_y^{\varepsilon_0\nu}}(k)e^{2\pi ik(\cdot)}\right\|^2_{L^2(I^{\varepsilon_0\nu})}.$$

Notice $g_y^{\varepsilon_0\nu}(t) = p_y^{\varepsilon_0\nu}(t) + q_y^{\varepsilon_0\nu}(t) = 0$ on its zero set $F_y^{\varepsilon_0\nu}$ which contains interval $I^{\varepsilon_0\nu}$. Combining (4.5), we have for any $y \in \Sigma$,

$$\begin{aligned}
|\widehat{f}(y)|^2 &\le \frac{1}{\varepsilon_0} \sum_{k \in \Lambda_y^{\varepsilon_0 \nu}} |\widehat{g_y^{\varepsilon_0 \nu}}(k)|^2 \\
&\le \frac{C_0}{\varepsilon_0} \| \sum_{k \in \Lambda_y^{\varepsilon_0 \nu}} \widehat{g_y^{\varepsilon_0 \nu}}(k) e^{2\pi i k (\cdot)} \|^2_{L^2(I^{\varepsilon_0 \nu})} \\
&= \frac{C_0}{\varepsilon_0} \| p_y^{\varepsilon_0 \nu} \|^2_{L^2(I^{\varepsilon_0 \nu})} \\
&= \frac{C_0}{\varepsilon_0} \| q_y^{\varepsilon_0 \nu} \|^2_{L^2(I^{\varepsilon_0 \nu})} \\
&\le \frac{C_0}{\varepsilon_0} \| q_y^{\varepsilon_0 \nu} \|^2 \\
&= \frac{C_0}{\varepsilon_0} \sum_{k \notin \Lambda_y^{\varepsilon_0 \nu}} |\widehat{g_y^{\varepsilon_0 \nu}}(k)|^2.
\end{aligned} \tag{4.6}$$

By Proposition 4.2.5,

$$\mathbf{E} \| q_y^{\varepsilon_0 \nu} \|^2 = \mathbf{E} \sum_{k \notin \Lambda_y^{\varepsilon_0 \nu}} |\widehat{g^{\varepsilon_0 \nu}}(k)|^2 \le 2 \int_{\Sigma^c} |\widehat{f}(x)|^2 dx.$$

By Chebyshev's inequality,

$$\begin{aligned}
&\mathbf{P}\{ \| q_y^{\varepsilon_0 \nu} \|^2 > 4 \int_{\Sigma^c} |\widehat{f}|^2 \} \\
\le &\mathbf{P}\{ \| q_y^{\varepsilon_0 \nu} \|^2 > 2 \mathbf{E} \| q_y^{\varepsilon_0 \nu} \|^2 \} \\
< &\frac{\mathbf{E} \| q_y^{\varepsilon_0 \nu} \|^2}{2 \mathbf{E} \| q_y^{\varepsilon_0 \nu} \|^2} \\
= &\frac{1}{2}.
\end{aligned}$$

Then,

$$\mathbf{P}\{ \| q_y^{\varepsilon_0 \nu} \|^2 \le 4 \int_{\Sigma^c} |\widehat{f}|^2 \} > \frac{1}{2}.$$

So there exists a $\nu_0 \in (1,2)$ such that

$$\| q_y^{\varepsilon_0 \nu_0} \|^2 \le 4 \int_{\Sigma^c} |\widehat{f}(x)|^2 dx.$$

Combine (4.6), we have

$$|\widehat{f}(y)|^2 \le \frac{C_0}{\varepsilon_0} \| q_y^{\varepsilon_0 \nu_0} \|^2 \le \frac{4 C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{f}(x)|^2 dx.$$

Since $y \in \Sigma$ is arbitrary,

$$\sup_{y\in\Sigma} |\widehat{f}(y)|^2 \leq \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{f}(x)|^2 dx,$$

where $\varepsilon_0 = \min\{\frac{d}{6D^+(\Lambda)}, \frac{nd}{6|E_\delta|}\}$ with $d = \frac{1}{n} - [b]D^+(\Lambda)$, and C_0 does not depend on $|E|$. □

Proof of theorem 4.2.1. For every function $f \in L^2(E)$ and every $y \in \Sigma$, define

$$h_y(t) := \frac{e^{2\pi iyt}}{2\delta} 1_{(\delta,\delta)}(t), \quad t \in \mathbb{R},$$

and

$$F_y(t) := f * h_y(t), \quad t \in \mathbb{R},$$

where $f * h_y$ denote the convolution of f and h_y. Then

$$\widehat{F_y}(x) = \widehat{f}(x) \frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}, \tag{4.7}$$

and $suppF_y \subseteq E_\delta$. By Lemma 4.2.9,

$$\sup_{x\in\Sigma} |\widehat{F_y}(x)|^2 \leq \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{F_y}(x)|^2 dx. \tag{4.8}$$

Let $x = y \in \Sigma$. Combining (4.7) and (4.8), we obtain

$$|\widehat{f}(y)|^2 = |\widehat{F_y}(y)|^2 \leq \sup_{x\in\Sigma} |\widehat{F_y}(x)|^2 \leq \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{F_y}(x)|^2 dx.$$

Integrating this inequality over Σ on both sides, we have

$$\begin{aligned}
\int_\Sigma |\widehat{f}(y)|^2 dy &\leq \int_\Sigma \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{F_y}(x)|^2 dx dy \\
&= \frac{4C_0}{\varepsilon_0} \int_\Sigma \int_{\Sigma^c} |\widehat{f}(x)|^2 |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dx dy \\
&= \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} \int_\Sigma |\widehat{f}(x)|^2 |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dy dx \\
&= \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{f}(x)|^2 \int_\Sigma |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dy dx \\
&\leq \frac{4C_0}{\varepsilon_0} \int_{\Sigma^c} |\widehat{f}(x)|^2 \int_{\mathbb{R}} |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dy dx \\
&= \frac{2C_0}{\varepsilon_0\delta} \int_{\Sigma^c} |\widehat{f}(x)|^2 dx.
\end{aligned}$$

Then we obtain

$$\frac{\varepsilon_0\delta}{\varepsilon_0\delta+2C_0}\left\|\widehat{f}\right\|^2_{L^2(\mathbb{R})}\leq\int_{\Sigma^c}|\widehat{f}(x)|^2dx.$$

Since $\varepsilon_0=\min\{\frac{d}{6D^+(\Lambda)},\frac{nd}{6|E_\delta|}\}$ with $d=\frac{1}{n}-[b]D^+(\Lambda)$, and C_0 does not rely on $|E|$ (see Lemma 4.2.9). Let $\delta=\frac{1}{2n}$, for large enough $|E|$ we have

$$C:=\frac{d}{(24C_0+1)(|E|+1)}\leq\frac{\varepsilon_0\delta}{\varepsilon_0\delta+2C_0}.$$

It follows that

$$C\left\|\widehat{f}\right\|^2_{L^2(\mathbb{R})}\leq\int_{\Sigma^c}|\widehat{f}(x)|^2dx,$$

where C^{-1} grows linearly with $|E|$ when $|E|$ is large. □

4.2.4 Proof of Theorem 4.2.2

In order to prove Theorem 4.2.2, we need the following new lemmas which play the same role with Lemmas 4.2.7 and 4.2.9 for the proof of Theorem 4.2.1.

Again, consider any function $f\in L^2(\mathbb{R})$ supported on the δ-neighbourhood E_δ of E for some $\delta>0$. Let $f_y:=M_{-y}f$ be the modulation of f, and $g_y^{\varepsilon\nu}$ be the random periodization of f_y.

Let $E_y^{\varepsilon\nu}:=\{t\in\mathbb{T}:g_y^{\varepsilon\nu}(t)\neq 0\}$ be the support of $g_y^{\varepsilon\nu}$, and $F_y^{\varepsilon\nu}:=\{t\in\mathbb{T}:g_y^{\varepsilon\nu}(t)=0\}$ be the zero set of $g_y^{\varepsilon\nu}$.

And denote $\Lambda_y^{\varepsilon\nu}:=\{k\in\mathbb{Z}:k\varepsilon\nu\in\Sigma-y\}$.

Lemma 4.2.10. *If* $\inf_k\{b_k\}\geqslant b$ *for some positive number* b*, then*

$$D^+(\Lambda_y^{\varepsilon\nu})\leq(1+\frac{2\varepsilon}{b})D^+(\Sigma).$$

Proof. Without loss of generality, consider the case when $y=0$. Let $\varepsilon\nu\Lambda_0^{\varepsilon\nu}:=\{k\varepsilon\nu:k\varepsilon\nu\in\Sigma,k\in\mathbb{Z}\}$ be the $\varepsilon\nu$ multiple of $\Lambda_0^{\varepsilon\nu}$. For any $r>0$ and $a\in\mathbb{R}$, by $\inf_k\{b_k\}\geqslant b>0$, we have finite number of intervals $(\lambda_k-\frac{b_k}{2},\lambda_k+\frac{b_k}{2})\subseteq\Sigma$ which intersect with interval $(a-r,a+r)$. Denote the length of those intersections by $\{l_1, l_2, \cdots, l_n\}$, then

$$\begin{aligned}
&\#\{\varepsilon\nu\Lambda_0^{\varepsilon\nu} \cap (a-r, a+r)\} \\
=&\#\{k\varepsilon\nu : k\varepsilon\nu \in \Sigma \cap (a-r, a+r)\} \\
\leq&\lceil \frac{l_1}{\varepsilon\nu} \rceil + \lceil \frac{l_2}{\varepsilon\nu} \rceil + \cdots + \lceil \frac{l_n}{\varepsilon\nu} \rceil \\
<&(\frac{l_1}{\varepsilon\nu} + 1) + (\frac{l_2}{\varepsilon\nu} + 1) + \cdots + (\frac{l_n}{\varepsilon\nu} + 1) \\
=&\frac{l_1 + l_2 + \cdots + l_n}{\varepsilon\nu} + n \\
=&\frac{|\Sigma \cap (a-r, a+r)|}{\varepsilon\nu} + n.
\end{aligned}$$

Furthermore, it's easy to see $(n-2)b \leq |\Sigma \cap (a-r, a+r)|$, so

$$n \leq \frac{|\Sigma \cap (a-r, a+r)|}{b} + 2.$$

Then

$$\begin{aligned}
D^+(\varepsilon\nu\Lambda_0^{\varepsilon\nu}) &= \limsup_{r\to\infty} \sup_a \frac{\#\{\varepsilon\nu\Lambda_0^{\varepsilon\nu} \cap (a-r, a+r)\}}{2r} \\
&\leq \limsup_{r\to\infty} \sup_a \frac{\frac{|\Sigma\cap(a-r,a+r)|}{\varepsilon\nu} + n}{2r} \\
&\leq \limsup_{r\to\infty} \sup_a \frac{\frac{|\Sigma\cap(a-r,a+r)|}{\varepsilon\nu} + \frac{|\Sigma\cap(a-r,a+r)|}{b} + 2}{2r} \\
&= \limsup_{r\to\infty} \sup_a \left((\frac{1}{\varepsilon\nu} + \frac{1}{b}) \frac{|\Sigma \cap (a-r, a+r)|}{2r} + \frac{2}{2r} \right) \\
&= (\frac{1}{\varepsilon\nu} + \frac{1}{b}) \limsup_{r\to\infty} \sup_a \frac{|\Sigma \cap (a-r, a+r)|}{2r} \\
&= (\frac{1}{\varepsilon\nu} + \frac{1}{b}) D^+(\Sigma).
\end{aligned}$$

By simple computations,

$$\begin{aligned}
D^+(\Lambda_0^{\varepsilon\nu}) &= \varepsilon\nu D^+(\varepsilon\nu\Lambda_0^{\varepsilon\nu}) \\
&\leq \varepsilon\nu(\frac{1}{\varepsilon\nu} + \frac{1}{b}) D^+(\Sigma) \\
&\leq (1 + \frac{2\varepsilon}{b}) D^+(\Sigma).
\end{aligned}$$

□

Lemma 4.2.11. *If $D^+(\Sigma) < \frac{1}{n}$, then there exists a constant C such that*

$$\sup_{y\in\Sigma} |\widehat{f}(y)|^2 \leq C \int_{\Sigma^c} |\widehat{f}(x)|^2 dx,$$

for any function $f \in L^2(\mathbb{R})$ supported on E_δ.

Proof. Let $g_y^{\varepsilon\nu}$ be the random periodization of $f \in L^2(\mathbb{R})$ supported on E_δ. We can rewrite $g_y^{\varepsilon\nu}$ as the sum of two functions.

$$\begin{aligned} g_y^{\varepsilon\nu}(t) &= \sum_{k\in\mathbb{Z}} \widehat{g_y^{\varepsilon\nu}}(k) e^{2\pi ikt} \\ &= p_y^{\varepsilon\nu}(t) + q_y^{\varepsilon\nu}(t), \end{aligned}$$

where $p_y^{\varepsilon\nu}(t) = \sum_{k\in\Lambda_y^{\varepsilon\nu}} \widehat{g_y^{\varepsilon\nu}}(k) e^{2\pi ikt}$, $q_y^{\varepsilon\nu}(t) = \sum_{k\notin\Lambda_y^{\varepsilon\nu}} \widehat{g_y^{\varepsilon\nu}}(k) e^{2\pi ikt}$.

When $y \in \Sigma$, $0 \in \Lambda_y^{\varepsilon\nu}$. Then for $y \in \Sigma$,

$$\begin{aligned} |\widehat{f}(y)|^2 &= |\widehat{f}_y(0)|^2 \\ &= \frac{1}{\varepsilon\nu} |\widehat{g_y^{\varepsilon\nu}}(0)|^2 \\ &\leq \frac{1}{\varepsilon} |\widehat{g_y^{\varepsilon\nu}}(0)|^2 \\ &\leq \frac{1}{\varepsilon} \sum_{k\in\Lambda_y^{\varepsilon\nu}} |\widehat{g_y^{\varepsilon\nu}}(k)|^2. \end{aligned} \tag{4.9}$$

By lemma 4.2.10, $D^+(\Lambda_y^{\varepsilon\nu}) \leq (1 + \frac{2\varepsilon}{b}) D^+(\Sigma)$. Let $\varepsilon \to 0^+$,

$$D^+(\Lambda_y^{\varepsilon\nu}) \leq (1 + \frac{2\varepsilon}{b}) D^+(\Sigma) \downarrow D^+(\Sigma).$$

By lemma 4.2.8, $F_y^{\varepsilon\nu}$ contains an interval $I^{\varepsilon\nu}$ with length $|I^{\varepsilon\nu}| > (1 - 2\varepsilon|E_\delta|)\frac{1}{n}$. Let $\varepsilon \to 0^+$,

$$|I^{\varepsilon\nu}| > (1 - 2\varepsilon|E_\delta|)\frac{1}{n} \uparrow \frac{1}{n}.$$

Since $D^+(\Sigma) < \frac{1}{n}$, let $d := \frac{1}{n} - D^+(\Sigma) > 0$, and $\varepsilon_0 := \min\{\frac{bd}{6D^+(\Sigma)}, \frac{nd}{6|E_\delta|}\}$. Then

$$D^+(\Lambda_y^{\varepsilon_0\nu}) \leq D^+(\Sigma) + \frac{d}{3} < \frac{1}{n} - \frac{d}{3} < |I^{\varepsilon_0\nu}|.$$

Again, by Theorem 2.6.5, there exists a constant $C_0 = C_0(D^+(\Sigma) + \frac{d}{3}, \frac{1}{n} - \frac{d}{3})$ such that

$$\sum_{k\in\Lambda_y^{\varepsilon_0\nu}} \left|\widehat{g_y^{\varepsilon_0\nu}}(k)\right|^2 \leq C_0 \left\|\sum_{k\in\Lambda_y^{\varepsilon_0\nu}} \widehat{g_y^{\varepsilon_0\nu}}(k)e^{2\pi ik(\cdot)}\right\|^2_{L^2(I^{\varepsilon_0\nu})}.$$

By $g_y^{\varepsilon_0\nu}(t) = p_y^{\varepsilon_0\nu}(t) + q_y^{\varepsilon_0\nu}(t) = 0$ on its zero set $F_y^{\varepsilon_0\nu}$ which contains the interval $I^{\varepsilon_0\nu}$. Combining (4.9), we have for every $y \in \Sigma$,

$$\begin{aligned}
|\widehat{f}(y)|^2 &\leq \frac{1}{\varepsilon_0} \sum_{k\in\Lambda_y^{\varepsilon_0\nu}} |\widehat{g_y^{\varepsilon_0\nu}}(k)|^2 \\
&\leq \frac{C_0}{\varepsilon_0} \| \sum_{k\in\Lambda_y^{\varepsilon_0\nu}} \widehat{g_y^{\varepsilon_0\nu}}(k)e^{2\pi ik(\cdot)}\|^2_{L^2(I^{\varepsilon_0\nu})} \\
&= \frac{C_0}{\varepsilon_0}\|p_y^{\varepsilon_0\nu}\|^2_{L^2(I^{\varepsilon_0\nu})} \\
&= \frac{C_0}{\varepsilon_0}\|q_y^{\varepsilon_0\nu}\|^2_{L^2(I^{\varepsilon_0\nu})} \\
&\leq \frac{C_0}{\varepsilon_0}\|q_y^{\varepsilon_0\nu}\|^2 \qquad (4.10) \\
&= \frac{C_0}{\varepsilon_0} \sum_{k\notin\Lambda_y^{\varepsilon_0\nu}} |\widehat{g_y^{\varepsilon_0\nu}}(k)|^2.
\end{aligned}$$

By Proposition 4.2.5,

$$\mathbf{E}\|q_y^{\varepsilon_0\nu}\|^2 = \mathbf{E} \sum_{k\notin\Lambda_y^{\varepsilon_0\nu}} |\widehat{g^{\varepsilon_0\nu}}(k)|^2 \leq 2\int_{\Sigma^c} |\widehat{f}(x)|^2 dx.$$

By Chebyshev's inequality,

$$\begin{aligned}
&\mathbf{P}\{\|q_y^{\varepsilon_0\nu}\|^2 > 4\int_{\Sigma^c}|\widehat{f}|^2\} \\
\leq&\mathbf{P}\{\|q_y^{\varepsilon_0\nu}\|^2 > 2\mathbf{E}\|q_y^{\varepsilon_0\nu}\|^2\} \\
<&\frac{\mathbf{E}\|q_y^{\varepsilon_0\nu}\|^2}{2\mathbf{E}\|q_y^{\varepsilon_0\nu}\|^2} \\
=&\frac{1}{2}.
\end{aligned}$$

Then,

$$\mathbf{P}\{\|q_y^{\varepsilon_0\nu}\|^2 \leq 4\int_{\Sigma^c}|\widehat{f}|^2\} > \frac{1}{2}.$$

So there exists a $\nu_0 \in (1,2)$ such that

$$\|q_y^{\varepsilon_0\nu_0}\|^2 \leq 4\int_{\Sigma^c}|\widehat{f}(x)|^2 dx.$$

Combining (4.10), we obtain

$$|\widehat{f}(y)|^2 \leq \frac{C_0}{\varepsilon_0}\|q_y^{\varepsilon_0\nu_0}\|^2 \leq \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c}|\widehat{f}(x)|^2 dx.$$

Since $y \in \Sigma$ is arbitrary,

$$\sup_{y\in\Sigma}|\widehat{f}(y)|^2 \leq \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c}|\widehat{f}(x)|^2 dx,$$

where $\varepsilon_0 = \min\{\frac{bd}{6D^+(\Sigma)}, \frac{nd}{6|E_\delta|}\}$ with $d = \frac{1}{n} - D^+(\Sigma) > 0$, and C_0 does not depend on $|E|$. □

Proof of theorem 4.2.2. For every function $f \in L^2(E)$ and every $y \in \Sigma$, define

$$h_y(t) := \frac{e^{2\pi i y t}}{2\delta}1_{(\delta,\delta)}(t), \quad t \in \mathbb{R},$$

and

$$F_y(t) := f * h_y(t), \quad t \in \mathbb{R}.$$

Then

$$\widehat{F_y}(x) = \widehat{f}(x)\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)},$$

and $suppF_y \subseteq E_\delta$. By Lemma 4.2.11

$$\sup_{x\in\Sigma} |\widehat{F_y}(x)|^2 \le \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c} |\widehat{F_y}(x)|^2 dx.$$

Let $x = y \in \Sigma$, we obtain

$$|\widehat{f}(y)|^2 = |\widehat{F_y}(y)|^2 \le \sup_{x\in\Sigma} |\widehat{F_y}(x)|^2 \le \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c} |\widehat{F_y}(x)|^2 dx.$$

Integrating the inequality over Σ on both sides, we get:

$$\begin{aligned}
\int_\Sigma |\widehat{f}(y)|^2 dy &\le \int_\Sigma \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c} |\widehat{F_y}(x)|^2 dxdy \\
&= \frac{4C_0}{\varepsilon_0}\int_\Sigma\int_{\Sigma^c} |\widehat{f}(x)|^2 |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dxdy \\
&= \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c}\int_{\Sigma} |\widehat{f}(x)|^2 |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dydx \\
&= \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c} |\widehat{f}(x)|^2 \int_{\Sigma} |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dydx \\
&\le \frac{4C_0}{\varepsilon_0}\int_{\Sigma^c} |\widehat{f}(x)|^2 \int_{\mathbb{R}} |\frac{\sin(2\pi\delta(x-y))}{2\pi\delta(x-y)}|^2 dydx \\
&= \frac{2C_0}{\varepsilon_0\delta}\int_{\Sigma^c} |\widehat{f}(x)|^2 dx.
\end{aligned}$$

Then we obtain

$$\frac{\varepsilon_0\delta}{\varepsilon_0\delta + 2C_0}\left\|\widehat{f}\right\|^2_{L^2(\mathbb{R})} \le \int_{\Sigma^c} |\widehat{f}(x)|^2 dx.$$

Since $\varepsilon_0 = \min\{\frac{bd}{6D^+(\Sigma)}, \frac{nd}{6|E_\delta|}\}$ with $d = \frac{1}{n} - D^+(\Sigma) > 0$, and C_0 does not depend on $|E|$ (see Lemma 4.2.11). Let $\delta = \frac{1}{2n}$, for large enough $|E|$ we have

$$C := \frac{d}{(24C_0+1)(|E|+1)} \le \frac{\varepsilon_0\delta}{\varepsilon_0\delta + 2C_0}.$$

It follows that

$$C\left\|\widehat{f}\right\|^2_{L^2(\mathbb{R})} \le \int_{\Sigma^c} |\widehat{f}(x)|^2 dx,$$

where C^{-1} grows linearly with $|E|$ when $|E|$ is large. □

Chapter 5

Uncertainty Principles in Framed Hilbert Spaces

5.1 Introduction

5.1.1 Uncertainty Principle for Gabor Riesz Bases

Let X and D be the multiplication and differentiation operators on their domain $\mathcal{D}(X)$ and $\mathcal{D}(D)$ (see Section 1.1). For any $f \in L^2(\mathbb{R})$ and any $a, b \in \mathbb{R}$, we define the notations:

$$\|(X-a)f\|_2^2 := \int_{\mathbb{R}} |(x-a)f(x)|^2 \, dx = \infty, \text{ when } f \notin \mathcal{D}(X),$$

$$\|(D-a)f\|_2^2 := \int_{\mathbb{R}} \left|(\xi-b)\widehat{f}(\xi)\right|^2 d\xi = \infty, \text{ when } f \notin \mathcal{D}(D).$$

Recall the classical uncertainty principle (Theorem 1.2.1): for any $f \in L^2(\mathbb{R})$ and any $a, b \in \mathbb{R}$,

$$\|(X-a)f\|_2^2 \, \|(D-b)f\|_2^2 \geq \frac{1}{16\pi^2} \, \|f\|_2^4 . \tag{5.1}$$

And the classical Balian-Low theorem (Theorem 1.3.2) could be restated as: Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ with $\alpha\beta = 1$. If the Gabor system $\{M_{m\beta}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ forms a Riesz basis for $L^2(\mathbb{R})$, then for any $a, b \in \mathbb{R}$,

$$\|(X-a)f\|_2^2 \, \|(D-b)f\|_2^2 = \infty, \tag{5.2}$$

or equivalently

$$\|(X-a)f\|_2^2 + \|(D-b)f\|_2^2 = \infty.$$

Comparing the similarity of (5.1) and (5.2), the Balian-Low theorem could be view as the uncertainty principle type of results for Gabor Riesz Bases.

5.1.2 Uncertainty Principle for General Riesz Bases

It is natural for mathematicians to release the Gabor system structures and work on more general bases in next step. So Meyer asks the following question: is it true that for any orthonormal basis $\{f_n\}_{n=1}^{\infty} \subseteq L^2(\mathbb{R})$ (which may not be a Gabor system) and any sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, we have

$$\sup_{n\in\mathbb{N}} \left(\|(X-a_n)f_n\|_2^2 + \|(D-b_n)f_n\|_2^2 \right) = \infty. \tag{5.3}$$

If this is true, then it would be a "generalized Balian-Low theorem" for general orthonormal bases. However the answer is No! Bourgain constructed an orthonormal basis $\{f_n\}_{n=1}^{\infty}$ of $L^2(\mathbb{R})$ such that (5.3) fails.

Based on this fact, the next question will be: can we impose some extra conditions to make (5.3) still hold. In 2011, Gröchenig and Malinnikova proved the following very similar equality holds for general Riesz bases.

Theorem 5.1.1 ([6], Theorem 1). *If $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis of $L^2(\mathbb{R})$ and $s > 1$, then*

$$\sup_{n\in\mathbb{N}} \left(\int_{\mathbb{R}} |x-a_n|^{2s} |f_n(x)|^2 dx + \int_{\mathbb{R}} |\xi - b_n|^{2s} |\widehat{f}_n(\xi)|^2 d\xi \right) = \infty,$$

for any sequences of real numbers $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$.

By studying Theorem 5.1.1, we get some hints to formulate the conditions for our "generalized Balian-Low theorem".

5.2 Generalized Balian-Low Theorem

5.2.1 Main Results

In order to introduce our main theorem, we set up the following definition and notation:

Definition 5.2.1. *We say a countable subset Λ of a metric space (X, d) decays annularly around point $a \in X$, if for any $\rho > 0$*

$$\lim_{r\to\infty} \frac{\#\{\Lambda \cap (B(a, r+\rho) - B(a, r))\}}{\#\{\Lambda \cap B(a, r)\}} = 0.$$

Let A and B be two operators on a Hilbert space $\mathcal{H}$, denote their domain by $\mathcal{D}(A) := \{f \in \mathcal{H} : Af \in \mathcal{H}\}$ and $\mathcal{D}(B) := \{f \in \mathcal{H} : Bf \in \mathcal{H}\}$. When $f, g \in \mathcal{D}(A) \cap \mathcal{D}(B)$, we define the notation

$$\langle i[B, A]f, g\rangle := i\,\langle Af, Bg\rangle - i\,\langle Bf, Ag\rangle\,.$$

We now state our generalized Balian-Low theorem:

Theorem 5.2.2. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be dual Riesz bases of $\mathcal{H}$, and $\Lambda := \{\lambda_n\}_{n=1}^{\infty}$ be a countable subset of some metric space (X, d) which decays annularly around point $a \in X$ and satisfies $\#\{\Lambda \cap B(a, r)\} < \infty$ for any $r > 0$. And let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be two symmetric operators such that for every $n \in \mathbb{N}$, $\langle i[B, A]g_n, f_n\rangle \geq c$ for some $c > 0$. If*

1. $\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_m-\lambda_n)>r} |\langle Ag_n, f_m\rangle|^2 = 0,$
2. $\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_m-\lambda_n)>r} |\langle Bg_n, f_m\rangle|^2 = 0.$

Then for any sequences of complex numbers $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$,

$$\sup_{n\in\mathbb{N}} \left(\|(A - a_n)f_n\|^2 + \|(B - b_n)f_n\|^2\right) = \infty.$$

Notice if we let $\Lambda := \mathbb{Z}^d$ embedded in the Euclidean metric space $(\mathbb{R}^d, |\cdot|)$, then Λ would satisfy all the requirements in Theorem 5.2.2. Based on this fact, we have the following corollary:

Corollary 5.2.3. *Let $\{f_n\}_{n\in\mathbb{Z}^d}$ be an orthonormal basis of $\mathcal{H}$. And let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be two symmetric operators such that for every $n \in \mathbb{Z}^d$, $\langle i[B, A]f_n, f_n\rangle \geq c$ for some $c > 0$. If*

1. $\lim_{r\to\infty} \sup_{n\in\mathbb{Z}^d} \sum_{|m-n|>r} |\langle Af_n, f_m\rangle|^2 = 0$,

2. $\lim_{r\to\infty} \sup_{n\in\mathbb{Z}^d} \sum_{|m-n|>r} |\langle Bf_n, f_m\rangle|^2 = 0$.

Then for any sequences of complex numbers $\{a_n\}_{n\in\mathbb{Z}^d}, \{b_n\}_{n\in\mathbb{Z}^d}$,

$$\sup_{n\in\mathbb{Z}^d} \left(\|(A-a_n)f_n\|^2 + \|(B-b_n)f_n\|^2 \right) = \infty.$$

5.2.2 Proof of Theorem 5.2.2

Proof. Suppose there exists $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \left(\|(A-a_n)f_n\|^2 + \|(B-b_n)f_n\|^2 \right) \le C < \infty.$$

We claim that a contradiction will follow from this assumption.

Since $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are dual Riesz bases, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} c &\le \langle i[A,B]g_n, f_n\rangle \\ &:= i\,\langle Bg_n, Af_n\rangle - i\,\langle Ag_n, Bf_n\rangle \\ &= i\sum_{m\in\mathbb{N}} \langle Bg_n, f_m\rangle \langle g_m, Af_n\rangle - i\sum_{m\in\mathbb{N}} \langle Ag_n, f_m\rangle \langle g_m, Bf_n\rangle \\ &= i\sum_{m\in\mathbb{N}} \langle Bg_n, f_m\rangle \langle g_m, Af_n\rangle - \langle Ag_n, f_m\rangle \langle g_m, Bf_n\rangle . \end{aligned}$$

Let $B := B(a,r)$ be the ball with the fixed center a (for which Λ decays annularly around) and radius $r > 0$, then

$$\begin{aligned} c\#\{\Lambda\cap B\} &\le i \sum_{\lambda_n\in\Lambda\cap B}\sum_{m\in\mathbb{N}} \langle Bg_n, f_m\rangle \langle g_m, Af_n\rangle - \langle Ag_n, f_m\rangle \langle g_m, Bf_n\rangle \\ &= i \sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B} \langle Bg_n, f_m\rangle \langle g_m, Af_n\rangle - \langle Ag_n, f_m\rangle \langle g_m, Bf_n\rangle \\ &+ i \sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c} \langle Bg_n, f_m\rangle \langle g_m, Af_n\rangle - \langle Ag_n, f_m\rangle \langle g_m, Bf_n\rangle . \end{aligned}$$

By A, B are symmetric, the first term vanishes. So we have

$$c\#\{\Lambda \cap B\} \leq i \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} \langle Bg_n, f_m \rangle \langle g_m, Af_n \rangle - \langle Ag_n, f_m \rangle \langle g_m, Bf_n \rangle .$$

Put absolute value sign inside, we obtain

$$\begin{aligned} c\#\{\Lambda \cap B\} &\leq \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Bg_n, f_m \rangle \langle g_m, Af_n \rangle - \langle Ag_n, f_m \rangle \langle g_m, Bf_n \rangle| \\ &\leq \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Bg_n, f_m \rangle \langle g_m, Af_n \rangle| + |\langle Ag_n, f_m \rangle \langle g_m, Bf_n \rangle| \\ &\leq \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Bg_n, f_m \rangle \langle g_m, Af_n \rangle| \\ &\quad + \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Ag_n, f_m \rangle \langle g_m, Bf_n \rangle| . \end{aligned} \tag{5.4}$$

Apply Cauchy-Schwarz twice for the first term, we have

$$\begin{aligned} &\sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Bg_n, f_m \rangle \langle g_m, Af_n \rangle| \\ &\leq \left(\sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Bg_n, f_m \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle g_m, Af_n \rangle|^2 \right)^{\frac{1}{2}} . \end{aligned}$$

Let $\rho > 0$ and denote $B(a, r + \rho)$ by B_ρ which is the ρ-neighbourhood of B. By the assumption, the first factor satisfies

$$
\begin{aligned}
&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Bg_n,f_m\rangle|^2\\
=&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B_\rho^c}|\langle Bg_n,f_m\rangle|^2+\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap(B_\rho\setminus B)}|\langle Bg_n,f_m\rangle|^2\\
\le&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+\sum_{\lambda_m\in\Lambda\cap(B_\rho\setminus B)}\sum_{\lambda_n\in\Lambda\cap B}|\langle g_n,Bf_m\rangle|^2\\
=&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+\sum_{\lambda_m\in\Lambda\cap(B_\rho\setminus B)}\sum_{\lambda_n\in\Lambda\cap B}|\langle g_n,(B-b_m)f_m\rangle|^2\\
\le&\sum_{\lambda_n\in\Lambda\cap B}\sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+\sum_{\lambda_m\in\Lambda\cap(B_\rho\setminus B)}C\,\|(B-b_m)f_m\|^2\\
\le&\ \#\{\Lambda\cap B\}\sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+\sum_{\lambda_m\in\Lambda\cap(B_\rho\setminus B)}C\\
=&\ \#\{\Lambda\cap B\}\sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+C\#\{\Lambda\cap(B_\rho\setminus B)\}.
\end{aligned}
$$

Again by the assumption, the second factor satisfies

$$
\begin{aligned}
&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle g_m,Af_n\rangle|^2=\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle g_m,(A-a_n)f_n\rangle|^2\\
\le&\sum_{\lambda_n\in\Lambda\cap B}C\,\|(A-a_n)f_n\|^2\le\sum_{\lambda_n\in\Lambda\cap B}C\le C\#\{\Lambda\cap B\}.
\end{aligned}
$$

Then for some $C>0$, we get an estimate for the first term,

$$
\begin{aligned}
&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Bg_n,f_m\rangle\langle g_m,Af_n\rangle|\\
\le&C\#\{\Lambda\cap B\}^{\frac12}\left(\#\{\Lambda\cap B\}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+\#\{\Lambda\cap(B_\rho\setminus B)\}\right)^{\frac12}.
\end{aligned}
\tag{5.5}
$$

Similarly, for some $C>0$, we get an estimate for the second term,

$$
\begin{aligned}
&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Ag_n,f_m\rangle\langle g_m,Bf_n\rangle|\\
\le&C\#\{\Lambda\cap B\}^{\frac12}\left(\#\{\Lambda\cap B\}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Ag_n,f_m\rangle|^2+\#\{\Lambda\cap(B_\rho\setminus B)\}\right)^{\frac12}.
\end{aligned}
\tag{5.6}
$$

So for any $r > 0$ and $\rho > 0$, combine (5.4) (5.5) (5.6), we have

$$\begin{aligned}
c &\leq \frac{1}{\#\{\Lambda \cap B\}} \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Bg_n, f_m \rangle \langle g_m, Af_n \rangle| \\
&\quad + \frac{1}{\#\{\Lambda \cap B\}} \sum_{\lambda_n \in \Lambda \cap B} \sum_{\lambda_m \in \Lambda \cap B^c} |\langle Ag_n, f_m \rangle \langle g_m, Bf_n \rangle| \\
&\leq C \left(\sup_{n \in \mathbb{N}} \sum_{\lambda_m \in \Lambda \cap B(\lambda_n, \rho)^c} |\langle Bg_n, f_m \rangle|^2 + \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\#\{\Lambda \cap B\}} \right)^{\frac{1}{2}} \\
&\quad + C \left(\sup_{n \in \mathbb{N}} \sum_{\lambda_m \in \Lambda \cap B(\lambda_n, \rho)^c} |\langle Ag_n, f_m \rangle|^2 + \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\#\{\Lambda \cap B\}} \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.7}$$

Let $\varepsilon > 0$, by conditions 1 and 2, we can find a $\rho > 0$ such that

$$\sup_{n \in \mathbb{N}} \sum_{\lambda_m \in \Lambda \cap B(\lambda_n, \rho)^c} |\langle Bg_n, f_m \rangle|^2 < \varepsilon^2, \quad \sup_{n \in \mathbb{N}} \sum_{\lambda_m \in \Lambda \cap B(\lambda_n, \rho)^c} |\langle Ag_n, f_m \rangle|^2 < \varepsilon^2. \tag{5.8}$$

So for such ρ, combine (5.7) and (5.8), we have

$$\frac{c}{C} \leq \left(\varepsilon^2 + \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\#\{\Lambda \cap B\}} \right)^{\frac{1}{2}} + \left(\varepsilon^2 + \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\#\{\Lambda \cap B\}} \right)^{\frac{1}{2}},$$

for any $r > 0$. Let $r \to \infty$, by Λ decays annularly around point a,

$$\begin{aligned}
\frac{c}{C} &\leq \lim_{r \to \infty} \left(\varepsilon^2 + \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\#\{\Lambda \cap B\}} \right)^{\frac{1}{2}} + \left(\varepsilon^2 + \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\#\{\Lambda \cap B\}} \right)^{\frac{1}{2}} \\
&= \left(\varepsilon^2 + 0 \right)^{\frac{1}{2}} + \left(\varepsilon^2 + 0 \right)^{\frac{1}{2}} = 2\varepsilon.
\end{aligned}$$

By ε is arbitrarily small, contradiction completes. □

5.2.3 Applications

Apply Corollary 5.2.3, we could give another proof of the classical Balian-Low theorem.

Proof of Theorem 1.3.2. For convenience, we let $a = b = 1$. The multiplication operator X and differentiation operator D are two symmetric operators which will play the role of operators A and B in Corollary 5.2.3 respectively.

Again, we will prove the theorem by contradiction. Suppose

$$\int_{\mathbb{R}} |xg(x)|^2 dx \int_{\mathbb{R}} |\xi\widehat{g}(\xi)|^2 d\xi = \|Xg\|_2^2 \|Dg\|_2^2 < \infty. \tag{5.9}$$

Denote $\{M_m T_n g\}_{m,n\in\mathbb{Z}}$ by $\{g_{m,n}\}_{m,n\in\mathbb{Z}}$, then for any $m, n \in \mathbb{Z}$

$$\begin{aligned}
\|Xg_{m,n}\|_2^2 &:= \int_{\mathbb{R}} |Xg_{m,n}(x)|^2 dx \\
&= \int_{\mathbb{R}} \left|xe^{2\pi imx} g(x-n)\right|^2 dx \\
&= \int_{\mathbb{R}} |x+n|^2 |g(x)|^2 dx \\
&\leq 2\int_{\mathbb{R}} |x|^2 |g(x)|^2 dx + 2\int_{\mathbb{R}} |n|^2 |g(x)|^2 dx \\
&= 2\|Xg\|_2^2 + 2|n|^2 \|g\|_2^2 < \infty,
\end{aligned}$$

which implies $\{g_{m,n}\}_{m,n\in\mathbb{Z}} \subseteq \mathcal{D}(X)$. Similarly, we could obtain $\{g_{m,n}\}_{m,n\in\mathbb{Z}} \subseteq \mathcal{D}(D)$. By generalized canonical commutation relation (see (1.6)), we have for any $m, n \in \mathbb{Z}$

$$\begin{aligned}
&\langle i[D,X]g_{m,n}, g_{m,n}\rangle \\
:=& i\langle Xg_{m,n}, Dg_{m,n}\rangle - i\langle Dg_{m,n}, Xg_{m,n}\rangle \\
=& \frac{1}{2\pi}\|g_{m,n}\|_2^2 \\
=& \frac{1}{2\pi} > 0.
\end{aligned}$$

By (1.2), we obtain

$$
\begin{aligned}
&\sum_{\|(k,l)-(m,n)\|>r} |\langle Xg_{m,n}, g_{k,l}\rangle|^2 \\
&= \sum_{\|(k,l)-(m,n)\|>r} |\langle XM_mT_ng, g_{k,l}\rangle|^2 \\
&= \sum_{\|(k,l)-(m,n)\|>r} |\langle M_mXT_ng, g_{k,l}\rangle|^2 \\
&= \sum_{\|(k,l)-(m,n)\|>r} |\langle M_m(T_nX + nT_n)g, g_{k,l}\rangle|^2 \\
&= \sum_{\|(k,l)-(m,n)\|>r} |\langle M_mT_nXg, g_{k,l}\rangle|^2 \\
&= \sum_{\|(k,l)-(m,n)\|>r} |\langle M_mT_nXg, M_kT_lg\rangle|^2 \\
&= \sum_{\|(k,l)-(m,n)\|>r} \left|\left\langle Xg, M_{(k-m)}T_{(l-n)}g\right\rangle\right|^2 \\
&= \sum_{\|(k,l)\|>r} |\langle Xg, M_kT_lg\rangle|^2 . \\
&= \sum_{\|(k,l)\|>r} |\langle Xg, g_{k,l}\rangle|^2 .
\end{aligned}
$$

By $\{g_{m,n}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis,

$$
\begin{aligned}
&\lim_{r\to\infty} \sup_{m,n\in\mathbb{Z}} \sum_{\|(k,l)-(m,n)\|>r} |\langle Xg_{m,n}, g_{k,l}\rangle|^2 \\
&= \lim_{r\to\infty} \sup_{m,n\in\mathbb{Z}} \sum_{\|(k,l)\|>r} |\langle Xg, g_{k,l}\rangle|^2 \\
&= \lim_{r\to\infty} \sum_{\|(k,l)\|>r} |\langle Xg, g_{k,l}\rangle|^2 = 0.
\end{aligned}
$$

Similarly,

$$\lim_{r\to\infty} \sup_{m,n\in\mathbb{Z}} \sum_{\|(k,l)-(m,n)\|>r} |\langle Dg_{m,n}, g_{k,l}\rangle|^2 = 0.$$

Apply Corollary 5.2.3, we have for any sequences $\{a_{m,n}\}_{m,n\in\mathbb{Z}}$ and $\{b_{m,n}\}_{m,n\in\mathbb{Z}}$,

$$\sup_{m,n\in\mathbb{Z}} \left(\|(X - a_{m,n})g_{m,n}\|_2^2 + \|(D - b_{m,n})g_{m,n}\|_2^2\right) = \infty \tag{5.10}$$

On the other hand, let $a_{m,n} = n$ and $b_{m,n} = m$ for any $m, n \in \mathbb{Z}$. Again by (1.2), we have

$$\begin{aligned}
&\|(X-n)g_{m,n}\|_2^2 + \|(D-m)g_{m,n}\|_2^2 \\
&= \|(X-n)M_m T_n g\|_2^2 + \|(D-m)M_m T_n g\|_2^2 \\
&= \|M_m (X-n) T_n g\|_2^2 + \|M_m D T_n g\|_2^2 \\
&= \|M_m T_n X g\|_2^2 + \|M_m T_n D g\|_2^2 \\
&= \|Xg\|_2^2 + \|Dg\|_2^2 ,
\end{aligned}$$

for any $m, n \in \mathbb{Z}$. By assumption (5.9),

$$\begin{aligned}
&\sup_{m,n\in\mathbb{Z}} \left(\|(X-n)g_{m,n}\|_2^2 + \|(D-m)g_{m,n}\|_2^2 \right) \\
&= \sup_{m,n\in\mathbb{Z}} \left(\|Xg\|_2^2 + \|Dg\|_2^2 \right) \\
&= \|Xg\|_2^2 + \|Dg\|_2^2 < \infty.
\end{aligned}$$

Which contradicts with (5.10). □

5.3 Balian-Low Theorem in Framed Hilbert Spaces

5.3.1 Uniformly Localized Sequences in Framed Hilbert Spaces

Our next result is based on *FHS*. In order to state the theorem, we introduce the following definitions. Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a *FHS*, and $\Lambda := \{\lambda_n\}_{n=1}^{\infty}$ be a countable subset of X.

Definition 5.3.1. *A sequence $\{f_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ is said to be **uniformly localized** on Λ if*

$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \int_{B(\lambda_n,r)^c} |\langle f_n, k_x\rangle|^2 \, d\sigma(x) = 0.$$

Recall the *ULP* (in Section 2.4.2), it is not hard to see if $\{k_x\}_{x\in(X,d,\sigma)}$ satisfies the *ULP*, then the sequence $\{k_{\lambda_n}\}_{n=1}^{\infty}$ in $\{k_x\}_{x\in(X,d,\sigma)}$ is always uniformly localized on its index set $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$.

Definition 5.3.2. *Let $w : [0, \infty) \to [0, \infty)$ be a weight function satisfies $w(d) \uparrow \infty$ as $d \to \infty$. A* *sequence* $\{f_n\}_{n=1}^{\infty}$ *of* $\mathcal{H}$ *is said to be* ***uniformly w-localized*** *on* Λ *if*

$$\sup_{n\in\mathbb{N}} \int_X |\langle f_n, k_x\rangle|^2 w(d(\lambda_n, x))d\sigma(x) < \infty.$$

The following proposition proves that uniform w-localization implies uniform localization.

Proposition 5.3.3. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of* $\mathcal{H}$*. If* $\{f_n\}_{n=1}^{\infty}$ *is uniformly* w*-localized on* Λ*, then it is uniformly localized on* Λ*.*

Proof. By $\{f_n\}_{n=1}^{\infty}$ is uniformly w-localized on Λ,

$$\sup_{n\in\mathbb{N}} \int_X |\langle f_n, k_x\rangle|^2 w(d(\lambda_n, x))d\sigma(x) < \infty,$$

for some weight function w. It follows that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \int_{B(\lambda_n,r)^c} |\langle f_n, k_x\rangle|^2 d\sigma(x) \\
=&\sup_{n\in\mathbb{N}} \int_{B(\lambda_n,r)^c} |\langle f_n, k_x\rangle|^2 \frac{w(d(\lambda_n, x))}{w(d(\lambda_n, x))} d\sigma(x) \\
\leq&\sup_{n\in\mathbb{N}} \int_{B(\lambda_n,r)^c} |\langle f_n, k_x\rangle|^2 \frac{w(d(\lambda_n, x))}{w(r)} d\sigma(x) \\
\leq&\frac{1}{w(r)} \sup_{n\in\mathbb{N}} \int_X |\langle f_n, k_x\rangle|^2 w(d(\lambda_n, x))d\sigma(x) \\
\lesssim&\frac{1}{w(r)}.
\end{aligned}$$

By the property of w,

$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \int_{B(\lambda_n,r)^c} |\langle f_n, k_x\rangle|^2 d\sigma(x) \lesssim \lim_{r\to\infty} \frac{1}{w(r)} = 0.$$

□

Definition 5.3.4. *Let* $w : [0, \infty) \to [0, \infty)$ *be a weight function satisfies* $w(d) \uparrow \infty$ *as* $d \to \infty$*. We say* w *is* ***localized*** *on* Λ *if*

$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_n,\lambda_m)\geq r} \frac{1}{w(d(\lambda_n, \lambda_m))} = 0.$$

One example of localized weights is the power weight on the lattice of integers. Let $\Lambda := \mathbb{Z}^n$

be the subset of the Euclidean metric space $(\mathbb{R}^n, |\cdot|)$, and let the weight function $w(d) := d^s$ be the power function with $s > n$. Easily see $w(d) \uparrow \infty$ as $d \to \infty$.

By Euclidean metric is equivalent to maximum metric, i.e., $|\mathbf{x}| \simeq \max_i |x_i|$,

$$\begin{aligned}\sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{1}{|\mathbf{m}|^s} &\lesssim \sum_{\mathbf{m}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{1}{\max\{|m_1|, |m_2|, \cdots, |m_n|\}^s} \\ &= \sum_{l=1}^{\infty} \frac{1}{l^s}((l+1)^n - l^n) \\ &\leq \sum_{l=1}^{\infty} \frac{(2^n-1)l^{n-1}}{l^s} \\ &\lesssim \sum_{l=1}^{\infty} \frac{1}{l^{s-n+1}} \\ &< \infty. \end{aligned} \tag{5.11}$$

It implies

$$\begin{aligned} &\lim_{r\to\infty} \sup_{\mathbf{n}\in\mathbb{Z}^n} \sum_{|\mathbf{n}-\mathbf{m}|\geq r} \frac{1}{|\mathbf{n}-\mathbf{m}|^s} \\ =&\lim_{r\to\infty} \sum_{|\mathbf{m}|\geq r} \frac{1}{|\mathbf{m}|^s} \\ =&0. \end{aligned}$$

So in the Euclidean metric space $(\mathbb{R}^n, |\cdot|)$, the weight function $w(d) = d^s$ with $s > n$ is localized on $\Lambda = \mathbb{Z}^n$.

5.3.2 Main Results

Let $(\mathcal{H}, \{k_x\}_{x\in(X,d,\sigma)})$ be a *FHS* which satisfies *ADP* and *UMD*, then our next result is the following:

Theorem 5.3.5. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be dual Riesz bases of $\mathcal{H}$ where the index set $\Lambda := \{\lambda_n\}_{n=1}^{\infty}$ is relatively separated with $0 < D^+(\Lambda)$. Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be two symmetric operators such that for every $n \in \mathbb{N}$, $\langle i[B, A]g_n, f_n\rangle \geq c$ for some $c > 0$. If there exist sequences of complex numbers $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and a weight function w such that the following conditions hold:*

1. $\sup_{n\in\mathbb{N}}\left(\|(A-a_n)g_n\|^2+\|(B-b_n)g_n\|^2\right)<\infty$*,*

2. $\{(A-a_n)g_n\}_{n=1}^{\infty}$ *and* $\{(B-b_n)g_n\}_{n=1}^{\infty}$ *are uniformly localized on* Λ*,*

3. $\{f_n\}_{n=1}^{\infty}$ *is uniformly* w*-localized on* Λ*,*

4. w *is localized on* Λ*, and* $w(2d)\lesssim w(d)$ *for every* $d\geq 0$*.*

Then

$$\sup_{n\in\mathbb{N}}\left(\|(A-a_n)f_n\|^2+\|(B-b_n)f_n\|^2\right)=\infty.$$

5.3.3 Prerequisites

Before we give a proof of Theorem 5.3.5, we need to do some preparations.

Proposition 5.3.6. *Let* (X,d,σ) *be a metric measure space satisfying ADP and UMD, and* Λ *be a relatively separated set in* X*. Then for every* $\rho>0$*,*

$$\limsup_{r\to\infty}\sup_{a\in X}\frac{\#\{\Lambda\cap(B(a,r+\rho)\setminus B(a,r))\}}{\sigma(B(a,r))}=0.$$

Proof. Since Λ is relatively separated, we have

$$\Lambda=\cup_{k=1}^{K}\Lambda_k,$$

where Λ_k is a separated set with the separation constant δ_k for any $k=1,\cdots,K$. Let $\delta:=\min_{1\leq k\leq K}\delta_k$, and $\{\lambda_i^k\}_{i\in I_k}:=\{\Lambda_k\cap B(a,r)\}$ for any $k=1,\cdots,K$.

By *UMD*, there exist $c(\frac{\delta}{2})>0$ and $D(r+\frac{\delta}{2})>0$ such that

$$\begin{aligned}&\#\{\Lambda\cap(B(a,r+\rho)\setminus B(a,r))\}\\ \leq&\sum_{k=1}^{K}\#\{\Lambda_k\cap(B(a,r+\rho)\setminus B(a,r))\}\\ \leq&\sum_{k=1}^{K}\frac{\sigma(B(a,r+\rho+\frac{\delta}{2})\setminus B(a,r-\frac{\delta}{2}))}{c(\frac{\delta}{2})}\\ =&\frac{K}{c(\frac{\delta}{2})}\sigma(B(a,r+\rho+\frac{\delta}{2})\setminus B(a,r-\frac{\delta}{2})).\end{aligned}$$

Combine ADP,

$$
\begin{aligned}
&\limsup_{r\to\infty}\sup_{a\in X}\frac{\#\{\Lambda\cap(B(a,r+\rho)\setminus B(a,r))\}}{\sigma(B(a,r))}\\
\leq&\limsup_{r\to\infty}\sup_{a\in X}\frac{K}{c(\frac{\delta}{2})}\frac{\sigma(B(a,r+\rho+\frac{\delta}{2})\setminus B(a,r-\frac{\delta}{2}))}{\sigma(B(a,r))}\\
=&\frac{K}{c(\frac{\delta}{2})}\limsup_{r\to\infty}\sup_{a\in X}\frac{\sigma(B(a,r+\rho+\frac{\delta}{2})\setminus B(a,r-\frac{\delta}{2}))}{\sigma(B(a,r))}\\
\leq&\frac{K}{c(\frac{\delta}{2})}\limsup_{r\to\infty}\sup_{a\in X}\frac{\sigma(B(a,r+\rho+\frac{\delta}{2})\setminus B(a,r-\frac{\delta}{2}))}{\sigma(B(a,r-\frac{\delta}{2}))}\\
=&\frac{K}{c(\frac{\delta}{2})}\limsup_{r\to\infty}\sup_{a\in X}\frac{\sigma(B(a,r+\rho+\delta)\setminus B(a,r))}{\sigma(B(a,r))}\\
=&0.
\end{aligned}
$$

$\square$

The next proposition is very important for us. We will use it to verify

$$\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle Ag_n,f_m\rangle|^2=0,$$

and

$$\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle Bg_n,f_m\rangle|^2=0,$$

hold in Theorem 5.3.5.

Proposition 5.3.7. *Let $w:[0,\infty)\to[0,\infty)$ be a weight function satisfies $w(d)\uparrow\infty$ as $d\to\infty$. Let $\{h_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$, and $\{f_n\}_{n=1}^\infty$ be a frame of $\mathcal{H}$. If the following conditions hold*

1. $\sup_{n\in\mathbb{N}}\|h_n\|^2<\infty$,

2. $\{h_n\}_{n=1}^\infty$ *is uniformly localized on* Λ,

3. $\{f_n\}_{n=1}^\infty$ *is uniformly* w*-localized on* Λ,

4. w *is localized on* Λ, *and* $w(2d)\lesssim w(d)$ *for every* d,

then

$$\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle h_n,f_m\rangle|^2=0.$$

Proof. Fix any $\lambda_n \in \Lambda$, and define the ball $B := B(\lambda_n, \frac{r}{2})$ for some $r > 0$. And let T_B, T_{B^c} be two concentration operators on $\mathcal{H}$ given by

$$T_B f := \int_B \langle f, k_x \rangle k_x d\sigma(x), \quad T_{B^c} f := \int_{B^c} \langle f, k_x \rangle k_x d\sigma(x).$$

Then for any $f \in \mathcal{H}$,

$$\begin{aligned} f &= \int_X \langle f, k_x \rangle k_x d\sigma(x) \\ &= \int_B \langle f, k_x \rangle k_x d\sigma(x) + \int_{B^c} \langle f, k_x \rangle k_x d\sigma(x) \\ &= T_B f + T_{B^c} f. \end{aligned}$$

So the identity map $I = T_B + T_{B^c}$. It follows that

$$\begin{aligned} |\langle h_n, f_m \rangle|^2 &= |\langle (T_B + T_{B^c}) h_n, f_m \rangle|^2 \\ &= |\langle T_B h_n, f_m \rangle + \langle T_{B^c} h_n, f_m \rangle|^2 \\ &\leq 2 |\langle T_B h_n, f_m \rangle|^2 + 2 |\langle T_{B^c} h_n, f_m \rangle|^2 . \end{aligned} \tag{5.12}$$

Estimate the first term to the right in (5.12). By condition 1 and 3,

$$\begin{aligned} |\langle T_B h_n, f_m \rangle| &= \left| \int_B \langle h_n, k_x \rangle \langle k_x, f_m \rangle d\sigma(x) \right| \\ &\leq \int_X |\langle h_n, k_x \rangle|^2 d\sigma(x) \int_B |\langle k_x, f_m \rangle|^2 d\sigma(x) \\ &= \|h_n\|^2 \int_B |\langle f_m, k_x \rangle|^2 \frac{w(d(\lambda_m, x))}{w(d(\lambda_m, x))} d\sigma(x) \\ &\leq \sup_{n \in \mathbb{N}} \|h_n\|^2 \sup_{x \in B} \frac{1}{w(d(\lambda_m, x))} \int_X |\langle f_m, k_x \rangle|^2 w(d(\lambda_m, x)) d\sigma(x) \\ &\lesssim \sup_{x \in B} \frac{1}{w(d(\lambda_m, x))}. \end{aligned} \tag{5.13}$$

Notice when $x \in B(\lambda_n, r/2)$ and $d(\lambda_m, \lambda_n) > r$, we have

$$\begin{aligned} d(\lambda_m, \lambda_n) &\leq d(\lambda_m, x) + d(x, \lambda_n) \\ &\leq d(\lambda_m, x) + r/2 \\ &\leq 2d(\lambda_m, x). \end{aligned} \tag{5.14}$$

Combine (5.13) and (5.14) , by condition 4 we obtain

$$\begin{aligned} &\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_m,\lambda_n)>r} |\langle T_B h_n, f_m\rangle|^2 \\ \lesssim &\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_m,\lambda_n)>r} \sup_{x\in B(\lambda_n,\frac{r}{2})} \frac{1}{w(d(\lambda_m, x))} \\ \leq &\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_m,\lambda_n)>r} \frac{1}{w(\frac{d(\lambda_m,\lambda_n)}{2})} \\ \lesssim &\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \sum_{d(\lambda_m,\lambda_n)>r} \frac{1}{w(d(\lambda_m, \lambda_n))} \\ = &0. \end{aligned} \tag{5.15}$$

Estimate the second term to the right in (5.12). By $\{f_n\}_{n=1}^{\infty}$ is a frame and $\{k_x\}_{x\in X}$ is a continuous Parseval frame,

$$
\begin{aligned}
&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle T_{B^c}h_n,f_m\rangle|^2\\
\le&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_m|\langle T_{B^c}h_n,f_m\rangle|^2\\
\lesssim&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\|T_{B^c}h_n\|^2\\
=&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\left\|\int_{B^c}\langle h_n,k_x\rangle k_x d\sigma(x)\right\|^2\\
=&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sup_{\|g\|=1}\left|\left\langle\int_{B^c}\langle h_n,k_x\rangle k_x d\sigma(x),g\right\rangle\right|^2\\
=&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sup_{\|g\|=1}\left|\int_{B^c}\langle h_n,k_x\rangle\langle k_x,g\rangle d\sigma(x)\right|^2\\
\le&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sup_{\|g\|=1}\int_{B^c}|\langle h_n,k_x\rangle|^2 d\sigma(x)\int_X|\langle k_x,g\rangle|^2 d\sigma(x)\\
=&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sup_{\|g\|=1}\int_{B^c}|\langle h_n,k_x\rangle|^2 d\sigma(x)\,\|g\|^2\\
=&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\int_{B^c}|\langle h_n,k_x\rangle|^2 d\sigma(x)\\
=&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\int_{B(\lambda_n,\frac{r}{2})^c}|\langle h_n,k_x\rangle|^2 d\sigma(x)\\
=&0,
\end{aligned}
\tag{5.16}
$$

where the last equality comes from condition 2. Combining (5.12), (5.15) and (5.16), we have

$$
\begin{aligned}
&\lim_{r\to\infty}\sup_n\sum_{d(\lambda_m,\lambda_n)>r}|\langle h_n,f_m\rangle|^2\\
\lesssim&\lim_{r\to\infty}\sup_n\sum_{d(\lambda_m,\lambda_n)>r}|\langle T_Bh_n,f_m\rangle|^2+|\langle T_{B^c}h_n,f_m\rangle|^2\\
=&0.
\end{aligned}
$$

□

5.3.4 Proof of Theorem 5.3.5

Proof. By conditions 1-4 of Theorem 5.3.5, applying Proposition 5.3.7, we have

$$\begin{aligned}&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle Ag_n,f_m\rangle|^2\\ =&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle (A-a_n)g_n,f_m\rangle|^2=0;\end{aligned}\tag{5.17}$$

and

$$\begin{aligned}&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle Bg_n,f_m\rangle|^2\\ =&\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\sum_{d(\lambda_m,\lambda_n)>r}|\langle (B-b_n)g_n,f_m\rangle|^2=0.\end{aligned}\tag{5.18}$$

Notice we have different assumptions on Λ from Theorem 5.2.2, we cannot use Theorem 5.2.2 to prove Theorem 5.3.5 directly. However, we could borrow the first half of the proof of Theorem 5.2.2. Again, we assume

$$\sup_{n\in\mathbb{N}}\left(\|(A-a_n)f_n\|^2+\|(B-b_n)f_n\|^2\right)<\infty,$$

and expect to derive a contradiction.

Let $B:=B(a,r)$ be any ball with the center $a\in X$ and the radius $r>0$, as in (5.4) we have

$$\begin{aligned}c\#\{\Lambda\cap B\}\le&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Bg_n,f_m\rangle\langle g_m,Af_n\rangle|\\ &+\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Ag_n,f_m\rangle\langle g_m,Bf_n\rangle|.\end{aligned}\tag{5.19}$$

Let $\rho>0$ and denote $B(a,r+\rho)$ by B_ρ which is the ρ-neighbourhood of B. As in (5.5), there exists some $C>0$ such that

$$\begin{aligned}&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Bg_n,f_m\rangle\langle g_m,Af_n\rangle|\\ \le&C\#\{\Lambda\cap B\}^{\frac12}\left(\#\{\Lambda\cap B\}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n,f_m\rangle|^2+\#\{\Lambda\cap(B_\rho\setminus B)\}\right)^{\frac12}.\end{aligned}\tag{5.20}$$

Similarly, as in (5.6), for some $C > 0$ we have

$$\begin{aligned}&\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Ag_n, f_m\rangle\langle g_m, Bf_n\rangle|\\ \leq& C\#\{\Lambda\cap B\}^{\frac{1}{2}}\left(\#\{\Lambda\cap B\}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Ag_n, f_m\rangle|^2+\#\{\Lambda\cap(B_\rho\setminus B)\}\right)^{\frac{1}{2}}.\end{aligned} \tag{5.21}$$

Divide (5.19) by $\sigma(B)$ on both sides. Combine (5.20) and (5.21), we have for every $a \in X$, $r > 0$ and $\rho > 0$,

$$\begin{aligned}&\frac{c\#\{\Lambda\cap B\}}{\sigma(B)}\\ \leq&\frac{1}{\sigma(B)}\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Bg_n, f_m\rangle\langle g_m, Af_n\rangle|\\ +&\frac{1}{\sigma(B)}\sum_{\lambda_n\in\Lambda\cap B}\sum_{\lambda_m\in\Lambda\cap B^c}|\langle Ag_n, f_m\rangle\langle g_m, Bf_n\rangle|\\ \lesssim&\left(\frac{\#\{\Lambda\cap B\}^2}{\sigma(B)^2}\sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n, f_m\rangle|^2+\frac{\#\{\Lambda\cap B\}}{\sigma(B)}\frac{\#\{\Lambda\cap(B_\rho\setminus B)\}}{\sigma(B)}\right)^{\frac{1}{2}}\\ +&\left(\frac{\#\{\Lambda\cap B\}^2}{\sigma(B)^2}\sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Ag_n, f_m\rangle|^2+\frac{\#\{\Lambda\cap B\}}{\sigma(B)}\frac{\#\{\Lambda\cap(B_\rho\setminus B)\}}{\sigma(B)}\right)^{\frac{1}{2}}.\end{aligned} \tag{5.22}$$

Let $\varepsilon > 0$, by (5.17) and (5.18), we can find a $\rho > 0$ such that

$$\sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Bg_n, f_m\rangle|^2<\varepsilon^2,\quad \sup_{n\in\mathbb{N}}\sum_{\lambda_m\in\Lambda\cap B(\lambda_n,\rho)^c}|\langle Ag_n, f_m\rangle|^2<\varepsilon^2. \tag{5.23}$$

So for such ρ, combine (5.22) and (5.23), we have

$$\frac{c\#\{\Lambda\cap B(a,r)\}}{\sigma(B(a,r))}\lesssim\left(\frac{\#\{\Lambda\cap B\}^2}{\sigma(B)^2}\varepsilon^2+\frac{\#\{\Lambda\cap B\}}{\sigma(B)}\frac{\#\{\Lambda\cap(B_\rho\setminus B)\}}{\sigma(B)}\right)^{\frac{1}{2}},$$

for any $a \in X$ and $r > 0$. Then by Proposition 5.3.6, we obtain for such ρ

$$\begin{aligned}
cD^+(\Lambda) &= c \limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda \cap B(a,r)\}}{\sigma(B(a,r))} \\
&\lesssim \limsup_{r\to\infty} \sup_{a\in X} \left(\frac{\#\{\Lambda \cap B\}^2}{\sigma(B)^2}\varepsilon^2 + \frac{\#\{\Lambda \cap B\}}{\sigma(B)} \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\sigma(B)} \right)^{\frac{1}{2}} \\
&\le \left(\limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda \cap B\}^2}{\sigma(B)^2}\varepsilon^2 + \limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda \cap B\}}{\sigma(B)} \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\sigma(B)} \right)^{\frac{1}{2}} \\
&\le \left(D^+(\Lambda)^2\varepsilon^2 + D^+(\Lambda) \limsup_{r\to\infty} \sup_{a\in X} \frac{\#\{\Lambda \cap (B_\rho \setminus B)\}}{\sigma(B)} \right)^{\frac{1}{2}} \\
&= \left(D^+(\Lambda)^2\varepsilon^2 + 0 \right)^{\frac{1}{2}} \\
&= \varepsilon D^+(\Lambda).
\end{aligned}$$

Since Λ is relatively separated with $D^+(\Lambda) > 0$, by Proposition 3.1, we also have $D^+(\Lambda) < \infty$. By ε is arbitrarily small, contradiction completes. ☐

5.3.5 Applications

Now we could give another proof of the main result in [6] applying Theorem 5.3.5.

Theorem 5.3.8 ([6], Lemma 3)**.** *Assume that $\{f_n\}_{n=1}^\infty$ is a Riesz basis for $L^2(\mathbb{R})$ with the biorthogonal basis $\{g_n\}_{n=1}^\infty$. If there exist sequences of real numbers $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ such that*

(a) $\int_{\mathbb{R}} |x - a_n|^{2s} |f_n(x)|^2 \, dx + \int_{\mathbb{R}} |\xi - b_n|^{2s} \left|\widehat{f}_n(\xi)\right|^2 d\xi \le S^2 < \infty$ *for every* n,

(b) $\int_{\mathbb{R}} |x - a_n|^{2s} |g_n(x)|^2 \, dx + \int_{\mathbb{R}} |\xi - b_n|^{2s} |\widehat{g}_n(\xi)|^2 d\xi \le T^2 < \infty$ *for every* n,

(c) $\Lambda := \{(a_n, b_n)\}_{n=1}^\infty \subseteq \mathbb{R}^2$ *is relatively separated with* $0 < D^+(\Lambda) < \infty$.

Then $s \le 1$.

Proof. Let $(L^2(\mathbb{R}), \{M_b T_a g\}_{(a,b)\in(\mathbb{R}^2, |\cdot|, m)})$ be our *FHS* which satisfies *ADP* and *UMD*, where $g(x) := e^{-\frac{\pi}{2}x^2}$ is normalized Gaussian function. And multiplication X and differentiation D will play the role of operators A and B in Theorem 5.3.5 respectively.

Again, we will prove the theorem by contradiction. Suppose all the above assumptions (a),(b),(c) hold and $s > 1$. Our proof strategy is the following: We will verify the following conditions 1-4 of Theorem 5.3.5 hold. Namely, there exists a weight function w such that

1. $\sup_{n\in\mathbb{N}}\left(\|(X-a_n)g_n\|^2+\|(D-b_n)g_n\|^2\right)<\infty$,

2. $\{(X-a_n)g_n\}_{n=1}^{\infty}$ and $\{(D-b_n)g_n\}_{n=1}^{\infty}$ are uniformly localized on Λ,

3. $\{f_n\}_{n=1}^{\infty}$ is uniformly w-localized on Λ,

4. w is localized on Λ, and $w(2d)\lesssim w(d)$ for every d.

Meanwhile, we will show

$$\sup_{n\in\mathbb{N}}\left(\|(X-a_n)f_n\|^2+\|(D-b_n)f_n\|^2\right)<\infty.$$

Then Theorem 5.3.5 will give us the desired contradiction.

Let $w(d):=d^{2s}$ be the power weight function with exponent $2s$. Obviously, $w(2d)\lesssim w(d)$ for every d. Since $\Lambda=\{(a_n,b_n)\}_{n=1}^{\infty}\subseteq\mathbb{R}^2$ is relatively separated and $s>1$, by the same argument of (5.11) in Section 5.3.1, the weight function w is localized on Λ. So condition 4 holds.

By assumption (a),

$$\begin{aligned}\|(X-a_n)f_n\|^2&=\int_{\mathbb{R}}|x-a_n|^2\,|f_n(x)|^2\,dx\\&\le\int_{\mathbb{R}}(|x-a_n|^{2s}+1)\,|f_n(x)|^2\,dx\\&=\int_{\mathbb{R}}|x-a_n|^{2s}\,|f_n(x)|^2\,dx+\int_{\mathbb{R}}|f_n(x)|^2\,dx\\&\le S^2+C,\end{aligned}$$

where C is the Riesz upper bound of $\{f_n\}_{n=1}^{\infty}$. And

$$\begin{aligned}\|(D-b_n)f_n\|^2&=\|\mathcal{F}(D-b_n)f_n\|^2\\&=\|(X-b_n)\mathcal{F}f_n\|^2\\&=\int_{\mathbb{R}}|\xi-b_n|^2\left|\widehat{f}_n(\xi)\right|^2d\xi\\&\le\int_{\mathbb{R}}(|\xi-b_n|^{2s}+1)\left|\widehat{f}_n(\xi)\right|^2d\xi\\&\le(S^2+C).\end{aligned}$$

So we have

$$\sup_{n\in\mathbb{N}}\left(\|(X-a_n)f_n\|^2+\|(D-b_n)f_n\|^2\right)<\infty. \tag{5.24}$$

Similarly by assumption (b), condition 1 holds, i.e.,

$$\sup_{n\in\mathbb{N}}\left(\|(X-a_n)g_n\|^2+\|(D-b_n)g_n\|^2\right)<\infty.$$

Denote M_bT_ag by $g_{b,a}$ for any $a,b\in\mathbb{R}$. Notice

$$\begin{aligned}
&\int_{\mathbb{R}^2}|\langle f_n,g_{b,a}\rangle|^2|a-a_n|^{2s}\,dadb\\
=&\int_{\mathbb{R}^2}|\mathcal{F}(f_nT_ag)(b)|^2|a-a_n|^{2s}\,dbda\\
=&\int_{\mathbb{R}^2}|(f_nT_ag)(b)|^2\,db\,|a-a_n|^{2s}\,da\\
=&\int_{\mathbb{R}^2}|f_n(b)|^2|g(b-a)|^2|a-b+b-a_n|^{2s}\,dbda\\
\lesssim&\int_{\mathbb{R}^2}|f_n(b)|^2|g(b-a)|^2\left(|a-b|^{2s}+|b-a_n|^{2s}\right)dadb\\
=&\int_{\mathbb{R}}|g(x)|^2|x|^{2s}\,dx\int_{\mathbb{R}}|f_n(x)|^2\,dx\\
+&\int_{\mathbb{R}}|g(x)|^2\,dx\int_{\mathbb{R}}|f_n(x)|^2|x-a_n|^{2s}\,dx.
\end{aligned} \tag{5.25}$$

Denote $M_{-a}T_b\widehat{g}$ by $\widehat{g}_{-a,b}$ for any $a,b\in\mathbb{R}$, similarly,

$$\begin{aligned}
&\int_{\mathbb{R}^2}|\langle f_n,g_{b,a}\rangle|^2|b-b_n|^{2s}\,dadb\\
=&\int_{\mathbb{R}^2}\left|\left\langle \widehat{f}_n,\widehat{g}_{-a,b}\right\rangle\right|^2|b-b_n|^{2s}\,dadb\\
\lesssim&\int_{\mathbb{R}}|\widehat{g}(\xi)|^2|\xi|^{2s}\,d\xi\int_{\mathbb{R}}\left|\widehat{f}_n(\xi)\right|^2d\xi\\
+&\int_{\mathbb{R}}|\widehat{g}(\xi)|^2\,d\xi\int_{\mathbb{R}}\left|\widehat{f}_n(\xi)\right|^2|\xi-b_n|^{2s}\,d\xi.
\end{aligned} \tag{5.26}$$

Combine (5.25) and (5.26), by $g \in \mathcal{S}(\mathbb{R})$ and assumption (a), we have

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} |\langle f_n, g_{b,a}\rangle|^2 \, d^{2s}((a,b),(a_n,b_n)) dadb \\
=&\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} |\langle f_n, g_{b,a}\rangle|^2 \left(|a-a_n|^2 + |b-b_n|^2\right)^s dadb \\
\lesssim&\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} |\langle f_n, g_{b,a}\rangle|^2 \left(|a-a_n|^{2s} + |b-b_n|^{2s}\right) dadb \\
<&\infty. \qquad (5.27)
\end{aligned}$$

Which implies condition 3 holds. Similarly by assumption (b), we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} |\langle g_n, g_{b,a}\rangle|^2 \, d^{2s}((a,b),(a_n,b_n)) dadb < \infty. \qquad (5.28)$$

In order to verify condition 2, let $k_{b,a} := M_b T_a X g$ be the continuous Gabor frame generated by the new window function Xg. Notice

$$\begin{aligned}
&\int_{B((a_n,b_n),r)^c} |\langle (X-a_n) g_n, g_{b,a}\rangle|^2 \, dadb \\
=&\int_{B((a_n,b_n),r)^c} |\langle g_n, (X-a_n) g_{b,a}\rangle|^2 \, dadb \\
=&\int_{B((a_n,b_n),r)^c} |\langle g_n, (X-a_n) M_b T_a g\rangle|^2 \, dadb \\
=&\int_{B((a_n,b_n),r)^c} |\langle g_n, (a-a_n) M_b T_a g + M_b T_a X g\rangle|^2 \, dadb \\
\leq&2\int_{B((a_n,b_n),r)^c} |\langle g_n, (a-a_n) M_b T_a g\rangle|^2 \, dadb \\
+&2\int_{B((a_n,b_n),r)^c} |\langle g_n, M_b T_a X g\rangle|^2 \, dadb \\
=&2\int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 \, |a-a_n|^2 \, dadb \\
+&2\int_{B((a_n,b_n),r)^c} |\langle g_n, k_{b,a}\rangle|^2 \, dadb. \qquad (5.29)
\end{aligned}$$

Estimate the first term in (5.29),

$$
\begin{aligned}
&\int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 |a-a_n|^2 \, dadb \\
\leq &\int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 \left(|a-a_n|^2 + |b-b_n|^2\right) dadb \\
= &\int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 d^2((a,b),(a_n,b_n)) dadb \\
= &\int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 \frac{d^{2s}((a,b),(a_n,b_n))}{d^{2s-2}((a,b),(a_n,b_n))} dadb \\
\leq &r^{2-2s} \int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 d^{2s}((a,b),(a_n,b_n)) dadb \\
\leq &r^{2-2s} \int_{\mathbb{R}^2} |\langle g_n, g_{b,a}\rangle|^2 d^{2s}((a,b),(a_n,b_n)) dadb.
\end{aligned}
$$

By (5.28), we have

$$
\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \int_{B((a_n,b_n),r)^c} |\langle g_n, g_{b,a}\rangle|^2 |a-a_n|^2 \, dadb = 0. \tag{5.30}
$$

Estimate the second term in (5.29),

$$
\begin{aligned}
&\int_{B((a_n,b_n),r)^c} |\langle g_n, k_{b,a}\rangle|^2 \, dadb \\
= &\int_{B((a_n,b_n),r)^c} |\langle g_n, k_{b,a}\rangle|^2 \frac{d^{2s}((a,b),(a_n,b_n))}{d^{2s}((a,b),(a_n,b_n))} dadb \\
\leq &r^{-2s} \int_{B((a_n,b_n),r)^c} |\langle g_n, k_{b,a}\rangle|^2 d^{2s}((a,b),(a_n,b_n)) dadb \\
\leq &r^{-2s} \int_{\mathbb{R}^2} |\langle g_n, k_{b,a}\rangle|^2 d^{2s}((a,b),(a_n,b_n)) dadb.
\end{aligned}
$$

Repeat the same argument of (5.25), (5.26) and (5.27), by $Xg \in \mathcal{S}(\mathbb{R})$ and assumption (b), we obtain

$$
\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} |\langle g_n, k_{b,a}\rangle|^2 d^{2s}((a,b),(a_n,b_n)) dadb < \infty.
$$

So we have

$$
\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \int_{B((a_n,b_n),r)^c} |\langle g_n, k_{b,a}\rangle|^2 \, dadb = 0. \tag{5.31}
$$

Combine (5.29), (5.30) and (5.31),

$$\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\int_{B((a_n,b_n),r)^c}\left|\langle (X-a_n)g_n, g_{b,a}\rangle\right|^2 dadb = 0.$$

Similarly,

$$\lim_{r\to\infty}\sup_{n\in\mathbb{N}}\int_{B((a_n,b_n),r)^c}\left|\langle (D-b_n)g_n, g_{b,a}\rangle\right|^2 dadb = 0.$$

Which imply condition 2 holds as well. Since we collect all the conditions 1-4, we could apply Theorem 5.3.5, and obtain

$$\sup_{n\in\mathbb{N}}\left(\left\|(X-a_n)f_n\right\|^2 + \left\|(D-b_n)f_n\right\|^2\right) = \infty.$$

It contradicts with (5.24)! □

Acknowledgment

First, I would like to thank my advisor, Dr. Mishko Mitkovski. Without his guidance and knowledge, I couldn't have done it. Thank you for leading me to begin my journey of analysis, and also thank you for providing suggestions and helps when I met a problem in daily life.

Besides, I want to say thank you to all of my professors at Clemson, especially to: Dr. Shitao Liu, Dr. Martin Schmoll, Dr. Jeong-Rock Yoon, and Dr. Brian Fralix. Thank all of you for teaching me with great patience. Thank you again to Dr. Liu for sharing your experience of studying abroad and treating me more like a friend. It was a huge support and encouragement to me.

Finally, I would like to express my gratitude to my family members. Thank you to my sister Xiaoming Wang for taking care of me all the time. Thank you to my parents. You give me everything you can give. I feel sorry about not being at home for a long time. Because of you, I could play math for fun.

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