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THE LOCAL DENSITY AND THE LOCAL WEAK DENSITY IN THE SPACE OF PERMUTATION DEGREE AND IN HATTORI SPACE

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Abstract: In this paper, the local density (ld) and the local weak density (lwd) in the space of permutation degree as well as the cardinal and topological properties of Hattori spaces are studied. In other words, we study the properties of the functor of permutation degree SP^n and the subfunctor of permutation degree SP^n_G, P is the cardinal number of topological spaces. Let X be an infinite T_1-space. We prove that the following propositions hold.
(1) Let $Y^n \subset X^n$; (A) if $d\,(Y^n) = d\,(X^n)$, then $d\,(SP^nY) = d\,(SP^nX)$; (B) if $lwd\,(Y^n) = lwd\,(X^n)$, then $lwd\,(SP^nY) = lwd\,(SP^nX)$.
(2) Let $Y \subset X$; (A) if $ld\,(Y) = ld\,(X)$, then $ld\,(SP^nY) = ld\,(SP^nX)$; (B) if $wd\,(Y) = wd\,(X)$, then $wd\,(SP^nY) = wd\,(SP^nX)$.
(3) Let n be a positive integer, and let G be a subgroup of the permutation group S_n. If X is a locally compact T_1-space, then SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces.
(4) Let n be a positive integer, and let G be a subgroup of the permutation group S_n. If X is an infinite T_1-space, then $n\,\pi\,w\,(X) = n\,\pi\,w\,(SP^nX) = n\,\pi\,w\left(SP^n_GX\right) = n\,\pi\,w\,(\exp_n X)$.

We also have studied that the functors SP^n, SP^n_G, and $\exp_n$ preserve any k-space. The functors SP^2 and SP^3_G do not preserve Hattori spaces on the real line. Besides, it is proved that the density of an infinite T_1-space X coincides with the densities of the spaces X^n, SP^nX, and $\exp_n X$. It is also shown that the weak density of an infinite T_1-space X coincides with the weak densities of the spaces X^n, SP^nX, and $\exp_n X$.

Keywords: Local density, Local weak density, Space of permutation degree, Hattori space, Covariant functors.

1. Introduction

In mathematical research in the modern world, a special place is occupied by the study of the topological properties of objects in various topological spaces. Research in general topology is topical, where the properties of topological spaces and their continuous mappings, operations on topological spaces and their mappings, as well as the classification of topological spaces are studied. This section of general topology uses concepts such as neighborhood, closure, compactness, density, separability, cardinal number, π-base of sets, sum, intersection, Tikhonov product, and others. An overview of the main stages in the development of set-theoretic topology is given in [1]. Some cardinal properties of topological spaces related to weak density were studied in [4]. In [5], some cardinal properties of Hattori spaces and their hyperspaces were studied. In [2, 6], some properties of topological spaces related to local density and local weak density in various topological spaces were studied.

Along with the concepts of local τ-density and local weak τ-density in various topological spaces, we are interested in such concepts as hereditary Souslin number, hereditary density, hereditary π-weight, hereditary Shanin number, hereditary pre-Shanin number, hereditary caliber, hereditary precaliber, hereditary weak density, hereditary Lindelöf number, and hereditary extent of topological spaces.

Denote by P the cardinal number of topological spaces. Let SP^n be a functor of permutation degree, and let SP^n_G be a subfunctor of the functor of permutation degree.

At the Prague topological symposium in 1981, V.V. Fedorchuk posed the following general problem in the theory of covariant functors [8] and thus created a new direction of research in this area of topology.

Problems. Let P be a geometrical or topological property, and let F be a covariant functor. If X has the property P, does $F(X)$ have the same property P? The opposite problem: for which functors $F(X)$ the space X has the property P if $F(X)$ has this property?

In [2], it was proved that the property of local density and the property of local weak density coincide for stratifiable spaces. These cardinal numbers are preserved under open mappings and are inverse invariant of a class of closed irreducible mappings.

In our present work, we prove that the following propositions are true for an infinite T_1-space X:

(1) if $ld\,(Y) = ld\,(X)$ for $Y \subset X$, then $ld\,(SP^nY) = ld\,(SP^nX)$;

(2) if $lwd\,(Y) = lwd\,(X)$ for $Y \subset X$, then $lwd\,(SP^nY) = lwd\,(SP^nX)$;

(3) if X is a locally compact T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n, then SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces;

(4) if X is an infinite T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n, then

$$n\,\pi\,w\,(X) = n\,\pi\,w\,(SP^nX) = n\,\pi\,w\,(SP^n_GX) = n\,\pi\,w\,(\exp_n X)\,.$$

We also prove that the functors SP^2 and SP^3_G do not preserve Hattori spaces on the real line. In addition, we prove that the density of an infinite T_1-space X coincides with the densities of the spaces X^n, SP^nX, and $\exp_n X$. We show that the weak density of an infinite T_1-space X coincides with the weak densities of the spaces X^n, SP^nX, and $\exp_n X$.

2. Auxiliary material

Recall some notation, definitions, and statements that are widely used in this paper. The permutation group of X is the group of all permutations (one-to-one and onto mappings $X \to X$). The permutation group of a set X is usually denoted by $S(X)$. If $X = \{1, 2, 3, \dots, n\}$, then $S(X)$ is denoted by S_n.

Let X^n be the nth power of a compact set X. The permutation group S_n of all permutations acts on the nth power X^n as the permutation of coordinates. The set of all orbits of this action with quotient topology is denoted by SP^nX. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Thus, two points $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n) \in X^n$ are equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$. The space SP^nX is called the n-permutation degree of a space X. An equivalent relation by which we obtain the space SP^nX is called the symmetric equivalence relation. The nth permutation degree is always a quotient of X^n. Thus, the quotient mapping is denoted as $\pi^s_n\colon X^n \to SP^nX$, where $\pi^s_n\big((x_1, x_2, \dots, x_n)\big) = \big[x = (x_1, x_2, \dots, x_n)\big]$ is an orbit of the point $x = (x_1, x_2, \dots, x_n) \in X^n$.

The concept of permutation degree has generalizations. Let SP^n_GX be any subgroup of the group S_n. Then it also acts on X^n as the group of permutations of coordinates. Consequently, it generates a G-symmetric equivalence relation on X^n. This quotient space of the product X^n under the G-symmetric equivalence relation is called the G-permutation degree of the space X and is denoted by SP^n_GX. The operation SP^n_G is also a covariant functor in the category of compact sets and is said to be a functor of G-permutation degree. If $G = S_n$, then $SP^n_G = SP^n$. If the group SP^n_GX consists of only one element, then $SP^n_GX = X^n$.

Let X be a T_1-space. The collection of all nonempty closed subsets of X is denoted by $\exp X$. The family B of all sets of the form

$$O\langle U_1, \ldots, U_n\rangle = \left\{F:\ F \in \exp X,\ F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset,\ \ i = 1, 2, \ldots, n\right\}$$

generates a topology on the set $\exp X$, where $U_1, \ldots, U_n$ is a family of open sets of X. This topology is called the Vietoris topology. The set $\exp X$ with the Vietoris topology is called the exponential space or the hyperspace of X [9]. Let X be a T_1-space. Denote by $\exp_n X$ the set of all closed subsets of X such that $\exp_n X = \{F \in \exp\ X : |\,F\,| \leq n\}$.

We use the following notation:

$$\exp_\omega X = \cup\{\exp_n X :\ n = 1, 2, \ldots\}, \quad \exp_c X = \{F \in \exp\ X :\ F \text{ is compact in } X\}.$$

It is clear that $\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X$ for any topological space X. Moreover, if $G_1 \subset G_2$ for subgroups G_1 and G_2 of the permutation group

$$\pi_n^s\big(\,(x_1, x_2, \ldots, x_n)\,\big) = \big[x = (x_1, x_2, \ldots, x_n)\,\big] \in X^n,$$

then we have the following chain of factorizations of functors [9]:

$$X^n \to SP^n_{G_1} X \to SP^n_{G_2} X \to SP^n X \to \exp_n X.$$

A subset D of a topological space X is called a dense set in X if $[D] = X$. Define the density $d\,(X)$ of X by $d\,(X)$ =min$\{|D| : D$ is a dense subset of $X\}$ [7].

We say that the local density of a topological space X is τ at a point x if τ is the smallest cardinal number such that x has a neighbourhood of density τ in X. The local density at a point x is denoted by $ld\,(x)$. The local density of a topological space X is defined as the supremum of all numbers $ld\,(x)$ for $x \in X$: $ld\,(X) = \sup\{ld\,(x) \colon x \in X\}$ [2, 6]. It is known that $ld\,(X) \leq d\,(X)$ for any topological space.

Example. Let $\mathbb{R}$ be the real line with discrete topology. In the discrete topological space $(\mathbb{R}, \tau_d)$, every point $x \in \mathbb{R}$ has the one-point neighbourhood $\{x\}$. It follows that $ld\,(\mathbb{R}, \tau_d) = 1$. On the other hand, the boundary set of any set is empty in a discrete space, and hence the only dense set is the space itself. This means that $d\,(\mathbb{R}, \tau_d) = |\mathbb{R}| = c$. Then $1 = ld\,(\mathbb{R}, \tau_d) < d\,(\mathbb{R}, \tau_d) = c$.

We say that the weak density of a topological space is $\tau \geq \aleph_0$ if τ is the smallest cardinal number such that there exists a π-base coinciding with τ centered systems of open sets, i.e., there is a π-base $B = \bigcup\{B_\alpha :\ \alpha \in A\}$, where B_α is a centered system of open sets for every $\alpha \in A$, $|A| = \tau$.

The weak density of a topological space X is denoted by $wd\,(X)$. If $d\,(X) = \tau \geq \aleph_0$, then $wd\,(X) \leq \tau$. Similarly, if Y is dense in a topological space X, then $wd\,(Y) = wd\,(X)$ [3]. The following theorem and proposition were proved in [3].

Theorem 1. *Let $\{X_\alpha : \alpha \in A\}$ be a family of topological spaces such that $wd\,(X_\alpha) \leq \tau \geq \aleph_0$ for every $\alpha \in A$, where $|A| \leq 2^\tau$. Then $wd\,\big(\prod_{\alpha\in A} X_\alpha\big) \leq \tau$.*

Proposition 1. *Assume that X and Y are topological spaces and there exists a continuous "onto" mapping $f\colon X \to Y$. Then $wd\,(Y) \leq wd\,(X)$.*

A topological space X is called a locally weak τ-dense space at a point $x \in X$ if τ is the smallest cardinal number such that x has a neighbourhood of weak density τ in X. The local weak density at a point x is denoted by $lwd\,(x)$. The local weak density of a topological space X is defined as the supremum of all numbers $lwd(x)$ for $x \in X$: $lwd\,(X) = \sup\{lwd\,(x) \colon x \in X\}$ [2, 6]. If X is a space of local density τ and $f\colon X \to Y$ is an open continuous "onto" mapping, then Y is a space of local density τ [12]. The quotient mapping $\pi_n^s\colon X^n \to SP^n X$ is a clopen continuous onto mapping [13].

The following two statements are from [11].

Proposition 2. *If X is a topological space, then $\exp_n X$ is dense in $\exp X$.*

Proposition 3. *X is separable if and only if $\exp X$ is separable.*

These propositions imply that, for any infinite T_1-space X, we have

$$lwd\,(X) = lwd\,(X^n) = lwd\,(SP^n X).$$

The following theorem was proved in [4].

Theorem 2. *Let X be an infinite T_1-space. Then $wd\,(X) = wd\,(\exp_n X) = wd\,(\exp X)$.*

To substantiate our results, we also use the following notation and definitions from [7].

An uncountable cardinal number τ is a caliber of a topological space if every family of cardinality τ consisting of nonempty open sets contains subfamily of the same cardinality with nonempty intersection. The caliber of a topological space X is denoted by $k\,(X)$.

The cardinal number $\min\{\tau\colon \tau^+ \text{ is a caliber of } X\}$ is called the Shanin number of X and is denoted by $sh\,(X)$.

A cardinal number $\tau > \aleph_0$ is called a precaliber of a space X if every family of cardinality τ consisting of nonempty open subsets of X contains a subfamily of cardinality τ with finite intersection. Define

$$pk\,(X) = \{\tau^+ : \tau \text{ is a precaliber of } X\}.$$

The cardinal number $psh\,(X) = \min\{\tau^+ : \tau \text{ is a precaliber of } X\}$ is called the pre-Shanin number. We always have $c\,(X) \leq psh\,(X) \leq sh\,(X) \leq d\,(X)$.

The Lindelöf number $l(X)$ of X is defined as $l(X) = \min\{\tau :$ every open cover of X has a refinement of cardinality $\leq \tau\} + \aleph_0$. If $l(X) = \aleph_0$, i.e., every open cover has a countable refinement, we say that X is a Lindelöf space.

The notion of cellularity (Souslin number) $c\,(X)$ of X is defined as $c\,(X) = \min\{\tau :$ every family of pairwise disjoint nonempty open subsets of X has cardinality $\leq \tau\} + \aleph_0$. If $c(X) = \aleph_0$, we say that X has the countable chain condition (Souslin property).

The spread $s(X)$ and the extent $e(X)$ are defined as follows: $s(X) = \sup\{|D| : D$ is a discrete subset of $X\} + \aleph_0$ and $e(X) = \sup\{|D| : D$ is a discrete closed subset of $X\} + \aleph_0$, respectively.

For a metrizable space X, we have $l\,(X) = d\,(X) = c\,(X) = s\,(X) = e\,(X)$.

For a cardinal function φ, we define the corresponding hereditary cardinal function $h\varphi = \sup\{\varphi(Y) : Y \subset X\}$. For example, we have the hereditary Souslin number $hc\,(X)$, the hereditary density $hd\,(X)$, the hereditary π-weight $h\pi w\,(X)$, and the hereditary Shanin number $hsh\,(X)$. Similar symbols we use to denote the hereditary pre-Shanin number, the hereditary caliber, the hereditary precaliber, the hereditary weak density, the hereditary Lindelöf number, and the hereditary extent of the space X, respectively: $hpsh\,(X)$, $hk\,(X)$, $hpk\,(X)$, $hwd\,(X)$, $hl\,(X)$, and $he\,(X)$.

It is easy to see that the hereditary Souslin number $hc\,(X)$ of a space X coincides with its spread $s\,(X)$.

Definition 1. *A topological space is a k-space if it is a quotient image of some topological space Y.*

Recall that a topological space is locally compact if, for every $x \in X$, there exists a neighbourhood U of x such that $[U]$ is a compact subspace of X.

In 2010, Hattori defined [10] the following topology on $\mathbb{R}$. Let $\mathbb{R}$ be the real line and $\mathbb{A} \subseteq \mathbb{R}$. The topology $\tau(\mathbb{A})$ on $\mathbb{R}$ is defined as follows:

(1) for each $x \in \mathbb{A}$, $\{(x-\varepsilon,\, x+\varepsilon) : \varepsilon > 0\}$ is the neighbourhood base at x;

(2) for each $x \in \mathbb{R}\backslash\mathbb{A}$, $\{[x,\, x+\varepsilon) : \varepsilon > 0\}$ is the neighbourhood base at x.

The space $(\mathbb{R},\, \tau\,(\mathbb{A}))$ is called [5] a Hattori space. Let $\tau_{\mathbb{E}}$ be the Euclidean topology on $\mathbb{R}$. Note that, for any $\mathbb{A}, \mathbb{B} \subseteq \mathbb{R}$, we have $\mathbb{A} \supseteq \mathbb{B}$ if $\tau\,(\mathbb{A}) \subseteq \tau\,(\mathbb{B})$, in particular, $\tau\,(\mathbb{R}) = \tau_{\mathbb{E}} \subseteq \tau\,(\mathbb{A})$ and $\tau\,(\mathbb{B}) \subseteq \tau(\emptyset) = \tau_S$. We set $P_{\text{top}}\,(\mathbb{R}) = \{\tau\,(\mathbb{A}) : \mathbb{A} \subseteq \mathbb{R}\}$ and define a partial order $\leq$ on $P_{\text{top}}\,(\mathbb{R})$ by the inclusion: $\tau\,(\mathbb{A}) \leq \tau\,(\mathbb{B})$ if $\tau\,(\mathbb{A}) \subseteq \tau\,(\mathbb{B})$.

3. Main results

Theorem 3. *Let X be an infinite T_1-space, and let Y^n be dense in X^n. Then SP^nY is also dense in SP^nX.*

P r o o f. Let Y^n be a dense subset of X^n, and let SP^nU be an arbitrary open set from SP^nX. Since the mapping $\pi_n^s : X^n \to SP^nX$ is continuous, the set $(\pi_n^s)^{-1}\,(SP^nU) \subset X^n$ is open. Thus, taking into account the density of Y^n in X^n, we conclude that $(\pi_n^s)^{-1}\,(SP^nU) \cap Y^n \neq \emptyset$. Therefore, there exists $y \in Y^n$ such that $y \in (\pi_n^s)^{-1}\,(SP^nU)$. Then $\pi_n^s(y) \in SP^nU$ (and $\pi_n^s(y) \in SP^nY$). Hence, we have $SP^nU \cap SP^nY \neq \emptyset$ for every open set SP^nU. This means that the set SP^nY is dense in SP^nX. Theorem 3 is proved. □

Corollary 1. *If X is an infinite T_1-space and Y^n is a subset of X^n such that $d\,(Y^n) = d\,(X^n)$, then $d\,(SP^nY) = d\,(SP^nX)$.*

Proposition 4. *Assume that X is an infinite T_1-space, n is a positive number, and G_1 and G_2 are subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then*

$$d\,(X) = d\,(X^n) = d\,\left(SP^n_{G_1}X\right) = d\,\left(SP^n_{G_2}X\right) = d\,(SP^nX) = d\left(\exp_n X\right).$$

P r o o f. Let X be an infinite T_1-space. Taking into account that

$$X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to \exp_n X$$

and the fact that continuous mappings do not increase the density of topological spaces, we directly obtain the inequalities

$$d\,(X) \geq d\,(X^n) \geq d\,\left(SP^n_{G_1}X\right) \geq d\,\left(SP^n_{G_2}X\right) \geq d\,(SP^nX) \geq d\,(\exp_n X).$$

By Propositions 2 and 3, we get $d\,(X) = d\left(\exp_n X\right)$, and hence

$$d\,(X) = d\,(X^n) = d\,\left(SP^n_{G_1}X\right) = d\,\left(SP^n_{G_2}X\right) = d\,(SP^nX) = d\,(\exp_n X).$$

Proposition 4 is proved. □

Theorem 4. *Let X be an infinite T_1-space, and let Y^n be a locally dense set in X^n. Then SP^nY is also locally dense in SP^nX.*

P r o o f. The set Y^n is locally dense in X^n. By definition, for any point $y \in Y^n$, there exists a neighbourhood $Oy \subset X^n$ such that Oy is dense in X^n. Then Theorem 3 implies that $SP^n(Oy)$ is also dense in SP^nX. On the other hand, the quotient mapping $\pi_n^s : X^n \to SP^nX$ is an open mapping. Therefore, $SP^n(Oy)$ is a neighbourhood of the point $\pi_n^s(y) \in SP^nY$. Then SP^nY is locally dense in SP^nX. Theorem 4 is proved. □

Corollary 2. *If X is an infinite T_1-space and $Y \subset X$ is such that $ld\,(Y) = ld\,(X)$, then*

$$ld\,(SP^nY) = ld\,(SP^nX)\,.$$

Theorem 5. *Let X be an infinite T_1-space. Then $wd\,(X) = wd\,(SP^nX)$.*

P r o o f. First, we will show that $wd\,(SP^nX) \le wd\,(X)$. Suppose that $wd\,(X) = \tau \ge \aleph_0$. Then $wd\,(X^n) = \tau$ by Theorem 1. The space SP^nX is a continuous image of the space X^n. Proposition 1 implies that $wd\,(SP^nX) \le \tau$.

Now we will prove that $wd\,(SP^nX) \ge wd\,(X^n)$. To this end, assume that $wd\,(SP^nX) = \tau \ge \aleph_0$. This means that there exists $SP^nB = \cup\{SP^nB_\alpha : \alpha \in A,\ |A| = \tau\}$ and this is a π-base in SP^nX, where $SP^nB_\alpha = \{SP^nU_s^\alpha : s \in A_\alpha\}$ is a centered system of nonempty open sets for every $\alpha \in A$.

We set

$$B_\alpha = \big\{ (\pi_n^s)^{-1}\,(SP^nU_s^\alpha) : s \in A_\alpha \big\}, \quad B = \cup\big\{B_\alpha : \alpha \in A\big\}.$$

Let us show that B_α is a centered system of nonempty open sets in X^n for every $\alpha \in A$. For every finite subfamily $\{SP^nU_{s_i}^\alpha\}_{i=1}^k$ of SP^nB_α, we have $\cap_{i=1}^k SP^nU_{s_i}^\alpha \ne \emptyset$. Then

$$\emptyset \ne (\pi_n^s)^{-1}\left(\cap_{i=1}^k SP^nU_{s_i}^\alpha\right) = \cap_{i=1}^k\big((\pi_n^s)^{-1}\,(SP^nU_{s_i}^\alpha)\big).$$

This shows that $B_\alpha = \big\{ (\pi_n^s)^{-1}\,(SP^nU_s^\alpha) : s \in A_\alpha \big\}$ is also a centered system of nonempty open sets in X^n. Now, we show that B is a π-base in X^n. Since

$$SP^nB = \cup\big\{SP^nB_\alpha : \alpha \in A,\ |A| = \tau\big\}$$

is a π-base of SP^nX, for every open subset SP^nU of SP^nX, there exists $SP^nU_s^\alpha \in SP^nB_\alpha \subset SP^nB$ such that $SP^nU_s^\alpha \subset SP^nU$. Since the quotient mapping $\pi_n^s : X^n \to SP^nX$ is open and onto, we have

$$(\pi_n^s)^{-1}\,(SP^nU_s^\alpha) \subset (\pi_n^s)^{-1}\,(SP^nU)\,.$$

This means that B is a π-base in X^n. Therefore, we have $wd\,(X^n) \le \tau$. Theorem 5 is proved. □

Corollary 3. *If X is an infinite T_1-space and $Y \subset X$ is such that $wd\,(Y) = wd\,(X)$, then*

$$wd\,(SP^nY) = wd\,(SP^nX)\,.$$

Theorem 6. *Let X be an infinite T_1-space, and let Y^n be locally weakly dense in X^n. Then SP^nY is locally weakly dense in SP^nX.*

P r o o f. Suppose that X is an infinite T_1-space and $Y^n \subset X^n$ is locally weakly dense. Then, for every point $y \in Y^n$, there exists a neighbourhood Oy such that Oy is weakly dense in X^n. According to Theorem 5, $SP^n(Oy) = \{\pi_n^s(y') : y' \in Oy\}$ is also weakly dense in SP^nX. This means that, for every point $\pi_n^s(y) \in SP^nY$, there exists $SP^n(Oy)$ such that it is weakly dense in SP^nX. This shows that SP^nY is locally weakly dense in SP^nX. Theorem 6 is proved. □

Corollary 4. *If X is an infinite T_1-space and $Y^n \subset X^n$ is such that $lwd\,(Y^n) = lwd\,(X^n)$, then $lwd\,(SP^nY) = lwd\,(SP^nX)$.*

Proposition 5. *Assume that X is an infinite T_1-space, n is a positive number, and G_1 and G_2 are subgroups of the permutation group S_n such that $G_1 \subset G_2$. Then*

$$wd\,(X) = wd\,(X^n) = wd\,\left(SP_{G_1}^nX\right) = wd\,\left(SP_{G_2}^nX\right) = wd\,(SP^nX) = wd\,(\exp_n X).$$

P r o o f. Let X be an infinite T_1-space. Taking into account that

$$X^n \to SP^n_{G_1}X \to SP^n_{G_2}X \to SP^nX \to \exp_n X$$

and the fact that continuous mappings do not increase the weak density of topological spaces, we directly obtain the inequalities

$$wd\,(X) \geq wd\,(X^n) \geq wd\,\left(SP^n_{G_1}X\right) \geq wd\,\left(SP^n_{G_2}X\right) \geq wd\,(SP^nX) \geq wd\,(\exp_n X).$$

According to Theorem 2, $wd\,(X) = wd\,(\exp_n X)$. Hence, we get

$$wd(X) = wd\,(X^n) = wd\,\left(SP^n_{G_1}X\right) = wd\,\left(SP^n_{G_2}X\right) = wd\,(SP^nX) = wd\,(\exp_n X).$$

Proposition 5 is proved. □

Proposition 6. *Assume that X is a locally compact T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n. Then SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces.*

P r o o f. Let X be a locally compact T_1-space. Then X^n is a locally compact space for each $n \in \mathbb{N}$. The spaces SP^nX, SP^n_GX, and $\exp_n X$ become quotient images of the space X^n. Therefore, SP^nX, SP^n_GX, and $\exp_n X$ are k-spaces. Proposition 6 is proved. □

Corollary 5. *The functors SP^n, SP^n_G, and $\exp_n$ preserve any k-space.*

Proposition 7. *Assume that X is an infinite T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n. Then $n\,\pi w\,(SP^nX) = n\,\pi w\,(X)$.*

P r o o f. It was proved in Proposition 4 that $d\,(SP^nX) = d\,(X)\,,\ n \in \mathbb{N}$. It is known that any dense set $M \subset X$ can be a π-net of this space. Hence, we have $n\,\pi w\,(SP^nX) = n\,\pi\,w\,(X)$. Proposition 7 is proved. □

Corollary 6. *Assume that X is an infinite T_1-space, n is a positive integer, and G is a subgroup of the permutation group S_n. Then*

$$n\,\pi\,w(X) = n\,\pi\,w(SP^nX) = n\,\pi\,w(SP^n_GX) = n\,\pi\,w(SP^n_{G_1}X) =$$
$$= n\,\pi\,w(SP^n_{G_2}X) = n\,\pi\,w(\exp_n X) = n\,\pi\,w(\exp_\omega X) = n\,\pi\,w(\exp X).$$

Theorem 7. *Let $\mathbb{A}$ be a subset of $\mathbb{R}$ such that $int\,(\mathbb{R}\backslash\mathbb{A}) \neq \emptyset$. Then the following nonequalities hold for the Hattori space $(\mathbb{R},\,\tau\,(\mathbb{A}))$ and the functor of permutation degree SP^2:*

(1) $s\,(\mathbb{R},\,\tau(\mathbb{A})) \neq s\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(2) $hd\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hd\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(3) $h\pi w\,(\mathbb{R},\,\tau(\mathbb{A})) \neq h\pi\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(4) $hsh\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hsh\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(5) $hc\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hc\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(6) $hk\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hk\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(7) $hpk\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hpk\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(8) $hpsh\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hpsh\,\left(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(9) $hwd\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hwd\,\big(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\big)$;

(10) $hl\,(\mathbb{R},\,\tau(\mathbb{A})) \neq hl\,\big(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\big)$;

(11) $he\,(\mathbb{R},\,\tau(\mathbb{A})) \neq he\,\big(SP^2(\mathbb{R},\,\tau(\mathbb{A}))\big)$.

P r o o f. It is known that the space SP^2X contains the squared Hattori space X^2. However, X^2 contains a discrete set of cardinality c. The other nonequalities can be easily checked. Theorem 7 is proved. □

Corollary 7. *The functor SP^2 does not preserve Hattori spaces on the real line.*

Corollary 8. *Let $\mathbb{A}$ be a subset of $\mathbb{R}$ such that $int\,(\mathbb{R}\backslash\mathbb{A}) \neq \emptyset$, and let G be an arbitrary subgroup of the group S_3. Then the following nonequalities hold for the Hattori space $(\mathbb{R},\,\tau\,(\mathbb{A}))$ and the functor of permutation degree SP^3_G:*

(1) $s\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq s\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(2) $hd\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hd\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(3) $h\,\pi\,w\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq h\,\pi\,w\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(4) $hsh\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hsh\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(5) $hc\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hc\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(6) $hk\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hk\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(7) $hpk\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hpk\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(8) $hpsh\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hpsh\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(9) $hwd\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hwd\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(10) $hl\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq hl\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$;

(11) $he\,(\mathbb{R},\,\tau\,(\mathbb{A})) \neq he\,\left(SP^3_G(\mathbb{R},\,\tau(\mathbb{A}))\right)$.

Corollary 9. *The functor SP^3_G does not preserve Hattori spaces on the real line.*

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