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Parameter estimation of p-dimensional Rayleigh distribution under weighted loss function

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Abstract

In this paper, p-dimensional Rayleigh distribution is considered. The classical maximum likelihood estimator has been obtained. Bayesian method of estimation is employed in order to estimate the scale parameter of p-dimensional Rayleigh distribution by using quasi and inverted gamma priors. The Bayes estimators of the scale parameter have been obtained under squared error and weighted loss functions.

Keywords: Bayesian method, p-dimensional Rayleigh distribution, quasi and inverted gamma priors, squared error and weighted loss functions.

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1 Introduction

The probability density function (pdf) of p-dimensional Rayleigh distribution is given by

$$f\left(x;\theta\right)=\frac{2}{\Gamma\left(p/2\right)}\frac{\theta^{-p/2}}{x^{p-1}}e^{-\left(x^{2}\right)/\theta} \quad ; x\geq 0,\ \theta>0. \tag{1}$$

(Cohen and Whitten [1]).

The distribution with pdf (1), in which p=1, sometimes called the folded Gaussian, the folded normal, or the half normal distribution. With p=2, the pdf of (1) is reduced to two-dimensional Rayleigh distribution. With p=3, the pdf of (1) is reduced to Maxwell-Boltzmann distribution. Let $x_1, x_2, ..., x_n$ be a random sample of size n having probability density function (1), then the likelihood function of (1) is given by (Rao and Pandey [2])

$$f\left(\underline{x};\theta\right)=\left(\frac{2}{\Gamma\left(p/2\right)}\right)^{n}\theta^{-np/2}\left(\prod_{i=1}^{n}x_i^{p-1}\right)e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i^2} \tag{2}$$

The log likelihood function is given by

$$log\ f\left(\underline{x};\theta\right)=n\,log\,2-n\,log\,\Gamma\left(p/2\right)-\frac{np}{2}log\,\theta+log\prod_{i=1}^{n}x_i^{p-1}-\frac{1}{\theta}\sum_{i=1}^{n}x_i^2 \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get

$$\hat{\theta}=\frac{2\sum_{i=1}^{n}x_i^2}{np} \tag{4}$$

2 Bayesian Method of Estimation

In Bayesian analysis the fundamental problem is that of the choice of prior distribution g (θ) and a loss function $L\left(\hat{\theta},\theta\right)$. The squared error loss function for the scale parameter θ are defined as

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^{2} \tag{5}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E(\theta) \tag{6}$$

This loss function is often used because it does not lead to extensive numerical computations but several authors (Zellner [3], Basu and Ebrahimi [4]) have recognized that the inappropriateness of using symmetric loss function. J.G.Norstrom [5] introduced an alternative asymmetric precautionary loss function. and also presented a general class of precautionary loss functions with quadratic loss function as a special case. Weighted loss function (Ahamad et al. [6]) is given a

$$L\left(\hat{\theta},\theta\right) = \frac{\left(\hat{\theta}-\theta\right)^2}{\theta} \tag{7}$$

The Bayes estimator under weighted loss function is denoted by $\hat{\theta}_W$ and is obtained as

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} \tag{8}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \ ; \theta > 0, \ d \geq 0, \tag{9}$$

where *d = 0* leads to a diffuse prior and *d = 1*, a non-informative prior.

(ii) Inverted gamma prior: The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and $\beta(>0)$ with probability density function given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} \ ; \theta > 0. \tag{10}$$

3 Bayes Estimators under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{\left(\frac{2}{\Gamma(p/2)}\right)^n \theta^{-np/2} \left(\prod_{i=1}^{n} x_i^{p-1}\right) e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} \theta^{-d}}{\int_0^\infty \left(\frac{2}{\Gamma(p/2)}\right)^n \theta^{-np/2} \left(\prod_{i=1}^{n} x_i^{p-1}\right) e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} \theta^{-d} d\theta}$$

$$= \frac{\theta^{-\left(\frac{np}{2}+d\right)} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2}}{\int_0^\infty \theta^{-\left(\frac{np}{2}+d\right)} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} d\theta}$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\left(\frac{np}{2}+d-1\right)}}{\Gamma\left(\frac{np}{2}+d-1\right)} \theta^{-\left(\frac{np}{2}+d\right)} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} \tag{11}$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_S = \frac{\sum_{i=1}^{n} x_i^2}{\left(\frac{np}{2}+d-2\right)} \tag{12}$$

Proof. From equation (6), on using (11),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\left(\frac{np}{2}+d-1\right)}}{\Gamma\left(\frac{np}{2}+d-1\right)} \int_0^\infty \theta^{-\left(\frac{np}{2}+d-1\right)} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+d-1}}{\Gamma\left(\frac{np}{2}+d-1\right)} \frac{\Gamma\left(\frac{np}{2}+d-2\right)}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+d-2}}$$

or, $\hat{\theta}_S = \dfrac{\sum_{i=1}^{n} x_i^2}{\left(\frac{np}{2}+d-2\right)}$.

Theorem 2. Assuming the weighted loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_W = \frac{\sum_{i=1}^{n} x_i^2}{\left(\frac{np}{2}+d-1\right)} \tag{13}$$

Proof. From equation (8), on using (11),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f\left(\theta / \underline{x}\right) d\theta\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+d-1}}{\Gamma\left(\frac{np}{2}+d-1\right)} \int_0^{\infty} \theta^{-\left(\frac{np}{2}+d+1\right)} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} d\theta\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+d-1}}{\Gamma\left(\frac{np}{2}+d-1\right)} \frac{\Gamma\left(\frac{np}{2}+d\right)}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+d}}\right]^{-1}$$

$$= \left[\frac{\frac{np}{2} + d - 1}{\sum_{i=1}^{n} x_i^2} \right]^{-1}$$

$$\Rightarrow \qquad \hat{\theta}_W = \frac{\sum_{i=1}^{n} x_i^2}{\left(\frac{np}{2} + d - 1 \right)}$$

4 Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f(\theta/\underline{x}) = \frac{\left(\frac{2}{\Gamma(p/2)} \right)^n \theta^{-np/2} \left(\prod_{i=1}^{n} x_i^{p-1} \right) e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}}{\int_0^\infty \left(\frac{2}{\Gamma(p/2)} \right)^n \theta^{-np/2} \left(\prod_{i=1}^{n} x_i^{p-1} \right) e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} d\theta}$$

$$= \frac{\theta^{-\left(\frac{np}{2}+\alpha+1\right)} e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^2\right)}}{\int_0^\infty \theta^{-\left(\frac{np}{2}+\alpha+1\right)} e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^2\right)} d\theta}$$

$$= \frac{\theta^{-\left(\frac{np}{2}+\alpha+1\right)} e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^2\right)}}{\Gamma\left(\frac{np}{2} + \alpha \right) \Big/ \left(\beta + \sum_{i=1}^{n} x_i^2 \right)^{\frac{np}{2}+\alpha}}$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^2 \right)^{\frac{np}{2}+\alpha}}{\Gamma\left(\frac{np}{2} + \alpha \right)} \theta^{-\left(\frac{np}{2}+\alpha+1\right)} e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^2\right)} \qquad (14)$$

Theorem 3. Assuming the squared error loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_S = \frac{\beta + \sum_{i=1}^{n} x_i^2}{\frac{np}{2} + \alpha - 1} \tag{15}$$

Proof. From equation (6), on using (14),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+\alpha}}{\Gamma\left(\frac{np}{2} + \alpha\right)} \int_0^\infty \theta^{-\left(\frac{np}{2}+\alpha\right)} e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^2\right)} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+\alpha}}{\Gamma\left(\frac{np}{2} + \alpha\right)} \frac{\Gamma\left(\frac{np}{2} + \alpha - 1\right)}{\left(\beta + \sum_{i=1}^{n} x_i^2\right)^{\frac{np}{2}+\alpha-1}}$$

or, $$\hat{\theta}_S = \frac{\beta + \sum_{i=1}^{n} x_i^2}{\frac{np}{2} + \alpha - 1}.$$

Theorem 4. Assuming the weighted loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_W = \frac{\beta + \sum_{i=1}^{n} x_i^2}{\frac{np}{2} + \alpha} \tag{16}$$

Proof. From equation (8), on using (14),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f(\theta/\underline{x})\, d\theta\right]^{-1}$$

$$=\left[\frac{\left(\beta+\sum_{i=1}^{n}x_i^2\right)^{\frac{np}{2}+\alpha}}{\Gamma\left(\frac{np}{2}+\alpha\right)}\int_0^{\infty}\theta^{-\left(\frac{np}{2}+\alpha+2\right)}e^{-\frac{1}{\theta}\left(\beta+\sum_{i=1}^{n}x_i^2\right)}d\theta\right]^{-1}$$

$$=\left[\frac{\left(\beta+\sum_{i=1}^{n}x_i^2\right)^{\frac{np}{2}+\alpha}}{\Gamma\left(\frac{np}{2}+\alpha\right)}\frac{\Gamma\left(\frac{np}{2}+\alpha+1\right)}{\left(\beta+\sum_{i=1}^{n}x_i^2\right)^{\frac{np}{2}+\alpha+1}}\right]^{-1}$$

$$=\left[\frac{\frac{np}{2}+\alpha}{\left(\beta+\sum_{i=1}^{n}x_i^2\right)}\right]^{-1}$$

or, $$\hat{\theta}_W=\frac{\beta+\sum_{i=1}^{n}x_i^2}{\frac{np}{2}+\alpha}\ .$$

5 Conclusion

In this paper, we have obtained a number of estimators of parameter. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (12) and (13) we have obtained the Bayes estimators under squared error and weighted loss function using quasi prior. In equation (15) and (16) we have obtained the Bayes estimators under squared error and weighted loss function using inverted gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

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