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On an Ill-Posed Problem for the Heat Equation

Roman E. Puzyrev[*]
Alexander A. Shlapunov[†]
Institute of Mathematics,
Siberian Federal University,
Svobodny, 79, Krasnoyarsk, 660041,
Russia

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A boundary value problem for the heat equation is studied. It consists of recovering a function, satisfying the heat equation in a cylindrical domain, via its values ant the values of its normal derivative on a given part of the lateral surface of the cylinder. We prove that the problem is ill-posed in the natural spaces of smooth functions and in the corresponding Hölder spaces; besides, additional initial data do not turn the problem to a well-posed one. Using Integral Representation's Method we obtain Uniqueness Theorem and solvability conditions for the problem.

Keywords: boundary value problems for heat equation, ill-posed problems, integral representation's method.

Introduction

Ill-posed problems are already known in Mathematical Physics for many years. A classical example of problems of this type is the famous Cauchy problem for Laplace Equation (see [1]). Beginning from the middle of XX century they appeared in applications (in Geophysics, Hydrodynamics, Theory of Electronic Signals etc.), see, for example, [2, 3]. One of the effective methods of solving ill-posed problems is the so-called Regularization Method (cf. [4, 5, 6, 7] in the theory of the Cauchy Problem for Elliptic Equations). A combination of Integral Representation's Method and Spectral Theory for self-adjoint operators in Hilbert Spaces was especially effective for ill-posed problems for Elliptic Equations (see [8, 9, 10, 11]).

It is well known that many methods for studying elliptic equations have the corresponding analogues for parabolic ones (see, for instance, [12, 13, 14]). Instead of classic boundary value problems for the Heat Equation we consider the ill-posed problem, consisting in finding a function satisfying the equation in a cylindrical domain via its values and the values of its normal derivative on a given part of the lateral surface of the cylinder. Using the Heat Potentials we prove Uniqueness Theorem and obtain solvability conditions for the problem.

1. The problem

Let Ω be a bounded domain (i.e. bounded open connected set) in n-dimensional real space $\mathbb{R}^n$ with the coordinates $x = (x_1, \ldots, x_n)$. As usual we denote by $\overline{\Omega}$ the closure of Ω, and we denote by $\partial\Omega$ its boundary. In the sequel we assume that $\partial\Omega$ is piece-wise smooth.

Consider a bounded open cylinder $\Omega_T = \{x \in \Omega,\ 0 < t < T\}$, having the altitude $T > 0$ and the base Ω, in $(n+1)$-dimensional real space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \{-\infty < t < +\infty\}$. Let also $\Gamma \subset \partial\Omega$

[*]effervesce@mail.ru
[†]ashlapunov@sfu-kras.ru

be a non empty connected open (in the topology of $\partial\Omega$) subset of $\partial\Omega$. Set $\Gamma_T = \Gamma \times (0,\ T)$ to obtain $\overline{\Gamma_T} = \overline{\Gamma} \times [0,\ T]$.

As usual, for $s \in \mathbb{Z}_+$ (here $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$) and an open subset $D \subset \mathbb{R}^m$ we denote $C^s(D)$ the set of all s times continuously differentiable functions in D. The standard topology of this metrisable space induces the uniform convergence on compact subsets in D together with all the partial derivatives up to order s.

For $S \subset \partial D$ we denote $C^s(D \cup S)$ the set of such functions from the space $C^s(D)$ that all their derivatives up to order s can be extended continuously onto $D \cup S$. The standard topology of this metrisable space induces the uniform convergence on compact subsets in $D \cup S$ together with all the partial derivatives up to order s. In particular, for bounded domains, $C^s(D \cup \partial D) = C^s(\overline{D})$ is a Banach space.

Apart from the standard functional spaces, we need also spaces reflecting the specific properties of parabolic equations in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \{-\infty < t < +\infty\}$. Namely, let $C^{1,0}(\Omega_T)$ be the set of continuous functions u in Ω_T, having in Ω_T the continuous partial derivatives u_{x_i}, and let $C^{2,1}(\Omega_T)$ denote the set of continuous functions in Ω_T, having in Ω_T the continuous partial derivatives u_{x_i}, $u_{x_i x_j}$, u_t. The standard topology of this metrisable space induces the uniform convergence on compact subsets in D together with all the partial derivatives used in its definition.

As before, for $S \subset \partial\Omega_T$ we denote by $C^{1,0}(\Omega_T \cup S)$ the set of such functions u from the space $C^{1,0}(\Omega_T)$ that their derivatives u_{x_i} can be extended continuously onto $\Omega_T \cup S$. The standard topology of this metrisable space induces the uniform convergence on compact subsets of $\Omega_T \cup S$ of both the functional sequences and the corresponding sequences of the first partial derivatives with respect to x_i. Clearly, $C^{1,0}(\Omega_T \cup \partial\Omega_T) = C^{1,0}(\overline{\Omega_T})$ is a Banach space.

Let now $\Delta_n = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ be the Laplace operator in $\mathbb{R}^n$ and $L_{n+1} = \dfrac{\partial}{\partial t} - \Delta_n$ stand for the heat operator in $\mathbb{R}^{n+1}$. It is well known that the Laplace operator is elliptic and the heat operator is parabolic.

Besides, let $\dfrac{\partial}{\partial \nu} = \sum\limits_{i=1}^{n} \nu_i \dfrac{\partial}{\partial x_i}$ denote the derivative at the direction of the exterior unit normal vector $\nu = (\nu_1, ..., \nu_n)$ to the surface $\partial\Omega$. As $\partial\Omega$ is piece-wise smooth, the normal vector $\nu = (\nu_1, ..., \nu_n)$ is defined almost everywhere on $\partial\Omega$.

Consider two problems for the Heat Equation. Let functions $u_0(x) \in C(\overline{\Omega})$, $u_1(x,t) \in C^{1,0}(\overline{\Gamma_T})$, $u_2(x,t) \in C(\overline{\Gamma_T})$ and $f(x,t) \in C(\overline{\Omega_T})$ be given.

Problem 1. *Find a function* $u(x,\ t) \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus (\partial\Omega \setminus \overline{\Gamma})_T)$ *satisfying the Heat Equation*

$$L_{n+1} u = f \ \text{in} \ \Omega_T \tag{1}$$

and boundary conditions

$$u(x,t) = u_1(x,\ t) \ \text{on} \ \overline{\Gamma_T}, \tag{2}$$

$$\frac{\partial u}{\partial \nu}(x,t) = u_2(x,\ t) \ \text{on} \ \overline{\Gamma_T}. \tag{3}$$

In particular, if $n = 1$ we have $\Omega_T = (0,1) \times (0,T)$ (i.e. with one spaces variable $x = x_1 \in (0,1)$ and time variable $t \in (0,T)$). In this case $\Gamma_T = \{0\} \times (0,\ T)$ and the following conditions correspond to Problem 1:

$$\dot{u}(x,t) - u''(x,t) = f(x,t) \ \text{в} \ (0,\ 1) \times (0,T), \tag{4}$$

$$u(0,t) = u_1(t), \quad 0 \leqslant t \leqslant T, \tag{5}$$

$$u'(0,t) = u_2,(t), \quad 0 \leqslant t \leqslant T \tag{6}$$

(traditionally, here we set $\dot{u} = \dfrac{\partial u}{\partial t}$, $u' = \dfrac{\partial u}{\partial x}$, $u'' = \dfrac{\partial^2 u}{\partial x^2}$).

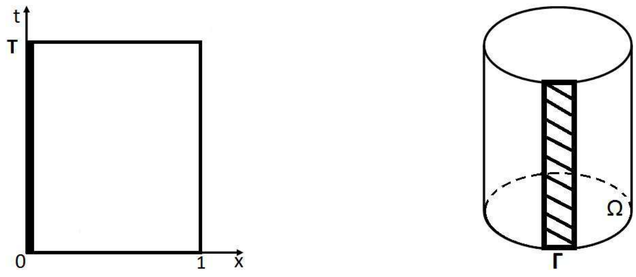

Fig. 1

Fig. 1 indicates the sets where the boundary data for Problem 1 are given in the cases of one and two space variables. The process of the heat conduction is described by the function $u(x, t)$ presenting the temperature at the space point x and the time t. Though, of course, it is known that the Heat Equation is not ideal to model the process of the heat conduction.

Problem 2. *Find* $u(x,\ t) \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus (\partial\Omega \setminus \overline{\Gamma})_T)$ *satisfying in* Ω_T *the Heat Equation* (1)*, the boundary conditions* (2)*,* (3) *and the initial condition*

$$u(x, 0) = u_0(x), \quad x \in \overline{\Omega}. \tag{7}$$

In the case of one space variable the following initial conditions corresponds to Problem 2:

$$u(x, 0) = u_0(x), \quad x \in [0, 1]. \tag{8}$$

Of course one should also take care on the compatibility of the data u_0, u_1, u_2: at least

$$u_0(x) = u_1(x, 0) \text{ on } \Gamma, \tag{9}$$

and, if $u_0 \in C^1(\overline{\Omega})$, even

$$\frac{\partial u_0}{\partial \nu}(x) = u_2(x, 0) \text{ on } \Gamma. \tag{10}$$

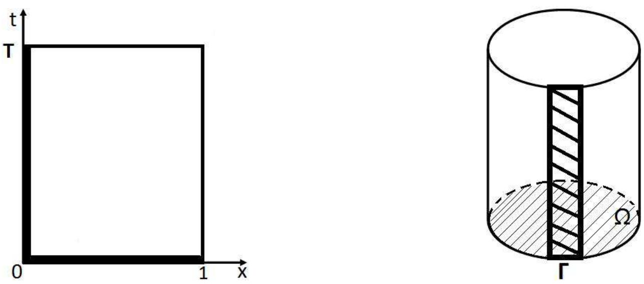

Fig. 2

Fig. 2 indicates the sets where the boundary and the initial data for Problem 2 are given in the cases of one and two space variables.

We note that in the classical theory of the (initial and) boundary problems for the Heat Equation equation (1), the initial condition (7) and the boundary condition $\alpha u + \beta \frac{\partial u}{\partial \nu} = u_3$ on the whole lateral surface $\partial\Omega_T$ of the cylinder Ω_T are usually considered. As a rule, such a problem is well-posed in the proper spaces (Hölder spaces, Sobolev spaces etc.), see, for instance, [12].

The motivation of Problems 1 and 2 is transparent. The first one describes the situation where for some reasons at each time $t \in [0, T]$ only part $\overline{\Gamma}$ of the boundary of the «body» $\overline{\Omega}$ is available for measurements (though the continuity up to the initial time $t = 0$ is postulated). The second one describes the situation where the whole body $\overline{\Omega}$ was available for measurement at the initial time $t = 0$ but data on $\partial\Omega \setminus \overline{\Gamma}$ were lost as a consequence of extremal temperature conditions for $0 < t < T$. Nevertheless we want to preserve the continuity up to the final time $t = T$ in both problems at least on $\Omega \cup \overline{\Gamma}$. It is often important for applications that solutions to Problem 1 and 2 would belong to $C^{1,0}(\overline{\Omega_T})$. Later we will indicate the corresponding cases for Problem 1 (see Corollary 2).

Let us show that both Problem 1 and Problem 2 are ill-posed.

Example 1. *Take a cube $Q_n = \{0 < x_j < 1, 1 \leqslant j \leqslant n\}$ as base Ω of the cylinder Ω_T. Let Γ be the face $\{x_n = 0\}$ of the cube Q_n. Then $\Gamma_T = Q_{n-1} \times (0, T)$. Fix $N \in \mathbb{N}$ and consider the sequence of solutions*

$$u_k(x,\ t) = \frac{e^{k^2(t-T)+kx_n}}{k^N} \in C^\infty(\overline{\Omega_T})$$

to problem (1), (2), (3), (7) with the data

$$f_k(x,t) = 0, \quad u_{0,k}(x) = \frac{e^{-k(kT-x_n)}}{k^N},$$

$$u_{1,k}(x_1, \dots, x_{n-1}, t) = \frac{e^{k^2(t-T)}}{k^N}, \quad u_{2,k}(x_1, \dots, x_{n-1}, t) = \frac{e^{k^2(t-T)}}{k^{N-1}}.$$

It is clear, that compatibility conditions (9), (10) hold and

$$f_k \underset{k\to\infty}{\longrightarrow} 0 \textit{ in } C^\infty(\overline{\Omega_T}), \quad u_{0,k} \underset{k\to\infty}{\longrightarrow} 0 \textit{ in } C^\infty(\overline{\Omega}),$$

$$u_{1,k} \underset{k\to\infty}{\longrightarrow} 0 \textit{ in } C^s(\overline{\Gamma_T}), \qquad u_{2,k} \underset{k\to\infty}{\longrightarrow} 0 \textit{ in } C^s(\overline{\Gamma_T}),$$

if $N > 2s + 1$. On the other hand, for all $x_n > 0$ and all $N \in \mathbb{N}$ we have:

$$u_k(x,\ T) = \frac{e^{k^2(T-T)+kx_n}}{k^N} = \frac{e^{kx_n}}{k^N} \underset{k\to\infty}{\longrightarrow} \infty.$$

Thus there is no continuity with respect to the data and hence Problem 2 is ill-posed. Obviously, Problem 1 is ill-posed, too.

Remark 1. *If we replace in settings of Problem 1 and 2 solution's space $C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus (\partial\Omega \setminus \overline{\Gamma})_T)$ with the space $C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\Omega_T \cup \overline{\Gamma_T} \cup (\Omega \times \{0\}))$ then Example 1 will be not fit to demonstrate that problems are ill-posed because the uniform convergence on compact sets from $\Omega_T \cup \overline{\Gamma_T} \cup (\Omega \times \{0\})$ will be granted for the sequence of solutions. However, in practice this would mean an infinite temperature (and hence a catastrophe) at the final time $t = T$.*

By the way, it is natural to replace solution's space $C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus (\partial\Omega \setminus \overline{\Gamma})_T)$ with $C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T)$ in the setting of Problem 1. But then the classical Hadamard example (see [1] or, for instance, [15, Ch. 1, §2]) for the Cauchy problem for the Laplace Equation in $\mathbb{R}^n$ shows that the problem for Heat Equation in $\mathbb{R}^{n+1}$ will be ill-posed (at least if $n \geqslant 2$). Of course, in this case the data for the example do not depend on t.

As both Problems 1 and 2 are ill-posed, we will study Problem 1 only because in addition to (1)-(3) to investigate Problem 2 one needs to know also initial condition (7).

2. Uniqueness Theorem

In this section we will prove that Problem 1 can not have more than one solution. If the surface Γ and the data of the problem are real analytic then the Cauchy-Kovalevsky Theorem implies that Problem 1 can not have more than one solution in class of (even formal) power series. However the theorem does not imply the existence of solutions to Problem 1 because it grants the solution in a small neighborhood of the surface Γ_T only (but not in a given domain Ω_T!).

To obtain a Uniqueness Theorem for Problem 1 we use an integral representation constructed with the use the fundamental solution

$$\Phi(x,t) = \begin{cases} \frac{1}{\left(2\sqrt{\pi t}\right)^n}\ e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0, \end{cases}$$

to Heat Operator L_{n+1}.

More precisely, consider the cylinder type domain $\Omega_{T_1,T_2} = \Omega_{T_2} \setminus \overline{\Omega_{T_1}}$ and a closed measurable set $S \subset \partial\Omega$. For functions $f \in C(\overline{\Omega_{T_1,T_2}})$, $v \in C(S_T)$, $w \in C(S_T)$, $h \in C(\overline{\Omega_{T_1,T_2}})$ we set

$$I_{\Omega,T_1}(h)(x,t) = \int\limits_{\Omega} \Phi(x-y,t)h(y,T_1)dy, \tag{11}$$

$$G_{\Omega,T_1}(f)(x,t) = \int\limits_{T_1}^{t}\int\limits_{\Omega} \Phi(x-y,\ t-\tau)f(y,\tau)dyd\tau, \tag{12}$$

$$V_{S,T_1}(v)(x,t) = \int\limits_{T_1}^{t}\int\limits_{S} \Phi(x-y,t-\tau)v(y,\tau)ds(y)d\tau, \tag{13}$$

$$W_{S,T_1}(w)(x,t) = -\int\limits_{T_1}^{t}\int\limits_{S} \frac{\partial}{\partial\nu_y}\Phi(x-y,t-\tau)w(y,\tau)ds(y)d\tau, \tag{14}$$

where ds is the volume form on S induced from $\mathbb{R}^n$. All these functions are called *Heat Potentials* with densities f, v, w and h respectively. In our situation these are convergent improper integrals depending on vector parameter $(x,t) \in \mathbb{R}^{n+1}$ (see, for instance, [12, Ch. 4, §1], [13, Ch. 3, §10], [14, Ch. 1, §3 and Ch. 5, §2]). The potential $I_{\Omega,T_1}(h)$ is sometimes called *Poisson type integral* for the Heat Operator, the functions $G_{\Omega,T_1}(f)$, $V_{S,T_1}(v)$, $W_{S,T_1}(w)$ are often referred to as *Heat Volume Potential*, *Heat Single Layer Potential* and *Heat Double Layer Potential* respectively.

The integral formula, that we need, is similar to the famous Green Formula for the Laplace Operator.

Lemma 1. *For all $0 < T_1 < T_2$ and all $u \in C^{2,1}(\Omega_{T_1,T_2}) \cap C^{1,0}(\overline{\Omega_{T_1,T_2}})$ with $L_{n+1}u \in C(\overline{\Omega_{T_1,T_2}})$ the following formula holds:*

$$\left.\begin{matrix} u(x,t),\ (x,t) \in \Omega_{T_1,T_2} \\ 0,\ (x,t) \notin \overline{\Omega_{T_1,T_2}} \end{matrix}\right\} = \left(I_{\Omega,T_1}(u) + G_{\Omega,T_1}(L_{n+1}u) + V_{\partial\Omega,T_1}\left(\frac{\partial u}{\partial\nu}\right) + W_{\partial\Omega,T_1}(u)\right)(x,t). \tag{15}$$

Proof. See, for instance, [16, Ch. 6, §12]. □

Theorem 1 (Uniqueness Theorem). *If Γ has at least one interior point (on $\partial\Omega$), and function $u \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T})$ satisfies (1), (2), (3) with $f \equiv 0$, $u_1 \equiv u_2 \equiv 0$ then $u \equiv 0$ in Ω_T.*

Proof. Under the hypothesis of the theorem there is an interior point x_0 on Γ. Then there is such a number $r > 0$ that $B(x_0,\ r) \cap \partial\Omega \subset \Gamma$ where $B(x_0,\ r)$ is ball in $\mathbb{R}^n$ with center at x_0 and radius r. Fix an arbitrary point $(x', t') \in \Omega_T$. It is clear that there is a domain $\Omega' \ni x'$ satisfying $\Omega' \subset \Omega$ and $\Omega' \cap \partial\Omega \subset \Gamma \cap B(x_0,\ r)$. Then $(x', t') \in \Omega'_{T_1,T_2}$ with some $0 < T_1 < T_2 < T$.

But $u \in C^{2,1}(\Omega'_{T_1,T_2}) \cap C^{1,0}(\overline{\Omega'_{T_1,T_2}})$ and $L_{n+1}u = 0$ in Ω'_{T_1,T_2} under the hypothesis of the theorem. Hence formula (15) implies:

$$\left.\begin{array}{r} u(x,t),\ (x,t) \in \Omega'_{T_1,T_2} \\ 0,\ (x,t) \notin \overline{\Omega'_{T_1,T_2}} \end{array}\right\} = I_{\Omega',T_1}(u)(x,t) + V_{\partial\Omega'\setminus\Gamma,T_1}\left(\frac{\partial u}{\partial\nu}\right)(x,t) + W_{\partial\Omega'\setminus\Gamma,T_1}(u)(x,t), \qquad (16)$$

because $u \equiv \dfrac{\partial u}{\partial \nu} \equiv 0$ on Γ_T.

Taking into account the character of the singularity of the kernel $\Phi(x-y, t-\tau)$ we conclude that the following properties are fulfilled for the integrals, depending on parameter, from the right hand side of identity (16):

$$I_{\Omega',T_1}(u) \in C^{2,1}(\{x \in \mathbb{R}^n, T_1 < t < T_2\}),$$
$$W_{\partial\Omega'\setminus\Gamma,T_1}(u), V_{\partial\Omega'\setminus\Gamma,T_1<t<T_2}\left(\frac{\partial u}{\partial\nu}\right) \in C^{2,1}(\{x \in \mathbb{R}^n \setminus (\partial\Omega' \setminus \Gamma), T_1 < t < T_2\})$$

(see, for instance, [12, Ch. 4, §1], [13, Гл. 3, §10] or [14, Ch . 1, §3 and Ch. 5, §2]). Moreover, as Φ is a fundamental solution to Heat Operator then $L_{n+1}(x,t)\Phi(x-y, t-\tau) = 0$ for $(x,t) \neq (y,\tau)$, and therefore, using Leibniz rule for differentiation of integrals depending on parameter we obtain:

$$L_{n+1}I_{\Omega',T_1}(u) = 0 \text{ in the domain } \{x \in \mathbb{R}^n, T_1 < t < T_2\},$$
$$L_{n+1}V_{\partial\Omega'\setminus\Gamma,T_1}\left(\frac{\partial u}{\partial\nu}\right) = L_{n+1}W_{\partial\Omega'\setminus\Gamma,T_1}(u) = 0 \text{ in } \Omega''_{T_1,T_2} = \{x \in \mathbb{R}^n \setminus (\partial\Omega' \setminus \Gamma), T_1 < t < T_2\}.$$

Hence the function

$$P(x,t) = I_{\Omega',T_1}(u)(x,t) + V_{\partial\Omega'\setminus\Gamma,T_1}\left(\frac{\partial u}{\partial\nu}\right)(x,t) + W_{\partial\Omega'\setminus\Gamma,T_1}(u)(x,t),$$

satisfies the heat equation

$$(L_{n+1}P)(x,t) = 0 \text{ in } \Omega''_{T_1,T_2}.$$

This implies that the function $P(x,t)$ is real analytic with respect to the space variable $x \in \mathbb{R}^n \setminus (\partial\Omega' \setminus \Gamma)$ for any $T_1 < t < T_2$ (see, for instance, [15, Ch. VI, §1, Theorem 1]). In particular, by the construction the function $P(x,t)$ is real analytic with respect to x in the ball $B(x_0, r)$ and it equals to zero for $x \in B(x_0, R) \setminus \overline{\Omega}$ for all $T_1 < t < T_2$. Therefore, the Uniqueness Theorem for real analytic functions yields $P(x,t) \equiv 0$ in Ω''_{T_1,T_2}, and in the cylinder Ω'_{T_1,T_2}, the containing point (x', t'). Now it follows from (16) that $u(x', t') = P(x', t') = 0$ and then, since the point $(x', t') \in \Omega_T$ is arbitrary we conclude that $u \equiv 0$ in Ω_T. The proof is complete. □

Corollary 1. *Problem 1 has no more that one solution.*

Proof. Let $v(x,\ t)$ and $w(x,\ t)$ be two solutions to Problem 1. Then function $u = (v - w) \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus (\partial\Omega \setminus \Gamma)_T)$ is a solution to the corresponding problem with $f = 0,\ u_1 = 0,\ u_2 = 0$. Using 1 we conclude that u is identically zero in Ω_T. □

Thus, the Uniqueness Theorem implies that the data of Problem 1 are suitable in order to uniquely define its solution. Moreover, the theorem clarify why the problem is ill-posed. The reason is the redundant data. Indeed, if Γ has at least one interior point (on $\partial\Omega$), then taking a smaller set $\Gamma' \subset \Gamma$ we again obtain a problem with no more than one solution.

3. Solvability Conditions

From now on we will study Problem 1 under the assumption that its data belong to Hölder spaces (cf., [14, Ch. 1, §1] for other boundary problems for parabolic equations). We recall that a function $u(x)$, defined on a set $M \in \mathbb{R}^m$, is called *Hölder continuous with a power* $0 < \lambda < 1$ on M, if there is such a constant $C > 0$ that

$$|u(x) - u(y)| \leqslant C|x-y|^{\lambda} \text{ for all } x, y \in M \tag{17}$$

($|x-y| = \sqrt{\sum\limits_{j=1}^{m}(x_j-y_j)^2}$ being Euclidean distance between points x and y in $\mathbb{R}^m$). Let $C^{\lambda}(\overline{\Omega_T})$ stand for the set of Hölder continuous functions with a power λ over $\overline{\Omega_T}$. Besides, let $C^{1+\lambda,\lambda}(\overline{\Omega_T})$ be the set of Hölder continuous functions with a power λ over $\overline{\Omega_T}$, having Hölder continuous derivatives u_{x_i}, $1 \leqslant i \leqslant n$, with the same power in $\overline{\Omega_T}$.

We choose a set Ω^+ in such a way that the set $D = \Omega \cup \Gamma \cup \Omega^+$ would be a bounded domain with piece-wise smooth boundary. It is possible since Γ is an open connected set. It is convenient to set $\Omega^- = \Omega$. For a function v on D_T we denote by v^+ its restriction to Ω^+ and, similarly, we denote by v^- its restriction to Ω. It is natural to denote limit values of $v^{\pm}$ on Γ_T, when they are defined, by $v^{\pm}_{|\Gamma_T}$.

Theorem 2 (Solvability criterion). *Let* $\Gamma \in C^{1+\lambda}$, $f \in C^{\lambda}(\overline{\Omega_T})$, $u_1 \in C^{1+\lambda,\lambda}(\overline{\Gamma_T})$, $u_2 \in C^{\lambda}(\overline{\Gamma_T})$. *Problem (1), (2), (3) is solvable in the space* $C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T)$ *if and only if there is a function* $F \in C^{2,1}(D_T)$ *satisfying the following conditions:*

1) $L_{n+1}F = 0$ *in* D_T,
2) $F = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1)$ *in* Ω^+_T.

Proof. Necessity. Let a function $u(x,t) \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T)$ satisfies (1), (2), (2). Consider the function

$$F = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1) - \chi_{\Omega_T} u.$$

in the domain D_T, where χ_M is a characteristic function of the set $M \subset \mathbb{R}^{n+1}$. By the very construction condition 2) is fulfilled for it.

Clearly, the function $u(x,t)$ belongs to the space $C^{1,2}(\overline{\Omega'_T})$ for each cylindrical domain Ω'_T with such a base Ω' that $\Omega' \subset \Omega$ and $\overline{\Omega'} \cap \partial\Omega \subset \Gamma$. Besides, $L_{n+1}u = f \in C^{\lambda}(\overline{\Omega'_T})$. Without loss of the generality we may assume that the interior part Γ' of the set $\overline{\Omega'} \cap \partial\Omega$ is non-empty.

We note that $\chi_{\Omega_T} u = \chi_{\Omega'_T} u$ in D'_T, where $D' = \Omega' \cup \Gamma' \cup \Omega^+$. Then using Lemma 2 we obtain:

$$F = G_{\Omega\setminus\overline{\Omega'},0}(f) + V_{\overline{\Gamma}\setminus\Gamma',0}(u_2) + W_{\overline{\Gamma}\setminus\Gamma',0}(u_1) - I_{\Omega',0}(u) \text{ in } D'_T. \tag{18}$$

Arguing as in the proof of Theorem 1 we conclude that each of the integrals in the right hand side of (18) satisfies homogeneous Heat Equation outside the corresponding integration set. In particular, we see that $L_{n+1}F = 0$ in D'_T. Obviously, for any point $(x,t) \in D_T$ there is a domain D'_T containing $(x,\ t)$. That is why $L_{n+1}F = 0$ in D_T, and hence F belongs to the space $C^{2,1}(D_T)$. Thus this function satisfies condition 1), too.

Sufficiency. Let there be a function $F \in C^{2,1}(D_T)$, satisfying conditions 1) and 2) of the theorem. Consider on the set D_T the function

$$U = G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1) - F. \tag{19}$$

As $f \in C^{\lambda}(\overline{\Omega_T})$ then the results of [14, Ch. 1, §3] imply

$$G_{\Omega,0}(f) \in C^{2,1}(\Omega^{\pm}_T) \cap C^{1,0}(D_T) \cap C(\overline{D}_T) \tag{20}$$

and, moreover,

$$L_{n+1}G^{-}_{\Omega,0}(f) = f \text{ in } \Omega_T, \quad L_{n+1}G^{+}_{\Omega,0}(f) = 0 \text{ in } \Omega^{+}_{T}. \tag{21}$$

Since $u_2 \in C^{\lambda}(\overline{\Gamma_T})$ then the results of [14, Ch. 5, §2] yield

$$V_{\overline{\Gamma},0}(u_2) \in C^{2,1}(\Omega^{\pm}_{T}) \cap C^{1,0}((\Omega^{\pm} \cup \Gamma)_T) \cap C(\overline{D_T} \setminus (\partial\Gamma)_T), \tag{22}$$

$$L_{n+1}V_{\overline{\Gamma},0}(u_2) = 0 \text{ in } \Omega_T \cup \Omega^{+}_{T}. \tag{23}$$

On the other hand, the behavior of the Double Layer Potential $W_{\overline{\Gamma},0}(u_1)$ is similar to the behavior of the normal derivative of Single Layer Potential $V_{\overline{\Gamma},0}(u_1)$. Hence

$$W_{\overline{\Gamma},0}(u_1) \in C^{2,1}(\Omega^{\pm}_{T}) \cap C(\overline{\Omega^{\pm}_{T}} \setminus (\partial\Omega^{\pm} \setminus \Gamma)_T), \tag{24}$$

$$L_{n+1}W_{\overline{\Gamma},0}(u_1) = 0 \text{ in } \Omega_T \cup \Omega^{+}_{T}. \tag{25}$$

Lemma 2. *Let $S \subset \overline{\Gamma} \in C^{1+\lambda}$. If $u_1 \in C^{1+\lambda,\lambda}(\overline{\Gamma_T})$, then the potential $W^{-}_{\overline{\Gamma},0}(u_1)$ belongs to the space $C^{1,0}(\Omega_T \cup S_T)$ if and only if $W^{+}_{\overline{\Gamma},0}(u_2) \in C^{1,0}(\Omega^{+}_{T} \cup S_T)$.*

Proof. It is similar to the proof of the analogous lemma for Newton Double Layer Potential (see, for instance, [9, lemma 1.1]). Actually, one needs to use Lemma 2 instead of the standard Green formula for the Laplace operator. □

Since $F \in C^{1,0}(D_T)$ then it follows from the discussion above that $W^{+}_{\overline{\Gamma},0}(u_2) \in C^{1,0}((\Omega^{+} \cup \Gamma)_T)$. Thus, formulas (19)–(25) and Lemma 2 imply that

$$U \in C^{2,1}(\Omega^{\pm}_{T}) \cap C^{1,0}((\Omega^{\pm} \cup \Gamma)_T) \cap C(\overline{\Omega^{\pm}_{T}} \setminus (\partial\Omega \setminus \Gamma)_T),$$

$$L_{n+1}U = \chi_{D_T} f \text{ in } \Omega_T \cup \Omega^{+}_{T}.$$

In particular, (1) is fulfilled for U^{-}.

Let us show that the function U^{-} satisfies (2) and (3).

Since $F \in C^{1,0}(D_T)$ we see that $\partial^{\alpha} F^{-} = \partial^{\alpha} F^{+}$ on Γ_T for $\alpha \in \mathbb{Z}_{+}$ with $|\alpha| \leqslant 1$ and

$$\partial^{\alpha} F^{+}_{|\Gamma_T} = \left(\partial^{\alpha} G^{+}_{\Omega,0}(f) + \partial^{\alpha} V^{+}_{\overline{\Gamma},0}(u_2) + \partial^{\alpha} W^{+}_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T}.$$

It follows from formulas (20) and (22) that Heat Volume Potential and Single Layer Potential are continuous if the point (x,t) passes over the surface Γ_T. Then

$$U^{-}_{|\Gamma_T} = W^{-}_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} - W^{+}_{\overline{\Gamma},0}(u_1)_{|\Gamma_T} = u_1.$$

because of the theorem on jump behavior of the Heat Double Layer Potential (see, for instance, [14, Ch. 5, §2, theorem 1]), i.e. equality (2) is valid for U^{-} .

Formula (20) means that that the normal derivative of the Heat Volume Potential is continuous if the point (x,t) passes over the surface Γ_T. Therefore

$$\frac{\partial U^{-}}{\partial \nu}_{|\Gamma_T} = \left(\frac{\partial}{\partial \nu} V^{-}_{\overline{\Gamma},0}(u_2)\right)_{|\Gamma_T} - \left(\frac{\partial}{\partial \nu} V^{+}_{\overline{\Gamma},0}(u_2)\right)_{|\Gamma_T} + \left(\frac{\partial}{\partial \nu} W^{-}_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T} - \left(\frac{\partial}{\partial \nu} W^{+}_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T}. \tag{26}$$

By theorem on jump behavior of the normal derivative of the Heat Single Layer Potential (see, for instance, [13, Ch. 3, §10, theorem 10.1])

$$\left(\frac{\partial}{\partial \nu} V^{-}_{\overline{\Gamma},0}(u_2)\right)_{|\Gamma_T} - \left(\frac{\partial}{\partial \nu} V^{+}_{\overline{\Gamma},0}(u_2)\right)_{|\Gamma_T} = u_2. \tag{27}$$

Finally, we need the following lemma which is an analogue of the famous Theorem on jump behavior of the normal derivative of the Newton's Double Layer Potential.

Lemma 3. *Let* $\Gamma \in C^{1+\lambda}$ *and* $u_2 \in C^{\lambda}(\overline{\Gamma_T})$. *If* $W^{-}_{\overline{\Gamma},0}(u_1) \in C^{1,0}((\Omega \cup \Gamma)_T)$ *or* $W^{+}_{\overline{\Gamma},0}(u_1) \in C^{1,0}((\Omega^{+} \cup \Gamma)_T)$ *then*

$$\left(\frac{\partial}{\partial \nu} W^{-}_{\overline{\Gamma},0}(u_1) - \frac{\partial}{\partial \nu} W^{+}_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T} = 0. \tag{28}$$

Proof. Really, let, for instance, $W^{-}_{\overline{\Gamma},0}(u_1) \in C^{1,0}((\Omega \cup \Gamma)_T)$. Then using Lemma 2 we obtain $W^{+}_{\overline{\Gamma},0}(u_1) \in C^{1,0}((\Omega^{+} \cup \Gamma)_T)$ and $\left(\frac{\partial}{\partial \nu} W^{\pm}_{\overline{\Gamma},0}(u_1)\right)_{|\Gamma_T} \in C(\Gamma_T)$.

Let $\phi \in C^{\infty}_0(D_T)$ be a function with compact support in D_T. Then Gauss–Ostrogradskii formula yields:

$$\int_{\Gamma_T} \phi \left(\frac{\partial}{\partial \nu} W^{-}_{\overline{\Gamma},0}(u_1) - \frac{\partial}{\partial \nu} W^{+}_{\overline{\Gamma},0}(u_1)\right) ds(x)dt = \tag{29}$$

$$\int_{\Omega_T \cup \Omega^{+}_T} \phi \Delta W_{\overline{\Gamma},0}(u_1)dxdt + \int_{\Omega_T \cup \Omega^{+}_T} (\nabla \phi)' \nabla W_{\overline{\Gamma},0}(u_1)dxdt =$$

$$\int_{\Omega_T \cup \Omega^{+}_T} \phi \frac{\partial}{\partial t} W_{\overline{\Gamma},0}(u_1)dxdt + \int_{\Omega_T \cup \Omega^{+}_T} (\nabla \phi)' \nabla W_{\overline{\Gamma},0}(u_1)dxdt$$

because $L_{n+1} W^{\pm}_{\overline{\Gamma},0}(u_1) = 0$ in $\Omega^{\pm}$ according to (25).

Again integrating by parts and using Theorem on jump behavior of Heat Double Layer Potential we see that

$$\int_{\Omega_T \cup \Omega^{+}_T} \phi \frac{\partial}{\partial t} W_{\overline{\Gamma},0}(u_1)dxdt + \int_{\Omega_T \cup \Omega^{+}_T} (\nabla \phi)' \nabla W_{\overline{\Gamma},0}(u_1)dxdt = \tag{30}$$

$$\int\limits_{\Omega_T \cup \Omega^{+}_T} \frac{\partial \phi}{\partial t} W_{\overline{\Gamma},0}(u_1)dxdt - \int\limits_{\Omega_T \cup \Omega^{+}_T} (\Delta \phi)' W_{\overline{\Gamma},0}(u_1)dxdt + \int_{\Gamma_T} \frac{\partial \phi}{\partial \nu} (W^{-}_{\overline{\Gamma},0}(u_1) - W^{+}_{\overline{\Gamma},0}(u_1))ds(x)dt =$$

$$\int_{\Gamma_T} \frac{\partial \phi}{\partial \nu} u_1 ds(x)dt - \int_{\Omega_T \cup \Omega^{+}_T} (L'_{n+1}\phi) W_{\overline{\Gamma},0}(u_1)dxdt.$$

But the kernel $\Phi(x-y, t-\tau)$ is a fundamental solution of the parabolic operator L'_{n+1} with respect to variables (y, τ). Hence

$$\int_{D_T} L'_{n+1}\phi(x,t)\Phi(x-y, t-\tau)dxdt = \phi(y,\tau), \quad (y,\tau) \in D_T.$$

Then the type of the singularity of the fundamental solution allows us to apply Fubini Theorem and to conclude that

$$\begin{aligned}\int_{\Omega_T \cup \Omega^{+}_T} (L'_{n+1}\phi) W_{\overline{\Gamma},0}(u_1)dxdt &= \int_{\Gamma_T} u_1 \frac{\partial}{\partial \nu} \int_{D_T} L'_{n+1}\phi(x,t)\Phi(x-y, t-\tau)dxdtds(y)d\tau = \\ &= \int_{\Gamma_T} \frac{\partial \phi}{\partial \nu} u_1 ds(y)d\tau.\end{aligned} \tag{31}$$

Finally, formulas (29)–(31) imply that

$$\int_{\Gamma_T} \phi \left(\frac{\partial}{\partial \nu} W^{-}_{\overline{\Gamma},0}(u_1) - \frac{\partial}{\partial \nu} W^{+}_{\overline{\Gamma},0}(u_1)\right) ds = 0$$

for all $\phi \in C^{\infty}_0(D_T)$. As such functions are dense in the Lebesgue space $L^1(K)$ for any compact $K \subset \Gamma_T$ then formula (28) holds true. □

Now using lemma 3 and formulas (26), (27), we conclude that $\frac{\partial U^-}{\partial \nu}_{|\Gamma_T} = u_2$, i.e. (3) is fulfilled for U^-.

Thus, function $u(x,t) = U^-(x,t)$ satisfies conditions (1)–(3). The proof is complete. □

It follows from formula (19) that properties of a solution to Problem 1 depend on properties of the extension F of the sum of heat potentials, described in Theorem 2.

Corollary 2. *Let $S \subset \partial\Omega \setminus \Gamma$. Under the hypotheses of Theorem 2, Problem 1 is solvable in the space*

$$C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus S_T)$$

if and only if there exists a function

$$F \in C^{2,1}(D_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \cap C(\overline{\Omega_T} \setminus S_T),$$

satisfying conditions 1) and 2) of Theorem 2.

In particular, if $S = \varnothing$ then corollary 2 gives criterion for the existence of solution to Problem 1 in the space $C(\overline{\Omega_T})$.

We note that Theorem 2 is an analogue of Theorem by Aizenberg and Kytmanov [8]) describing solvability conditions of the Cauchy problem for the Cauchy–Riemann system (cf. also [9] in the Cauchy Problem for Laplace Equation or [7] in the Cauchy problem for general elliptic systems). Formula (19), obtained in the proof of Theorem 2, gives the unique solution to Problem 1. Clearly, if we will be able to write the extension F of the sum of potentials $G_{\Omega,0}(f) + V_{\overline{\Gamma},0}(u_2) + W_{\overline{\Gamma},0}(u_1)$ from Ω^+ onto D_T as a series with respect to special functions or a limit of parameter depending integrals then we will get a Carleman type formula for solutions to Problem 1 (cf. [8]). However this is a topic for another paper. In the present article we will discuss polynomial and formal solutions only.

4. Polynomial Solutions and Dense Solvability

It is not difficult to prove dense solvability of Problem 1 in the case where Γ is an open connected set of the hyperplane $\{x_n = 0\}$.

Lemma 4. *If Γ is an open connected set if the hyperplane $\{x_n = 0\}$ the Problem 1 is densely solvable.*

Proof. First let us prove that if in this case the data of Problem 1 are polynomials then the problem is solvable and its solution is a polynomial.

Indeed, Problem 1 is easily can be reduced to the following one:

$$L_{n+1} v = g \text{ in } \Omega_T \tag{32}$$

$$v(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \overline{\Gamma_T}, \tag{33}$$

$$\frac{\partial v}{\partial x_n}(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ на } \overline{\Gamma_T}. \tag{34}$$

with $g(x,t) = f(x,t) - (L_n u_1)(x_1, \ldots, x_{n-1}, t) - x_n (L_n u_2)(x_1, \ldots, x_{n-1}, t)$.

Besides, $u(x,t) = v(x,t) + u_1(x_1, \ldots, x_{n-1}, v) + x_n u_2(x_1, \ldots, x_{n-1}, t)$.

Now consider data $g^{(j,\alpha)}(x,t) = t^j x^\alpha$ with a multi-index $\alpha \in \mathbb{Z}_+^n$.

If $0 \leqslant \alpha_1 + \ldots \alpha_{n-1} \leqslant 1$, we easily obtain (unique) polynomial solutions

$$v^{(j,\alpha)}(x,t) = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j,\alpha_n)}(x_n, t), \; \alpha_n, j \in \mathbb{Z}_+, \tag{35}$$

to problem (32)–(34) where $w^{(0,k)}(y,t) = -\dfrac{y^{k+2}k!}{(k+2)!}$, $w^{(1,k)}(y,t) = -\dfrac{ty^{k+2}k!}{(k+2)!} - \dfrac{y^{k+4}k!}{(k+4)!}$, $k \in \mathbb{Z}_+$, $y \in \mathbb{R}$ and, by the induction with respect to $j \in \mathbb{Z}_+$,

$$w^{(j,k)}(y,t) = -\sum_{\mu=0}^{j} \frac{t^{j-\mu} y^{k+2\mu+2} k! j!}{(k+2\mu+2)!(j-\mu)!},\ k \in \mathbb{Z}_+,\ y \in \mathbb{R}. \tag{36}$$

To finish the arguments we use the induction with respect to $|\alpha'| \in \mathbb{Z}_+$ where $\alpha' = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$. Namely, let for $s \geqslant 2$ and all α' with $|\alpha'| = s$ the solutions to the problem are polynomial. If $|\alpha'| = s+1$ then $L_{n+1}\left(x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}} w^{(j,\alpha_n)}(x_n,t)\right) = t^j x^\alpha - w^{(j,\alpha_n)}(x_n,t)\Delta_{n-1}\left(x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}\right)$. Clearly, the degree of the polynomial $p_{j,\alpha}(x,t) = w^{(j,\alpha_n)}(x_n,t)\Delta_{n-1}\left(x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}\right)$ with respect to $x' \in \mathbb{R}^{n-1}$ equals to $s-1$. Then, by the induction, problem (1)–(3) with data $p_{j,\alpha}(x,t)$ admits a polynomial solution, say, $r_{j,\alpha}(x,t)$. Therefore the solution $v^{(j,\alpha)}(x,t)$ to problem (1)–(3) with data $g^{(j,\alpha)}(x,t) = t^j x^\alpha$, $|\alpha'| = s+1$, is given as follows: $v^{(j,\alpha)}(x,t) = x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}} w^{(j,\alpha_n)}(x_n,t) + r_{j,\alpha}(x,t)$, i.e. it is a polynomial, too.

Now Problem 1 with zero boundary data in the case $\Gamma \subset \{x_n = 0\}$ is densely solvable because any continuous function g on the compact set $\overline{\Omega_T}$ can be approximated by polynomials. But the reducing to zero boundary data was organized in such a way that one easily sees, in this case Problem 1 is densely solvable for non-zero boundary data, too. □

The dense solvabilty of Problem 1 in general setting is natural to expect if the set $\partial\Omega \setminus \overline{\Gamma}$ has at least one interior point in $\partial\Omega$ (cf. [10] in the Cauchy Problem for elliptic equations).

Finally, we note that polynomial solutions indicated in the proof of Lemma 4 can be used in order to construct formal solutions to Problem 1. For example, if $n = 1$ and the data for problem (4)–(6) are written as (formal) power series $f(x,t) = \sum\limits_{k,j=0}^{\infty} c_{k,j} x^k t^j$, $u_1(t) = \sum\limits_{j=0}^{\infty} a_j t^j$, $u_2(t) = \sum\limits_{j=0}^{\infty} b_j t^j$ then (35), (36) imply that its formal solution is given by the power series

$$u(x,t) = \sum_{k,j=0}^{\infty} d_{k,j} x^k t^j \text{ with}$$

$$d_{k,j} = \begin{cases} a_j, & k = 0,\\ b_j, & k = 1,\\ \left((j+1)a_{j+1} - c_{0,j}\right)/2, & k = 2,\\ \left((j+1)b_{j+1} - c_{1,j}\right)/6, & k = 3,\\ -\sum_{p=0}^{k-2} \sum\limits_{i \geqslant j, 2i+p = k+2j} c_{p,i} \dfrac{p!i!}{k!j!}, & k \geqslant 4. \end{cases}$$

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Об одной некорректной задаче для уравнения теплопроводности

Роман Е. Пузырев
Александр А. Шлапунов

В работе исследована одна краевая задача для уравнения теплопроводности. Она состоит в восстановлении функции, удовлетворяющей уравнению теплопроводности в цилиндрической области, по заданным ее значениям и значениям ее нормальной производной на части боковой поверхности цилиндра. Доказано, что задача является некорректной в естественных для нее пространствах гладких функций и соответствующих пространствах Гельдера, а добавление к условиям начальных данных не превращает задачу в корректную. С помощью метода интегральных представлений получены теорема единственности, условия разрешимости и формулы для решения задачи.

Ключевые слова: краевые задачи для уравнения теплопроводности, некорректные задачи, метод интегральных представлений.