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From Classical Logic to Fuzzy Logic and Quantum Logic: A General View

S. Nădăban

Sorin Nădăban*
Aurel Vlaicu University of Arad, Department of Mathematics and Computer Science, Elena Drăgoi 2, RO-310330 Arad, Romania
*Corresponding author: snadaban@gmail.com

Abstract

The aim of this article is to offer a concise and unitary vision upon the algebraic connections between classical logic and its generalizations, such as fuzzy logic and quantum logic. The mathematical concept which governs any kind of logic is that of lattice. Therefore, the lattices are the basic tools in this presentation. The Hilbert spaces theory is important in the study of quantum logic and it has also been used in the present paper.

Keywords: fuzzy logic, quantum logic, orthomodular lattice, Hilbert space, quantum mechanics, effect algebras.

1 Aristotelian logic

What is logic? Logic is the science that studies thinking. As in the process of thinking there are two factors, information and emotion, we can talk about a domain of human thought called rational thinking, characterized by the fact that the obtained results are out of the factor called emotion. Therefore we can say that logic is the science that studies rational thinking and we will name it rational thinking logic. The first research about logic was made in the Ancient Greece. The results were systematized by the Greek philosopher Aristotle, in a treatise of logic called Organon. For this reason the rational thinking logic is called Aristotelian logic. In this logic a sentence is either true or false and to one true sentence value 1 is associated while to a false sentence value 0 is associated. Because of this classical logic is also called bivalent logic (having two truth values - true or false).

The principles that stay at the basis of this logic are:

1. **Principle of Identity**: each concept is identical to itself $A = A$.
2. **Principle of Noncontradiction**: one sentence cannot be both true and false at the same time.
3. **Principle of the Excluded Middle**: a sentence is either true or false. "There cannot be an intermediate between contradictions, but of one subject we must either affirm or deny any one predicate" *(Aristotle's Metaphysics)*.

4. **Principle of Double Negation**: A double negation is an affirmative statement, $\bar{\bar{A}} = A$. This principle was stated as a law of propositional calculus by B. Russell and A.N. Whitehead in Principia Mathematica.

5. **Principle of Sufficient Reason**: states that everything must have a reason or a cause. This principle led, along the years, to many controversies and various interpretations. Starting from this principle we get to *modus ponens*, the most important rule of inference, which can be summarized as: if P implies Q and P is true, then Q must be also true. In this situation proposition P is called "sufficient proposition" of proposition Q.

I have not yet used the term of *mathematical logic*. The mathematical logic is the result of a process of mathematical modelling of the rational thinking logic elements, a process that started in the 17^{th} century, by the German mathematician and philosopher G. Leibniz and it was continued in the 19^{th} century by the English mathematicians George Boole and Augustus de Morgan and also by many others along the time. In conclusion, Mathematics, through mathematical modelling of rational logic, brought about an essential contribution to the development of logic. On the other hand, the mathematical logic has become a fundamental instrument in the building of the mathematical universe.

2 The lattices of classical logic

In 1854, George Boole discovered a connection between the laws of logic and some laws of algebraic calculus. Thus, he introduced the algebra that was named after him, the Boolean algebra. In this section we briefly review the theory of lattices and we will present two important examples: the lattice of all propositions about a universe of discourse and the lattice of all subsets of an universal set, which, as we will see, can be identified. For other results and notions in the theory of general lattices we refer to [3], [10], [14].

Definition 1. *A lattice is a 4-tuple* $(L, \leq, \wedge, \vee)$ *such that* $(L, \leq)$ *is a partial order set (shortly poset) and for all* $x, y \in L$ *there exists a greatest lower bound (meet or infimum)* $x \wedge y$ *and a least upper bound (join or supremum)* $x \vee y$.

Proposition 2. *If* $(L, \leq, \wedge, \vee)$ *is a lattice, then:*

1. $x \vee y = y \vee x$; $x \wedge y = y \wedge x$ *(commutativity);*

2. $x \vee (y \vee z) = (x \vee y) \vee z$; $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *(associativity);*

3. $x \vee (x \wedge y) = x$; $x \wedge (x \vee y) = x$ *(absorption);*

4. $x \vee x = x$; $x \wedge x = x$ *(idem potency).*

Definition 3. *A bounded lattice is a lattice* $(L, \leq, \wedge, \vee)$ *with two elements* $0 \leq 1$, *the least and the greatest elements of* L.

Proposition 4. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice. Then:*

$$x \vee 0 = x, x \wedge 0 = 0, x \vee 1 = 1, x \wedge 1 = 1, (\forall)x \in L.$$

Definition 5. *A lattice* $(L, \leq, \wedge, \vee)$ *is called distributive if*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), (\forall)x, y, z \in L.$$

Remark 6. *Condition from previous definition is equivalent to:*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), (\forall)x, y, z \in L.$$

Definition 7. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice.*

Let $x \in L$. *An element* $x' \in L$ *is called the complement of* x *if* $x \vee x' = 1$ *and* $x \wedge x' = 0$.

The lattice $(L, \leq, \wedge, \vee, 0, 1)$ *is called complemented if* $(\forall)x \in L$ *has a complement in* L. *The lattice* $(L, \leq, \wedge, \vee, 0, 1)$ *is called uniquely complemented if* $(\forall)x \in L$ *has a unique complement in* L.

Theorem 8. *If $(L, \leq, \wedge, \vee, 0, 1)$ is distributive and complemented then it is uniquely complemented.*

Definition 9. *A bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$ is called Boolean (Boolean algebra or Boolean lattice) if it is distributive and complemented.*

In the theory of Boolean algebra, the complement is usually denoted by "$'$", but we can also meet the symbol "$\neg$". In some papers the term negation is used for the complement, but there are different types of negations which are studied in more general context of non-distributive lattices (see [5]). Therefore, in order to exclude any possible confusion, when we use the term "complement" this will have the meaning of Definition 7. As we will see, in the case of fuzzy logic, we do not have a complement but we do have a negation.

Theorem 10. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a Boolean lattice. Then:*

1. $(x \vee y)' = x' \wedge y'$; $(x \wedge y)' = x' \vee y'$ *(De Morgan law);*
2. $(x')' = x$ *(involution);*

Example 11. *Let $\mathcal{F}$ be the family of all proposition about an universe of discourse. If $x, y \in \mathcal{F}$ we will denote "x implies y" by $x \leq y$. It is natural to admit that:*

1. *x implies x;*
2. *if x implies y and y implies z, then x implies z;*
3. *if x implies y and y implies x, then x and y are logically equivalent.*

Thus $(\mathcal{F}, \leq)$ is a poset. The conjunction of two propositions x, y is denoted by $x \wedge y$ and it is the true sentence if x and y are both true and it is false otherwise. The disjunction of two propositions x, y is denoted by $x \vee y$ and it is the true sentence if, at least one of the sentences x and y is true and false otherwise. Thus $(\mathcal{F}, \leq, \wedge, \vee)$ is a lattice.

We note that $(\mathcal{F}, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice, where 0 is a proposition that is always false (called a contradiction) and 1 is a proposition that is always true (called a tautology).

The complement of the proposition x is the proposition x' which is true if x is false and it is false if x is true.

Example 12. *Let X be a nonempty set. The power set $\mathcal{P}(X)$ (the family of all subsets of X) is a lattice in which the order relation is inclusion and the operations on $\mathcal{P}(X)$ are union and intersection. Moreover, $(\mathcal{P}(X), \leq, \wedge, \vee, 0, 1)$ is a bounded lattice, where the smallest element 0 is the empty set and the largest element 1 is the set X itself. The complement of A is $\mathcal{C}_X(A)$.*

In the theory of the lattices the concept of logical implication is a very important one. This concept is associated with an operation over two logical values. In the classical logic $x \to y$ means that $x \leq y$ which is logically equivalent to $x' \vee y$ sau cu $x = x \wedge y$. In the case of non-Boolean lattices we can also have other ways of defining the logical implication.

3 Orthomodular lattices

In this section we will present some aspects of orthomodular lattices, because they represent the basic structure in quantum logic. For other results and notions in the theory of general lattices we refer to [3], [10], [14].

Definition 13. *A lattice $(L, \leq, \wedge, \vee)$ is called modular if:*

$$x \leq y \Rightarrow x \vee (z \wedge y) = (x \vee z) \wedge y, (\forall) z \in L.$$

Theorem 14. *Every distributive lattice is modular.*

Definition 15. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A unary operation $\perp \colon L \to L$ is called orthocomplementation if $(\forall) x, y \in L$ we have:*

1. $x^{\perp^{\perp}} = x$ *(involution);*
2. $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$ *(antitone);*
3. $x \wedge x^{\perp} = 0$ *(non-contradiction).*

The 7-tuple $(L, \leq, \wedge, \vee, 0, 1, \perp)$ *is called orthocomplemented lattice.*

Theorem 16. *If* $(L, \leq, \wedge, \vee, 0, 1, \perp)$ *is an orthocomplemented lattice, then:*

1. $0^{\perp} = 1$; $1^{\perp} = 0$ *(boundary condition);*
2. $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$; $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$ *(De Morgan laws);*
3. $x \vee x^{\perp} = 1$ *(excluded middle).*

Corollary 17. *If L is an orthocomplemented lattice, then L is complemented.*

Corollary 18. *If L is an orthocomplemented and distributive, then L is Boolean.*

Definition 19. *An orthomodular lattice is an orthocomplemented lattice such that it satisfies:*
$x \leq y \Rightarrow x \vee (x^{\perp} \wedge y) = y, (\forall)x, y \in L$ *(orthomodular law).*

Theorem 20. *If L is Boolean algebra, than L is modular, orthomodular and orthocomplemented.*

In the theory of lattices the concepts of negation and implication on an orthomodular lattice have an important role. We will restrain to presenting only some notions.

Definition 21. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice.*

1. *A function $\neg : L \to L$ is a negation on L if:*
   (a) $x \leq y \Rightarrow \neg y \leq \neg x$ *(antitone);*
   (b) $x \leq \neg\neg x, (\forall)x \in L$ *(weak double negation).*
2. *A negation $\neg$ is called intuitionistic negation on L if $x \wedge \neg x = 0$ (non-contradiction).*
3. *A negation $\neg$ is called fuzzy negation on L if $\neg 1 = 0$ (boundary condition).*
4. *A negation $\neg$ is called de Morgan negation on L if $x = \neg\neg x, (\forall)x \in L$ (involutory).*
5. *A de Morgan negation $\neg$ is an ortho negation on L if $x \wedge \neg x = 0, (\forall)x \in L$ (non-contradiction).*
6. *An ortho negation $\neg$ is an orthomodular negation on L if $x \leq y \Rightarrow x \vee (\neg x \wedge y) = y$ (orthomodular).*

The concept of complement and the concept of negation are fundamentally different. Complementation is a characteristic of a lattice while negation is a function defined on a lattice. The necessity of a notion of "implication" is obvious. We cannot talk of a "logic" without having such a notion. "I would argue that a 'logic' without an implication function susceptible of reasonable interpretation is radically incomplete, and indeed, hardly qualifies as a theory of deduction" (see [28]). Any definition we give to the concept of implication, it should satisfy the condition

$$x \to y = 1 \Leftrightarrow x \leq y,$$

condition that we could consider natural also in orthomodular lattice not only in the case of Boolean algebras. But, as we have seen in the previous section, in the classical logic we have the implication $x \to y \Leftrightarrow \neg x \vee y$. In a orthomodular lattice an implication defined in such a way would not satisfy anymore the condition $x \to y = 1 \Leftrightarrow x \leq y$. In 1970 the following implication

$$x \to y \Leftrightarrow \neg x \vee (x \wedge y)$$

was seen to satisfy the above condition. It was called Sasaki hook or quantum implication. We highlight that, in the case that the lattice is distributive, namely a Boolean lattice, the Sasaki hook becomes equal to $\neg x \vee y$. Using Sasaki hook, many authors reformulated the theory of orthomodular lattices and called them "quantum logic". We also specify that, in 1974, G. Kalmbach showed that along the Sasaki hook there are four more "quantum implications", which satisfy the above condition and reduce themselves to $\neg x \vee y$ in the case of Boolean algebras. (see [19]). A much more general definition for the concept of implication can be the following

Definition 22. *[10] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with negation. A function $\rightarrow$ on L is called implication if:*

1. $\{x \leq y\} \Rightarrow x \rightarrow y \;\; \leq \;\; x \vee y, (\forall)x, y \in L;$
2. $x \wedge (x \rightarrow y) \;\; \leq \;\; \neg x \vee y, (\forall)x, y \in L.$

4 Fuzzy logic

As we have seen, the fundamental concept of the classical logic is that of the proposition. This is an affirmation that we can decide, without any ambiguity, whether it is true or false. For example, “10 is an even number” is a true sentence. Everything functions well, because the sentences are clearly formulated. But, for many sentences from everyday language, like “John is tall” or “the room is clean” there is no clear answer as true or false. Traditional mathematics avoided such situations, or, in order to avoid ambiguity, we give a more precise definition like “tall is more than 1,72 m”. Such an approach has the disadvantage that a man of 1,718 m is not considered tall whereas a man of 1,72 m is.

A more efficient way to treat such ambiguities is offered by fuzzy logic, introduced in 1965 by L.A. Zadeh in his famous paper [26]. Fuzzy logic and fuzzy set theory have led to a new domain in mathematics and has found applications in several other fields.

Thus, if X is an arbitrary set, by fuzzy set in X we understand an application $\mu : X \rightarrow [0, 1]$. This representing a generalization of the classical subsets which can be identified with their characteristic functions defined on X, but with values in $\{0, 1\}$. The value $\mu(x)$ is regarded as the degree of membership of x to fuzzy set, or the value of truth with which x belongs to the fuzzy set.

The classical operations of union, intersection and complementarity were redefined by L.A. Zadeh in the context of fuzzy sets. Thus, if μ, ν are fuzzy sets in X there union, noted $\mu \vee \nu$, their intersection, notated $\mu \wedge \nu$ and μ's complementarity notated $\mathcal{C}\mu$, are fuzzy sets in X defined by

$$(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}, (\forall)x \in X;$$

$$(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}, (\forall)x \in X;$$

$$(\mathcal{C}\mu)(x) = 1 - \mu(x), (\forall)x \in X.$$

For μ, ν fuzzy sets in X, we will say that $\mu \leq \nu$ if $\mu(x) \leq \nu(x), (\forall)x \in X$. We will note by 0 si 1 the constant functions on X, defined by $0(x) = 0, (\forall)x \in X$ and, respectively, by $1(x) = 1, (\forall)x \in X$. We agree to further note by $\mathcal{F}(X)$ the family of all fuzzy sets in X.

We have that $(\mathcal{F}(X), \leq \wedge, \vee, 0, 1)$ is a distributive lattice and $\mu \rightarrow \mathcal{C}\mu = 1 - \mu$ is a fuzzy negation, but is not an ortho negation. The only laws that are not satisfied are the law of non-contradiction and the excluded-middle law because $\mu \wedge \mathcal{C}\mu \neq 0$ and $\mu \vee \mathcal{C}\mu \neq 1$.

After L.A. Zadeh introduced the concept of fuzzy set, many generalizations have rapidly appeared. Recently, in the paper [4], 21 variants of fuzzy sets have been listed and the connections among them have been presented. Later, in paper [12], it is shown that in the two-dimensional case, several of the lattices of truth values considered here are pairwise isomorphic, and so are the corresponding families of fuzzy sets. Further on, we will present only some of these generalizations.

J.A. Goguen [9] introduces in 1967 the notion of L-fuzzy sets, when the membership functions take values in a partially ordered set L, most often a lattice.

L.A. Zadeh [27] introduces in 1975 the concept of interval-valued fuzzy set. This is characterized by the membership function $\mu : X \rightarrow \mathcal{D}([0, 1])$, where $\mathcal{D}([0, 1])$ represents the family of all closed subintervals of the unit interval $[0, 1]$.

K.T. Atanassov [1] introduces in 1986 the concept of intuitionistic fuzzy sets, where each element is characterized by a membership function μ, as in the case of fuzzy sets, but also by a non-membership function ν such that $0 \leq \mu(x) + \nu(x) \leq 1, (\forall)x \in X$. Later, in 1989, K.T. Atanassov and G. Gargov [2] introduce interval valued intuitionistic fuzzy sets, which are characterized by two functions $\mu, \nu : X \to \mathcal{D}([0,1])$ such that $0 \leq \sup_{x\in X} \mu(x) + \sup_{x\in X} \nu(x) \leq 1$.

As a generalization of the classic sets, fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuiyionistic fuzzy sets, F. Smarandache [24] proposes in 1999 the concept of neutrosophic set. A neutrosophic set in X is defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

where $T_A(x), I_A(x), F_A(x)$ are subsets of $]0^-, 1^+[$ and represent the truth-membership function, indeterminacy-membership function and falsity-membership function such that

$$0^- \leq \sup_{x\in X} T_A(x) + \sup_{x\in X} I_A(x) + \sup_{x\in X} F_A(x) \leq 3^+,$$

where

$$a^- = \{a - \epsilon \ : \ \epsilon \in \mathbb{R}^*, \epsilon \text{ is infinitesimal}\}$$

$$b^+ = \{b + \epsilon \ : \ \epsilon \in \mathbb{R}^*, \epsilon \text{ is infinitesimal}\}.$$

For applications we can consider that $T_A(x), I_A(x), F_A(x)$ are subsets of $[0,1]$.

Pythagorean fuzzy sets were proposed by R.R. Yager in 2013 (see [25]). A Pythagorean fuzzy set is characterized by the functions $\mu, \nu : X \to [0,1]$ which give us the degree of membership and degree of non-membership, respectively and $0 \leq (\mu(x))^2 + (\nu(x))^2 \leq 1, (\forall)x \in X$. The function $\pi : X \to [0,1]$ defined by $\pi(x) = \sqrt{1 - [(\mu(x))^2 + (\nu(x))^2]}$ is called the degree of indeterminacy. In the following table we explain the differences between Pytagorean fuzzy sets and intuitionistic fuzzy sets (see [7]):

Intuitionistic fuzzy sets	Pythagorean fuzzy sets
$0 \leq \mu + \nu \leq 1$	$0 \leq \mu^2 + \nu^2 \leq 1$
$\pi = 1 - (\mu + \nu)$	$\pi = \sqrt{1 - [\mu^2 + \nu^2]}$
$\mu + \nu + \pi = 1$	$\mu^2 + \nu^2 + \pi^2 = 1$

We cannot end this section without discussing some aspects regarding the triangular norms (briefly t-norms). These were introduced by A K. Menger [18] and developed by B. Schweizer and A. Sklar [23]. They have become interesting in the fuzzy logic because they maintain the main properties of conjunction and they serve as generalization of it. Associated with t-norm is the triangular conorm (t-conorm) which has the main properties of disjunction. Our basic reference for t-norms is [13].

Definition 23. *[23]. A binary operation*

$$* : [0,1] \times [0,1] \to [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall)a, b \in [0,1]$;
2. $a * 1 = a, (\forall)a \in [0,1]$;
3. $(a * b) * c = a * (b * c), (\forall)a, b, c \in [0,1]$;
4. *If* $a \leq c$ *and* $b \leq d$*, with* $a, b, c, d \in [0,1]$*, then* $a * b \leq c * d$.

Definition 24. *A triangular conorm S (t-conorm) is a binary operation*

$$\circ : [0,1] \times [0,1] \to [0,1]$$

such that:

1. $a \circ b = b \circ a, (\forall)a, b \in [0,1]$;

2. $a \circ 0 = a, (\forall)a \in [0,1]$;

3. $(a \circ b) \circ c = a \circ (b \circ c), (\forall)a, b, c \in [0,1]$;

4. *If* $a \leq c$ *and* $b \leq d$, *with* $a, b, c, d \in [0,1]$, *then* $a * b \leq c * d$.

Proposition 25. $\circ$ *is a t-conorm if and only if there exists a t-norm* $*$ *such that*

$$x \circ y = 1 - ((1-x) * (1-y)), (\forall)x, y \in [0,1].$$

$*$ *is called the dual of t-conorm* $\circ$ *and* $\circ$ *is the dual of t-norm* $*$.

The most important pairs of t-norm and corresponding t-conorms are:

1. the minimum t-norm $a \wedge b = \min\{a, b\}$ and the maximum t-conorm $a \vee b = \max\{a, b\}$;
2. $a \cdot b$ the usual multiplication in $[0,1]$ and the probabilistic sum $a \circ_\pi b = a + b - ab$;
3. the Lukasiewicz t-norm $a *_L b = \max\{a + b - 1, 0\}$ and the Lukasiewicz t-conorm $a \circ_L b = \min\{a + b, 1\}$;
4. the drastic product

$$a *_D b = \begin{cases} a & \text{if} & b = 1 \\ b & \text{if} & a = 1 \\ 0 & otherwise & \end{cases}$$

and the drastic sum

$$a \circ_D b = \begin{cases} a & \text{if} & b = 0 \\ b & \text{if} & a = 0 \\ 0 & otherwise & \end{cases}.$$

We note that

$$a *_D b \leq a *_L b \leq a \cdot b \leq a \wedge b, (\forall)a, b \in [0,1]$$

and for each t-norm $*$ we have that

$$a *_D b \leq a * b \leq a \wedge b, (\forall)a, b \in [0,1].$$

On the other hand,

$$a \vee b \leq a \circ_\pi b \leq a \circ_L b \leq a \circ_D b, (\forall)a, b \in [0,1]$$

and for each t-conorm $\circ$ we have:

$$a \vee b \leq a \circ b \leq a \circ_D b, (\forall)a, b \in [0,1].$$

Finally, if $*$ is a t-norm, $\circ$ is a t-conorm and μ, ν are fuzzy sets in X, then the intersection of μ and ν is the fuzzy set $\mu \cap_* \nu$ defined by $(\mu \cap_* \nu)(x) = \mu(x) * \nu(x)$. The union of μ and ν is the fuzzy set $\mu \cup_\circ \nu$ defined by $(\mu \cup_\circ \nu)(x) = \mu(x) \circ \nu(x)$.

5 From quantum mechanics to quantum logic

Although quantum mechanics was discovered since 1925 by W. Heisenberg, M. Born and P. Jordan and formulated mathematically as "matrix mechanics", and later, in 1927, E. Schrödinger developed another mathematic technique known as "wave mechanics", John von Neumann in two years (1927-1929) developed the mathematical frame of the theory. He realized that the natural context is the Hilbert space theory and the operators among them.

In quantum mechanics each physical system is associated a separable Hilbert space H and the states of the physical system are described by unitary vectors $x \in H$. A physical quantity that can be measured is called observable. These will be represented by self-adjoint operators, not necessarily bounded, on the Hilbert space H, which in general do not commute. From here a thorough investigation of linear operators on Hilbert spaces has been conducted by John von Neumann, reaching its highlight in 1929 with the spectral theorem for unbounded self-adjoint operators.

To the observable A, which is a self-adjoint operator, the spectral measure is associated $P_A : Bor(\mathbb{R}) \to \mathcal{A}(H)$. If A is from $\mathcal{B}(H)$ (bounded operator) then

$$Ax = \left(\int_{\mathbb{R}} t dP_A(t) \right) x, (\forall) x \in H.$$

If A is from $\mathcal{C}(H)$ (closed operator) then

$$Ax = \lim_{k \to \infty} \left(\int_{-k}^{k} t dP_A(t) \right) x, (\forall) x \in D(A),$$

where

$$D(A) = \left\{ x \in H : \int_{\mathbb{R}} t^2 d < P_A(t)x, x >< \infty \right\}.$$

This spectral measure induces a probability measure

$$\mu_{x,A} : Bor(\mathbb{R}) \to [0,1] \text{ defined as } \mu_{x,A}(S) =< P_A(S)x, x >,$$

which John von Neumann regarded as the observable represented by A to take values in the set S when the state of the system is represented by the vector x. In many situations, the physical quantity which will be measured can have only particular values. This happens, for example, in the case of energy. In this case, we will have $\sigma(A) = \{\lambda_k\}_{k=1}^{\infty}$, where λ_k are eigenvalues of A. Let $P_k = P_A(\{\lambda_k\})$. Then for $S \in Bor(\mathbb{R})$ we have

$$\mu_{x,A}(S) =< P_A(S)x, x >=< \sum_{\lambda_k \in S} P_A(\{\lambda_k\})x, x >=< \sum_{\lambda_k \in S} P_k x, x >= \sum_{\lambda_k \in S} < P_k x, x >=$$

$$= \sum_{\lambda_k \in S} < P_k^2 x, x >= \sum_{\lambda_k \in S} < P_k x, P_k^* x >= \sum_{\lambda_k \in S} < P_k x, P_k x >= \sum_{\lambda_k \in S} \|P_k x\|^2 .$$

In particular, the probability of the observable A to take the value λ_k is $\mu_{x,A}(\{\lambda_k\}) = \|P_k x\|^2$.

If the observable A has an arbitrary spectrum, then we will not be able to talk about the probability of a value. In this situation, we will talk about the probability of obtaining values in the interval $[a, b]$, probability that will be $\mu_{x,A}([a, b])$.

In the axiomatization of quantum mechanics, the next definition is essential.

Definition 26. *We suppose that the physical system is in state x and let A an observable having the spectral measure P_A. Then for any $S \in Bor(\mathbb{R})$, after performing a measurement on the observable A, which gave the value $\lambda \in S$, the state of the physical system will change and will be given by a new state vector*

$$y = \frac{P_A(S)x}{\|P_A(S)x\|}.$$

Corollary 27. *$\mu_{y,A}(L) = \mu_{x,A}(S \cap L)/\mu_{x,A}(S)$, $(\forall) L \in Bor(\mathbb{R})$.*

Let's suppose now that the physical system is in state x and consider two observables A and B with the spectral measures P_A and P_B. Suppose a measurement was performed on the observable A, which gave the value $\lambda \in S$, $S \in Bor(\mathbb{R})$. According to the previous definition, the state of the physical system becomes

$$y = \frac{P_A(S)x}{\|P_A(S)x\|}.$$

If we perform now a measurement on the observable B, which gives the value $v \in L$, $L \in Bor(\mathbb{R})$, the state of the physical system becomes

$$z = \frac{P_B(L)y}{\|P_B(L)y\|} = \frac{P_B(L)P_A(S)x}{\|P_B(L)P_A(S)x\|}.$$

Definition 28. *The observables A and B are called simultaneous or compatible if the state of the system after performing a measurement on the observable A, followed by a measurement on the observable B, coincides with the state of the system obtained by first performing a measurement on the observable B, followed by a measurement on the observable A.*

Remark 29. *The observables A and B are simultaneous if and only if*

$$\frac{P_B(L)P_A(S)x}{\|P_B(L)P_A(S)x\|} = \frac{P_A(S)P_B(L)x}{\|P_A(S)P_B(L)x\|}$$

for any state x and any $S, L \in Bor(\mathbb{R})$.

Proposition 30. *The observables A and B are simultaneous if and only if the self-adjoint operators A and B commute.*

We suppose now that the operators A and B don't commute. We make the following notations:

$$m(A,x) = \int_{\mathbb{R}} \lambda d < P_A(\lambda)x, x >=< Ax, x >$$

$$v(A,x) = \int_{\mathbb{R}} (\lambda - m(A,x))^2 d < P_A(\lambda)x, x >= \|Ax\|^2 - < Ax, x >,$$

where $x \in D(A)$.

For the arbitrary A and B we introduce their commutator $[A, B] := AB - BA$ defined on $D([A, B]) = D(AB) \cap D(BA)$.

Theorem 31. (Heisenberg's inequality). *For any $x \in D([A, B])$ we have*

$$v(A,x) \cdot v(B,x) \geqslant \frac{| < [A,B]x, x > |^2}{4}.$$

The results obtained by John von Neumann were presented by him in an extent form in the book "Mathematical Foundation of Quantum Mechanics", published in 1932 and which rapidly became the book that has been, since then, considered the basis in mathematical foundation of the quantum mechanics. In the third chapter of this book John von Neumann proposes for the first time the quantum logic: "the relation between the properties of physical system on the one hand, and the projections on the other hand, makes possible a sort of logical calculus with these". In fact, the closed linear subspaces of a Hilbert space H are in one-to-one correspondence with the projection over them. John von Neumann, preoccupied by the logic and the algebraic structure of the quantum mechanics, starts cooperation with G. Birkhoff with whom, in 1936, publishes "The Logic of Quantum Mechanics", paper that can be considered the official birth of quantum logic. The family of all closed linear subspaces of a Hilbert space H, denoted $Lat(H)$ and called Hilbert lattice associated to H, was G. Birkhoff's and J. von Neumann's proposal for the algebraic structures that organizes the sentences. This is a different structure from that of classic logic where the sentences were organized in the power set $\mathcal{P}(X)$ which form, as we have seen, a Boolean algebra which satisfies the distributive law. $Lat(H)$ is a lattice in which the inclusion relation is the usual inclusion between subspaces, the operation $\wedge$ corresponds to the intersection of subspaces, but the operation $\vee$ is the smallest closed subspace that contains the union of the two subspaces. $Lat(H)$ is a bounded lattice, where $1 := H$ and $0 = \emptyset$. We remind the fact that two elements $x, y \in H$ are called orthogonal and we note $x \perp y$ if $< x, y >= 0$. For $M \in Lat(H)$ we will define $M^\perp = \{x \in H \ : \ x \perp y, (\forall) y \in M\}$. It is easy to check that the application $M \mapsto M^\perp$ is an othocomplementation on the lattice $Lat(H)$. Moreover, K. Husini [11] was the first to show, in 1937, that $Lat(H)$ satisfies the orthomodular law:

$$S \subset M \Rightarrow S \vee (S^\perp \wedge M) = M, (\forall) S, M \in Lat(H).$$

Thus $Lat(H)$ is an orhomodular lattice.

Definition 32. *If L is an orthomodular lattice, we say that $a, b \in L$ are orhogonal and we denote $a \perp b$ if $a \subset b^\perp$.*

Definition 33. *An orhomodular lattice L is called σ-complete if the join of every countable pairwise orthogonal sequence exists in L, namely if $a_i \leq a_j^{\perp}$ for $i \neq j$, then the join $\bigvee_i a_i$ exists in L.*

Definition 34. *An othomodular lattice which is σ-complete is called quantum logic.*

Proposition 35. *$Lat(H)$ is a quantum logic.*

Definition 36. *A probability measure on a quantum logic is a mapping $s : L \to [0,1]$ such that:*

1. *$s(1) = 1$;*
2. *$s\left(\bigvee_i a_i\right) = \sum_i s(a_i)$, for any sequence of pairwise orthogonal elements of L.*

From another point of view, as we have already mentioned, the family of all closed linear subspaces of a Hilbert space H is in one-to-one correspondence with the family of all projections over them. More precisely, the projectors can be considered as the quantum analogue of the characteristic functions. Given a projection $P_k : H \to K$, we can ask ourselves if the system is in K. This question can have several answers, not just "yes" or "no". If the answer is "yes" then we will have $< P_k x, x >= 1$, because $\|x\| = 1$. If the answer is "no" then we will have $x \in K^{\perp}$, so $< P_k x, x >= 0$. But it is possible that x is of form $x = x_1 + x_2$, where $x_1 \in K^{\perp}$ and $x_2 \in K$. Then

$$< P_k x, x >=< x_2, x >=< x_2, x_2 >= \|x_2\|^2 \in [0,1] .$$

In conclusion, the number $< P_k x, x >$ is the probability of the state x to be in K.

6 Some connections between fuzzy logic and quantum logic

6.1 Quantum logic view as an example of fuzzy logic

In paper [17] the author views quantum logic as an example of fuzzy logic. In order to justify this affirmation we consider a Hilbert space H. The family of all fuzzy sets in H, denoted $\mathcal{F}(H)$ is a lattice. Each closed linear subspace in H corresponds, as we have seen, to an elementary sentence in quantum logic. On the other hand, to each closed linear subspace M of H it corresponds an orthogonal projection P_M onto M. The set of the orthogonal projections is a lattice equipped with the relation or partial order $P \leq R$ if $< Px, x >\leq< Rx, x >, (\forall) x \in H$. Operations $P \wedge R$, $P \vee R$, $P^{\perp} = 1 - P$ are conjunction, disjunction and negation in quantum logic. To each closed linear subspace $M \subseteq H$ we can associate a fuzzy set $\mu_M : H \to [0,1], \mu_M(x) =< P_M x, x >$. The fuzzy sets $\{\mu_M, M \subseteq H\}$ form a fuzzy logic.

6.2 Mączyński's representation theorem for quantum logic

Although at a first glance, the quantum structure, which is an abstract algebraic structure, seems to have nothing in common with fuzzy logic, M. Mączyński [15], [16] proved a theorem of representation of any quantum logic as a family of $[0,1]$-valued functions, which can be regarded as the membership functions of some fuzzy sets (see also [22]).

Theorem 37. *[16] If S is a non-empty set and L a family of applications from S into $[0,1]$ such that:*

1. *$0 \in L$ (0 is the null function),*
2. *$a \in L \Rightarrow 1 - a \in L$,*
3. *$(\forall)\{a_i\}_{i \in I} \subset L$ (I finite or countable): $a_i + a_j \leq 1$, for $i \neq j \Rightarrow \sum_{i \in I} a_i \in L$,*

then L is a quantum logic with respect to the natural partial order of real function, with orthocomplementation $a^{\perp} = 1 - a$. Each point $u \in S$ induces a probability measure m_u on $(L, \leq, \perp)$, where $m_u(a) = a(u), (\forall) a \in L$ and the family of measures $\{m_u : u \in S\}$ is ordering.

Conversely, if $(L, \leq, \perp)$ in a quantum logic and S is an ordering set of probability measures, then any $a \in L$ induces a function $\tilde{a} : S \to [0,1], \tilde{a}(m) = m(a), (\forall) m \in S$. The set of such functions has properties (1)-(3) and $(\tilde{L}, \leq, \perp)$ is isomorphic with $(L, \leq, \perp)$.

We also note that condition (1) means that $\emptyset \in L$, while (2) shows that, if a fuzzy set is in L then its standard fuzzy complement also belongs to L. Only condition (3) cannot be expressed in terms of fuzzy sets. Nevertheless, a solution was found in the paper [22].

6.3 Fuzzy sets models of quantum logics

Starting with late '80s of the previous century, the noticed similarities between the operations with fuzzy sets and the operations of quantum logic have led to the desire of creating fuzzy models for quantum logic (see [20]). It was clear from the very beginning that the othocomplementation must be modelled through standard fuzzy set complementation. The problems appear to relations $\mu \wedge \mu^{\perp} = \emptyset$, $\mu \vee \mu^{\perp} = X$, which are not satisfied by fuzzy sets but which take place in Birkhoff-von Neumann quantum logic. What if we replace operations $\vee$ and $\wedge$ introduced by Zadeh with those of Lukasiewics? On the other hand, the fact that the operations $\vee$ and $\wedge$ introduced by Zadeh are distributive means that they cannot be used in the construction of fuzzy models for quantum logic. The problem was solved by J. Pykacz in paper[21].

Theorem 38. *[21] Any quantum logic L with an ordering set of probability measures S can be isomorphically represented in the form $\mathcal{L}(S)$ of fuzzy subsets of S satisfying the following conditions:*

1. *$\mathcal{L}(S)$ contains the empty set $\emptyset$;*
2. *$\mathcal{L}(S)$ is closed with respect to standard fuzzy set complementation;*
3. *$\mathcal{L}(S)$ is closed with respect to countable Lukasiewicz unions of pairwise weakly disjoint sets, i.e. if $A_i \sqcap A_j = \emptyset$ for $i \neq j$, then $\bigsqcup_i A_i \in \mathcal{L}(S)$;*
4. *$(\forall) A \in \mathcal{L}(S) : A \sqcap A = \emptyset \Rightarrow A = \emptyset$.*

Conversely, any family $\mathcal{L}(U)$ of fuzzy subsets of an arbitrary universe U satisfying conditions (1)-(4) is a quantum logic partially ordered by the inclusion of fuzzy sets, with the fuzzy set complementation $\mu' = 1 - \mu$ as orthocomplementation, the orthogonality of the elements coinciding with their weak disjointness, and an ordering set of probability measures generated by points of the universe U according to the formula $S_x(A) = \mu_A(x), (\forall) x \in U$.

$\mathcal{L}(U)$ will be called fuzzy quantum logic.

6.4 Effect algebras and fuzzy sets

More recently, in 1994, in the study of the algebraic foundations of the quantum logic, some more general algebraic structures called effect algebras have been introduced by D. Foulis and M. Bennet [8].

Definition 39. *[8] An effect algebra is a system $(E, \oplus, 0, 1)$ consisting on a set E which contains two distinct elements $0, 1$ called zero and unit and a binary operation partially defined $\oplus$ such that the following axioms are satisfied:*

1. *If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;*
2. *If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and*
$$a \oplus (b \oplus c) = (a \oplus b) \oplus c;$$
3. *$(\forall) x \in E, (\exists)! x' \in E$ such that $x \oplus x'$ is defined and $x \oplus x' = 1$;*
4. *If $1 \oplus a$ is defined, then $a = 0$.*

Remark 40. *The way of defining the concept of effect algebra allows us to easily notice that effect algebras are generalizations of other algebraic structures. Thus, if condition (4) is replaced by a more powerful condition*

$$(4')\textit{If } a \oplus a \textit{ is defined, then } a = 0$$

we obtain an ortho-algebra. If to conditions (1)-(4) we add the condition

$$(5)\ \textit{If } a \oplus b, b \oplus c, a \oplus c \textit{ are defined, then } (a \oplus b) \oplus c \textit{ is defined}$$

we obtain an orthomodular poset. If, we further add

$$(6)(\forall)x, y \in E, (\exists)a, b, c \in E \textit{ such that } b \oplus c, a \oplus c \textit{ are defined and } x = a \oplus c, y = b \oplus c$$

then E becomes an Boolean algebra.

We also specify that the operation $\oplus$ allows us to introduce a relation of partial order:

$$a \leq b \textit{ iff } (\exists)c \in E\ :\ a \oplus c \textit{ is defined and } a \oplus c = b\ .$$

In 1996, A. Dvurečenskij [6] showed that a system of fuzzy sets $L \subset [0, 1]^X$ satisfying the following conditions:

D1 $X \in L$,

D2 $A \in L \Rightarrow 1_X - A \in L$,

D3 If $A + B \leq 1_X$, then $A + B \in L$,

can be organized into an effect algebra $(L, \oplus, \emptyset, X)$ if we assume that $A \oplus B$ is defined if $A \oplus B \leq 1_X$ and we will set $A \oplus B = A + B$.

Furthermore, A. Dvurečenskij showed that if we add the condition:

$$\textbf{D4}\ \ A + A \leq 1_X \Rightarrow A = \emptyset$$

then L becomes an orthoalgebra.

We also note that in order that L becomes an orthomodular poset it must satisfy (D1), (D2) and

$$\textbf{D5}\ \text{If } A + B \leq 1_X, B + C \leq 1_X, A + C \leq 1_X \Rightarrow A + B + C \in L\ .$$

The results obtained by A. Dvurečenskij were generalized by J. Pykacz [22].

7 Conclusions

In this paper, our aim is to briefly present an evolution of logics from ancient times until present days and to highlight the main connections among classical logic, fuzzy logic and quantum logic. Of course, such a review is limited and it couldn't be otherwise. Many other research directions have been, unfortunately, left aside.

What we intend is for this approach to be continued and to highlight several other connections between fuzzy logic and quantum logic.

Conflict of interest

The author declares no conflict of interest.

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