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Some Generalized Difference Cesàro Double Sequence Space of Fuzzy Real Number Defined by Double Orlicz Function

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Abstract

In this paper we introduce some generalized difference Cesàro double sequence space of fuzzy real number by using double orlicz function . and we study their different properties like completeness , solidity, symmwtricity etc. . Also we obtain some inclusion relation involving these double sequence space.

Keywords :double orlicz function , Cesàro double sequence, metric space ,fuzzy number.

1 Introduction

The concepts of fuzzy sets theory was introduced by L.A.Zadeh in 1965[1] . Later Esi in 2006 [2] , Tripathy and Dutta (2007,2008)[3],[4] and many other discussed sequences of fuzzy number .

The difference sequence spaces $\ell_\infty(\Delta), C(\Delta), C_0(\Delta)$ introduced by Kizmaz (1981)[5],as follows:

$\mathbb{Z}(\Delta) = \{(x_r) \in w: (\Delta x_r) \in \mathbb{Z}\}$,

For $\mathbb{Z} = \ell_\infty , C_0 \; and \; C$. where $(\Delta x_r) = (x_r - x_{r+1}) , \forall \, r \in N$.

New type of difference sequence space introduced by Tripathy and Esi (2005)[6] , as follows , for fixed $m \in N$

$\mathbb{Z}(\Delta_m) = \{(x_r) \in w: (\Delta_m x_r) \in \mathbb{Z}\}$,

For $\mathbb{Z} = \ell_\infty , C_0 \; and \; C$. where $(\Delta_m x_r) = (x_r - x_{r+m}) , \forall \, r \in N$.

Generalized difference sequence space introduced by Tripathy and Esi (2006)[7] , as follows , for $n \geq 1 , m \geq 1$

$\mathbb{Z}(\Delta_m^n) = \{(x_r) \in w: (\Delta_m^n x_r) \in \mathbb{Z}\}$,

For $\mathbb{Z} = \ell_\infty , C_0 \; and \; C$. where $(\Delta_m^n x_r) = \sum_{q=0}^{n} (-1)^q x_{r+qm}$.

New types of difference Cesàro sequence space as $C_p(\Delta_m^n), C_\infty(\Delta_m^n) , O_p(\Delta_m^n)$, $O_\infty(\Delta_m^n) \; and \; \ell_\infty(\Delta_m^n) \; for \; 1 \leq p < \infty$ introduced by Tripathy and Esi and Tripathy (2005)[8] .

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New types of difference Cesàro sequence space of fuzzy real number $C_p^F(\Delta_m^n)$, $C_\infty^F(\Delta_m^n)$, $O_p^F(\Delta_m^n)$, $O_\infty^F(\Delta_m^n)$, *and* $\ell_\infty^F(\Delta_m^n)$ *for* $1 \leq p < \infty$ introduced by Tripathy and Borgohain (2015) [9] .

Battor and Neamah [10] introduced the concept of a double Orlicz function M which is defined as following:

M:$[0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$ where $M(x,y) = \big(M_1(x), M_2(y)\big)$ such that $M_1: [0,\infty) \to [0,\infty)$ and $M_2: [0,\infty) \to [0,\infty)$ where M_1, M_2 be two Orlicz function which are continuous , non – decreasing , even , convex and satisfy the following condition :

$M_1(0) = 0$, $M_2(0) = 0 \Rightarrow M(0,0) = \big(M_1(0), M_2(0)\big) = (0,0)$.

$M_1(x) > 0$, $M_2(y) > 0 \Rightarrow M(x,y) = \big(M_1(x), M_2(y)\big) > (0,0)$ for all $x, y > 0$, so $M(x,y) > (0,0)$, mean that $M_1(x) > 0$, $M_2(y) > 0$.

$M_1(x) \to \infty$, $M_2(y) \to \infty$ as $x, y \to \infty \Rightarrow M(x,y) = \big(M_1(x), M_2(y)\big) \to (\infty,\infty)$, as $(x,y) \to (\infty,\infty)$. So , $M(x,y) \to (\infty,\infty)$ means that $M_1(x) \to \infty$, $M_2(y) \to \infty$.

Also , a double Orlicz function $M(X,Y) = \big(M_1(X), M_2(Y)\big)$, satisfied Δ_2- condition for all values of X,Y , if there exist a constant $k > 0$ such that $M_1(2X) \leq kM_1(X), M_2(2Y) \leq kM_2(Y)$, for all $X, Y \geq 0$,

(i.e) $M(2X,2Y) = \big(M_1(2X), M_2(2Y)\big) \leq \big(kM_1(X), kM_2(Y)\big) = k\big(M_1(X), M_2(Y)\big) = kM(X,Y)$, for all $X, Y \geq 0$.

2 Definitions and background

Definition 2.1 A double sequence (X,Y)=(X_{kl}, Y_{kl}) of fuzzy numbers on R^{2n}, is said to be converge in R^{2n}, if there exist fuzzy numbers $(X_0, Y_0) \in R^{2n}$ such that $\forall\ \varepsilon > 0$, $\exists\ k_0, l_0 \in N$ $\ni \bar{d}\big((X_{kl}, Y_{kl}), (X_0, Y_0)\big) \leq (\varepsilon, \varepsilon)$, for all k$\geq k_0$, $l \geq l_0$.

Definition 2.2. A double sequence (X,Y)=(X_{kl}, Y_{kl}) of fuzzy numbers on R^{2n} is said to be a double Cauchy sequence, if $\forall\varepsilon$ >0 there exists $n_0 \in N$ such that $\bar{d}\left(\big(X_{ij}, Y_{ij}\big), (X_{kl}, Y_{kl})\right) \leq (\varepsilon, \varepsilon)$, for all i$\geq k \geq n_0$, j$\geq l \geq n_0$.

Definition 2.3. A double sequence space $2E^F$ is said to be solid if $(F_{kl}, H_{kl}) \in 2E^F$, whenever $(X_{kl}, Y_{kl}) \in 2E^F$ and $|(F_{kl}, H_{kl})| \leq |(X_{kl}, Y_{kl})|$, for all $k, l \in N$.

Definition 2.4. A double sequence space $2E^F$is said to be symmetric if $S(X_{kl}, Y_{kl}) \subset 2E^F$ whenever $(X_{kl}, Y_{kl}) \in 2E^F$, where $S(X_{kl}, Y_{kl})$ denotes the set of all permutations of the elements of (X_{kl}, Y_{kl}), that is , $(X_{kl}, Y_{kl}) = \{\big(X_{\pi(r)\pi(s)}, Y_{\pi(r)\pi(s)}\big): \pi\ is$ a permutation of N $\}$.

Definition 2.5. A double sequence space $2E^F$ is said to be convergence-free if $(F_{kl}, H_{kl}) \in 2E^F$ whenever $(X_{kl}, Y_{kl}) \in 2E^F$ and $(X_{kl}, Y_{kl}) = (\bar{0}, \bar{0})$implies that $(F_{kl}, H_{kl}) = (\bar{0}, \bar{0})$.

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Definition 2.6. A double sequence space $2E^F$ is said to be monotone**.** if $2E^F$ contains the canonical preimages of all its step spaces.

Definition 2.7. A fuzzy real number X on R^{2n} is a function $X: R^{2n} \to I = [0,1]$ associating with $t \in R^{2n}$, with its grade of membership X(t) .

The class of fuzzy real number on R^{2n} is denoted by $R^{2n}(I)$. For $0 < \alpha \le 1$, the α-level set $[X]^\alpha = \{t \in R^{2n}: X(t) \ge \alpha\}$.

Remark 2.1. A class of double sequences $2E^F$ is solid implies that $2E^F$ is monotone.

Tripathy and Borgohain [9] introduced new types of sequence space of fuzzy real number defined by Orlicz function, we use that to defined the following difference Cesàro double sequence spaces of fuzzy real numbers, for a double Orlicz function ,

Let $m, n \ge 0$ be fixed integer and $1 \le p < \infty$.

$$\left(C_p^2\right)_F(M, \Delta_m^n) = \left\{(X_{kl}, Y_{kl}) \in w_F^2 : \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{kl}, \bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{kl}, \bar{0})}{\rho}\right)\right)\right)^p < \infty,\ \text{forsome}\ \rho > 0\right\}.$$

$$\left(C_\infty^2\right)_F(M, \Delta_m^n) = \left\{(X_{kl}, Y_{kl}) \in w_F^2 : sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{kl}, \bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{kl}, \bar{0})}{\rho}\right)\right)\right) < \infty,\ \text{for some}\ \rho > 0\right\}.$$

$$\left(\ell_p^2\right)_F(M, \Delta_m^n) = \left\{(X_{kl}, Y_{kl}) \in w_F^2 : \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{kl}, \bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{kl}, \bar{0})}{\rho}\right)\right)^p < \infty,\ \text{for some}\ \rho > 0\right\}.$$

$$\left(O_p^2\right)_F(M, \Delta_m^n) = \left\{(X_{kl}, Y_{kl}) \in w_F^2 : \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{kl}, \bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{kl}, \bar{0})}{\rho}\right)\right)\right)^p < \infty,\ \text{for some}\ \rho > 0\right\}.$$

$$\left(O_\infty^2\right)_F(M, \Delta_m^n) = \left\{(X_{kl}, Y_{kl}) \in w_F^2 : sup_{i,j}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{kl}, \bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{kl}, \bar{0})}{\rho}\right)\right) < \infty,\ \text{for some}\ \rho > 0\right\}.$$

Where $\Delta_m^n X_{kl} = \sum_{q_1=0}^{n}\sum_{q_2=0}^{n}(-1)^{q_1+q_2}\binom{n}{q_1}\binom{n}{q_2}X_{k+q_1m,l+q_2m} - X_{k+m,l} - X_{k,l+m}$.

$\Delta_m^n Y_{kl} = \sum_{q_1=0}^{n}\sum_{q_2=0}^{n}(-1)^{q_1+q_2}\binom{n}{q_1}\binom{n}{q_2}Y_{k+q_1m,l+q_2m} - Y_{k+m,l} - Y_{k,l+m}$. then ,

$$(\Delta_m^n X_{kl}, \Delta_m^n Y_{kl}) = \left(\sum_{q_1=0}^{n}\sum_{q_2=0}^{n}(-1)^{q_1+q_2}\binom{n}{q_1}\binom{n}{q_2}X_{k+q_1m,l+q_2m} - X_{k+m,l} - X_{k,l+m}\ ,\sum_{q_1=0}^{n}\sum_{q_2=0}^{n}(-1)^{q_1+q_2}\binom{n}{q_1}\binom{n}{q_2}Y_{k+q_1m,l+q_2m} - Y_{k+m,l} - Y_{k,l+m}\right) \quad (1)$$

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3 Main Results

Theorem 3.1. Let $1 \leq p < \infty$. then ,

(i) $\left(C_p^2\right)_F(M)$ is a completemetric space with the metric ,

$$\eta_1(X,Y) = inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}$$

(ii) The space $(C_\infty^2)_F(M)$ is a complete metric space with respect to the metric ,

$$\eta_2(X,Y) = inf\left\{\rho > 0 : sup_{i,j}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right)\right) \leq 1\right\}$$

.

(iii) The space $\left(\ell_p^2\right)_F(M)$ is a complete metric space with respect to the metric ,

$$\eta_3(X,Y) = inf\left\{\rho > 0 : \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(M_1\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(iv) $\left(O_p^2\right)_F(M)$ is a completemetric space with the metric ,

$$\eta_4(X,Y) = inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}$$

(v) $(O_\infty^2)_F(M,\Delta_m^n)$ is a complete metric space with the metric ,

$$\eta_5(X,Y) = inf\left\{\rho > 0 : sup_{i,j}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(X_{kl},Y_{kl})}{\rho}\right)\right) \leq 1\right\}.$$

Proof. We prove the result for $\left(C_p^2\right)_F(M)$. The prove for the other cases is similarly .

Let (X^i,Y^i) be a double Cauchy sequence in $\left(C_p^2\right)_F(M)$ such that $(X^i,Y^i) = \left(X_{nq}^i, Y_{nq}^i\right)_{n,q=1}^{\infty}$, for $i \in N$.

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Let $\varepsilon > 0$ be given . For a fixed $x_0 > 0$, choose $r > 0$ such that $M_1\left(\frac{rx_0}{2}\right) \geq 1$, $M_2\left(\frac{rx_0}{2}\right) \geq 1$ that is , $M\left(\frac{rx_0}{2},\frac{rx_0}{2}\right) = \left(M_1\left(\frac{rx_0}{2}\right), M_2\left(\frac{rx_0}{2}\right)\right) \geq (1,1)$.Then there exits a positive integer $n_0 = n_0(\varepsilon)$ shch that

$\eta_1\left((X^i, X^j), (Y^i, Y^j)\right) < \left(\frac{\varepsilon}{rx_0}, \frac{\varepsilon}{rx_0}\right)$, for all i,j $\geq n_0$.

By the definition of η_1 , we gett ;

$$inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}\left(x_{kl}^i, x_{kl}^j\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(Y_{kl}^i, Y_{kl}^j\right)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\} <$$

$(\varepsilon, \varepsilon)$, for all i,j $\geq n_0$. (2)

Which implies ,

$$\left(M_1\left(\frac{\bar{d}\left(x_{kl}^i, x_{kl}^j\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(Y_{kl}^i, Y_{kl}^j\right)}{\rho}\right)\right) \leq 1, \text{ for all i, j } \geq n_0 \ . \quad (3)$$

$$\Rightarrow \left(M_1\left(\frac{\bar{d}\left(x_{kl}^i, x_{kl}^j\right)}{\eta_1(x^i, x^j)}\right), M_2\left(\frac{\bar{d}\left(Y_{kl}^i, Y_{kl}^j\right)}{\eta_1(Y^i, Y^j)}\right)\right) \leq (1,1) \leq \left(M_1\left(\frac{rx_0}{2}\right), M_2\left(\frac{rx_0}{2}\right)\right) \text{, for all i, j } \geq$$

n_0 . By continuity of M so M_1, M_2 , we have

$\bar{d}\left((X_{kl}^i, X_{kl}^j), (Y_{kl}^i, Y_{kl}^j)\right) \leq \left(\frac{rx_0}{2}.\eta_1(X^i, X^j), \frac{rx_0}{2}.\eta_1(Y^i, Y^j)\right)$,for all i,j $\geq n_0$.

$\Rightarrow \bar{d}\left((X_{kl}^i, X_{kl}^j), (Y_{kl}^i, Y_{kl}^j)\right) < \left(\frac{rx_0}{2}.\frac{\varepsilon}{rx_0}, \frac{rx_0}{2}.\frac{\varepsilon}{rx_0}\right) = \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$,for all i,j $\geq n_0$.

$\Rightarrow \bar{d}\left((X_{kl}^i, X_{kl}^j), (Y_{kl}^i, Y_{kl}^j)\right) < \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$,for all i,j $\geq n_0$.

$\Rightarrow (X_{kl}^i, Y_{kl}^i)$ is a double Cauchysequence in $R^{2n}(I)$and so it convergent in $R^{2n}(I)$ by the completeness property of $R^{2n}(I)$.

Also , $\lim_{i\to\infty} X_{kl}^i = X_{kl}$, $\lim_{i\to\infty} Y_{kl}^i = Y_{kl}$, for each $k, l \in N$.

So , $\lim_{i\to\infty}(X_{kl}^i, Y_{kl}^i) = (X_{kl}, Y_{kl})$, for each $k, l \in N$

Now , taking $j \to \infty$ anf fixing i and using the continuity of M so M_1, M_2 , it follows from (3) ,

$$\left(M_1\left(\frac{\bar{d}\left(x_{kl}^i, X_{kl}\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(Y_{kl}^i, Y_{kl}\right)}{\rho}\right)\right) \leq 1, \text{ ,for some } \rho > 0 \ .$$

Now on taking the infimum of such ρ ' s and using (2) we get ,

$$inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}\left(x_{kl}^i, X_{kl}\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(Y_{kl}^i, Y_{kl}\right)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\} <$$

$(\varepsilon, \varepsilon)$, $\forall$ i $\geq n_0$.

$\Rightarrow \eta_1\left((X^i, X), (Y^i, Y)\right) < (\varepsilon, \varepsilon)$, $\forall$ i $\geq n_0$.

i.e. $\lim_i (X^i, Y^i) = (X, Y)$.

Now , we prove that $(X, Y) \in \left(C_p^2\right)_F(M)$

We have , $\bar{d}((X_{kl},\bar{0}),(Y_{kl},\bar{0})) \le \bar{d}\left((X_{kl}^i,X_{kl}),(Y_{kl}^i,Y_{kl})\right) + \bar{d}\left((X_{kl}^i,\bar{0}),(Y_{kl}^i,\bar{0})\right)$.

Since M so M_1, M_2 is continuous,non – decreasing,so we get ,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(X_{kl},\bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(Y_{kl},\bar{0})}{\rho}\right)\right)^p \le$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}\left(X_{kl}^i,X_{kl}\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(Y_{kl}^i,Y_{kl}\right)}{\rho}\right)\right)^p +$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}\left(X_{kl}^i,\bar{0}\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(Y_{kl}^i,\bar{0}\right)}{\rho}\right)\right)^p < \infty .(finite)$$

$\Rightarrow (X,Y) \in \left(C_p^2\right)_F(M)$.

Hence $\left(C_p^2\right)_F(M)$ is a complete metric space .■

Proposition3.2. The classes of double sequences $\left(C_p^2\right)_F(M,\Delta_m^n)$, $\left(0_p^2\right)_F(M,\Delta_m^n)$, $(C_\infty^2)_F(M,\Delta_m^n)$, $(0_\infty^2)_F(M,\Delta_m^n)$ and $\left(\ell_p^2\right)_F(M,\Delta_m^n)$, for $1 \le p < \infty$, are metric space with respect to the metric ,

$f(X,Y) = \sum_{k=1}^{nm}\sum_{l=1}^{nm}\bar{d}(X_{kl}-\bar{0}) + \eta(\Delta_m^n X_{kl},\Delta_m^n Y_{kl})$.

Proof. The proof is a routine work , so omitted .

Theorem3.3. Let Z(M) be a complete metric space with respect to the metric η , the space $Z(M,\Delta_m^n)$ is a complete metric space with metric ,

$f(X,Y) = \sum_{k=1}^{nm}\sum_{l=1}^{nm}\bar{d}(X_{kl}-\bar{0}) + \eta(\Delta_m^n X_{kl},\Delta_m^n Y_{kl})$.

Where Z=$\left(C_p^2\right)_F$, $\left(0_p^2\right)_F$, $(C_\infty^2)_F$, $(0_\infty^2)_F$, $\left(\ell_p^2\right)_F$.

Proof. Let$\left(X^i,Y^i\right)$ be a Cauchysequence in Z(M,Δ_m^n) such that $\left(X^i,Y^i\right) = \left(X_{nq}^i,Y_{nq}^i\right)_{n,q=1}^{\infty}$.

We have for $\varepsilon > 0$, there exits a positive integer $n_0 = n_0(\varepsilon)$ shch that,

$f\left(\left(X^i,X^j\right),\left(Y^i,Y^j\right)\right) < (\varepsilon,\varepsilon)$,∀ i, j $\ge n_0$.

By definition of f , we get ;

$\sum_{r=1}^{nm}\sum_{s=1}^{nm}\bar{d}\left(\left(X_{rs}^i,X_{rs}^j\right),\left(Y_{rs}^i,Y_{rs}^j\right)\right) + \eta\left(\left(\Delta_m^n X_{rs}^i,\Delta_m^n X_{rs}^j\right),\left(\Delta_m^n Y_{rs}^i,\Delta_m^n Y_{rs}^j\right)\right) < (\varepsilon,\varepsilon)$, for all i, j $\ge n_0$. (4)

Which implies, $\sum_{k=r}^{nm}\sum_{l=s}^{nm}\bar{d}\left(\left(X_{rs}^i,X_{rs}^i\right),\left(Y_{rs}^i,Y_{rs}^i\right)\right) < (\varepsilon,\varepsilon)$, for all i, j $\ge n_0$.

$\Rightarrow \bar{d}\left(\left(X_{rs}^i,X_{rs}^j\right),\left(Y_{rs}^i,Y_{rs}^j\right)\right) < (\varepsilon,\varepsilon)$, for all i, j $\ge n_0$, $r,s = 1,2,\ldots,mn$.

Hence $\left(X_{rs}^i,Y_{rs}^i\right)$ is a double Cauchy sequence in $R^{2n}(I)$, so it is convergent in $R^{2n}(I)$, by the completeness property of $R^{2n}(I)$, for $r,s = 1,2,\ldots,mn$.

Let $\lim_{i\to\infty} X_{rs}^i = X_{rs}$, $\lim_{i\to\infty} Y_{rs}^i = Y_{rs}$, for $r,s = 1,2,\ldots,mn$.

So , $\lim_{i\to\infty}\left(X_{rs}^i,Y_{rs}^i\right) = (X_{rs},Y_{rs})$, for $r,s = 1,2,\ldots,mn$. (5)

Next we have , $\eta\left(\left(\Delta_m^n X_{rs}^i,\Delta_m^n X_{rs}^j\right),\left(\Delta_m^n Y_{rs}^i,\Delta_m^n Y_{rs}^j\right)\right) < (\varepsilon,\varepsilon)$ for all i, j $\ge n_0$.

Which implies that $(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i)$ is a double Cauchysequence in $Z(M)$,since M is continuous function and so it is convergent in $Z(M)$, by the completeness property of $Z(M)$.

Let $\lim_i(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i) = (F_{rs}, H_{rs})\ (say)$, in $Z(M)$, for each $r, s \in N$.

We have prove that , , $\lim_i(X^i, Y^i) = (X, Y)\ and\ (X, Y) \in Z(M, \Delta_m^n)$

For $r, s = 1$, we have ,from(1) and (5)

$\lim_{i\to\infty}(X_{mn+1,mn+1}^i, Y_{mn+1,mn+1}^i) = (X_{mn+1,mn+1}, Y_{mn+1,mn+1})$, for $m \geq 1, n \geq 1$.

Proceeding in this way of induction , we get , $\lim_{i\to\infty}(X_{rs}^i, Y_{rs}^i) = (X_{rs}, Y_{rs})$, for each $r, s \in N$.

Also , $\lim_i(\Delta_m^n X_{rs}^i, \Delta_n^m Y_{rs}^i) = (\Delta_m^n X_{rs}, \Delta_n^m Y_{rs})$, for each $r, s \in N$.

Now , taking $j \to \infty$ and fixing i it follows from (4) ,

$\sum_{r=1}^{nm}\sum_{s=1}^{nm} \bar{d}\left((X_{rs}^i, X_{rs}), (Y_{rs}^i, Y_{rs})\right) + \eta\left((\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}), (\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})\right) < (\varepsilon, \varepsilon)$, for all i $\geq n_0$.

Which implies, $f\left((X^i, X), (Y^i, Y)\right) < (\varepsilon, \varepsilon)$, for all i $\geq n_0$.

i.e $\lim_i(X^i, Y^i) = (X, Y)$.

Now , it is to prove that $(X, Y) \in Z(M, \Delta_m^n)$.

We have , $f((\Delta_m^n X_{rs}, \bar{0}), (\Delta_m^n Y_{rs}, \bar{0})) \leq f\left((\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}), (\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})\right) + f\left((\Delta_m^n X_{rs}^i, \bar{0}), (\Delta_m^n Y_{rs}^i, \bar{0})\right) < \infty$.

$\Longrightarrow\ (X, Y) \in Z(M, \Delta_m^n)$.

Hence $Z(M, \Delta_m^n)$ is a completemetric space .■

Proposition3.4. The classes of space $Z(M, \Delta_m^n)$, where z=$(C_p^2)_F$, $(0_p^2)_F$, $(C_\infty^2)_F$, $(0_\infty^2)_F$, $(\ell_p^2)_F$, for $1 \leq p < \infty$, are not monotone and such are , not solid for $m, n \geq 1$.

Proof. To prove that take the following example , let consider proof for $(C_p^2)_F(M, \Delta_m^n)$,

Example3.5. Let $(X_{kl}, Y_{kl}) = (\overline{kl}, \overline{kl})$ for all $k, l \in N$.

Let $m = 3$ and $n = 2$. Let $M(X, Y) = (|X|, |Y|)$ for all$(X, Y) \in [0, \infty) \times [0, \infty)$.

Then , we have $\bar{d}((\Delta_3^2 X_{kl}, \bar{0}), (\Delta_3^2 Y_{kl}, \bar{0})) = (0,0)$, for all $k, l \in N$.

Hence , we get , for $1 \leq p < \infty$,

$$\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_3^2 X_{kl}, \bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_3^2 Y_{kl}, \bar{0})}{\rho}\right)\right)^p\right)^{\frac{1}{p}} < \infty, \text{ for some } \rho > 0.$$

Which implies that , $(X_{kl}, Y_{kl}) \in (C_p^2)_F(M, \Delta_3^2)$.

Let $JI = \{kl: kl \text{ is even}\} \subseteq N$. Let (F_{kl}, H_{kl}) be the canonical pre-image of $(X_{kl}, Y_{kl})_{JI}$ for ,the subsequence JI of N. Then ,

$$(F_{kl}, H_{kl}) = \begin{cases}(X_{kl}, Y_{kl}), for\ JI\ odd \\ (\bar{0}, \bar{0}), for\ JI\ even\end{cases}$$

But $(F_{kl}, H_{kl}) \notin (C_p^2)_F(M, \Delta_3^2)$.

Hence the space are not monotone as such not solid . ■

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Remark3.6. For,m=0 or n=0 the, space $\left(C_p^2\right)_F(M)$, $(C_\infty^2)_F(M)$ are neither solid and not monotone, where as the space $\left(\ell_p^2\right)_F(M)$, $\left(O_p^2\right)_F(M)$, $(O_\infty^2)_F(M)$ are solid ,and hence are monotone .

Proposition3.7. The classes, of space $\mathbb{Z}(M,\Delta_m^n)$, where $\mathbb{Z}$ $=\left(C_p^2\right)_F$, $\left(O_p^2\right)_F$, $(C_\infty^2)_F$, $(O_\infty^2)_F$ and $\left(\ell_p^2\right)_F$, for $1 \le p < \infty$, are,not symmetric , for $m,n \ge 1$.

Proof. To prove that take the following example , let consider proof for $(C_\infty^2)_F(M,\Delta_m^n)$.

Example3.8. Let $m = 4$ and $n = 1$. Let $M(X,Y) = (|X|,|Y|)$, $\forall (X,Y) \in [0,\infty) \times [0,\infty)$.

Consider the double sequence (X_{kl},Y_{kl}) defined by , for $k = l$

$(X_{kl},Y_{kl}) = (1,1)$ for all $k,l \in N$.

$(X_{kl},Y_{kl}) = (\bar{0},\bar{0})$ otherwise

Then , we have, $\bar{d}\left((\Delta_4 X_{kl},\bar{0}),(\Delta_4 Y_{kl},\bar{0})\right) = (0,0)$, for all $k,l \in N$.

Hence , we get ,

$$sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_4 X_{kl},\bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_4 Y_{kl},\bar{0})}{\rho}\right)\right)\right) < \infty, \text{ for some } \rho > 0 .$$

Which implies that , $(X_{kl},Y_{kl}) \in (C_\infty^2)_F(M,\Delta_4)$.

Consider the rearranged sequence (F_{kl},H_{kl}) of (X_{kl},Y_{kl}) such that $(F_{kl},H_{kl}) = \left((X_{11},Y_{11}),(X_{22},Y_{22}),(X_{44},Y_{44}),(X_{33},Y_{33}),(X_{99},Y_{99}),(X_{55},Y_{55}),\ldots\right)$,

Such that $\bar{d}\left((\Delta_4 F_{kl},\bar{0}),(\Delta_4 H_{kl},\bar{0})\right) \approx (k-(k-1)^2, k-(k-1)^2) \approx (k^2,k^2)$, for all $k = l$,and $\bar{d}\left((\Delta_4 F_{kl},\bar{0}),(\Delta_4 H_{kl},\bar{0})\right) = (\bar{0},\bar{0})$ otherwise .

Which shows ,

$$sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_4 F_{kl},\bar{0})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_4 H_{kl},\bar{0})}{\rho}\right)\right)\right) = \infty, \text{ for some fixed } \rho > 0 .$$

Hence , $(F_{kl},H_{kl}) \notin (C_\infty^2)_F(M,\Delta_4)$. It follows that space are not symmetric . ■

Proposition3.9. The classes, of space $\mathbb{Z}(M,\Delta_m^n)$, where $\mathbb{Z}$ $=\left(C_p^2\right)_F$, $\left(O_p^2\right)_F$, $(C_\infty^2)_F$, $(O_\infty^2)_F$ and $\left(\ell_p^2\right)_F$, for $1 \le p < \infty$, are not convergence-free , for $m,n \ge 1$.

Proof. To prove that take the following example , let consider proof for $(C_\infty^2)_F(M,\Delta_m^n)$

Example3.10. Let $m = 3$ and $n = 1$. Let $M(X,Y) = (X^3,Y^3)$, $\forall (X,Y) \in [0,\infty) \times [0,\infty)$.

Consider the double sequence (X_{kl},Y_{kl}) defined by , for $k = l$;

$$(X_{kk},Y_{kk})(t)\begin{cases}(1+k^2t,1+k^2t), & for\ t \in \left[-\frac{1}{k^2},0\right],\\ (1-k^2t,1-k^2t), & for\ t \in \left[0,\frac{1}{k^2}\right],\\ (0,0), & \text{otherwise}\end{cases}$$

$(X_{kl},Y_{kl}) = (\bar{0},\bar{0})$ otherwise .

Then , for $k = l$,

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$$(\Delta_3 X_{kk}, \Delta_3 Y_{kk})(t)\begin{cases}\left(1+\frac{k^2(k+3)^2}{2k^2+6k+9}t, 1+\frac{k^2(k+3)^2}{2k^2+6k+9}t\right), & for\ t\in\left[-\frac{2k^2+6k+9}{k^2(k+3)^2},0\right],\\ \left(1-\frac{k^2(k+3)^2}{2k^2+6k+9}t, 1-\frac{k^2(k+3)^2}{2k^2+6k+9}t\right), & for\ t\in\left[\frac{2k^2+6k+9}{k^2(k+3)^2},0\right],\\ (0,0), & \text{otherwise}\end{cases}$$

And $(\Delta_3 X_{kl}, \Delta_3 Y_{kl})(t) = (\bar{0},\bar{0})$ otherwise .
Such that,

$$\bar{d}\big((\Delta_3 X_{kl},\bar{0}),(\Delta_3 Y_{kl},\bar{0})\big) = \left(\frac{2k^2+6k+9}{k^2(k+3)^2}, \frac{2k^2+6k+9}{k^2(k+3)^2}\right) = \left(\frac{1}{k^2}+\frac{1}{(k+3)^2}, \frac{1}{k^2}+\frac{1}{(k+3)^2}\right).$$

Hence , we get ,

$$sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_3 X_{kl},\bar{0})}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_3 Y_{kl},\bar{0})}{\rho}\right)\right)\right) < \infty,\ \text{for some } \rho > 0.$$

Which implies that , $(X_{kl}, Y_{kl}) \in (C_\infty^2)_F(M,\Delta_3)$.
Now , let (F_{kl}, H_{kl}) be a double sequence defined as , for $k = l$

$$(F_{kk}, H_{kk})(t) = \begin{cases}\left(1+\frac{t}{k^2}, 1+\frac{t}{k^2}\right), & for\ t\in[-k^2,0],\\ \left(1-\frac{t}{k^2}, 1-\frac{t}{k^2}\right), & for\ t\in[0,k^2],\\ (0,0), & \text{otherwise}\end{cases}$$

$(F_{kl}, H_{kl}) = (\bar{0},\bar{0})$ otherwise .
So that , for $k = l$,

$$(\Delta_3 X_{kk}, \Delta_3 Y_{kk})(t)\begin{cases}\left(1+\frac{t}{2k^2+6k+9}, 1+\frac{t}{2k^2+6k+9}\right), & for\ t\in[-2k^2+6k+9,0],\\ \left(1-\frac{t}{2k^2+6k+9}, 1-\frac{t}{2k^2+6k+9}\right), & for\ t\in[2k^2+6k+9,0],\\ (0,0), & \text{otherwise}\end{cases}$$

And $(\Delta_3 F_{kl}, \Delta_3 H_{kl})(t) = (\bar{0},\bar{0})$ otherwise .
But , $\bar{d}\big((\Delta_3 F_{kl},\bar{0}),(\Delta_3 H_{kl},\bar{0})\big) = (2k^2+6k+9, 2k^2+6k+9)$.
which implies that ,

$$sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_3 F_{kl},\bar{0})}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_3 H_{kl},\bar{0})}{\rho}\right)\right)\right) = \infty,\ \text{for some } \rho > 0.$$

Thus , $(F_{kl}, H_{kl}) \notin (C_\infty^2)_F(M,\Delta_3)$.
Hence $(C_\infty^2)_F(M,\Delta_m^n)$ is not convergence-free in general . ■

Theorem3.11.

(a) $(\ell_p^2)_F(M,\Delta_m^n) \subset (O_p^2)_F(M,\Delta_m^n) \subset (C_p^2)_F(M,\Delta_m^n)$ and the inclusions are strict.

(b) $\mathbb{Z}(M,\Delta_m^{n-1}) \subset \mathbb{Z}(M,\Delta_m^n)$(in general $\mathbb{Z}(M,\Delta_m^i) \subset \mathbb{Z}(M,\Delta_m^n)$, for i=1,2,3,……,n-1),where$\mathbb{Z}$=$(C_p^2)_F$, $(O_p^2)_F$, $(C_\infty^2)_F$, $(O_\infty^2)_F$ and $(\ell_p^2)_F$, $for\ 1 \le p < \infty$.

(c) $(O_\infty^2)_F(M,\Delta_m^n) \subset (C_\infty^2)_F(M,\Delta_m^n)$ and the inclusions is strict .

Proof . The proofs of (a) and (c) are routine works . so omitted.
(b)Let $(X_{kl}, Y_{kl}) \in (C_\infty^2)_F(M,\Delta_m^{n-1})$. Then we have ,

$$sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^{n-1} X_{kl},\bar{0})}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^{n-1} Y_{kl},\bar{0})}{\rho}\right)\right)\right) < \infty,\ \text{for some } \rho > 0.$$

Now we have ,

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$sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^{n-1}X_{kl}-\Delta_m^{n-1}X_{k,l+1},\bar{0})}{2\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{kl}-\Delta_m^{n-1}Y_{k,l+1},\bar{0})}{2\rho}\right)\right)\right)\leq$
$\frac{1}{2}sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^{n-1}X_{kl},\bar{0})}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{kl},\bar{0})}{\rho}\right)\right)\right)+$
$\frac{1}{2}sup_{i,j}\frac{1}{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}\left(M_1\left(\frac{\bar{d}(\Delta_m^{n-1}X_{k,l+1},\bar{0})}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^{n-1}Y_{k,l+1},\bar{0})}{\rho}\right)\right)\right)<\infty$.

Proceeding in this way , we have , $\mathbb{Z}(M,\Delta_m^i)\subset\mathbb{Z}(M,\Delta_m^n)$, for i=1,2,3,……,n-1), where$\mathbb{Z}=\left(C_p^2\right)_F$, $\left(O_p^2\right)_F$, $(C_\infty^2)_F$, $(O_\infty^2)_F$ and $\left(\ell_p^2\right)_F$, $1\leq p<\infty$. ■

Theorem3.12.
(a) if $1\leq p<q$,then,
(i) $\left(C_p^2\right)_F(M,\Delta_m^n)\subset\left(C_q^2\right)_F(M,\Delta_m^n)$
(ii) $\left(\ell_p^2\right)_F(M,\Delta_m^n)\subset\left(\ell_q^2\right)_F(M,\Delta_m^n)$
(b) $\left(C_p^2\right)_F(M)\subset\left(C_p^2\right)_F(M,\Delta_m^n)$, for all $m\geq 1$ and $n\geq 1$.

Proof. The proof is a routine work , so omitted .

Theorem3.13. Let $M=(M_1,M_2)$ and $\grave{M}=(M_3,M_4)$ be a double orlicz function satisfying Δ_2- condition .Then , for Z=$\left(C_p^2\right)_F$, $\left(O_p^2\right)_F$, $(C_\infty^2)_F$, $(O_\infty^2)_F$,$\left(\ell_p^2\right)_F$ and Z_*=$\left(C_p^2\right)_F$, $\left(O_p^2\right)_F$, $(C_\infty^2)_F$, $(O_\infty^2)_F$, $\left(\ell_p^2\right)_F$, for $1\leq p<\infty$,
(i) $Z(M,\Delta_m^n)\subset Z(\grave{M}\circ M,\Delta_m^n)$.
(ii) $Z(M,\Delta_m^n)\cap Z(\grave{M},\Delta_m^n)\subseteq Z(M+\grave{M},\Delta_m^n)$.

Proof.
(i) Let $(X_{kl},Y_{kl})\in Z(M,\Delta_m^n)$ such that $X_{kl}\in Z_*(M_1,\Delta_m^n)$, $Y_{kl}\in Z_*(M_2,\Delta_m^n)$.
Consider $\varepsilon>0$, there exists $\eta>0$ such that $(\varepsilon,\varepsilon)=\grave{M}(\eta,\eta)$
Then , $\left[M_1\left(\frac{\bar{d}(\Delta_m^nX_{kl},L_1)}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^nY_{kl},L_2)}{\rho}\right)\right]<(\eta,\eta)$, for some $\rho>0$ and $L_1,L_2\in R^2(I)$.

Let (F_{kl},H_{kl})=$\left[M_1\left(\frac{\bar{d}(\Delta_m^nX_{kl},L_1)}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^nY_{kl},L_2)}{\rho}\right)\right]<(\eta,\eta)$, for some $\rho>0$ and $L_1,L_2\in R^2(I)$.

Since $\grave{M}$ is continuous and non-decreasing , we have ,
$\grave{M}(F_{kl},H_{kl})=\grave{M}\left[M_1\left(\frac{\bar{d}(\Delta_m^nX_{kl},L_1)}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\grave{\Delta}_m^nY_{kl},L_2)}{\rho}\right)\right]<\grave{M}(\eta,\eta)=(\varepsilon,\varepsilon)$, for some $\rho>0$ and $L_1,L_2\in R^2(I)$.

Which implies that , $(X_{kl},Y_{kl})\in Z(\grave{M}\circ M,\Delta_m^n)$. ■

(ii) Let $(X_{kl},Y_{kl})\in Z(M,\Delta_m^n)\cap Z(\grave{M},\Delta_m^n)$.
Then $\left[M_1\left(\frac{\bar{d}(\Delta_m^nX_{kl},L_1)}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^nY_{kl},L_2)}{\rho}\right)\right]<(\varepsilon,\varepsilon)$, for some $\rho>0$ and $L_1,L_2\in R^2(I)$.
And $\left[M_3\left(\frac{\bar{d}(\Delta_m^nX_{kl},L_1)}{\rho}\right)\vee M_4\left(\frac{\bar{d}(\Delta_m^nY_{kl},L_2)}{\rho}\right)\right]<(\varepsilon,\varepsilon)$, for some $\rho>0$ and $L_1,L_2\in R^2(I)$.
The majority of the proof comes from the equality ,

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of$\left(M+\grave{M}\right)\left(\frac{\bar{d}(\Delta_m^n X_{kl},L_1)}{\rho},\frac{\bar{d}(\Delta_m^n X_{kl},L_1)}{\rho}\right)=\left[M_1\left(\frac{\bar{d}(\Delta_m^n X_{kl},L_1)}{\rho}\right)\vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{kl},L_2)}{\rho}\right)\right]+$
$\left[M_3\left(\frac{\bar{d}(\Delta_m^n X_{kl},L_1)}{\rho}\right)\vee M_4\left(\frac{\bar{d}(\Delta_m^n Y_{kl},L_2)}{\rho}\right)\right]<(\varepsilon,\varepsilon)+(\varepsilon,\varepsilon)=(2\varepsilon,2\,\varepsilon)$. for some $\rho>0$.
Which implies $(X_{kl},Y_{kl})\in\ Z\left(M+\grave{M},\Delta_m^n\right)$. ■

Conflict of Interests.

There are non-conflicts of interest .

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الخلاصة

في هذا البحث قدمنا بعض تعميم اختلاف لفضاءات المتتابعات الثنائية من الاعداد الحقيقية الضبابية المعرفة بواسطة دالة أورليسز المضاعفة. وندرس خصائصها المختلفة مثل الكمال والصلابة والتناظر وما إلى ذلك. كما حصلنا على بعض علاقة التضمين التي تتضمن فضاء المتتابعات الثنائية .