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An Explicit Carleman Formula for the Dolbeault Cohomology

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We study formulas which recover a Dolbeault cohomology class in a domain of $\mathbb{C}^n$ through its values on an open part of the boundary. These are called Carleman formulas after the mathematician who first used such a formula for a simple problem of analytic continuation. For functions of several complex variables our approach gives the simplest formula of analytic continuation from a part of the boundary. The extension problem for the Dolbeault cohomology proves surprisingly to be stable at positive steps if the data are given on a concave piece of the boundary. In this case we construct an explicit extension formula.

Keywords: $\bar{\partial}$-operator, cohomology, integral formulas.

Introduction

Carleman formulas first appeared in function theory to reconstruct an analytic function in a domain through its values on a boundary piece. The problem of analytic continuation is well known to be ill posed, if the piece is different from the whole boundary, and it is treated by explicit formulas of complex analysis. While the classical construction for analytic continuation by Goluzin and Krylov (1933) is rather transparent, the simplest Carleman formula in one variable was apparently first written in the book [1]. It corresponds to the case where the data are given on a smooth curve in a disc which divides the disc into two domains and does not meet its center, the continuation being to the part which does not contain the center of the disc.

More precisely, let D be an open disc around the origin and $\mathcal{S}$ a smooth curve in $D \setminus \{0\}$ dividing the disc into two domains. Write $\mathbb{C}D$ for the domain that does not contain the origin. Then, for each function $u \in C(\overline{\mathcal{D}})$ holomorphic in $\mathcal{D}$, the formula holds

$$u(z) = \lim_{N\to\infty} \frac{1}{2\pi\imath} \int\limits_{\mathcal{S}} u(\zeta)\, \frac{\left(\dfrac{z}{\zeta}\right)^N}{\zeta - z} d\zeta \tag{0.1}$$

whenever $z \in \mathcal{D}$.

The limit in (0.1) is attained uniformly in z on compact subsets of $\overline{\mathcal{D}} \setminus \mathcal{S}$, as is easy to check.

In Sections 3 and 25 of [1] an approach of A.Kytmanov is mentioned which works for homogeneous domains in $\mathbb{C}^n$ and leads to very simple Carleman formulas. However, by the very nature this approach is through function theory of one complex variable.

In this paper we extend the explicit formula (0.1) to functions of several complex variables. Our formula gains in interest if we realise that it is much simpler than any Carleman formula in

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the monograph [1] summarising the development of the area before 1992. Since then there has been no progress in studying explicit constructions of analytic continuation in several complex variables while such formulas are of great importance in mathematics. As but one application we mention the Cauchy problem for solutions of analytic partial differential equations, both linear and nonlinear.

More precisely, let B the unit ball around the origin in $\mathbb{C}^n$ and $\mathcal{S}$ a smooth surface in $B \setminus \{0\}$ which divides the ball into two domains. Denote by $\mathcal{D}$ the domain which does not contain the origin. Then, given any function $u \in C(\overline{\mathcal{D}})$ holomorphic in $\mathcal{D}$, the formula holds

$$\begin{aligned} u(z) &= \int\limits_{\partial\mathcal{S}\times[0,1]} u(\zeta)\, K_1\Big((1-t)v_0 + tv_1\Big) - \\ &\quad - \lim_{N\to\infty} \int\limits_{\mathcal{S}} u(\zeta) \Big(K_1(v_1) - \Big(1 - \Big(\frac{\langle\overline{\zeta}, z\rangle}{|\zeta|^2}\Big)^{N+1}\Big)^n K_1(v_0)\Big) \end{aligned} \tag{0.2}$$

for all $z \in \mathcal{D}$, where $v_0 = \dfrac{\overline{\zeta}}{\langle\overline{\zeta}, \zeta - z\rangle}$ and $v_1 = \dfrac{\overline{\zeta - z}}{|\zeta - z|^2}$.

For a definition of $K_1(v) = K_1^{(0)}(v)$ we refer the reader to Section 1. This is a determinant with values in differential forms of bidegree $(n, n-1)$ first introduced by Koppelman in [6]. In particular, $K_1(v_1)$ is the Martinelli-Bochner kernel and $K_1(v_0)$ is the Cauchy-Fantappiè kernel for the ball. In the case $n = 1$ formula (0.2) just amounts to (0.1).

As written above, formula (0.2) can be interpreted in a much more general context, with B replaced by any linearly convex domain. In this case an appropriate support function $v_0(\zeta)$ for $\mathcal{D}$ is chosen for points $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$ at which the domain $\mathcal{D}$ is linearly convex. Analytic continuation occurs from $\mathcal{S}$ to a ball with center at the origin or a more general pseudoconvex domain. The situation is drastically different if $\mathcal{D}$ is a bounded domain in $\mathbb{C}^n$ whose boundary consists of a part of the sphere ∂B and a smooth surface $\mathcal{S}$ lying in the complement of $\overline{B}$. In this latter case one continues analytic function from $\mathcal{S}$ to a ball with center at the point at infinity. A suitable barrier function for $\mathcal{D}$ is required for those boundary points at which the domain $\mathcal{D}$ is linearly concave. Such a function v_0 depends on $z \in \mathcal{D}$ rather than on $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$. This problem is studied in [8]. The point of the construction is certainly in the explicit form of v_0, for the existence is equivalent to the well-known uniqueness theorem for holomorphic functions. Along with geometric properties of $\partial\mathcal{D} \setminus \mathcal{S}$ one invokes a familiar theorem of Hefer (1941) to find a function v_0 holomorphic in z and ζ.

Holomorphic functions represent the cohomology of the Dolbeault complex at step 0. On a strictly q-concave domain, the role of holomorphic functions is thus played by the Dolbeault cohomology at step q which is of infinite dimension. For $q > 0$ the Dolbeault cohomology at step q can have a very complicated structure, e.g., be not separated, even for strictly q-concave domains $\mathcal{D}$. To get rid of this lack, one often considers the so-called reduced cohomology being the quotient over the closed range, cf. [9].

The Carleman formula (0.2) extends to the differential forms of bidegree (p, q), with $0 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant n$. If u is a $\bar{\partial}$-closed form of bidegree (p, q) in $\mathcal{D}$ continuous up to the boundary, then

$$\begin{aligned} u(z) &= - \int\limits_{\partial\mathcal{S}\times[0,1]} u(\zeta) \wedge (-1)^{p+q+1} K_{q+1}^{(p)}\Big((1-t)v_0 + tv_1\Big) - \int\limits_{\mathcal{S}} u(\zeta) \wedge K_{q+1}^{(p)}(v_1) \\ &\quad + \bar{\partial}\, h_q^{(p)} u\,(z) \end{aligned} \tag{0.3}$$

for all $z \in \mathcal{D}$, where $h_q^{(p)}$ is a $\bar{\partial}$-homotopy operator on differential forms of bidegree (p, q) in $\mathcal{D}$, see Section 4.

Formula (0.3) implies first of all a uniqueness theorem for the Dolbeault cohomology. Namely, if u is a $\bar{\partial}$-closed differential form of bidegree (p, q) in $\mathcal{D}$ continuous up to the boundary and vanishing on $\mathcal{S}$, then $u = \bar{\partial}\, h_q^{(p)} u$ in $\mathcal{D}$, i.e., the Dolbeault cohomology class of u is zero. This result has a very transparent explanation. Namely, if u vanishes on $\mathcal{S}$ then the extension of u to the closed ball $\overline{B}$ given by zero in $\overline{B} \setminus \overline{\mathcal{D}}$ is obviously continuous on $\overline{B}$ and $\bar{\partial}$-closed in B. Since the Dolbeault cohomology of the ball is zero at the positive steps, it follows that the continuation of u is $\bar{\partial}$-exact in B, and so u is $\bar{\partial}$-exact in $\mathcal{D}$. This is actually an original motivation of Carleman formulas for the Dolbeault cohomology in [7]. Formula (0.3) gives more, namely, if the restriction of u to $\mathcal{S}$ is $\bar{\partial}_b$-exact, then u is exact in $\mathcal{D}$.

It is much more surprising that (0.3) contains no passage to the limit. This shows that the reduced Dolbeault cohomology depends continuously on its restriction to $\mathcal{S}$. Another way of stating this observation is to say that if $(u_j)_{j \in \mathbb{N}}$ is a sequence of $\bar{\partial}$-closed differential forms of bidegree (p, q) in $\mathcal{D}$, with $q \geq 1$, which are continuous up to $\partial\mathcal{D}$ and converge to zero uniformly on $\mathcal{S}$, then the reduced Dolbeault cohomology of u_j converges to zero. Thus, the Cauchy problem for the reduced Dolbeault cohomology with data on $\mathcal{S}$ is normally solvable at positive steps, which can never happen at step 0 unless $\mathcal{S}$ contains the Shilov boundary of $\mathcal{D}$. This testifies certain hyperbolicity of elliptic complexes, by which is meant the well-posedness of the Cauchy problem.

Let us dwell upon the contents of the paper. In Section 1 we give a slight development of the integral formula of Koppelman [6] for differential forms. In Section 2 we show an explicit Carleman formula for holomorphic functions of several variables which generalises (0.2). In Section 3 we indicate how these techniques apply to Reinhardt domains in $\mathbb{C}^n$. In Section 4 we prove formula (0.3) for the Dolbeault cohomology and extend it to more general domains. In Section 5 we discuss the Cauchy problem for the Dolbeault cohomology with data on a boundary piece in the framework of inverse problems. Our paper can be thought of as a good completion of [1].

1. Integral Representations of Differential Forms

For n-dimensional vectors $v_1, \ldots, v_N$ with entries in a ring and nonnegative integers $n_1, \ldots, n_N$ with $n_1 + \ldots + n_N = n$, we denote by $D_{n_1,\ldots,n_N}(v_1, \ldots, v_N)$ the determinant of order n whose first n_1 columns are v_1, the next n_2 columns are v_2 etc., the last n_N columns are v_N. We compute the determinant by columns, i.e., we define $\det(v_{ij}) = \sum_I (-1)^{\varepsilon_I} v_{i_1 1} \ldots v_{i_n n}$ where ε_I denotes the signature of the permutation $I = (i_1, \ldots, i_n)$ of the integers $(1, \ldots, n)$.

Let $v = v(z, \zeta, t)$ be a smooth function on $O \times [0, 1]$ with values in $\mathbb{C}^n$, O being an open set not intersecting the diagonal $\{z = \zeta\}$ in $\mathbb{C}n_z \times \mathbb{C}n_\zeta$. Fix $0 \leqslant p \leqslant n$. Consider the double differential forms $K_q^{(p)}(v)$ of bidegree $(p, q-1)$ in z and $(n-p, n-q)$ in ζ, t given by

$$
\begin{aligned}
K_q^{(p)}(v) \; = \; & \frac{(-1)^{q+(n-p)(q-1)}}{(2\pi i)^n n!} \binom{n}{p} \binom{n-1}{q-1} \times \\
\times \;\; & D_{p,n-p}(\partial z, \partial \zeta) \wedge D_{1,q-1,n-q}\left(v, \bar{\partial}_z v, (\bar{\partial}_\zeta + d_t) v\right), \qquad (1.1)
\end{aligned}
$$

for $1 \leqslant q \leqslant n$, and $K_0^{(p)} \equiv K_{n+1}^{(p)} \equiv 0$.

The double forms (1.1) were first introduced by Koppelman [6]. Here we rehearse some elementary properties of these forms.

Lemma 1.1. *For each smooth function f on $O \times [0, 1]$, we have the equality $K_q^{(p)}(fv) = f^n K_q^{(p)}(v)$.*

Proof. Indeed, if ∂ is one of the differentials $\bar\partial_z$, $\bar\partial_\zeta$ and d_t, then the Leibniz formula yields $\partial(fv) = (\partial f)v + f\partial v$. As the vector $(\partial f)v$ is proportional to v, it gives no contribution to the last determinant on the right-hand side of (1.1). This proves the lemma. □

In particular, if v satisfies $\langle v, \zeta - z\rangle \neq 0$ pointwise on the set $O \times [0,1]$, then

$$K_q^{(p)}\left(\frac{v}{\langle v, \zeta - z\rangle}\right) = \frac{1}{\langle v, \zeta - z\rangle^n} K_q^{(p)}(v)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard bilinear form $\mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$. Thus, when considering a vector-valued function v with the property that $\langle v, \zeta - z\rangle \neq 0$ on the set $O \times [0,1]$, after multiplication by a nonzero function we may actually assume that $\langle v, \zeta - z\rangle = 1$.

Lemma 1.2. *Suppose v satisfies $\langle v, \zeta - z\rangle = 1$ on $O \times [0,1]$. Then, the equality holds*

$$\left(\bar\partial_\zeta + d_t\right) K_{q+1}^{(p)}(v) = (-1)^{p+q}\, \bar\partial_z K_q^{(p)}(v). \tag{1.2}$$

Proof. See for instance Lemma 1.2 in [2] and elsewhere. □

Note that if $v_j = v_j(z,\zeta)$, $j = 0,1$, are smooth functions on O with values in $\mathbb{C}^n$, both satisfying $\langle v_j, \zeta - z\rangle = 1$ on O, then the linear homotopy $v_t = (1-t)v_0 + tv_1$ between them still satisfies $\langle v_t, \zeta - z\rangle = 1$ on the set $O \times [0,1]$. The lemma below allows nonlinear homotopies, too.

Lemma 1.3. *Let v satisfy $\langle v, \zeta - z\rangle = 1$ on $O \times [0,1]$. Write v_0 and v_1 for the values of v at $t = 0$ and $t = 1$, respectively. Then*

$$K_{q+1}^{(p)}(v_1) - K_{q+1}^{(p)}(v_0) = \bar\partial_z I_{q+1}^{(p)}(v) - (-1)^{p+q}\bar\partial_\zeta I_{q+2}^{(p)}(v), \tag{1.3}$$

on the set O, where $I_i^{(p)}(v) = (-1)^{p+(i-1)} \int_0^1 (\partial/\partial t)\rfloor K_{i-1}^{(p)}(v)\, dt.$

Proof. It suffices to integrate equality (1.2) over $t \in [0,1]$ and take into account that

$$\bar\partial_\zeta \int_0^1 (\partial/\partial t)\rfloor K_{q+1}^{(p)}(v)\, dt = -\int_0^1 (\partial/\partial t)\rfloor \bar\partial_\zeta K_{q+1}^{(p)}(v)\, dt$$

because $\bar\partial_\zeta$ and d_t anticommute. □

There is a universal solution to the equation $\langle v, \zeta - z\rangle = 1$ outside of the diagonal in $\mathbb{C}^n_z \times \mathbb{C}^n_\zeta$, given by

$$v_1(z,\zeta) = \frac{\overline{\zeta - z}}{|\zeta - z|^2}$$

for $z \neq \zeta$. Under this choice of v, the double forms $K_q^{(p)}(v)$ fit together to give a fundamental solution of convolution type to the Dolbeault complex on $\mathbb{C}^n$.

Lemma 1.4. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}n$ with a piecewise smooth boundary and $u \in C^1(\Lambda^{p,q}T^*_{\mathbb{C}}\overline{\mathcal{D}})$. Then,*

$$-\int_{\partial\mathcal{D}} u \wedge K_{q+1}^{(p)}(v_1) + \int_{\mathcal{D}} \bar\partial u \wedge K_{q+1}^{(p)}(v_1) + \bar\partial \int_{\mathcal{D}} u \wedge K_q^{(p)}(v_1) = \chi_{\mathcal{D}} u, \tag{1.4}$$

where $\chi_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$.

Proof. Cf. the original paper of Koppelman [6]. For a thorough treatment we also refer the reader to [2]. □

2. A Carleman Formula for Holomorphic Functions

More precisely, let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary. This domain is called linearly convex at a boundary point $\zeta \in \partial\mathcal{D}$ if there exists a complex hyperplane $H_\zeta = \{z \in \mathbb{C}^n : \langle v, z-\zeta\rangle = 0\}$ through ζ which does not meet $\mathcal{D}$.

Pick an open set $\mathcal{S}$ on the boundary of $\mathcal{D}$, such that $\mathcal{D}$ is linearly convex at each point of $\partial\mathcal{D} \setminus \mathcal{S}$. We thus get a distribution H_ζ of hyperplanes in $T_{\mathbb{C}}\overline{\mathcal{D}}$ parametrised by the points $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$.

Assume that $v(\zeta)$ extends to a smooth function in the closure of $\mathcal{D}$, such that no hyperplane H_ζ with $\zeta \in \overline{\mathcal{D}}$ passes through a fixed point $a \in \mathbb{C}^n$. In other words, $\langle v(\zeta), a-\zeta\rangle \neq 0$ holds for all $\zeta \in \overline{\mathcal{D}}$.

Set $v_0(z,\zeta) = \dfrac{v(\zeta)}{\langle v(\zeta), \zeta - z\rangle}$, thus obtaining a smooth function of $(z,\zeta) \in \mathbb{C}^n \times \overline{\mathcal{D}}$ away from the null set of the denominator $\langle v(\zeta), \zeta - z\rangle$. By assumption, v_0 is smooth on the set $\mathcal{D} \times (\partial\mathcal{D} \setminus \mathcal{S})$, and so we readily find

$$I_2((1-t)v_0 + tv_1) = \frac{(-1)^n}{(2\pi\imath)^n} d\zeta \wedge \sum_{k=0}^{n-2} D_{1,1,k,n-2-k}\Big(v_0, v_1, \bar\partial_\zeta v_0, \bar\partial_\zeta v_1\Big)$$

for all $(z,\zeta) \in \mathcal{D} \times (\partial\mathcal{D} \setminus \mathcal{S})$.

Theorem 2.1. *Under the above assumptions, if $u \in C(\overline{\mathcal{D}})$ is holomorphic in $\mathcal{D}$, then*

$$\begin{aligned} u(z) &= -\int_{\partial\mathcal{S}} u(\zeta)\, I_2\Big((1-t)v_0 + tv_1\Big) - \\ &\quad - \lim_{N\to\infty} \int_{\mathcal{S}} u(\zeta)\Big(K_1(v_1) - \Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n K_1(v_0)\Big) \end{aligned} \tag{2.1}$$

for all $z \in \mathcal{D}$ satisfying $\sup\limits_{\zeta\in\partial\mathcal{D}\setminus\mathcal{S}} \Big|\dfrac{\langle v(\zeta), z-a\rangle}{\langle v(\zeta), \zeta-a\rangle}\Big| < 1.$

Proof. On applying the Bochner-Martinelli formula (cf. (1.4) for $p=q=0$) we obtain

$$u(z) = -\int_{\partial\mathcal{D}} u(\zeta)\, K_1(v_1)$$

for $z \in \mathcal{D}$. Write the integral on the right-hand side as the sum of two integrals, the first of the two being over $\mathcal{S}$ and the second being over $\partial\mathcal{D} \setminus \mathcal{S}$. For $z \in \mathcal{D}$ and $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$, we use (1.3) for $p=q=0$, to get $K(v_1) = K(v_0) - \bar\partial_\zeta I_2((1-t)v_0 + tv_1)$. This yields

$$\begin{aligned} u(z) &= -\int_{\mathcal{S}} u(\zeta)\, K_1(v_1) - \int_{\partial\mathcal{D}\setminus\mathcal{S}} u(\zeta)\Big(K_1(v_0) - \bar\partial_\zeta I_2((1-t)v_0 + tv_1)\Big) = \\ &= -\int_{\partial\mathcal{S}} u(\zeta)\, I_2((1-t)v_0 + tv_1)\Big) - \int_{\mathcal{S}} u(\zeta)\, K_1(v_1) - \int_{\partial\mathcal{D}\setminus\mathcal{S}} u(\zeta)\, K_1(v_0) \end{aligned} \tag{2.2}$$

for each $z \in \mathcal{D}$, which is due to Stokes' formula. It remains to transform the last integral.

By Lemma 1.1,

$$K_1(v_0) = \Big(\frac{1}{\langle v, \zeta - z\rangle}\Big)^n K_1(v)$$

and furthermore

$$\begin{aligned}\frac{1}{\langle v, \zeta - z\rangle} &= \frac{1}{\langle v, \zeta - a\rangle}\,\frac{1}{1 - \dfrac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}} = \\ &= \lim_{N\to\infty} \Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)\,\frac{1}{\langle v, \zeta - z\rangle},\end{aligned}$$

the limit exists because $\Big|\dfrac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big| < 1$ holds for all $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$. Hence it follows that

$$K_1(v_0) = \lim_{N\to\infty} \Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n K_1(v_0), \tag{2.3}$$

each member of the sequence being smooth on the closure of $\mathcal{D}$, for no hyperplane $\langle v(\zeta), z-\zeta\rangle = 0$ passes through a whenever $\zeta \in \overline{\mathcal{D}}$.

Our next goal is to show that each member of the sequence in (2.3) is a $\bar\partial$-closed differential form in $\mathcal{D}$. Since the differential forms are smooth on $\overline{\mathcal{D}}$, it suffices to verify this only for those $\zeta \in \mathcal{D}$ which satisfy $\langle v(\zeta), \zeta - z\rangle \neq 0$. When differentiating the form

$$f_N = \Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n K_1(v_0),$$

we take into account that $\bar\partial_\zeta K_1(v_0) = 0$, which is a consequence of Lemma 1.2. It follows that

$$\begin{aligned}\bar\partial f_N &= \bar\partial_\zeta \Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n \wedge K_1(v_0) = \\ &= n\Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^{n-1} (-1)(N+1)\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^N \bar\partial_\zeta \frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle} \wedge K_1(v_0)\end{aligned}$$

and

$$\begin{aligned}\bar\partial_\zeta \frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle} \wedge K_1(v_0) &= \frac{\langle \bar\partial_\zeta v, z-a\rangle\langle v, \zeta-a\rangle - \langle v, z-a\rangle\langle \bar\partial_\zeta v, \zeta - a\rangle}{\langle v, \zeta-a\rangle^2} \wedge K_1(v_0) = \\ &= (-1)^{n-1}\frac{(n-1)!}{(2\pi\imath)^n}\,\frac{\langle v, z-a\rangle\langle v, \zeta-a\rangle - \langle v, z-a\rangle\langle v, \zeta-a\rangle}{\langle v, \zeta-a\rangle^2\langle v, \zeta - z\rangle^n}\, d\zeta \wedge \bigwedge_{j=1}^n \bar\partial_\zeta v_j = 0,\end{aligned}$$

as desired.

Combining (2.3) with Stokes' formula yields

$$\begin{aligned}\int_{\partial\mathcal{D}\setminus\mathcal{S}} u(\zeta)\, K_1(v_0) &= \lim_{N\to\infty} \int_{\partial\mathcal{D}\setminus\mathcal{S}} u(\zeta)\, f_N = \\ &= \lim_{N\to\infty} \Big(\int_{\partial\mathcal{D}} u(\zeta)\, f_N - \int_{\mathcal{S}} u(\zeta)\, f_N\Big) = -\lim_{N\to\infty}\int_{\mathcal{S}} u(\zeta)\, f_N,\end{aligned}$$

for every uf_N is $\bar\partial$-closed in $\mathcal{D}$ and continuous up to the boundary. On substituting this into (2.2) we arrive at (2.1). $\square$

For the domain $\mathcal{D}$ and piece $\mathcal{S}$ considered in the Introduction, we simply choose $v(\zeta) = \bar\zeta$ and $a = 0$, obtaining

$$\sup_{\zeta\in\partial\mathcal{D}\setminus\mathcal{S}} \Big|\frac{\langle\bar\zeta, z\rangle}{\langle\bar\zeta, \zeta\rangle}\Big| \leqslant |z|$$

which is less than 1 for all $z \in \mathcal{D}$. This gives formula (0.2).

3. Computations for Reinhardt Domains

In this section we indicate how to explicitly construct domains $\mathcal{D}$ which satisfy the assumptions of Theorem 2.1. Let B be a Reinhardt domain in $\mathbb{C}^n$ given in the form $B = \{z \in \mathbb{C}^n : \varrho(\zeta) < 0\}$, with $\varrho(\zeta) := h(|\zeta_1|^2, \dots, |\zeta_n|^2) - 1$ and $h(v)$ a C^2 function of n real variables homogeneous of degree m. Suppose $0 \in B$. The complex tangent hyperplane of the boundary ∂B at a point ζ is the set of all $z \in \mathbb{C}^n$, such that

$$\langle \varrho'_\zeta, z - \zeta \rangle = \langle \varrho'_\zeta, z \rangle - \sum_{j=1}^{n} \frac{\partial h}{\partial v_j} |\zeta_j|^2 = \langle \varrho'_\zeta, z \rangle - m \ (\varrho(\zeta) + 1)$$

vanishes, the last equality being a consequence of Euler's theorem and the homogeneity of h. Hence it follows that $T_{\mathbb{C},\zeta}(\partial B)$ never meets the origin, for $\langle \varrho'_\zeta, \zeta \rangle = m$ if $\zeta \in \partial B$.

Since $\langle \varrho'_\zeta, \zeta \rangle = m(\varrho(\zeta) + 1)$ is valid on all of $\mathbb{C}^n$, the modulus of $\langle \varrho'_\zeta, \zeta \rangle$ just amounts to m if and only if $\zeta \in \partial B$. From $0 \in B$ it follows that $|\langle \varrho'_\zeta, \zeta \rangle| < m$ for all $\zeta \in B$. We now choose a smooth hypersurface $\mathcal{S}$ in B which, together with a closed piece of ∂B, bounds a subdomain $\mathcal{D}$ of B with piecewise smooth boundary, such that

$$|\langle \varrho'_\zeta, z \rangle| < m \tag{3.1}$$

whenever $z \in \mathcal{D}$ and $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$. We can assume, by shrinking $\mathcal{D}$ if necessary, that $\langle \varrho'_\zeta, \zeta \rangle \neq 0$ for all $\zeta \in \overline{\mathcal{D}}$.

Put

$$v_0(z, \zeta) = \frac{\varrho_\zeta}{\langle \varrho_\zeta, \zeta - z \rangle},$$

which is a smooth function of $(z, \zeta) \in \mathbb{C}^n \times \overline{\mathcal{D}}$ away from the null set of the denominator $\langle \varrho_\zeta, \zeta - z \rangle$.

Corollary 3.1. *Under the above assumptions, if $u \in C(\overline{\mathcal{D}})$ is holomorphic in $\mathcal{D}$, then*

$$\begin{aligned} u(z) &= -\int_{\partial\mathcal{S}} u(\zeta)\, I_2\Big((1-t)v_0 + t v_1\Big) - \\ & \quad - \lim_{N\to\infty} \int_{\mathcal{S}} u(\zeta) \Big(K_1(v_1) - \Big(1 - \Big(\frac{\langle \varrho_\zeta, z\rangle}{\langle \varrho_\zeta, \zeta\rangle}\Big)^{N+1}\Big)^n K_1(v_0)\Big) \end{aligned}$$

for all $z \in \mathcal{D}$.

Proof. Indeed, from (3.1) we conclude that

$$\sup_{\zeta \in \partial\mathcal{D}\setminus\mathcal{S}} \Big| \frac{\langle \varrho_\zeta, z\rangle}{\langle \varrho_\zeta, \zeta\rangle} \Big| < 1$$

for each $z \in \mathcal{D}$. It remains to use (2.1). □

4. Formulas for the Dolbeault Cohomology

We now return to the general setting of Theorem 2.1. The same proof still goes when we drop the assumptions $q = 0$, thus studying $\bar{\partial}$-closed differential forms of bidegree (p, q) in $\mathcal{D}$.

Assume that $\mathcal{D}$ is a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary and $\mathcal{S} \subset \partial\mathcal{D}$ an open set, such that $\mathcal{D}$ is linearly convex at each point of $\partial\mathcal{D} \setminus \mathcal{S}$. Just as in Section 2, we define

$$v_0(z, \zeta) = \frac{v(\zeta)}{\langle v(\zeta), \zeta - z \rangle},$$

which is a smooth function of $(z,\zeta) \in \mathbb{C}^n \times \overline{\mathcal{D}}$ away from the null set of the denominator $\langle v(\zeta), \zeta - z\rangle$. By assumption, v_0 is smooth on the set $\mathcal{D} \times (\partial\mathcal{D} \setminus \mathcal{S})$ whence

$$\begin{aligned} I_q^{(p)}((1-t)v_0 + tv_1) &= \frac{(-1)^{n+(n-p+1)q}}{(2\pi i)^n n!} \binom{n}{p} \times \\ &\times\ D_{p,n-p}(\partial z, \partial\zeta) \wedge \sum_{k=0}^{n-q} \binom{n-2-k}{q-2} D_{1,1,q-2,k,n-q-k}\Big(v_0, v_1, \bar\partial_z v_1, \bar\partial_\zeta v_0, \bar\partial_\zeta v_1\Big) \end{aligned}$$

for $2 \leqslant q \leqslant n$. One also defines $I_1^{(p)} \equiv I_{n+1}^{(p)} \equiv 0$.

Using the double differential form $I_{q+1}^{(p)}((1-t)v_0 + tv_1)$, we may introduce a $\bar\partial$-homotopy operator

$$h_q^{(p)} u\,(z) = -\int\limits_{\partial\mathcal{D}\setminus\mathcal{S}} u \wedge I_{q+1}^{(p)}\Big((1-t)v_0 + tv_1\Big) + \int\limits_{\mathcal{D}} u \wedge K_q^{(p)}(v_1), \quad z \in \mathcal{D},$$

on differential forms u of bidegree (p,q) in $\mathcal{D}$ continuous up to the part $\partial\mathcal{D} \setminus \mathcal{S}$ of the boundary. The interest of the operator $h_q^{(p)}$ lies in the fact that we obtain $\bar\partial\, h_q^{(p)} u = u$ in $\mathcal{D}$, provided u is $\bar\partial$-closed in $\mathcal{D}$ and vanishes (or is merely $\bar\partial_b$-exact) on $\mathcal{S}$.

Theorem 4.1. *If u is a $\bar\partial$-closed differential form of bidegree (p,q) in $\mathcal{D}$ continuous up to the boundary, then*

$$u(z) = -\int\limits_{\partial\mathcal{S}} u(\zeta) \wedge I_{q+2}^{(p)}\Big((1-t)v_0 + tv_1\Big) - \int\limits_{\mathcal{S}} u(\zeta) \wedge K_{q+1}^{(p)}(v_1) + \bar\partial\, h_q^{(p)} u\,(z) \tag{4.1}$$

for all $z \in \mathcal{D}$.

Proof. This follows by the same way as in Theorem 2.1, the only difference being in the fact that we apply Lemmas 1.4 and 1.3 with $q = 0$ replaced by arbitrary $q \geq 1$. If $q > 0$, then the double form $K_{q+1}^{(p)}(v_0)$ vanishes for $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$ because v_0 is holomorphic in z on the set $\langle v(\zeta), \zeta - z\rangle \neq 0$. Therefore, we need not approximate it uniformly in $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$ by $\bar\partial$-closed differential forms on the closure of $\mathcal{D}$, which simplifies the proof. □

Since formula (4.1) does not contain any limit passage, it demonstrates rather strikingly that the Cauchy problem for the Dolbeault cohomology in $\mathcal{D}$ with data on $\mathcal{S}$ is stable, if posed in appropriate function spaces. In particular, this includes a uniqueness result.

Corollary 4.2. *Let u be a differential form of bidegree (p,q) and of class C^1 on the closure of $\mathcal{D}$. If moreover u is $\bar\partial$-closed in $\mathcal{D}$ and $\bar\partial_b$-exact on $\mathcal{S}$, then u is $\bar\partial$-exact in $\mathcal{D}$.*

Proof. Assume that $u = \bar\partial_b \wp$ on $\mathcal{S}$ where $\wp$ is the restriction to $\mathcal{S}$ of a smooth $(p,q-1)$-form in a neighbourhood of $\overline{\mathcal{S}}$. Let us transform the right-hand side of (4.1). On the boundary of $\mathcal{S}$ which belongs to $\partial\mathcal{D} \setminus \mathcal{S}$ we can invoke decomposition (1.3) to obtain

$$\begin{aligned} -\int\limits_{\partial\mathcal{S}} u \wedge I_{q+2}^{(p)}(v_t) &= \int\limits_{\partial\mathcal{S}} \wp \wedge (-1)^{p+q-1} \bar\partial_\zeta I_{q+2}^{(p)}(v_t) = \\ &= \int\limits_{\partial\mathcal{S}} \wp \wedge \Big(K_{q+1}^{(p)}(v_1) - K_{q+1}^{(p)}(v_0) - \bar\partial_z I_{q+1}^{(p)}(v_t)\Big), \end{aligned}$$

where we write $v_t = (1-t)v_0 + tv_1$ for short. On the other hand, integrating by parts and using Lemma 1.2 we get

$$-\int\limits_{\mathcal{S}} u \wedge K_{q+1}^{(p)}(v_1) = -\int\limits_{\mathcal{S}} \bar\partial\wp \wedge K_{q+1}^{(p)}(v_1) = -\int\limits_{\partial\mathcal{S}} \wp \wedge K_{q+1}^{(p)}(v_1) - \bar\partial \int\limits_{\mathcal{S}} \wp \wedge K_q^{(p)}(v_1)$$

for all $z \in \mathcal{D}$. Adding these two equalities yields

$$
\begin{aligned}
-\int\limits_{\partial\mathcal{S}} u(\zeta) \wedge I^{(p)}_{q+2}(v_t) - \int\limits_{\mathcal{S}} u(\zeta) \wedge K^{(p)}_{q+1}(v_1) = & \\
= \; -\int\limits_{\partial\mathcal{S}} \wp \wedge K^{(p)}_{q+1}(v_0) + \bar\partial \left(-\int\limits_{\partial\mathcal{S}} \wp \wedge I^{(p)}_{q+1}(v_t) - \int\limits_{\mathcal{S}} \wp \wedge K^{(p)}_{q}(v_1) \right) &
\end{aligned}
$$

for $z \in \mathcal{D}$.

Note that the double form $K^{(p)}_{q+1}(v_0)$ vanishes identically away from the set of singularities of v_0, if $q > 0$. Indeed, the determinant (1.1) contains at least one column $\bar\partial_z v_0$, if $q-1>0$, and $\bar\partial_z v_0 \equiv 0$ because v_0 is holomorphic in z. It follows from Theorem 4.1 that

$$
u = \bar\partial \left(-\int\limits_{\partial\mathcal{S}} \wp \wedge I^{(p)}_{q+1}(v_t) - \int\limits_{\mathcal{S}} \wp \wedge K^{(p)}_{q}(v_1) + h^{(p)}_q u \right)
$$

in $\mathcal{D}$, proving the corollary. □

5. The Cauchy Problem for the Dolbeault Cohomology

In this section we interpret the above results within the abstract framework of [4] developed in [9].

Let B be a domain in $\mathbb{C}^n$ and $\mathcal{S}$ a smooth surface in B which divides B into two domains, one of the two being $\mathcal{D}$ and the other $B \setminus \overline{\mathcal{D}}$. Fix $0 \leqslant p \leqslant n$. Write $\Omega^{p,q}(B)$ for the space of all C^∞ differential forms of bidegree (p,q) in B, where $0 \leqslant q \leqslant n$. These spaces are gathered in a complex $\Omega^{p,\cdot}(B)$ on B endowed with the differential $\bar\partial$. This is just the Dolbeault complex with coefficients in Ω^p, where Ω^p stands for the sheaf of all holomorphic p-form on B. The well-known Dolbeault theorem says that $H^q(\Omega^{p,\cdot}(B)) \cong H^q(B, \Omega^p)$ holds for all $0 \leqslant p \leqslant n$, where $H^q(\Omega^{p,\cdot}(B))$ is the cohomology of the complex $\Omega^{p,\cdot}(B)$ at step q and $H^q(B,\Omega^p)$ the qth cohomology of B with coefficients in Ω^p.

Similarly we introduce complexes $\Omega^{p,\cdot}(\mathcal{D} \cup \mathcal{S})$ and $\Omega^{p,\cdot}(B \setminus \mathcal{D})$ whose spaces consist of the differential forms on $\mathcal{D}\cup\mathcal{S}$ and $B\setminus\mathcal{D}$, respectively, smooth up to $\mathcal{S}$. As usual, by $\Omega^{p,\cdot}(\mathcal{S})$ is meant the induced complex on $\mathcal{S}$ with differential $\bar\partial_{\mathcal{S}}$ called the tangential Cauchy-Riemann operator on $\mathcal{S}$. For a deeper discussion of this complex we refer the reader to the original paper [4] or to Section 3.1.5 in [9].

To shorten notation, we write B^- and B^+ instead of $\mathcal{D} \cup \mathcal{S}$ and $B \setminus \mathcal{D}$, respectively. These are closed subsets of B whose intersection is $\mathcal{S}$.

Choose a parametrix $\mathcal{P}$ for the complex $\Omega^{p,\cdot}(B)$ which is given by properly supported pseudodifferential operators of order -1 in B. Such a parametrix is easily obtained from the standard fundamental solution of convolution type for the Dolbeault complex on all of $\mathbb{C}^n$ by a familiar construction with partition of unity in B. We thus get $\mathcal{P}\bar\partial + \bar\partial\mathcal{P} = 1 - \mathcal{R}$ on each space $\Omega^{p,q}(B)$ (actually on all currents in B).

The remainder $\mathcal{R}$ is a properly supported smoothing operator on B, and so its composition with $[\mathcal{S}]^{0,1}$, the $(0,1)$-current in B corresponding to integration over $\mathcal{S}$, is well defined.

Theorem 5.1. *The sequence of Dolbeault cohomology spaces*

$$
\begin{array}{lllll}
\ldots \overset{\mathcal{R}\circ[\mathcal{S}]^{0,1}}{\to} & H^q(\Omega^{p,\cdot}(B)) & \overset{\delta}{\to} & H^q(\Omega^{p,\cdot}(B^-)) \oplus H^q(\Omega^{p,\cdot}(B^+)) & \overset{\delta}{\to} \; H^q(\Omega^{p,\cdot}(\mathcal{S})) \\
 \overset{\mathcal{R}\circ[\mathcal{S}]^{0,1}}{\to} & H^{q+1}(\Omega^{p,\cdot}(B)) & \overset{\delta}{\to} & \ldots &
\end{array}
\tag{5.1}
$$

is exact.

Proof. See [4] and Theorem 4.3.14 in [9] which explicitly describes the binding homomorphism in (5.1). □

The mapping δ in the first segment of complex (5.1) is given by restricting cohomology classes on B to those on B^- and B^+. What is of greater interest in complex analysis is the mapping δ in the second segment of (5.1). For $\bar{\partial}$-closed differential forms $u_\pm$ of bidegree (p,q) on $B^\pm$, we denote by $[u_\pm]$ their cohomology classes in $H^q(\Omega^{p,\cdot}(B^\pm))$. Then, δ assigns $[t(u_+)-t(u_-)]$ to the pair $([u_-,u_+])$, where $t(u_\pm)$ is the complex tangential part of $u_\pm$ on $\mathcal{S}$ and $[t(u_+)-t(u_-)]$ the cohomology class of the $\bar{\partial}_{\mathcal{S}}$-closed form $t(u_+)-t(u_-)$ in $H^q(\Omega^{p,\cdot}(\mathcal{S}))$. In [4] this sequence is called the Mayer-Vietoris sequence, a designation which stems from homological algebra.

Corollary 5.2. *Suppose both $H^q(\Omega^{p,\cdot}(B))$ and $H^{q+1}(\Omega^{p,\cdot}(B))$ are finite dimensional. Then*

$$H^q(\Omega^{p,\cdot}(B^-)) \oplus H^q(\Omega^{p,\cdot}(B^+)) \xrightarrow{\delta} H^q(\Omega^{p,\cdot}(\mathcal{S})) \tag{5.2}$$

is a Fredholm mapping.

Proof. On the one hand, it readily follows from Theorem 5.1 that the null space of (5.2) is isomorphic to $\delta(H^q(\Omega^{p,\cdot}(B)))$, and so it is of finite dimension. On the other hand, the range of (5.2) is isomorphic, by Theorem 5.1, to the null space of the mapping

$$H^q(\Omega^{p,\cdot}(\mathcal{S})) \overset{\mathcal{R}\circ[\mathcal{S}]^{0,1}}{\to} H^{q+1}(\Omega^{p,\cdot}(B)),$$

and hence the range is of finite codimension. □

If $q=0$, the space $H^q(\Omega^{p,\cdot}(B))$ is never finite dimensional, and so the additive Riemann problem fails to be Fredholm for holomorphic functions of several variables. The nonzero cohomology classes in $H^{q+1}(\Omega^{p,\cdot}(B))$ prove to be obstructions to representing any $\bar{\partial}_{\mathcal{S}}$-closed differential form of bidegree (p,q) on $\mathcal{S}$ as the difference of two $\bar{\partial}$-closed differential forms, one of the two in B^+ and the other in B^-.

We now turn to the Cauchy problem for the Dolbeault cohomology in $\mathcal{D}$ with data on $\mathcal{S}$. Given a boundary class $[u_0] \in H^q(\Omega^{p,\cdot}(\mathcal{S}))$, find a class $[u] \in H^q(\Omega^{p,\cdot}(\mathcal{D}\cup\mathcal{S}))$ satisfying $t([u]) = [u_0]$. Obviously, the uniqueness or the existence of solutions to the Cauchy problem just amounts to the injectivity or the surjectivity of the homomorphism

$$H^q(\Omega^{p,\cdot}(\mathcal{D}\cup\mathcal{S})) \xrightarrow{\delta} H^q(\Omega^{p,\cdot}(\mathcal{S})). \tag{5.3}$$

Theorem 5.1 reduces these questions to the vanishing of some Dolbeault cohomology.

Corollary 5.3. *Assume that $H^q(\Omega^{p,\cdot}(B)) = 0$. Then the mapping (5.3) is injective.*

Proof. Let $[u] \in H^q(\Omega^{p,\cdot}(\mathcal{D}\cup\mathcal{S}))$ satisfy $t([u]) = 0$. Pick any representative $u \in \Omega^{p,q}(\mathcal{D}\cup\mathcal{S})$ of this class with $\bar{\partial}u = 0$ in $\mathcal{D}$ and $t(u) = 0$ on $\mathcal{S}$. Consider the current $\tilde{u}$ of bidegree (p,q) in B which coincides with u in $\mathcal{D}\cup\mathcal{S}$ and vanishes away from the closure of $\mathcal{D}$ in B. It is easy to verify that $\bar{\partial}\tilde{u} = 0$ on all of B. Since the cohomology $H^q(\Omega^{p,\cdot}(B))$ is zero, there is a current $\wp$ of bidegree $(p,q-1)$ in B, such that $\bar{\partial}\wp = \tilde{u}$ in B. An easy manipulation with the homotopy formula shows that $\tilde{u} = \bar{\partial}(\mathcal{P}\tilde{u} + \mathcal{R}\wp)$ in B. The current $\mathcal{R}\wp$ is actually a C^∞ form in B, for the operator $\mathcal{R}$ is smoothing. On the other hand, the restriction of $\mathcal{P}\tilde{u}$ to $\mathcal{D}$ extends to a C^∞ form on $\mathcal{D}\cup\mathcal{S}$, for $\mathcal{P}$ has the transmission property with respect to the hypersurface $\mathcal{S}$. Hence it follows immediately that the class of u in $H^q(\Omega^{p,\cdot}(\mathcal{D}\cup\mathcal{S}))$ is zero, as desired. □

Note that, for $q=0$, the cohomology $H^0(\Omega^{p,\cdot}(B))$ is nonzero while (5.3) is injective.

Corollary 5.4. *Suppose $H^{q+1}(\Omega^{p,\cdot}(B)) = 0$ and $H^q(\Omega^{p,\cdot}(B\setminus\mathcal{D}) = 0$. Then (5.3) is surjective.*

Proof. This follows immediately from the exactness of Mayer-Vietoris sequence (5.1). □

The conditions $H^q(\Omega^{p,\cdot}(B)) = 0$ and $H^{q+1}(\Omega^{p,\cdot}(B)) = 0$ are satisfied for the domain B considered in Section 4.. If moreover $\mathcal{D}$ fulfills $H^q(\Omega^{p,\cdot}(B \setminus \mathcal{D}) = 0$, then combining Theorem 4.1 and Corollaries 5.3 and 5.4 we conclude that, given any $u_0 \in \Omega^{p,q}(\mathcal{S})$ continuous up to the boundary of $\mathcal{S}$ and satisfying $\bar{\partial}_{\mathcal{S}} u_0 = 0$, the cohomology class of

$$u(z) = -\int_{\partial\mathcal{S}} u_0(\zeta) \wedge I_{q+2}^{(p)}\Big((1-t)v_0 + tv_1\Big) - \int_{\mathcal{S}} u_0(\zeta) \wedge K_{q+1}^{(p)}(v_1), \quad z \in \mathcal{D},$$

in $H^q(\Omega^{p,\cdot}(\mathcal{D} \cup \mathcal{S})$ gives a unique solution to the Cauchy problem $t([u]) = [u_0]$ on $\mathcal{S}$.

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Точная формула Карлемана для когомологий Дольбо

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Изучаются формулы, которые восстанавливают класс когомологий Дольбо в областях из $\mathbb{C}^n$ по их значениям на открытой части границы. Они называются формулами Карлемана по имени математика, который нашел их первым в простейшем случае для проблемы аналитического продолжения. Для функций многих комплексных переменных наш подход дает простейшую формулу для аналитического продолжения с части границы. Проблема продолжения для когомологий Дольбо неожиданно устойчива в положительных классах, если начальные данные даются на вогнутой части границы. В этом случае дается точная формула продолжения.

Ключевые слова: $\bar{\partial}$-оператор, когомологии, интегральные формулы.