УДК 538.9, 538.94

Drag Force and Hess-Fairbank Effect in the One-Dimensional Bose Gas

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Different aspects of superfluidity in one dimension (1D) are discussed in this paper. It is shown that the Hess-Fairbank effect takes place in 1D at zero temperature. In particular, the rotational inertia of an interacting 1D Bose gas in a finite ring is zero. Nevertheless, our results indicate that the frictionless motion of impurities depends sensitively on the strength of interactions in the gas. In general, this is possible only in a limit of weak interactions. We obtain the phase diagram of frictionless motion for the gas immersed into a moving shallow optical lattice. In this case the drag force, a quantitative measure of superfluidity, can be zero at specific values of lattice velocity and the gas density even for strong interactions.

Keywords: one-dimensional Bose-gas, superfluidity, dynamic structure factor.

Introduction

Whether a system is superfluid or not depends very much on how superfluidity is defined. The term superfluidity, in fact, is commonly used for a variety of phenomena (see, *e.g.*, [1–4]) that were first observed in superfluid Helium such as non-classical rotational inertia (Hess-Fairbank effect), quantization of vortices, dragless motion of impurities and metastability of ring currents. Since each of these phenomena may be taken as "defining" a transition to superfluidity, it is important to ask under what circumstances they occur together. As was pointed out by Leggett [2] the metastability of ring currents and nonclassical rotational inertia are two fundamental superfluid phenomena of yet very different nature. While the latter is an equilibrium property, the former is a dynamic one.

The 3D weakly-interacting Bose gas has all the superfluid properties mentioned above, which can be inferred from the existence of the order parameter, wave function of the Bose-Einstein

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condensate (BEC). By contrast, there is no BEC in the repulsive 1D Bose gas even at zero temperature, provided interactions are independent of velocities of particles [5,6]. This can be easily proved with the Bogoliubov "$1/q^2$" theorem [5] that predicts a $1/q^2$ divergence at small momentum in the average occupation number n_q for nonzero temperature and $1/q$ divergence for zero temperature. Thus, for studying superfluidity in one dimension, we should use other methods. It is a long-standing question whether the 1D Bose gas can support persistent currents with macroscopic lifetimes [6].

This system has been realized with ultracold bosonic atoms in tightly confining linear traps [7,8] (ring traps are also under development [9]), in which the boson interactions are effectively described [10,11] by the contact potential $V(x) = g_B \delta(x)$ of the Lieb-Liniger (LL) model [12] (the Lieb-Liniger parameter c relates to our notations by $g_B = \hbar^2 c/m$). The interaction strength is quantified by the dimensionless parameter $\gamma = mg_B/(\hbar^2 n)$, where n is the linear density and m is the mass. For $\gamma \to \infty$, the model is known as the Tonks-Girardeau (TG) gas and can be mapped to an ideal *Fermi* gas. For $\gamma \ll 1$, the Bogoliubov model of weakly interacting bosons is recovered.

Experimental investigation of the superfluid properties of the 1D Bose gas by observing the motion of impurities is at an early stage [8] and theoretical predictions are not yet comprehensive. Sonin [13] found that ring currents can be metastable except for infinitely strong interactions. Kagan et al. [14] also concluded that persistent currents could be observable on experimental time scales and Büchler et al. [15] found the 1D Bose gas able to sustain supercurrents even in the presence of a strong defect. Astrakharchik and Pitaevskii [16] considered the drag force on a moving heavy impurity within Luttinger liquid theory and predicted a power-law dependence on the velocity for small velocities (see also Ref. [17]). These results contain an unknown prefactor preventing the calculation of the actual value of the drag force and are in any case not applicable at larger velocities. The motion of an impurity of finite mass was considered in the TG gas [18] but for finite values of γ this problem is still unresolved.

The approach developed in this paper is based on recent advances in the understanding [19–23] of the dynamics of the LL model. We calculate the rate of energy dissipation of ring currents at zero temperature in the presence of a small integrability-breaking perturbation. In particular, the phase diagram is obtained for the 1D Bose gas in moving shallow lattices. Although our results suggest that the 1D Bose gas can support metastable currents only in the weakly interacting regime where $\gamma \ll 1$, the superfluid fraction is shown to be 1 at zero temperature regardless of γ according to the nonclassical rotational inertia for a finite ring (perfect Hess-Fairbank effect).

1. Landau Criterion of Superfluidity and Hess-Fairbank Effect

In the LL model the total momentum is a good quantum number, and periodic boundary conditions quantize it in units of $2\pi\hbar/L$, where L is the ring circumference. The low-lying spectrum of $N = nL$ bosons as shown in Fig. 1 has local minima [24] (note a misprint in Eq. (7) for the density-density correlator: the sign before the second therm should be minus) at the supercurrent states I ($I = 0, 1, 2, \ldots$) with momenta $p_I = 2\pi n\hbar I$ and excitation energies

$$\varepsilon_I = p_I^2/(2Nm). \tag{1}$$

These correspond to Galilean transformations of the ground state with velocities $v_I =$

$p_I/(Nm)$. The minima do not depend on interactions and tend to zero in the limit of large system size.
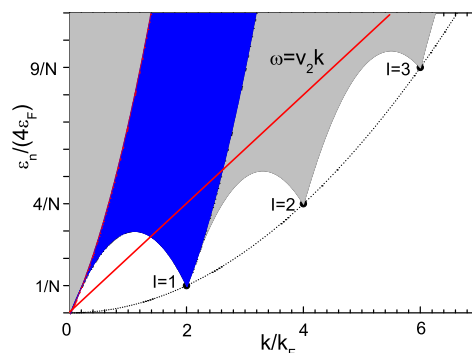

Fig. 1. (Color online) Schematic of the excitation spectrum of the 1D Bose gas in a perfectly isotropic ring. The supercurrent states I lie on the parabola $\hbar^2 k^2/(2Nm)$ (dotted line). Excitations occur in the shaded area; the discrete structure of the spectrum is not shown for simplicity. The blue (dark) area represents particle-hole excitations [26]. Motion of the impurity with respect to the gas causes transitions from the ground state to the states lying on the straight (red) line

2.   Drag Force as a Criterion of Superfluidity

Suppose that the gas is initially rotating with the linear velocity $-v_I$ and then is braked with an "obstacle," created, *e.g.*, by a laser beam [25]. In the frame where the gas is at rest, the obstacle moves with velocity v_I. In a superfluid we expect no energy dissipation, and thus zero drag force (the current is persistent). Energy conservation dictates that the transitions from the ground state caused by the moving obstacle with velocity v, lie on the line $\varepsilon = vp$. According to Landau, if the excitation spectrum lies above this line, the motion cannot excite the system, which is then regarded superfluid. The Landau critical velocity (when the line touches the spectrum) equals $v_c = \varepsilon_1/p_1 = v_1/2$. This implies that any supercurrent state with $I \geqslant 1$ is unstable since $v_I > v_c$. However, in 3D similar supercurrent states exist, which apparently leads to the absence of current metastability. The paradox can be resolved by considering not only the spectrum but also *probabilities* of excitations. Below we argue that in the 3D case, the probability to excite supercurrents is vanishingly small, while in the 1D case it depends on the strength of bosonic interactions. A related issue is the Hess-Fairbank effect: when the walls of a toroidal container are set in rotation adiabatically with a small tangential velocity v_D, a superfluid stays at rest while a normal fluid follows the container. This effect leads to a nonclassical rotational inertia of superfluid systems, which can be used to determine the superfluid fraction [27]. For the 1D Bose gas, rotation of the annular trap amounts to shifting the excitation spectrum to $\varepsilon - v_D p$ as shown in Fig. 2. It is assumed that an unspecified relaxation mechanism allows the system to relax to the ground state in the frame where *the trap is at rest*. The low-lying LL excitation spectrum is a convex function of momentum for $0 \leqslant p \leqslant p_1$ [26], and, hence, the momentum zero state remains the ground state for $|v_D| < v_c$. This leads to the Hess-Fairbank effect for the 1D Bose gas for arbitrary repulsive interactions γ completely determined by the low-lying energy
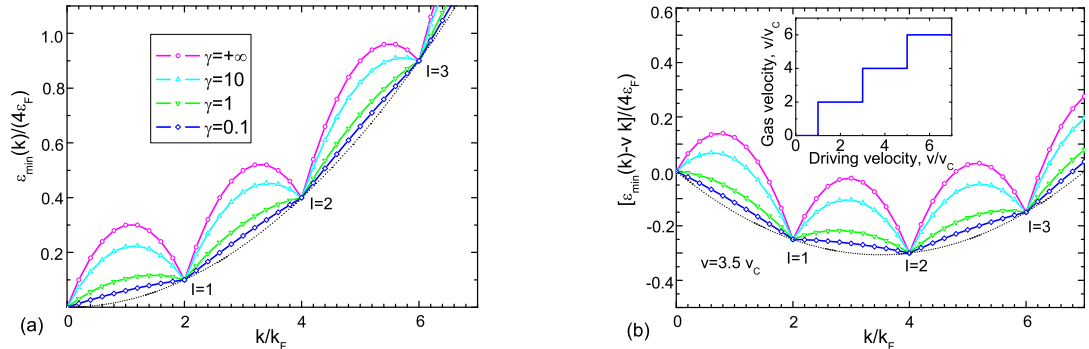
Fig. 2. (Color online) **a**) Low-lying excitation spectrum $\varepsilon_{\min}(k)$ for $N = 10$ particles for 1D repulsive bosons. At $\gamma \ll 1$, the low-lying spectrum verges towards that of the ideal Bose gas, which lies on the straight segments between points $I = 1, 2, 3, \ldots$. **b**) Quantization of current velocity for 1D repulsive bosons under influence of a moving trap. Shown are the low-energy excitations of the 1D Bose gas in the moving frame $\varepsilon_{\min}(k) - v_D \hbar k$, calculated from the Bethe-ansatz equations [12] for different values of the coupling strength (compare with $\varepsilon_{\min}(k)$ of Fig. 1). Inset: The velocity of the gas at equilibrium changes abruptly at integer values of driving velocity, since the gas occupies the state with lowest energy. In particular, the system is at rest when the driving velocity is less than v_c. Here, $k_F \equiv \pi n$ and $\varepsilon_F \equiv \hbar^2 k_F^2/(2m)$

spectrum [2], the 1D Bose gas for arbitrary repulsive interactions $\gamma > 0$ [28, 29]. According to this *equilibrium* property which is has a 100% superfluid fraction and zero rotational inertia at zero temperature.

By contrast to the Hess-Fairbank effect, metastability of currents is not an equilibrium effect and transition probabilities have to be considered. The dissipation rate as energy loss per unit time $\dot{E}$ of an obstacle (or heavy impurity) moving with velocity v relative to the gas can be related to the drag force F_v acting on the impurity by $\dot{E} = -F_v v$. For weak impurities with interaction potential $V_i(x)$ the drag force is related to the dynamic structure factor (DSF) in linear-response theory [16, 30, 31]:

$$F_v(v) = \int\limits_0^{+\infty} dk\, k |\tilde{V}_i(k)|^2 S(k, kv)/L, \tag{2}$$

where $\tilde{V}_i(k)$ is the Fourier transform of the impurity potential. The DSF $S(k, \omega)$ describes the transition probability between the ground state $|0\rangle$ and excited states $|m\rangle$ with energy transfer $\hbar\omega$ and momentum transfer $\hbar k$ caused by a density perturbation, and can be written as

$$S(k, \omega) = \sum_m |\langle 0|\delta\hat{\rho}_k|m\rangle|^2 \delta(\hbar\omega - E_m + E_0), \tag{3}$$

where $\delta\hat{\rho}_k = \sum_j e^{-ikx_j} - N\Delta(k)$ is the Fourier component of the density operator, $\Delta(k) = 1$ at $k = 0$ and $\Delta(k) = 0$ otherwise. Several results for the DSF in the LL model have recently become available [19, 21, 22]. It can be measured in cold gases by Bragg scattering [32, 33].

Numerical values of DSF calculated with the ABACUS algorithm [21] are shown in Fig. 3. The probability to create multiparticle excitations lying outside of the region $\omega_-(k) \leqslant \omega \leqslant \omega_+(k)$

are identically zero (below ω_-) or very small (above ω_+). Transitions from the ground state caused by a moving obstacle with velocity v occur along the straight (red) line. Drag force (2) is thus a generalization of the Landau criterion for superfluidity. Indeed, if the excitation spectrum of a generic system lies above the line $\omega = vk$ then it is superfluid; in this case the drag force (2) equals zero. The drag force thus proves to be fundamental and can be considered as a quantitative measure of superfluidity.
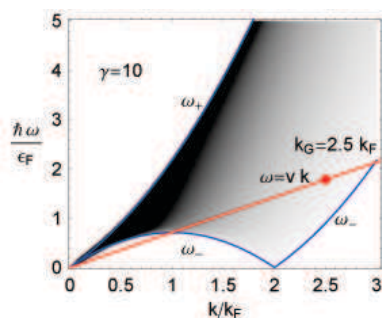

Fig. 3. (Color online) Dynamic structure factor of the 1D Bose gas from [21] for $N = 100$. Dimensionless values of $S(k,\omega)\varepsilon_F/N$ are shown in shades of gray between 0 (white) and 0.7 (black). The full (blue) lines represent the limiting dispersion relations $\omega_\pm(k)$ and the straight (red) line is the line of integration in Eq. (2). Only one point at $k = k_G$, shown in full (red) circle, contributes to the integral when the perturber is a shallow cosine potential with a reciprocal vector k_G

One can calculate the drag force from Eq. (2) by using a simple interpolating expression for the DSF [34]. The expression is well-applicable for all ranges of the parameters k, ω, and γ with increasing accuracy at large γ. A more detailed discussion can be found in Ref. [34].

3.   Shallow Optical Lattices

Equation (2) can be verified experimentally for different types of obstacles: for a local impurity we have $V_i(x) = g_i\delta(x)$ and all the points at the line $\omega = vk$ contribute to the drag force (see Fig. 3), while for a shallow lattice with the perturbing potential $V_i(x) = g_L \cos(2\pi x/a)$, only one point $(k_G, k_G v)$ in the k-ω plane does. Here $k_G \equiv 2\pi/a$ is the reciprocal lattice vector. Indeed, substituting the Fourier transform into Eq. (2) yields

$$F_v = \pi g_L^2 k_G S(k_G, k_G v)/2. \tag{4}$$

The filling factor of the lattice, that is, the number of particles per unit site, is given by $\alpha = 2\pi n/k_G$. Equation (4) can be exploited even in the case of a cigar-shaped quasi-1D gas of bosons at large number of particles, because the boundary conditions do not play a role in the thermodynamic limit. It gives us the momentum transfer per unit time from a moving shallow lattice, which can be measured experimentally [35–37].

The values of drag force (4) obtained from the interpolating expression for DSF [34] is shown in Fig. 4. At $k_G = 2\pi n$, corresponding to the Mott insulator state in a deep lattice, and at $\gamma \gg 1$, the drag force takes non-zero values for arbitrary $v \leqslant \omega_+(k_G)/k_G$. However, at small γ, its non-zero values practically localize in vicinity of $v = \omega_+(k_G)/k_G$. The frictionless motion at

some values of the parameters v, γ, α is consistent with the presence of persistent currents in the 1D Bose-Hubbard model [38–43]. As discussed in Sec. 2., the drag force can be considered as a measure for superfluidity. Then Fig. 4 represents the phase diagrams in the v-γ and v-$1/\alpha$ planes. They are similar to that of Polkovnikov *et al.* [40]. One can see from the diagrams that there is no sharp transition from superfluid to isolated phase in 1D. Note that in the latter paper, the superfluidity was examined in terms of quantum phase slips [44]. So, the both quasiparticle and quantum phase slip description lead to the same results.
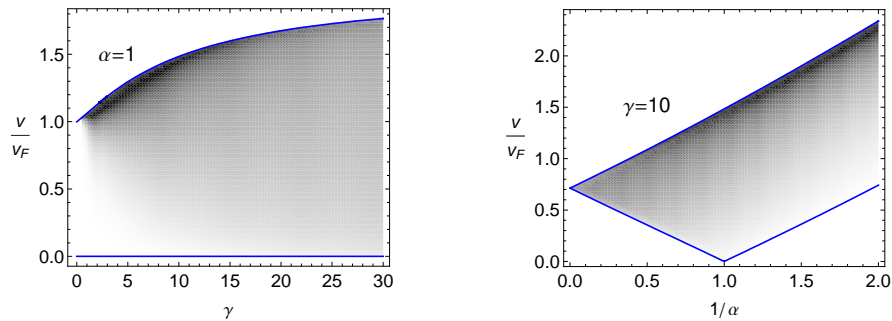

Fig. 4. (Color online) Zero temperature phase diagram for superfluid-insulator transition of the Bose gas in a moving shallow lattice. Dimensionless drag force $F_v 2\varepsilon_F/(\pi g_L^2 k_F N)$ versus the lattice velocity (in units $v_F \equiv \hbar k_F/m$) and the interaction strength γ (left panel) and versus the lattice velocity and the inverse filling factor (right panel). The dimensionless values are represented in shades of gray between zero (white) and 1.0 (black). The solid (blue) lines correspond to the DSF borders $\omega_+(k)$ and $\omega_-(k)$, respectively

Conclusion

Concluding, although the 1D Bose gas with finite repulsive inter-particle interaction shows superfluid phenomena of the *equilibrium type*, we show that in general its ability to support *dynamic* superfluid phenomena such as persistent ring currents is limited to a regime of very weak interactions; for a periodic potential, braking the gas, the persistent currents can be observed even in the TG regime at specific values of the velocity and density.

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Сила сопротивления и эффект Хесса-Фэрбэнка в одномерном бозе-газе

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Жан-Себастьян Ко
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В работе обсуждаются различные аспекты сверхтекучести в одномерном бозе-газе. Показано, что в одномерном случае при нулевой температуре проявляется эффект Хесса-Фэрбэнка. В частности, момент инерции одномерного неидеального бозе-газа, заключенного в кольцо, равен нулю. Несмотря на это, полученные результаты показывают, что отсутствие трения при движении примеси в газе определяется силой взаимодействия между бозонами. В общем случае это возможно только в пределе слабого взаимодействия. Мы получили фазовую диаграмму свободного от трения движения газа в оптической решетке. В этом случае сила сопротивления, являющаяся количественной мерой сверхтекучести, может отсутствовать даже для сильного взаимодействия при некоторых значениях скорости решетки и средней газовой плотности.

Ключевые слова: одномерный бозе-газ, сверхтекучесть, динамический структурный фактор.