Journal of Siberian Federal University. Mathematics & Physics 2010, 3(3), 357–368

УДК 517.9

Classical and Quantum-mechanical Description of the Arnol'd Diffusion in a System with 2.5 Degrees of Freedom

Alexander I. Malyshev*
Larisa A. Chizhova
Physics Department,
Nizhny Novgorod State University,
Gagarin 23, Nizhny Novgorod, 603950,
Russia

Received 10.04.2010, received in revised form 10.05.2010, accepted 20.06.2010

We study a universal phenomenon of nonlinear dynamics — the Arnol'd Diffusion — in a model system with 2.5 degrees of freedom. In the model an influence of three main resonances which take place in a phase space of the system is considered. The results obtained during classical and quantum-mechanical observation are compared. It was shown that a dependence of a rate of the quantum Arnol'd diffusion on parameters of the model behave alike classical one, however a value of the diffusion rate using methods of quantum mechanics lesser then that in classical case approximately at one of the order. It was found that presence of a threshold by the perturbation parameters is not necessarily feature of the Arnol'd diffusion. Also it was shown that there can occur a hybrid process in the quantum system in weak chaotic regime what doesn't have classical analogue — diffusion along resonance plus oscillations across overlapped resonances.

Keywords: nonlinear resonance, Arnol'd diffusion, quantum chaos.

Introduction

In this paper we study one of the most interesting phenomenon of nonlinear dynamics — the Arnol'd diffusion. Observation of such king of diffusion is possible only in multidimensional nonlinear systems. The matter is that in systems with degrees of freedom greater then two, stochastic layers of different resonances cross with each other creating a stochastic web. A trajectory within this web can travel from one resonance to another one even in case of small perturbation. Such motion has diffusion character and initially was predicted by Arnol'd [1].

To date, the Arnol'd diffusion was studied in details for enough wide range of systems. For instance, there is the problem of the particle motion between two planes, one of which is corrugated [2], and the problem of two weakly coupled oscillators, one of which is placed into the external time-periodic field [3]. In [4] the authors were able to visualize the Arnol'd web formed by many intersecting stochastic layers of resonances for the hydrogen atom placed into the crossed static electric and magnetic fields. From the practical point of view, the Arnol'd diffusion may be dangerous for long-time stability of motion of charged particles in high energy storage rings [5], such as in LHC.

So far, a possibility of observation of the Arnol'd diffusion in quantum system was unclear. On the one hand the Schrödinger equation is linear and this gave doubts of existence of a quantum chaos at all [6]. From the other hand, a correspondence principle shows that in a semiclassical regime when a system scale becomes comparable with de Broil wavelength the quantum mechanics transitions to classical one.

*Malyshev@phys.unn.ru

A first attempt of using a semiclassical method for description of the Arnol'd diffusion was made in [7] in the frame of a stochastic pump model. Later in [8, 9] a quantum analogue of the Arnol'd diffusion along a coupling resonance was studied through analysis of a system of two harmonic oscillators where one of them had been driven by an external oscillating force.

In this paper the Arnol'd diffusion is studied in a system of two weak coupling oscillators, one of which is placed into the external oscillating field. We observe the diffusion and time-evolution of the system using methods of classical and quantum mechanics. Therefore, it becomes possible to see not only likenesses and differences but to compare an amount of the effect as well.

1. Classical Arnol'd Diffusion: the Mechanism and Rate Estimation

For our observation, let's consider the following system: let a particle moves in 2D infinitely potential well size $a \times a$:

$$U(x,y) = \begin{cases} 0, \text{ if } |x| < a/2 \text{ and } |y| < a/2, \\ \infty, \text{ if } |x| \geqslant a/2 \text{ and } |y| \geqslant a/2. \end{cases} \tag{1}$$

This well is placed in external periodical field $U_{ext}(x,t) = -f_0 x \cos \Omega t$, where $f_0 << 1$. The weak interaction between two degrees of freedom is represented by the potential:

$$V(x,y) = -\mu(x+y)\left(xy - \frac{1}{2}\left(\frac{4a}{\pi^2}\right)^2\right). \tag{2}$$

Here μ is a small parameter. The reason for the choice of the interaction potential in this form will be described below.

For simplicity let's put the particle mass equal to unity. Thus, the Hamiltonian of the system inside the potential well is the following:

$$H(x,y,p_x,p_y,t) = \frac{p_x^2 + p_y^2}{2} - f_0 x \cos \Omega t - \mu(x+y)\left(xy - \frac{1}{2}\left(\frac{4a}{\pi^2}\right)^2\right). \tag{3}$$

The Hamiltonian of the unperturbed system ($f_0 = \mu = 0$) in action-angle variables is well known:

$$H_0(I_x, I_y) = \frac{\pi^2}{2a^2}\left(I_x^2 + I_y^2\right), \tag{4}$$

and the dependence of the x-coordinate (as well as y) on time can be represented, for example, as Fourier series:

$$x(t) = \frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\left((2n+1)\omega_x t\right)}{(2n+1)^2} = \frac{4a}{\pi^2}\left(\cos\omega_x t + \frac{\cos 3\omega_x t}{9} + \frac{\cos 5\omega_x t}{25} + \dots\right) \tag{5}$$

Here the frequency ω_x is determined as usual:

$$\omega_x = \frac{\partial H_0}{\partial I_x} = \frac{\pi^2}{a^2} I_x. \tag{6}$$

One can see that the amplitude a_n of higher harmonics sufficiently strongly decreases with an increase of n. Therefore, in action-angle variables one can approximately represent the position

of the x-oscillator by the expression $x \approx 4a/\pi^2 \cdot \cos\theta_x$. Let's rewrite the Hamiltonian (3) in action-angle variables leaving the slowest terms:

$$H(I_x, I_y, \theta_x, \theta_y, t) = \frac{\pi^2}{2a^2}\left(I_x^2 + I_y^2\right) - f_0 \frac{2a}{\pi^2}\cos(\theta_x - \Omega t) -$$

$$-\frac{\mu}{4}\left(\frac{4a}{\pi^2}\right)^3 \Big(\cos(2\theta_x - \theta_y) + \cos(\theta_x - 2\theta_y)\Big). \tag{7}$$

One can see from Eq. (7) that in the phase space of studied system, besides the resonance with the external field $\omega_x = \Omega$, there are two coupling resonances due to weak interaction between two degrees of freedom — $\omega_x = 2\omega_y$ and $\omega_y = 2\omega_x$. The location of these resonances as well as isoenergetic curve in the frequency plane are presented at Fig. 1. This figure provides quite clear illustration of Arnol'd diffusion mechanism. If $\mu = 0$, then by placing the initial conditions in domain of resonance with an external field the system behaves in accordance with well known laws of resonance dynamics. As for the vertical direction the state of the system does not change, because in this case $I_y = const$. Weak interaction between two spatial degrees of freedom ($\mu << 1$) giving rise to two coupling resonances (this is why we chose the potential of interaction in the form (2)). Their influence leads to additional chaos in the region of stochastic layer of the resonance $\omega_x = \Omega$, and what is of main importance it also leads to slow diffusion change of I_y, i.e. to the Arnol'd diffusion along the resonance with the external field.

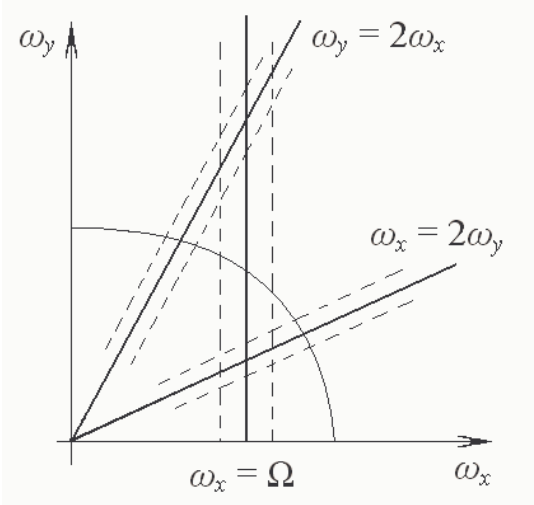

Fig. 1. The main resonances of the system (7) with their widths and isoenergetic curve in the frequency plane

In the frame of the introduced approximation, i.e. when the dynamics of the particle is being considered in the system with only three basic resonances present, it's possible to make an analytical estimation of the Arnol'd diffusion rate along the resonance with the external field [10]. As the purpose of the article above all is to study the dynamics of the quantum system, here we give only some results needed for the subsequent exposition.

For the analytical estimation of the rate of the Arnol'd diffusion one needs to calculate the dispersion of the energy in vertical direction $\overline{(\Delta E_y)^2}$ for a half-period of the phase oscillation T_a and divide it by value T_a itself [3]. Thus,

$$D_{theor} = \frac{\overline{(\Delta E_y)^2}}{T_a}. \tag{8}$$

Here $E_y = \pi^2 I_y^2/2a^2$. The initial conditions shall to be chosen inside of the stochastic layer of the resonance $\omega_x = \Omega$.

In paper [10] it was shown that half-width of the stochastic layer developed on the two different branches of the separatrix of the resonance with the external field from which a period of phase oscillations is dependent can be described by the following expressions:

$$h_{sl}^{+} = \mu \left(\frac{4\lambda}{\pi}\right)^3 \frac{a}{\Omega^2} \cdot \frac{\lambda^2}{\sinh(\pi\lambda)} e^{\frac{\pi\lambda}{2}} \quad \text{and} \quad h_{sl}^{-} = \mu \left(\frac{4\lambda}{\pi}\right)^3 \frac{a}{\Omega^2} \cdot \frac{2\lambda^2(\lambda^2-2)}{3\sinh(\pi\lambda)} e^{\frac{\pi\lambda}{2}} \tag{9}$$

where $\lambda = \Omega\sqrt{a/2f_0}$. One can see that where are two expressions related to separatrix's branches. Signs "+" and "−" indicate an upper and lower branch of the separatrix respectively. When the trajectory is located in one of the branch of the stochastic layer of the resonance with the external field, the system behavior effectively is determinded by one of the coupling resonance while the influence from another is exponentially small. The interesting fact is that the influence of the two coupling resonances, $\omega_x = 2\omega_y$ and $\omega_y = 2\omega_x$, with similar characteristics on the creation of stochastic layers of the resonance with the external field as well as on the Arnol'd diffusion is different. This is different from the case described by Chirikov [3], where the influence of two resonances with an external field on creating stochastic layer of coupling resonance was similar.

Assuming the hypothesis that dynamics in the both parts of the stochastic layer is separated, it can be shown that the diffusion coefficient along the resonance with the external field is given by:

$$D_{theor} = \left(\frac{a\Omega}{\pi}\right)^4 \cdot \frac{\Omega}{\lambda^7} \cdot \frac{2\left(h_{sl}^{+}\right)^2 + \left(h_{sl}^{-}\right)^2/2}{\ln\left(32e/h_{sl}^{+}\right) + \ln\left(32e/h_{sl}^{-}\right)}. \tag{10}$$

This estimation has a good agreement with the result of numerical experiments which will be discussed later in the last chapter of this paper.

2. Quantum-mechanical Observation of the Arnol'd Diffusion

In this section, we formulate a quantum-mechanical analog of the Arnol'd diffusion problem which was being considered previously. For that, it needs to solve nonstationary Schrödinger equation for the system with the Hamiltonian analogous to classical one (3):

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2} - f_0 x \cos\Omega t - \mu(x+y)\left(xy - \frac{1}{2}\left(\frac{4a}{\pi^2}\right)^2\right) + U(x,y), \tag{11}$$

where $U(x,y)$ describes 2D infinitely potential well (see (1)). The solution of the problem splits up into two phases. Firstly, it needs to build up quantum states belonging to the resonance with the external field in case the interaction between two degrees of freedom is excluded (i.e. $\mu = 0$). Then, it needs to study a time evolution of these states when $\mu \neq 0$ to observe whether or not the quantum Arnol'd diffusion along the resonance with the external field occurs in the system.

2.1. Quantum States on the Resonance with the External Field

From Hamiltonian operator one can see that if $\mu = 0$, then variables x and y splits up and therefor dynamics of the system in these two directions shall be independent. This means that for building quantum states of the resonance under study, we may consider only 1D infinitely potential well (in x-direction) placed into the external field. In y-direction there are known states of unperturbed 1D potential well.

Thus, we have to solve nonstationary Schrödinger equation with the following Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2} - f_0 x \cos \Omega t + U(x), \tag{12}$$

where

$$U(x) = \begin{cases} 0, \text{ if } |x| < a/2, \\ \infty, \text{ if } |x| \geqslant a/2. \end{cases} \tag{13}$$

If a wave function is chosen in terms of eigenfunctions of the unperturbed system, i.e. $\psi(x,t) = \sqrt{2/a} \sum\limits_n c_n(t) \sin\Big(\pi n(x + a/2)/a\Big)$, this equation is given by:

$$i\hbar_0 \frac{dc_n}{dt} = E_n^0 c_n - f_0 \cos \Omega t \cdot \sum_m x_{nm} c_m. \tag{14}$$

Here $\hbar_0$ is the dimensionless Plank constant and $E_n^0 = \pi^2 \hbar_0^2 n^2 / 2a^2$ — energy spectrum of the particle in the potential well (13).

Let's consider time evolution of the system near a resonance level n_0, i.e. when the expression $\hbar_0 \Omega = E_{n_0+1}^0 - E_{n_0}^0$ is satisfied. A value of the n_0 we consider much greater than one that corresponds to semiclassical regime [11,12]. Near the resonance level n_0 an expression for energy E_n^0 can be written as:

$$E_n^0 = E_{n_0}^0 + \hbar_0 \Omega (n - n_0) + \frac{\pi^2 \hbar_0^2}{2a^2} (n - n_0)^2. \tag{15}$$

After introducing new slow variables as $c_n(t) = a_n(t) \exp\Big(-i(E_{n_0}^0/\hbar_0 + \Omega(n - n_0))t\Big)$, the equation (14) becomes:

$$i\hbar_0 \frac{da_n}{dt} = \frac{\pi^2 \hbar_0^2}{2a^2} (n - n_0)^2 a_n - \frac{f_0}{2} \cdot \sum_m x_{nm} a_m \Big(e^{i\Omega(n-m+1)t} + e^{i\Omega(n-m-1)t} \Big). \tag{16}$$

One can see that the slowest terms in the sum by m correspond to $m = n \pm 1$. Further, only such summands will be considered in the frame of resonance approximation. Note that this approximation corresponds to taking into account only the first harmonic in the classical Fourier series (5).

As Hamiltonian (12) is periodic in time with period of the external field $T = 2\pi/\Omega$, it's convenient to use Floquet theory and to introduce the evolution operator (see, for example [5]). It's known [8, 9, 13, 14] that for building a matrix of the evolution operator it needs to solve the equation (14) through one period of the external field. The diagonal reduction of this matrix gives opportunity to find quasienergy spectrum (QE) and quasienergy functions (QF) of the system.

An example of the QE-spectrum is shown on the Fig. 2(a). In the frame of the resonance approximation it's Mathieu spectrum [15]. In a structure of the QE-spectrum, one can select three parts. The lower part corresponds to states near center of the resonance. In this area a distance between two energy levels equals to a frequency of a small phase oscillation on the resonance accurate within factor $\hbar_0$. This distance decreases enough slow with increasing of the QE value. The area where these levels become thickened matches with a classical separatrix of the resonance. The upper weak split high energy levels correspond to states what don't belong to the resonance.

For studying QE-functions A_n^Q in the basis of unperturbed system let's build a dependence of $\bar{n}_Q = \sum\limits_n (n - n_0)|A_n^Q|^2$ on square root of dispersion $\sigma_Q^2 = \sum\limits_n (n - n_0 - \bar{n}_Q)^2 |A_n^Q|^2$. A character of the diagram shown on the Fig. 2(b) demonstrates a structure of QE-function of the isolated

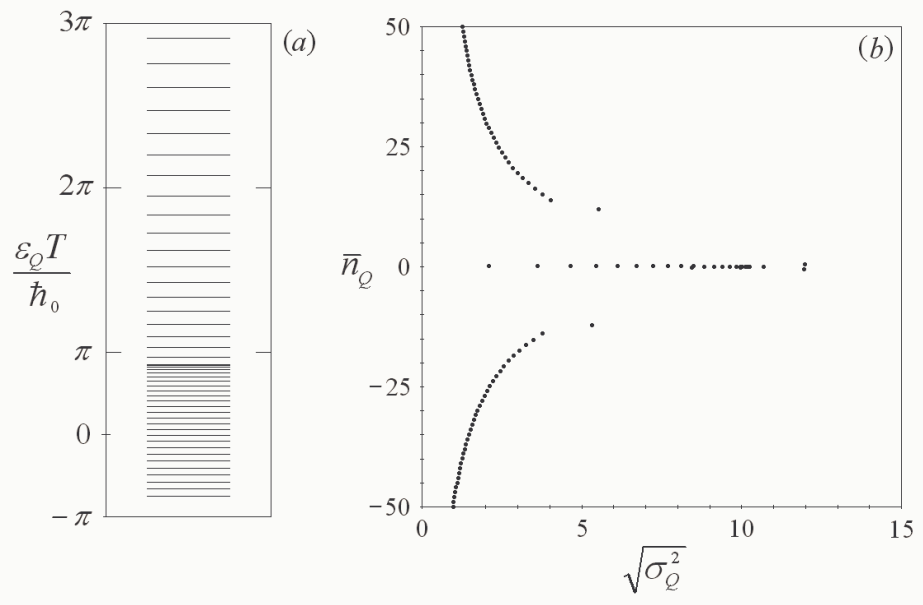

Fig. 2. Fragment of QE-spectrum of the system (12) (a) and the diagram which characterizes a structure of QE-functions (b). Here and later $\Omega = a = 1$, $n_0 = 200$, $f_0 = 0.001$ and $\hbar_0 = 1/\pi^2 n_0$

quantum nonlinear resonance [15]. Here, a middle branch corresponds to QF of oscillating regime (under-separatrix levels in the Mathieu spectrum), upper and lower branches match with QF of rotational regime (above-separatrix levels). Dots placed on the right edge of the middle branch correspond to states from the area of the separatrix of the resonance and hence these states have maximum delocalization.

Note that a similar system was studied in paper [16] where a character of a QE spacing distribution was specifically analyzed in dependence on the perturbation value.

An observation of the resonance of the 1D system with the external field allows to build QE-states and to define initial conditions which should be used for the next step of the problem solution, i.e. during the studying of the evolution of 2D system with the external field.

2.2. Evolution of a 2D System in the External Field. Quantum Arnol'd Diffusion

Let's consider the system with Hamiltonian (11). A solution of the nonstationary Schrödinger equation we will calculate in a basis of eigenstates of a unperturbed system:

$$\psi(x,y,t) = \frac{2}{a}\sum_{n,m} c_{n,m}(t)\sin\left(\frac{\pi n}{a}\left(x+\frac{a}{2}\right)\right)\sin\left(\frac{\pi m}{a}\left(y+\frac{a}{2}\right)\right). \tag{17}$$

In this case the Schrödinger equation is given by:

$$i\hbar_0\frac{dc_{n,m}}{dt} = \left(E_n^0 + E_m^0\right)c_{n,m} - f_0\cos\Omega t\cdot\sum_{n'} x_{nn'}c_{n'm} - \mu\sum_{n',m'} V_{nm,n'm'}c_{n'm'}. \tag{18}$$

Matrix elements $V_{nm,n'm'}$ are responsible for transitions between states due to weak interaction between two degrees of freedom. A value of the matrix elements extremely depends on ration

between state numbers. If $n' \neq n$ and $m' \neq m$, it takes the following form:

$$V_{nm,n'm'} = \frac{32a^3}{\pi^4} \cdot \frac{\Big((-1)^{n+m+n'+m'} - 1\Big) nmn'm'}{\Big(n^2 - n'^2\Big)^2 \Big(m^2 - m'^2\Big)^2}. \tag{19}$$

If $n' = n$ and $m' \neq m$, then

$$V_{nm,nm'} = \frac{a^3}{3\pi^6} \cdot \frac{\Big((-1)^{m+m'} - 1\Big) mm'}{\Big(m^2 - m'^2\Big)^2} \left(\pi^4 - \frac{6\pi^2}{n^2} - 96\right). \tag{20}$$

The matrix elements $V_{nm,n'm}$ have the form analogous to (20) and diagonal elements $V_{nm,nm}$ equal to zero.

Let's solve the equation (18) in the frame of resonance approximation, as it was provided in section 2.1. For that, one can enter new variables

$$c_{nm}(t) = a_{nm}(t) \exp\Big(-i\Big(\big(E^0_{n_0} + E^0_{m_0}\big)/\hbar_0 + \Omega(n - n_0) + \Omega(m - m_0)\Big)t\Big).$$

As it was done previously (see Eqs. (14)-(16)) in a first sum let's leave two terms corresponding to changing of the state number n on unity and in a second sum let's leave four terms with $n' = n \pm 1$ and $m' = m \mp 2$ as well as with $n' = n \pm 2$ and $m' = m \mp 1$.

We determine initial conditions in a form $c_{n,m}(0) = A^Q_n \delta_{m,m_0}$, where A^Q_n — QE functions which correspond to quantum nonlinear resonance with the external field (see section 3.1), and $m_0 = n_0$, where n_0 was introduced earlier.

For characterization of time evolution of state let's calculate a dispersion of energy $\overline{(\Delta E_m)^2} = \hbar_0^2 \Omega^2 \Delta_m^2$ in every moment of time. Here $\Delta_m^2 = \sum\limits_m (m - m_0 - \bar{m})^2 \sum\limits_n |c_{nm}(t)|^2$ and $\bar{m} = \sum\limits_m (m - m_0) \sum\limits_n |c_{nm}(t)|^2$. It's easy to see that this value indicates a distribution probability of quantum states on m, i.e. it describes their extent along the resonance with the external field.

Examples of dependencies $\Delta_m^2(t)$ for three different initial conditions is shown on Fig. 3: (a) corresponds to initial condition on the bottom state of the Mathieu spectrum (resonance center), (b) corresponds to initial state from above-separatrix region, (c) — initial state from the region of the separatrix of the resonance. The Fig. 3 shows that there is different behavior in time-evolution for different initial conditions. For states near the center of the resonance and above separatrix, the value of energy variance performs quasi-periodical oscillation near some mean value. But for initial states from near-separatrix region, i.e. for states from stochastic layer, the value of energy variance increases in time. One can admit that this growing is linear that clearly indicates a diffusion character of dynamics. Therefore it's obvious to assume that this phenomenon is similar to the classical Arnol'd diffusion, i.e. *quantum Arnol'd diffusion*. The diffusion rate can be defined through angle coefficient of incline of the line which approximates the dependence $\Delta_m^2(t)$ in an area of linear increasing. This coefficient can be calculated as:

$$D = \frac{\overline{(\Delta E_m)^2}(t)}{t} = \frac{\hbar_0^2 \Omega^2}{T} \cdot \frac{\Delta_m^2(t/T)}{t/T} \tag{21}$$

Dependencies of coefficients of the classical and quantum Arnol'd diffusion as a function of $1/\sqrt{f_0}$ are shown at Fig. 4. An analytical estimation of the classical diffusion rate (10) is shown here by bold curve. The calculation of the coefficient of the classical Arnol'd diffusion in numerical experiments was made using method described by Chirikov [3]:

$$D_n = \frac{1}{\Delta t_n} \overline{(\Delta \bar{E}_y)^2} \tag{22}$$

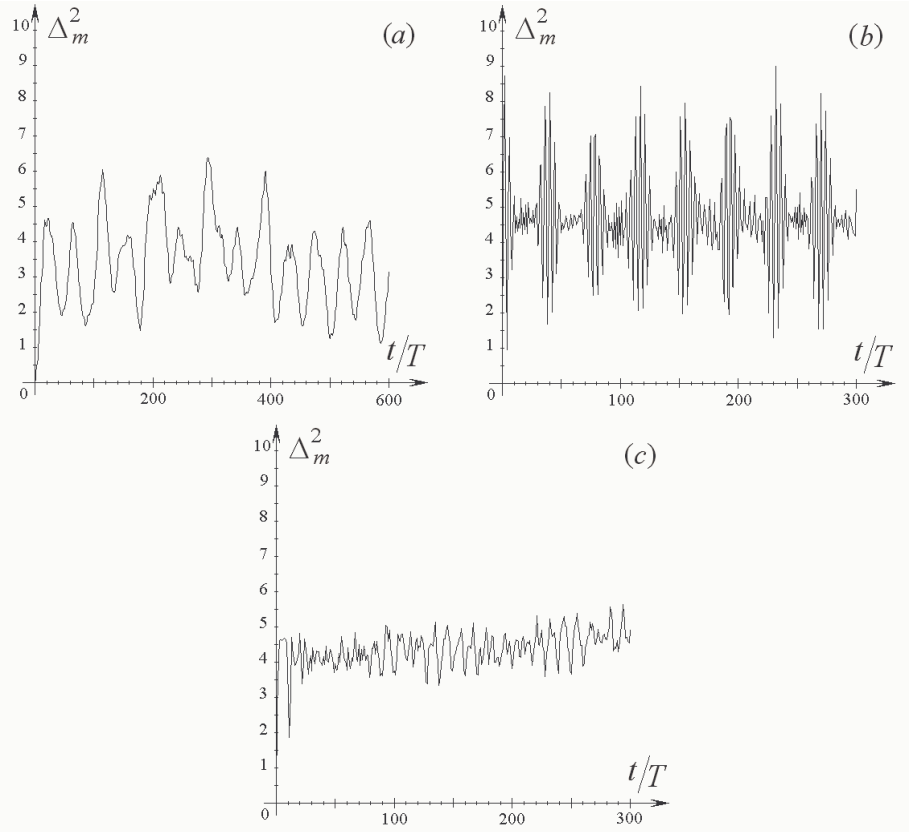

Fig. 3. Dispersion of the distribution along the resonance with the external field versus time for different initial states. Curves (a)-(c) correspond to an initial state near the center of the resonance, above the separatrix, and from the separatrix layer, respectively. Here $f_0 = 0.0095$, $\mu = 0.02$

Here $\bar{E}_y$ is the value of energy $E_y = \pi^2 I_y^2/2a^2$, averaged in time interval $\Delta t_n = 10^n \cdot T$. $\Delta\bar{E}_y$ is the difference of averaged values of energy on a neighbor intervals. Second overline means an averaging of $(\Delta\bar{E}_y)^2$ on many intervals. This procedure allows reducing the effect of random fluctuations of the energy and emphasizing of the increasing of its variance. The whole time of calculation in numerical experiments was 10^4 of periods of the external field. For every value of f_0 and for set of ten initial conditions from the stochastic layer of the resonance the two diffusion coefficients D_2 and D_3 were calculated. After averaging of computed data and calculation the confidence interval, the dependencies of diffusion coefficients on f_0 can be presented as it shown on Fig. 4.

In quantum case the relation (21) was used for calculating of the diffusion coefficient. For every value of f_0 the calculation was made for four initial states from the area of the stochastic layer of the resonance.

Let's now discuss the results of our observations. First of all, note that there is a good agreement between results of the numerical simulation of the classical Arnol'd diffusion and analytical estimation of its rate for those f_0 where diffusion process takes place (i.e. where diffusion coefficients D_2 and D_3 are close with taking in account the confident interval [3]). In this area the dependencies of $\log D_2$ and $\log D_3$ on f_0 have almost linear behavior. Hence

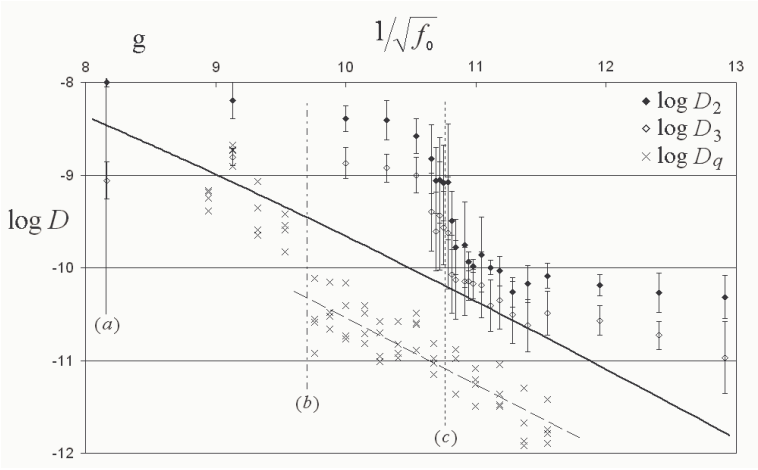

Fig. 4. Results of calculations of the Arnol'd diffusion coefficient in classical (D_2 and D_3) and quantum (D_q) cases. Solid incline curve corresponds to analytical estimation of the diffusion rate (10). Here $\mu = 0.02$. See comments in text

the diffusion rate is proportional to $\exp(-A/\sqrt{f_0})$ (see Eq. (10)) which was firstly derived by Chirikov for three-resonance model [3].

Note also that the dots what correspond to coefficients of quantum Arnol'd diffusion can be approximated by line which is parallel to the curve of the analytical estimation. The value of the rate of quantum diffusion is smaller then classical one at least on an order. This fact was observed also in [8, 9] while studying the quantum Arnol'd diffusion along a coupling resonance.

In above works it was also shown that one of the interesting property of the quantum Arnol'd diffusion is a threshold by parameters of perturbation (so-called "Shuryak border" [6]). Obviously that for observation of the quantum diffusion it needs to develop the regime of a weak quantum chaos that associated with a huge number of quantum states inside the stochastic layer. Initially it was shown by Shuryak [6]. However in the frame of the current work the threshold hasn't been found. From Fig. 4 it's easy to see that the classical and quantum diffusion stops practically in the same region $1/\sqrt{f_0} \approx 11.5$. In the area of lesser values of the amplitude f_0, coefficients D_2 and D_3 become noticeably different, i.e. an evolution of the classical system assumes quasiperiodical character. As for quantum case the dependence $\Delta_m^2(t)$ in this region stops increasing linearly. Absence of the threshold can be explained if we note that in an area of $1/\sqrt{f_0} \approx 11.5$ the number of quantum states inside the interval of classical stochastic layer is much greater than unity. Thus, one of the new results of the current work is that for a certain ratio between parameters f_0 and μ the quantum Arnol'd diffusion can be insensitive to "Shuryak border".

Vertical line (a) on Fig. 4 means theoretical estimation of the amplitude of the external field which corresponds to the overlapping moment of the resonance $\omega_x = \Omega$ with coupling resonance $\omega_y = 2\omega_x$. Dash line (c) shows an onset of resonances overlap during numerical simulation. This border is somehow relative because a travalling of the trajectory to the other resonance can occur for lesser values of the amplitude of the external field and it depends on initial conditions. A part of initial conditions which prevents to resonance overlapping growths with increasing of f_0. Line (c) corresponds to a border value of the amplitude of the external field when all initial conditions (within stochastic layer) prevent to resonance overlapping through 10^4 periods of the field. Crossing of this border prevents (see Fig. 4) to evident difference between coefficients D_2 and D_3 and this means that the classical Arnol'd diffusion "stops". Basically the diffusion as dynamical process still exists, however during time of observation $(10^4 \cdot T)$ the more stronger

process influences on the final result. This process is an oscillation across the stochastic layers of the overlapped resonances. Note that it could be possible to reduce the time of observation and effectively shift the border (c) to the left, but this would make impossible a uniform calculation of characteristics D_2 and D_3.

Discussed effect reflects on the results related to the quantum system. The matter is that the time of observation in the numerical experiments was no more then $10^3 \cdot T$. During simulation it was found that more to the right of the border (c) (see Fig. 4) the characteristic time of the quantum Arnol'd diffusion is about 500-1000 periods of the external field. And then the diffusion begins saturating and the dependence $\Delta_m^2(t)$ losses its linear character. The similar effect was observed in [8,9] and one can explain that as manifestation of dynamical localization alike Anderson localization in 1D random potential. To the left from (c) the dependence of Δ_m^2 on time qualitatively is not changing. However for f_0 close to $1/\sqrt{f_0} \approx 10$ together with the linear growth an oscillating process with period about several hundreds of T appears. This oscillating process can be explained through weak overlapping of quantum resonances: a part of probability distribution "takes" an edge of the coupling resonance $\omega_y = 2\omega_x$. Described situation are presented on the Fig. 5 where the probability distribution in unperturbed basis, $|c_{nm}(t)|^2$, at different moments of time is shown for $1/\sqrt{f_0} \approx 9.76$. An appropriate graphic of dependency of Δ_m^2 on time is also shown at the bottom of the figure.

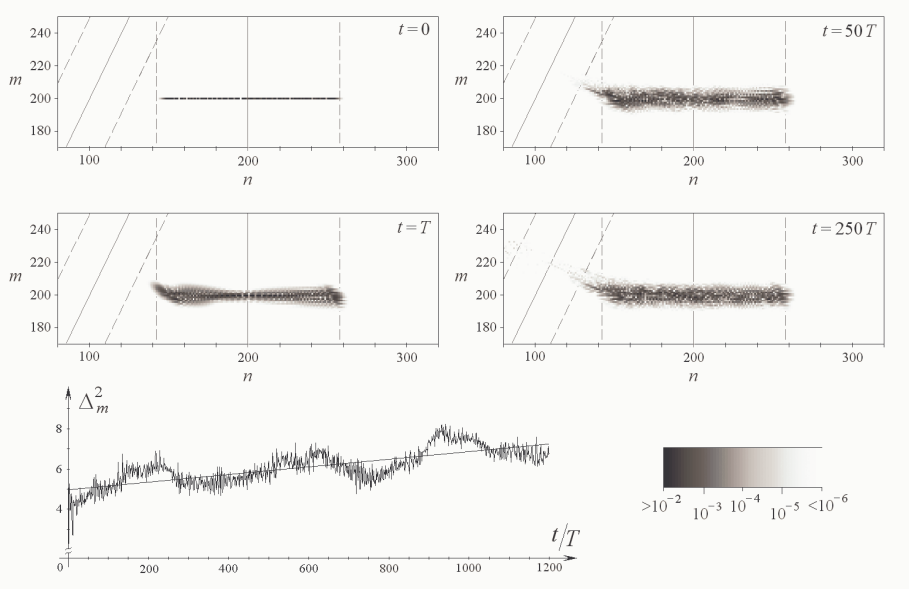

Fig. 5. Probability distribution of $|c_{nm}(t)|^2$ in unperturbed basis for different moments of time for initial state near the separatrix of the resonance with the external field. There are resonance with the external field, coupling resonance $\omega_y = 2\omega_x$ and their widths also shown. In the bottom part of the figure the dependence of dispersion Δ_m^2 on time is shown for the same initial state. Here $f_0 = 0.0105$ ($1/\sqrt{f_0} \approx 9.76$), $\mu = 0.02$

Thereby, in the interval from $1/\sqrt{f_0} \approx 10$ to the vertical line border (b) there is a superposition of two processes. On one hand, there is a linear growth of dispersion of probability distribution along the resonance with the external field, i.e. quantum diffusion along it. From the other hand, there are weak oscillations across the coupling resonance. This regime doesn't have a classical analogue as in case of resonance overlapping in classical system the Arnol'd diffusion becomes invisible in compare with the diffusion across stochastic layers of the overlapped resonances.

For quantum system one can talk about regime of *weak resonance overlapping*, when only small part of probability distribution, corresponded to one resonance, "takes" a small part of states corresponded to the neighbor one. In this regime, the quantum Arnol'd diffusion along the resonance and the diffusion across stochastic layers of the *weak overlapped* resonances become comparable.

In the area more to the left from (b) on the Fig. 4 the linear growth of $\Delta_m^2(t)$ (during several hundreds periods of the external field) becomes invisible in comparison with stronger oscillating process related to the resonance overlapping. Period of oscillations becomes shorter with growing of f_0 and in the area of $1/\sqrt{f_0} \approx 9$ it is lesser then $100T$. Linear increasing of dispersion which can be seen lasts no longe then this time. That's why the calculation of D_q in this area was made according to data through first 100-150 periods of the external field. Thus, as in classical case the jump of the rate of the quantum diffusion in the area to the left from (b) is related to a change of the evolution character due to the resonance overlapping and doesn't refer to the process of the pure Arnol'd diffusion.

Summary

Let us shortly summarize the main result of this work.

For classical approach to the system we found that the two coupling resonances influence on the creation of stochastic layer of the resonance with external field as well as on the Arnol'd diffusion in different ways. It is different from situation which was described by Chirikov [3]. In his model the influence of two resonances with an external field on creation of stochastic layer of coupling resonance was similar.

The dependence of the coefficient of the Arnol'd diffusion along the resonance with the external field on parameters of the system could be obtained analytically. And this estimation has a good agreement with numerical results.

In a frame of quantum-mechanical approach it was found that in some area of amplitudes of the external field the rate of the Arnol'd diffusion has the same character as in classical case. However the value of the coefficient of the quantum Arnol'd diffusion lesser than classical one approximately on one of the order.

The new result here is that for some ratio between parameters f_0 and μ the quantum Arnol'd diffusion can be insensible to Shuryak border which existence was mentioned in previous works [8, 9]. Thereby, a presence of the threshold by perturbation parameters isn't necessary characteristic of the quantum Arnol'd diffusion.

For the first time the quantum system with 2.5 degrees of freedom has been considered in the regime of overlapped resonances. For the condition of the weak chaos it's possible to obtain existence of the "hybrid" process — diffusion along the resonance with the external field which accompanies with oscillations across overlapped resonances. This situation obviously doesn't have a classical analogue.

References

[1] V.I. Arnol'd, *DAN USSR*, **156**(1964), 9 (in Russian).

[2] A.J. Lichtenberg, M.A. Lieberman, Regular and Chaotic Dynamics, Springer-Verlag, New York, 1992.

[3] B.V. Chirikov, *Phys. Rep.*, **52**(1979), 263.

[4] J. von Milchevski, G.H.F. Diercksen, T. Uzer, Phys. Rev. Lett. **76**(1996), 2890.

[5] L.E. Reichl, The Transition to Chaos: Conservative Classical Systems and Quantum Manifestation, Springer-Verlag, New York, 1992.

[6] E.V. Shuryak, *Zh. Eksp. Teor. Fiz.*, **71**(1976), 2039 (in Russian).

[7] D.M. Leitner, P.G. Wolynes, *Phys. Rev. Lett.*, **79**(1997), 55.

[8] V.Ya. Demikhovskii, F.M. Izrailev, A.I. Malyshev, *Phys. Rev. Lett.*, **88**(2002), 154101; *Phys. Rev. E*, **66**(2002), 036211.

[9] V.Ya. Demikhovskii, F.M. Izrailev, A.I. Malyshev, *Phys. Lett. A*, **352**(2006), 491; arXiv:cond-mat/0610390.

[10] A.I. Malyshev, L.A. Chizhova, *Izv. Vuzov. Prikl. Nelin. Din.*, **17**(2009), no. 1, 46 (in Russian).

[11] G.M. Zaslavsky, Physics of Chaos in Hamiltonian Systems, Imperial College Press, London, 1998.

[12] G.M. Zaslavsky, Chaos in Dynamic Systems, Harwood Acad. Publ., New York, 1985.

[13] V.Ya. Demikhovsky, D.I. Kamenev, G.A. Luna-Acosta, *Phys. Rev. E*, **59**(1999), 294.

[14] M.E. Flatté, M. Holthaus, *Ann. of Phys.*, **245**(1996), 113.

[15] G.P. Berman, A.R. Kolovsky, *Usp. Fiz. Nauk*, **162**(1992), no. 4, 95 (in Russian).

[16] W.A. Lin, L.E. Reichl, *Phys. Rev. A*, **37**(1988), 3972.

Классическое и квантово–механическое описание диффузии Арнольда в системе с 2.5 степенями свободы

Александр И. Малышев
Лариса А. Чижова

Работа посвящена изучению универсального явления нелинейной динамики —- диффузии Арнольда — в модельной системе с 2.5 степенями свободы. При учете влияния лишь трех основных резонансов, действующих в фазовом пространстве, проводится сопоставление результатов, полученных при классическом и квантово-механическом подходах. Показано, что зависимость скорости квантовой диффузии Арнольда от параметров задачи имеет характер, подобный классическому, при этом по абсолютной величине коэффициент квантовой диффузии отстает примерно на порядок. Обнаружено, что наличие порога по параметрам возмущения, на который указывалось ранее, не является обязательной чертой квантовой диффузии Арнольда. Также в работе показано, что в квантовой системе в режиме слабого перекрытия резонансов возможно и протекание гибридного процесса, не имеющего классического аналога, — диффузии вдоль резонанса, на которую накладывается процесс колебаний поперек перекрывшихся резонансов.

Ключевые слова: нелинейный резонанс, диффузия Арнольда, квантовый хаос.