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Mathematical Magic: A Study of Number Puzzles

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Cover Page Footnote

This research was conducted at Maryville College in Maryville, Tennessee as a Senior Thesis and was awarded the status of Exemplary Senior Study. As such, it can be found in the Library at Maryville College. I want to thank my thesis advisor, Dr. Jesse Smith, for all time and help with the research, and Dr. Chase Worley and others for helping with revisions.

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Mathematical Magic: A Study of Number Puzzles

By *Nicasio Velez*

Abstract. Within this paper, we will briefly review the history of a collection of number puzzles which take the shape of squares, polygons, and polyhedra in both modular and nonmodular arithmetic. Among other results, we develop construction techniques for solutions of both Modulo and regular Magic Squares. For other polygons in nonmodular arithmetic, specifically of order 3, we present a proof of why there are only four Magic Triangles using linear algebra, disprove the existence of the Magic Tetrahedron in two ways, and utilizing the infamous 3-SUM combinatorics problem we disprove the existence of the Magic Octahedron.

1 Introduction

1.1 Historical Note

As the Magic Square is the starting point for this paper and the inspiration, we will begin with a bit of history. The puzzle, defined below, is rooted in many years of history, and legend intertwines ancient Chinese and Taoist symbolism with Benjamin Franklin and many mathematicians and nonmathematicians alike [3].

Definition 1.1. A **Magic Square** is a square divided into n rows and n columns–where n is referred to as the order. Each slot must then be filled with integers from 1 to n^2 with no repeated values with the goal of each row, column and the main and minor diagonals to sum to the same value. This value is commonly referred to as the **magic constant** or **magic sum**. The **input set** is set of numbers to be placed in the slots of the Magic Square.

The Magic Square begins with two legends believed to take place in 23rd century BCE, both centering around an inscription of a Magic Square on a turtle shell. This specific square and the patterns within, referred to as the Lo Shu square can be seen below in Figure 1. From this legend, there is no written history until 4th century BCE, where the knowledge of the Taoist symbolism and many of the religious connections of the Magic Square are found. Fast forwarding a few centuries, Benjamin Franklin writes

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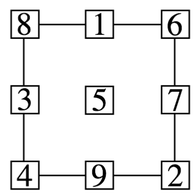
in a correspondence of, "the most magically magic of any magic square ever made by any magician" that he created in the 1700s. Then continuing forward to the 1800s, a Magic Square appears carved into the Facade of Passion on La Sagrada Familia–the infamously unfinished church that began its construction in 1882 [7, 3]. Though this specific solution violates the definition by repeating values, the symbolic use is much more prevalent in this square.

Figure 1: The LoShu Magic Square



In more recent history, complete books have been written in attempt to condense the plethora of research on Magic Squares. A few notable books being *Magic Squares and Cubes* written by WS Andrews in 1917 and *The Zen of Magic Squares, Circles, and Stars* written by Clifford Pickover [1, 5]. As suggested by the Pickover book title, the study of Magic Squares has expanded to other shapes as well. Both the team of mathematicians, Trump and Boyer, and another mathematician Weinstein have researched Magic Cubes and Hypercubes [10, 11]. Beyond circles and stars, this puzzle has even been extended into triangles and other polygons as well [9]. The latter extension will be the focus in Section 2.

1.2 Magic Square

Now that a bit of history has been discussed, this section will address previous research on the Magic Square. First, the puzzle significantly decreases in difficulty if one knows the magic constant. This can be calculated simply enough.

Theorem 1.2. *The equation for the magic constant,* S_n*, of an order n Magic Square is as follows.*

$$\mathrm{S}_n = \frac{n(n^2+1)}{2} \tag{1}$$

Proof. Using Gauss's formula for the sum of consecutive integers, we know the sum of the input set–thus the entire square–will be $\frac{n^2(n^2+1)}{2}$. This sum must be divided equally between the n rows, thus each row must sum to $\frac{n(n^2+1)}{2}$. □

With this, let's address the first orders. The order 1 square is trivial as it is one slot. The second order, however, cannot exist. If we let the top two slots of the Magic Square be A and B and the bottom be C and D, one can see that $A + B = A + C$. Thus $B = C$ and there is no solution (as repeated values are not permitted).

The third order is when solutions start to emerge. First we must define an isometry of solutions. If a solution is rotated by any multiple of $\frac{\pi}{2}$ radians, reflected, or any combination thereof a new solution is not created. Any solution that is a symmetric motion will be referred to as an isometry.

Theorem 1.3. Up to equivalence, the Lo Shu square is the unique magic square of order 3.

Proof. Using Theorem 1.2, we know the Magic Constant, S_3, must be 15. Note, the middle value is represented in four sums. Writing all combinations of three unique integers that sum to 15, it is evident that this happens with only one value: 5. Thus 5 must be the middle value. Similar logic can deduce the values for the edges and corners of the square. Each corner is represented in three sums and each edge in only two sums. □

Many mathematicians have created construction techniques for squares of order 3 and higher. WS Andrews has a construction technique for odd order squares and cites a construction technique for even order squares attributed to Euler in *Magic Squares and Cubes* [1, 2]. As far as uniqueness, it has been proven that there exist 880 Magic Squares of order 4, a number discovered by Bernard Frenicle de Bessy in 1693. For the fifth order, without excluding rotations and reflections, Richard Schroeppel wrote a program in 1973 that found $275,305,224$ Magic Squares of order 5. Schroeppel states the number of unique order 5 Magic Squares can be found simply by dividing this number by 4 to account for isometries, but many argue that this doesn't account for other types of beyond rotations and reflections. For example, in an order 5 solution, one can exchange the first and last columns to create another solution. Many consider this a new solution, while others do not–this is not considered an isometry by our previous definition. This argument of how to classify isomtetries and what exactly makes a new solution for higher orders has thus far prevented Magic Squares of orders larger than 5 to be classified. [3]

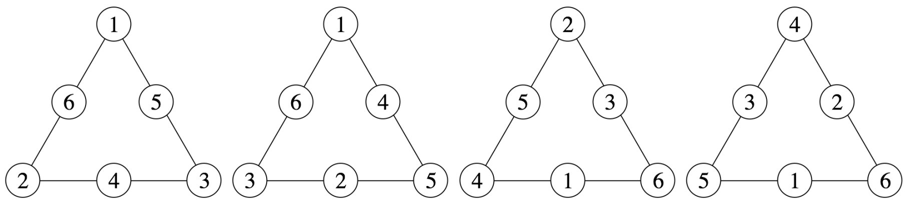
2 Magic Polygons

There is plenty of research on Magic Cubes and even Magic Hypercubes; thus in this section we will focus on polygons and solids that are not squares or their multi-dimensional equivalents.

Definition 2.1. A **Magic Polygon** is created by fixing a regular m-gon with a designated order, n. Each edge is divided into n slots so that the first and last slots contain the vertices. The $m(n-1)$ slots are to be filled uniquely with integers $1, 2, \ldots, m(n-1)$ so that the sum of each of the m edges is equivalent. This sum is referred to as the **magic constant**.

Below are examples of Magic Triangles of order 3, which we will soon prove are the only solutions to the order 3 puzzle

Figure 2: The four solutions to the third order Magic Triangle



Note, applying this definition to a square with order higher than 2 will not create the Magic Square discussed previously. It will create what is referred to as a Magic Perimeter Square as there are no slots in the middle of the polygon. Also notice, by definition a Magic Polygon of order 1 does not exist as the definition requires a slot on each vertex. Thus we must always have order 2 or greater.

Similar to the Magic Squares, we must address equivalences of Magic Polygons of order n. Any flip, rotation by $\frac{2\pi}{n}$, or reflection of a solution is isometric to the original hence do not yield a new solution.

Remark. There do not exist any Magic Polygons of the second order.

Proof. Consider a regular Magic Polygon of order 2. Let one vertex be represented as x_a. Let x_b be an adjacent vertex to x_a and x_c a vertex adjacent to x_b. Thus x_a and x_b create one edge and x_b and x_c create another. Thus $x_a + x_b = x_b + x_c$ implying x_a and x_c are equivalent. □

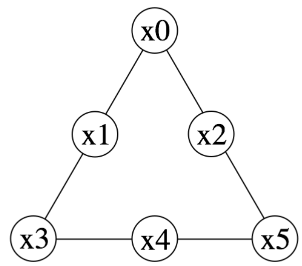
Notice this proof is a generalization of the proof of nonexistence of the order 2 Magic Square as the second order Magic Square fulfills the definition of a Magic Polygon.

2.1 Magic Triangles

Here we will explore the third order Magic Triangle. In a paper written in the early 1970s, Terell Trotter proved there exist only four Magic Triangles of order 3 by exhaustion [**?**]. Below we will present an alternative proof that utilizes matrix algebra to minimize the cases. For a good resource on linear algebra refer to Gilbert Strang's textboook, *Introduction to Linear Algebra*, or Professor Strang's online lectures from MIT [6].

Theorem 2.2. *Up to equivalence, there exist only four order* 3 *Magic Triangles.*

Figure 3: Variable view of the order 3 Magic Triangle



Proof. The slots of the Magic Triangle will be labeled from top to bottom and left to right as $x_0, x_1, \ldots, x_5$ (Figure 3). To begin we will split into two cases, $x_0 = 1$ and $x_1 = 1$ (the 1 occupying an edge or a vertex). Note that any Magic Polygon is a system of linear equations.

Case 1: $x_0 = 1$: We begin by creating a matrix and row reducing.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & S \\ 1 & 0 & 1 & 0 & 0 & 1 & S \\ 0 & 0 & 0 & 1 & 1 & 1 & S \\ 1 & 1 & 1 & 1 & 1 & 1 & 21 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2S+21 \\ 0 & 0 & 1 & 0 & 0 & 1 & S-1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3S-22 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2S+22 \end{bmatrix}$$

The first six rows represent the edges, the penultimate is the sum of the set, and the last is $x_0 = 1$.

From here we can see that $x_4 = -2S + 22$, thus it must be even. Ergo $x_4 = 2, 4$, or 6.

Subcase 1.2: $x_4 = 4$: Thus $x_0 = 1, x_4 = 4$, and $S = 9$. Using the reduced matrix, we have $x_3 + x_5 = 5$ and the possible pairs are $1,4$ and $2,3$ (order does not matter due to commutativity). The pair 1 and 4 will result in a repetition of 1. So we proceed with the pair $2,3$.

Subsubcase 1.2.1: $x_5 = 2$: Manipulating the input set and matrix derived equations results in the following solution.

$$x_0 = 1 \; x_1 = 5 \; x_2 = 6 \; x_3 = 3 \; x_4 = 4 \; x_5 = 2$$

Subsubcase 1.2.2: $x_5 = 3$: Similar logic as the previous case results in the following.

$$x_0 = 1 \; x_1 = 6 \; x_2 = 5 \; x_3 = 2 \; x_4 = 4 \; x_5 = 3$$

This is isometric to the previous subsubcase.

Subcase 1.3: $x_4 = 6$: Thus we have $x_0 = 1, x_4 = 6$, and $S = 8$. From the matrix we have $x_1 - x_5 = 5$ and the only possible pair from the input set that sums to 5 is $x_1 = 6$ and $x_5 = 1$. This is a repetition of 1, therefore there is no solution for this case.

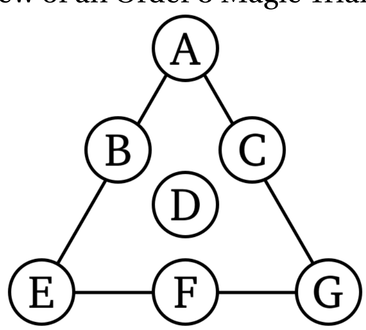
The logic follows for the last subcase, $x_4 = 2$ and throughout second case, $x_1 = 1$ (the matrix is presented below for verification). This results in the four, unique solutions. (see Figure 2).

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & S \\ 1 & 0 & 1 & 0 & 0 & 1 & S \\ 0 & 0 & 0 & 1 & 1 & 1 & S \\ 1 & 1 & 1 & 1 & 1 & 1 & 21 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 2S-21 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -3S+41 \\ 0 & 0 & 0 & 1 & 1 & 0 & -S+20 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2S-20 \end{bmatrix}.$$

□

Now, if we explore the triangle we can see why the Magic Polygon is designed with slots only on the edges of the polygon and not within the shape. Consider an order 3 Magic Triangle with a slot in the middle and an input set of $\{1,2,3,4,5,6,7\}$ (Figure 4). There is no solution. This can be proven with similarly logic to that of the combinatorics proof of Theorem 1.3. Let Figure 4 represent such a triangle.

Figure 4: Variable view of an Order 3 Magic Triangle with Middle Slot



Proof. For this triangle to be a solution, each combination of 3 slots that can be connected with a line are to be equivalent. So we can deduce $A+D+F = B+D+G = E+D+C$, or $A+F = B+G = E+C$. Now consider all combinations of 2 numbers from our input set, $\{1,2,3,4,5,6,7\}$. We need a number so that there are at least 3 ways to sum 2 elements of our input set to yield this number. The only possibilities are $7, 8$ and 9.

Case 1: $A+D=7$: Thus our pairs (A,D), (B,G), and (E,C) are either $(1,6)$, $(2,5)$ or $(3,4)$. Thus our sum must be $1+6+7 = 14$. Without loss of generality, let $A = 1$. Then $F = 6$ and $E+G$ must equal 8. Again, without loss of generality, let $E = 5$ and $G = 3$. Thus $A+B+E$ will equal 10 at most.

Now consider $A = 6$. Then $F = 1$ and $E+G$ must equal 12 which is not possible from our remaining values.

Case two would be $A + D = 8$ thus our triangle must have sum 12. Case 3 would be $A + D = 9$ with a sum of 10. For each of these cases we reach the same contradiction with similar logic. $\square$

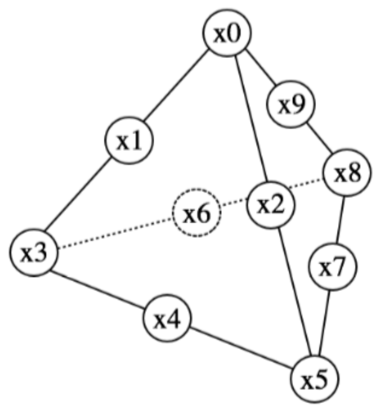
Higher order Magic Triangles are explored in some detail in *Normal Magic Triangle of Order n* published in the Journal of Recreational Mathematics by Trotter in 1972 [9].

2.2 Magic Tetrahedron

Moving to higher dimensions, this section addresses the third order Magic Tetrahedron. We can create higher dimensional equivalents to the Magic Polygon using regular polyhedra. To begin we consider a regular tetrahedron. To create a Magic Tetrehdron we must extend of definition of Magic Polygon by simply distributing the slots across all edges of the solid.

We begin once more, with order 3 (Figure 5).

Figure 5: Variable view of a Magic Tetrahedra



Theorem 2.3. There is no solution to a third order Magic Tetrahedron.

Proof. We construct the puzzle as depicted in Figure 5. Once again, we will split into cases. Without loss of generlaity, $x_0 = 1$ or $x_1 = 1$.

Case 1: $x_0 = 1$: Now, we create a matrix and row reduce.

$$\left[\begin{array}{cccccccccccc} 1&1&0&1&0&0&0&0&0&0&S \\ 1&0&1&0&0&1&0&0&0&0&S \\ 0&0&0&1&1&1&0&0&0&0&S \\ 0&0&0&1&0&0&1&0&1&0&S \\ 1&0&0&0&0&0&0&0&1&1&S \\ 0&0&0&0&0&1&0&1&1&0&S \\ 1&0&0&0&0&0&0&0&0&0&1 \\ 1&1&1&1&1&1&1&1&1&1&45 \end{array}\right] \sim \left[\begin{array}{ccccccccccc} 1&0&0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0&0&\frac{45-2S}{2} \\ 0&0&1&0&0&0&0&-1&0&1&S-2 \\ 0&0&0&1&0&0&0&-1&0&0&\frac{4S-47}{2} \\ 0&0&0&0&1&0&0&0&0&1&\frac{45-2S}{2} \\ 0&0&0&0&0&1&0&1&0&-1&1 \\ 0&0&0&0&0&0&1&1&0&-1&\frac{49-4S}{2} \\ 0&0&0&0&0&0&0&0&1&1&S-1 \end{array}\right]$$

Thus we have $x_1 + x_7 = \frac{45-2S}{2}$. Note x_1 and x_7 must be positive integers, the sum of which will be a whole number. Ergo we have a whole number equal to an odd number divided by 2. Thus a contradiction and there does not exist a solution to this case.

Case 2: $x_1 = 1$: Applying the same logic yields the a similar contradiction.

Hence there does not exist a solution to the third order Magic Tetrahedron. □

The above result was found independently, but upon further research another proof was discovered in the Journal of Recreational Mathematics by Charles W. Trigg and is more general than the above [8]. Trigg uses a variable to represent the smallest number and then adds to it to create the input set. The following is his proof from 1971 [8].

Proof. Let a be the smallest number in the input set of a Magic Tetrahedron. Thus the sum of the input set is $a + (a+1) + (a+2) + \ldots + (a+9) = 5(2a+9)$. Let $\sum V$ and $\sum M$ represent the sum of the verticies and the sum of the midpoints respectively, ergo:

$$\sum V + \sum M = 5(2a+9) \tag{2}$$

As well notice if you add up all the edges, the vertices will be represented in 3 different edges. So if we add all the edges up you will get the following equation where S is the Magic Constant.

$$3\sum V + \sum M = 6S \tag{3}$$

Now we can simplify Equation 3 with Equation 2.

$$\begin{aligned} 2\sum V + \sum V + \sum M = 2\sum V + 5(2a+9) &= 6S \quad \text{(Equation 2)} \\ \Rightarrow 2\sum V &= 6S - 5(2a+9) \end{aligned}$$

Notice $2\sum V$ is even and $6S - 5(2a+9)$ is odd. Therefore we have a contradiction.

□

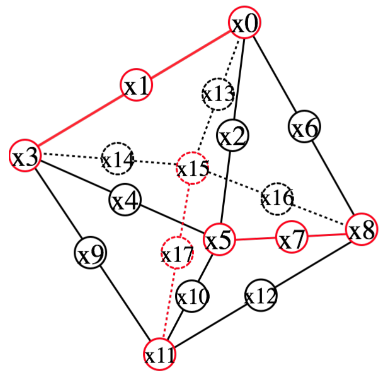
There is not much research on higher order Magic Tetrahedra. However, in 1971 C. W. Trigg published some results on antimagic Tetrahedra, which using the same construction as Magic Tetrahedra but instead of each edge summning to the same value, each edge must sum to consecutive values [8].

2.3 Magic Octahedron

In attempt at more variables and permutations of number we decided to extend the puzzle to the octahderon instead of exploring higher order Magic Tetrahedra. Once again, we attempted the third order.

Our Magic Octahedron is created similarly to the tetrahedron. Each edge is split into the n-slots of our designated order with the desire that each edge sum to the same value.

Figure 6: Variable View of the Magic Octahedron



Theorem 2.4. There does not exist a solution to the Magic Octahedron of order three.

To prove this we will use the 3SUM problem from combinatorics. The idea is to find a certain number of unique combinations of three unique integers that sum to the same value. We used computer programming to find solutions as needed [4]. As every Magic Octahedron has three distinct edges. Figure 6 is an example of such as the colored edges are distinct meaning they do not share a common slot.

Proof. We begin similarly by dividing into two cases once again and without loss of generality, we let $x_0 = 1$ or $x_1 = 1$. We will just look at the first case as the second case applies the same logic.

Case 1: $x_0 = 1$: Referencing figure 6 we observe that

$$x_0 + x_{13} + x_{15} = \mathrm{S}.$$

Thus,

$$x_{13} + x_{15} = \mathrm{S} - 1.$$

From this we can create a range of values for S using the input set. Thus,

$$2 + 3 = 5 \leq x_{13} + x_{15} \leq 17 + 18 = 35.$$

This gives us, $6 \leq \mathrm{S} \leq 36$. Using the program we created there does not exist three sets of three integers that sum to any value in this range. □

The code for both this proof and other programs used throughout are linked in Section 3.1.

3 Modulo Magic

In this section let us redefine a Magic Square using Modular arithmetic and explore how this changes the puzzle.

Definition 3.1. A **Modulo Magic Square** is a Magic Square using the integers modulo n^2, denoted here as $\mathbb{Z}_{n^2}$, as the input set where n is the order of the square.

To parallel the Magic square, our input set will be from [1] to $[n^2]$, rather than expressing [0] to $[n^2-1]$ as convention.

3.1 A Brief Summary of the Code

To test whether Modulo Magic Squares exist–beyond the fact that regular Magic Squares are inherently Modulo Magic–we created code utilizing backtracking. The main concept of the code is to test permutations by comparing the row, column and diagonal sums to one another to find Modulo Magic Squares.

The program was done in two ways, once using two-dimensional vectors and another using one-dimensional vectors. The original program used two-dimensional vectors as they are more easily visualized as matrices, but as the order increased the storage became much too large and slowed the program. Thus, the one-dimensional equivalent was created. The logic is the same with a few minor coding adjustments.

Both of the programs operate by recieving a user designated order and calculating all the possibilities for the magic constant (how is to be discussed in the next section). The program starts by creating permutations of the input set testing each row, column and diagonal as it is filled. If one of these rows, columns or diagonals does not sum as desired, backtracking is employed. The backtracking allows for quicker processing as we will not have to test every possible permutation of the square. More details and comments on the code are presented here:
https://cs.maryvillecollege.edu/~nvelez/SeniorStudy/code.html.

3.2 Results

To decide if a Modulo Magic Square exists (modulo n^2), the first question becomes, what is the magic constant now? Over the integers, the solution to our equation for the magic constant is uniquely determined. However, in $\mathbb{Z}_{n^2}$, there are many more possibilities. The magic constant must satisfy the following equation where S represents the magic constant as calculated by the Equation 1 (the equation for the magic constant of a regular Magic Square).

$$nS - \frac{n^2(n^2+1)}{2} = 0 \tag{4}$$

Theorem 3.2. There are n possible magic constants for an order n Modulo Magic Square.

Proof. We must consider two cases: when n is odd, and n is even.

Case (1): n is odd:

$$\begin{aligned}\left[nS - \frac{n^2(n^2+1)}{2}\right] &= [0]\\ [n][S]-[n^2]\left[\frac{n^2+1}{2}\right] &= [0]\\ [n][S]-[0]\left[\frac{n^2+1}{2}\right] &= [0]\\ [n][S] = [0] &= [n^2]\end{aligned}$$

Thus we need $[nS] = [0] = [n^2]$ for the equation to hold true. This will hold true for any multiple of n. There are n multiples of n in $\mathbb{Z}_{n^2}$. Therefore there are n distinct solutions, $\{0n, 1n, \ldots, (n-1)n\}$.

Case (2): n is even:

$$\begin{aligned}\left[nS - \frac{n^2(n^2+1)}{2}\right] &= [0]\\ [n][S]-\left[\frac{n^2}{2}\right]\left[n^2+1\right] &= [0]\\ [n][S] &= \left[\frac{n^2}{2}\right]\end{aligned}$$

There are n solutions to this as well, $\{\frac{n}{2}, \frac{n}{2}+n, \ldots, \frac{n}{2}+(n-1)n\}$.

Therefore there exist n distinct values for the magic constant when n is odd and n distinct values when n is even. □

Given this we discovered constructions of both Modulo and regular Magic Squares using modular arithmetic.

Theorem 3.3. Adding some integer from the set $\{1, 2, \ldots, n^2-1\}$, say $[a]$, to every slot of a Modulo Magic Square will create another Modulo Magic Square with the magic constant $[an]$ greater than the original square.

Proof. Say we add the integer $[a]$ to all slots of an order n Modulo Magic Squares, then the largest $[a]$ values will become the smallest $[a]$ values (modulo n^2) due to the cycling of modular arithmetic. This will keep us in the input set, $\mathbb{Z}_{n^2}$ and increase the sum by $[a]$, n times for the n slots. Thus we have an order n Modulo Magic Square with the magic constant being $[an]$ greater than the original square. □

If we combine Theorems 3.2 and 3.3 we see given a solution to Modulo Magic Square using the smallest magic constant we can generate a solution to each possible magic constant. This is done by adding n to each slot for a Modulo Magic Square of odd order and adding $\frac{n}{2}$ to a Modulo Magic Square of even order. As well, by taking advantage of the cycling of modular arithmetic we saw in the last theorem, we can use regular Magic Squares of a certain setup to create more regular Magic Squares.

Theorem 3.4. Given an order n regular Magic Square with the n largest values represented such that one is in each row, column and main diagonal, if we add n to each slot and take the sum modulo n^2, we will create another regular Magic Square.

Proof. When adding n to each slot of a $n \times n$ Magic Square, we are adding n^2 to each row, column and diagonal and thus to the magic constant. For the n largest numbers of the set, adding n will cause them to cycle back to the n smallest numbers when applying modular arithmetic and operating in $\mathbb{Z}_{n^2}$. Thus the sums of the row, column and/or diagonal that contain this slot will have increased n^2 by the initial addition, but also decreased by n^2. If each row, column and diagonal contains one and only one of these cycling slots, then their respective sums will increase by n^2 and decrease by n^2 leaving all sums the same. Thus we have generated a new Magic Square. $\square$

An example of this theorem on an order five Magic Square is presented below. The original order 5 square is presented first in the table. The square is outlined with the sums. Each double line indicates the beginning of the next square.

Table 1: Order 5 Modular Construction

+0	25	13	1	19	7	65
	16	9	22	15	3	65
	12	5	18	6	24	65
	8	21	14	2	20	65
	4	17	10	23	11	65
65	65	65	65	65	65	65
+5	30	18	6	24	12	90
	21	14	27	20	8	90
	17	10	23	11	29	90
	13	26	19	7	25	90
	9	22	15	28	16	90
90	90	90	90	90	90	90
mod25	5	18	6	24	12	65
	21	14	2	20	8	65
	17	10	23	11	4	65
	13	1	19	7	25	65
	9	22	15	3	16	65
65	65	65	65	65	65	65

4 Future Work

Due to the lack of research on Magic Polygons in three-dimensions, there is a lot yet to be explored. One idea would to try Modulo Magic Polygons and we went as far as to create a program to check if the third order Modulo Magic Tetrahedron and Modulo Magic Octahedron existed (also at the url in Section 3.1). The program returns “no solution” to both of these puzzles. We hope to be able to prove why and to find more Magic Polygons or Modulo Magic Polygons. As well, polygons are mostly researched as regular polygons or as platonic solids. It could potentially lead to more results and interesting trends if Archimedean solids were used rather than just platonic solids. How would the difference of faces and their respective number of slots affect the possibilities for solutions? How would this compare to the solid of these individual polygons in two dimensions? We could even research these polygons in higher dimensions. There has been a more recent shift into researching fourth dimensional Magic Hypercubes, so this idea can be extended and explored in relation to other polygons.

The Modulo Magic Square has a plethora of research opportunities. There are trends and connections to Abstract Algebra to be made–as well as opportunities to find more constructions. Instead of order three Magic Squares being defined in $\mathbb{Z}_{n^2}$, are the solutions of other groups of order n^2? One idea we considered more recently would be using the extended Quaternions–a group of order 16–to solve the 4×4 Magic Square. Solutions to Magic Squares using finite groups of order q^2 with q being a product of a prime as an input set could also be explored.

As the history of this puzzle has shown, when one construction becomes fully understood there are numerous ways to redefine the constraints, the shape, the layout of the slots, etc. to find more opportunities. This paper provides research in two different ideas, but from these ideas spark many more.

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