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On Differential Equations Associated with Perturbations of Orthogonal Polynomials on the Unit Circle

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Abstract: In this contribution, we propose an algorithm to compute holonomic second-order differential equations satisfied by some families of orthogonal polynomials. Such algorithm is based in three properties that orthogonal polynomials satisfy: a recurrence relation, a structure formula, and a connection formula. This approach is used to obtain second-order differential equations whose solutions are orthogonal polynomials associated with some spectral transformations of a measure on the unit circle, as well as orthogonal polynomials associated with coherent pairs of measures on the unit circle.

Keywords: Orthogonal polynomials on the unit circle; holonomic differential equations; spectral transformations; coherent pairs of measures

MSC: 42C05; 33C47

1. Introduction

A sequence of polynomials $\{p_n\}_{n\geq 0}$ is said to be orthogonal if

$$\int_E p_n(x)p_m(x)d\mu(x) = K_n\delta_{m,n}, \quad K_n > 0, \quad m,n \geq 0,$$

where μ is a nontrivial probability measure with support on some interval $E \subset \mathbb{R}$ and $\delta_{m,n}$ is the Kronecker delta. The so-called classical sequences of orthogonal polynomials (Bessel, Hermite, Jacobi, and Laguerre, with the first family corresponding to an orthogonality measure that is not positive) constitute the most broadly applied and thoroughly studied systems of orthogonal polynomials.

The classical families satisfy many properties and were first characterized by E. Routh [1]. Later, S. Bochner [2] focused on the second-order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y + \lambda y = 0, \tag{1}$$

where $a_0(x), a_1(x)$, and $a_2(x)$ are polynomials, and determined all of its possible polynomial solutions. S. Bochner arrived to the conclusion that the polynomials $a_0(x), a_1(x)$ and $a_2(x)$ must have degrees at most 0, 1, and 2, respectively. He also found that the Hermite, Jacobi, and Laguerre families are the only polynomial solutions for (1) with a corresponding measure that is positive. Moreover, such sequences satisfy the so-called Hahn's property: their derivatives also constitute an orthogonal family. It is interesting that some of these families have been studied in the context of symmetry Lie algebras. For instance, in [3–6], the authors studied how the differential equations satisfied by the classical families can be obtained from the second-order Casimir elements of the corresponding symmetry algebra and of some of its subalgebras. Furthermore, some operators defined on Lie algebras are used in [7] to obtain differential properties of some special functions, including the Jacobi polynomials.

On the other hand, in the last decades, some canonical examples of spectral transformations of orthogonality measures have been studied in the literature: the Christoffel transformation, consisting in a polynomial modification of the measure; the Uvarov transformation, defined by the addition of a Dirac's delta measure; and the Geronimus transformation, where the orthogonality measure is divided by a polynomial, and a Dirac's delta measure is added. For instance, in [8], the author is interested in the relations between the associated Stieltjes functions. Furthermore, he shows that all linear spectral transformations of Stieltjes functions can be obtained as a finite product of the three canonical transformations mentioned above. On the other hand, in [9], the authors consider the relations between these perturbations and LU factorizations of the corresponding Jacobi matrices, in the more general framework of orthogonality with respect to linear functionals. The study of differential equations of higher order satisfied by orthogonal polynomials was initiated by H. L. Krall in [10] and later on A. M. Krall (see [11]) showed that the sequences orthogonal with respect to an Uvarov perturbation of the classical families satisfy a fourth order differential equation. The orthogonal families associated with these perturbations are often called *classical-type* orthogonal polynomials. It is important to notice that they are not longer classical, and the polynomial coefficients in the corresponding differential equations (which are called *holonomic* equations) can even depend on the degree n (see [12]), in sharp contrast with the classical case. More recently, the study of holonomic equations for classical-type orthogonal polynomials, as well as their application for developing electrostatic models for the corresponding zeros, has been developed in [13–17], among many others.

The study of differential equations whose solutions are orthogonal polynomials on the unit circle (OPUC) has not drawn equal attention as in the real case. One possible reason for this is that the only classical family (i.e., the only sequence of orthogonal polynomials whose derivatives are also orthogonal) is $\{z^n\}_{n\geq 0}$, orthogonal with respect to the Lebesgue measure (see [18]). Some differential properties for OPUC, however, are known in the literature. For instance, in [19,20], a differential relation for OPUC is obtained under certain conditions on the orthogonality measure, and it is later used to obtain a second-order differential equation with degree-depending rational functions as coefficients. Differential properties for semi-classical orthogonal polynomials have also been analyzed in the literature (see [21] and references therein). More recently, some examples of OPUC satisfying second-order differential equations with varying polynomial coefficients were studied in [22], and differential equations for para-orthogonal polynomials on the unit circle, along with an electrostatic interpretation of their zeros, were obtained in [23].

The study of differential equations for spectral transformations of OPUC remains an open problem. This contribution is oriented in that direction. The manuscript is organized as follows. In Section 2, we present an algorithmic procedure to obtain second-order differential equations for arbitrary sequences of polynomials, by using an approach considered in [19] that utilizes three main ingredientes: a recurrence relation, a structure relation, and a connection formula. Later on, this procedure is applied to obtain second-order differential equations whose solutions are orthogonal polynomials associated with spectral transformations of OPUC in Section 3, and also for orthogonal polynomials associated with coherent pairs of measures supported on the unit circle in Section 4.

2. A General Approach to Obtain Second Order Differential Equations

Assume $\{\Phi_n\}_{n\geqslant 0}$ is a sequence of polynomials satisfying the structure relation

$$[\Phi_n(z)]' = a(z;n)\Phi_n(z) + b(z;n)\Phi_{n-1}(z), \quad n \geq 1, \tag{2}$$

and the three term recurrence formula

$$\Phi_{n+1}(z) = \beta(z;n)\Phi_n(z) + \gamma(z;n)\Phi_{n-1}(z), \quad n \geq 1, \tag{3}$$

where $a(z;n)$, $b(z;n)$, $\beta(z;n)$, and $\gamma(z;n)$ are some rational functions. In this section, we present a model to obtain (holonomic) second-order differential equations satisfied by a sequence of polynomials constructed in terms of $\{\Phi_n\}_{n\geqslant 0}$. We will denote such a sequence by $\{\Psi_n\}_{n\geqslant 0}$, and it can be constructed via

$$\Psi_n(z) = A_1(z;n)\Phi_n(z) + B_1(z;n)\Phi_{n-1}(z), \quad n \geq 0, \tag{4}$$

where $A_1(z;n)$ and $B_1(z;n)$ are (in general) rational functions in z, and $\Phi_{-1} \equiv 0$. It is important to notice that such model has been already used in the case when the sequence $\{\Phi_n\}_{n\geqslant 0}$ is orthogonal with respect to a positive measure supported on the real line, as it is well known that it satisfies relations of the form (2) and (3). Typically, $\{\Psi_n\}_{n\geqslant 0}$ is also an orthogonal sequence, associated with some perturbation of the orthogonality measure. Indeed, this approach has been used to obtain electrostatic interpretations of the zeros of polynomials orthogonal with respect to Sobolev-type perturbations of classical and semiclassical orthogonal polynomials (see, for instance, [13–17]). We point out that, although we are interested in the cases where the constructed sequence is again orthogonal with respect to some measure, this approach remains valid even if the constructed sequence is not orthogonal.

The three main ingredients involved are the structure relation (2), the recurrence relation (3), and the connection formula (4). Without loss of generality, both sequences $\{\Phi_n\}_{n\geq 0}$ and $\{\Psi_n\}_{n\geq 0}$ can be assumed to be monic. We begin by stating some lemmas that will be used to obtain the second-order ODE, satisfied by the sequence $\{\Psi_n\}_{n\geq 0}$. We first obtain $[\Psi_n(z)]'$, the derivative of the perturbed polynomials $\Psi_n(z)$ with respect to the variable z, in terms of two consecutive monic polynomials $\Phi_n(z)$ and $\Phi_{n-1}(z)$ from $\{\Phi_n\}_{n\geq 0}$.

Lemma 1. *For the monic sequences $\{\Psi_n\}_{n\geq 0}$ and $\{\Phi_n\}_{n\geq 0}$ we have*

$$[\Psi_n(z)]' = C_1(z;n)\Phi_n(z) + D_1(z;n)\Phi_{n-1}(z), \tag{5}$$

where

$$\begin{aligned} C_1(z;n) &= A_1'(z;n) + A_1(z;n)a(z;n) + B_1(z;n)\frac{b(z;n-1)}{\gamma(z;n-1)}, \\ D_1(z;n) &= B_1'(z;n) + A_1(z;n)b(z;n) + B_1(z;n)\left(a(z;n-1) - \frac{\beta(z;n-1)}{\gamma(z;n-1)}\right). \end{aligned} \tag{6}$$

Proof. First, we shift the index in (2) as $n \to n-1$,

$$[\Phi_{n-1}(z)]' = a(z;n-1)\Phi_{n-1}(z) + b(z;n-1)\Phi_{n-2}(z)$$

and we use (3) to obtain $[\Phi_{n-1}(z)]'$ in terms of $\Phi_n(z)$ and $\Phi_{n-1}(z)$ as follows,

$$[\Phi_{n-1}(z)]' = \frac{b(z;n-1)}{\gamma(z;n-1)}\Phi_n(z) + \left(a(z;n-1) - \frac{\beta(z;n-1)}{\gamma(z;n-1)}\right)\Phi_{n-1}(z). \quad (7)$$

Next, taking derivatives in both sides of (4), we obtain

$$\begin{aligned}[\Psi_n(z)]' &= A_1'(z;n)\Phi_n(z) + A_1(z;n)[\Phi_n(z)]' \\ &\quad + B_1'(z;n)\Phi_{n-1}(z) + B_1(z;n)[\Phi_{n-1}(z)]'.\end{aligned}$$

Finally, we replace (2) and (7) into the above formula, and after some easy computations, the Lemma follows. □

Next, we express $\Psi_{n-1}(z)$ and $[\Psi_{n-1}(z)]'$ in terms of $\Phi_n(z)$ and $\Phi_{n-1}(z)$ as well.

Lemma 2. *The sequences* $\{\Psi_n\}_{n\geq 0}$ *and* $\{\Phi_n\}_{n\geq 0}$ *satisfy*

$$\Psi_{n-1}(z) = A_2(z;n)\Phi_n(z) + B_2(z;n)\Phi_{n-1}(z), \quad (8)$$
$$[\Psi_{n-1}(z)]' = C_2(z;n)\Phi_n(z) + D_2(z;n)\Phi_{n-1}(z), \quad (9)$$

where

$$\begin{aligned} A_2(z;n) &= \frac{B_1(z;n-1)}{\gamma(z;n-1)}, \\ B_2(z;n) &= A_1(z;n-1) - B_1(z;n-1)\frac{\beta(z;n-1)}{\gamma(z;n-1)}, \\ C_2(z;n) &= \frac{D_1(z;n-1)}{\gamma(z;n-1)}, \\ D_2(z;n) &= C_1(z;n-1) - D_1(z;n-1)\frac{\beta(z;n-1)}{\gamma(z;n-1)}, \end{aligned} \quad (10)$$

where $C_1(z;n-1)$ *and* $D_1(z;n-1)$ *are given in (6).*

Proof. (8) follows at once from (4) and (3). For (9), we use (5) and (3). □

Remark 1. *Note that the coefficients in (6) and (10) depend only on the following known quantities; the coefficients* $A_1(z;n)$, $B_1(z;n)$ *in (4), the coefficients* $a(z;n)$, $b(z;n)$ *in (2) and* $\beta(z;n)$, *and* $\gamma(z;n)$ *of the three-term recurrence formula (3) satisfied by* $\{\Phi_n\}_{n\geq 0}$.

Notice that (4) and (8) define a linear system. Its solution implies the following Lemma.

Lemma 3.

$$\Phi_n(z) = \frac{B_2(z;n)}{\Lambda(z;n)}\Psi_n(z) - \frac{B_1(z;n)}{\Lambda(z;n)}\Psi_{n-1}(z), \quad (11)$$
$$\Phi_{n-1}(z) = \frac{-A_2(z;n)}{\Lambda(z;n)}\Psi_n(z) + \frac{A_1(z;n)}{\Lambda(z;n)}\Psi_{n-1}(z), \quad (12)$$

where $\Lambda(z;n)$ *is the determinant*

$$\Lambda(z;n) = A_1(z;n)B_2(z;n) - A_2(z;n)B_1(z;n).$$

Notice that the previous determinant $\Lambda(z;n)$ may not be well defined (it could also be zero), so the proposed algorithm fails in these cases. In what follows, we assume that all the expressions are well defined.

Theorem 1 (Ladder operators)**.** *Define* $\mathfrak{b}_n$ *and* $\mathfrak{b}_n^{\dagger}$ *by*

$$\begin{aligned} \mathfrak{b}_n &= \Xi(z;n,2)I - D_z, \\ \mathfrak{b}_n^{\dagger} &= \Theta(z;n,1)I + D_z, \end{aligned}$$

where I *and* D_z *denote the identity and z-derivative operators, respectively. Then,*

$$\begin{aligned} \mathfrak{b}_n[\Psi_n(z)] &= \Xi(z;n,1)\Psi_{n-1}(z), \\ \mathfrak{b}_n^{\dagger}[\Psi_{n-1}(z)] &= \Theta(z;n,2)\Psi_n(z), \end{aligned}$$

with $\Xi(z;n,k)$ *and* $\Theta(z;n,k)$*,* $k = 1,2$*, given by*

$$\Xi(z;n,k) = \frac{1}{\Lambda(z;n)} \begin{vmatrix} C_1(z;n) & A_k(z;n) \\ D_1(z;n) & B_k(z;n) \end{vmatrix}, \tag{13}$$

$$\Theta(z;n,k) = \frac{1}{\Lambda(z;n)} \begin{vmatrix} C_2(z;n) & A_k(z;n) \\ D_2(z;n) & B_k(z;n) \end{vmatrix}, \tag{14}$$

Proof. Replacing (11) and (12) into (5) and (9), respectively, we get the ladder equations

$$\begin{aligned} [\Psi_n(z)]' = & \left[\frac{C_1(z;n)B_2(z;n)}{\Lambda(z;n)} - \frac{D_1(z;n)A_2(z;n)}{\Lambda(z;n)}\right]\Psi_n(z) \\ & + \left[\frac{A_1(z;n)D_1(z;n)}{\Lambda(z;n)} - \frac{C_1(z;n)B_1(z;n)}{\Lambda(z;n)}\right]\Psi_{n-1}(z), \end{aligned}$$

$$\begin{aligned} [\Psi_{n-1}(z)]' = & \left[\frac{C_2(z;n)B_2(z;n)}{\Lambda(z;n)} - \frac{A_2(z;n)D_2(z;n)}{\Lambda(z;n)}\right]\Psi_n(z) \\ & + \left[\frac{A_1(z;n)D_2(z;n)}{\Lambda(z;n)} - \frac{C_2(z;n)B_1(z;n)}{\Lambda(z;n)}\right]\Psi_{n-1}(z). \end{aligned}$$

Therefore, we have

$$\Xi(z;n,2)\Psi_n(z) - [\Psi_n(z)]' = \Xi(z;n,1)\Psi_{n-1}(z), \tag{15}$$

$$\Theta(z;n,1)\Psi_{n-1}(z) + [\Psi_{n-1}(z)]' = \Theta(z;n,2)\Psi_n(z), \tag{16}$$

and the result follows. □

Theorem 2 (Holonomic equation)**.** *The sequence* $\{\Psi_n\}_{n\geq 0}$ *satisfies the second-order differential equation*

$$[\Psi_n(z)]'' + \mathcal{R}(z;n)[\Psi_n(z)]' + \mathcal{S}(z;n)\Psi_n(z) = 0, \tag{17}$$

where

$$\begin{aligned}\mathcal{R}(z;n) &= \Theta(z;n,1) - \Xi(z;n,2) - \frac{[\Xi(z;n,1)]'}{\Xi(z;n,1)},\\ \mathcal{S}(z;n) &= \Xi(z;n,2)\left[\frac{[\Xi(z;n,1)]'}{\Xi(z;n,1)} - \Theta(z;n,1)\right] - [\Xi(z;n,2)]'.\end{aligned}$$

Proof. The proof is a consequence of the previous theorem. A well known procedure (see [19]) is to apply $\mathfrak{b}_n^{\dagger}$ to the equation satisfied by $\mathfrak{b}_n$, i.e.,

$$\mathfrak{b}_n^{\dagger}\left[\frac{1}{\Xi(z;n,1)}\mathfrak{b}_n[\Psi_n(z)]\right] = \mathfrak{b}_n^{\dagger}[\Psi_{n-1}(z)] = \Theta(z;n,2)\Psi_n(z).$$

Next, using the expression for $\mathfrak{b}_n^{\dagger}$, the left hand side becomes

$$\begin{aligned}\mathfrak{b}_n^{\dagger}\left[\frac{1}{\Xi(z;n,1)}\mathfrak{b}_n[\Psi_n(z)]\right] &= \frac{\Theta(z;n,1)}{\Xi(z;n,1)}\mathfrak{b}_n[\Psi_n(z)] + D_z\left[\frac{1}{\Xi(z;n,1)}\mathfrak{b}_n[\Psi_n(z)]\right]\\ &= \frac{\Theta(z;n,1)\Xi(z;n,2)}{\Xi(z;n,1)}\Psi_n(z) - \frac{\Theta(z;n,1)}{\Xi(z;n,1)}[\Psi_n(z)]'\\ &\quad -\frac{[\Xi(z;n,1)]'\Xi(z;n,2)}{[\Xi(z;n,1)]^2}\Psi_n(z) + \frac{[\Xi(z;n,1)]'}{[\Xi(z;n,1)]^2}[\Psi_n(z)]'\\ &\quad +\frac{[\Xi(z;n,2)]'}{\Xi(z;n,1)}\Psi_n(z) + \frac{\Xi(z;n,2)}{\Xi(z;n,1)}[\Psi_n(z)]' - \frac{1}{\Xi(z;n,1)}[\Psi_n(z)]''\\ &= \frac{-1}{\Xi(z;n,1)}[\Psi_n(z)]''\\ &\quad + \left[\frac{[\Xi(z;n,1)]'}{[\Xi(z;n,1)]^2} + \frac{\Xi(z;n,2)}{\Xi(z;n,1)} - \frac{\Theta(z;n,1)}{\Xi(z;n,1)}\right][\Psi_n(z)]'\\ &\quad + \left[\frac{\Xi(z;n,2)\Theta(z;n,1)}{\Xi(z;n,1)} - \frac{\Xi(z;n,2)[\Xi(z;n,1)]'}{[\Xi(z;n,1)]^2} + \frac{[\Xi(z;n,2)]'}{\Xi(z;n,1)}\right]\Psi_n(z),\end{aligned}$$

which is a second-order differential equation for Ψ_n. After some easy computations, the Theorem follows. □

An Alternative Expression for the Differential Equation

The algorithmic approach considered before allows us to consider some interesting facts about the way in which the operators and differential equations are obtained for a particular sequence. First, notice that the resulting differential equation depends only on the functions $\beta(z;n)$ and $\gamma(z;n)$ in (3); $a(z;n)$ and $b(z;n)$ in (2), which contain information about the sequence $\{\Phi_n\}_{n\geq 0}$; and $A_1(z;n)$ and $B_1(z;n)$ in (4), which contain information concerning the relation between $\{\Psi_n\}_{n\geq 0}$ and $\{\Phi_n\}_{n\geq 0}$.

The wide generality of the proposed algebraic process to obtain the second-order differential equations for the sequence $\{\Psi_n(z)\}_{n\geq 0}$ can be used to obtain the second-order differential equations satisfied by the *original* sequence $\{\Phi_n(z)\}_{n\geq 0}$. Indeed, it is sufficient to take $A_1(z;n) = 1$ and $B_1(z;n) = 0$ in (4). In this particular case, we see that (5) trivially reduces to (2), and therefore $C_1(z;n) = a(z;n)$ and

$D_1(z;n) = b(z;n)$. Moreover, (8) forces us to take $A_2(z;n) = 0$ and $B_2(z;n) = 1$. Accordingly, we find that (9) has now simpler coefficients

$$\begin{aligned} C_2(z;n) &= \frac{b(z;n-1)}{\gamma(z;n-1)}, \\ D_2(z;n) &= a(z;n-1) - b(z;n-1)\frac{\beta(z;n-1)}{\gamma(z;n-1)}. \end{aligned}$$

Under these hypothesis, from Lemma 3 we have $\Lambda(z;n) = 1$, and therefore, for the original sequence $\{\Phi_n(z)\}_{n\geq 0}$, we find

$$\begin{aligned} \mathcal{R}(z;n) &= -a(z;n-1) - a(z;n) + b(z;n-1)\frac{\beta(z;n-1)}{\gamma(z;n-1)} - \frac{b'(z;n)}{b_n(z;n)}, \\ \mathcal{S}(z;n) &= a(z;n)\left[\frac{b'(z;n)}{b(z;n)} + a(z;n-1)\right] \\ &\quad - \frac{b(z;n-1)\,[a(z;n)\beta(z;n-1) + b(z;n)]}{\gamma(z;n-1)} - a'(z;n). \end{aligned}$$

This reasoning leads us to the following conclusion. As the four expressions $a(z;n)$, $b(z;n)$, $\beta(z;n)$, and $\gamma(z;n)$ depend only on the sequence $\{\Phi_n\}_{n\geq 0}$, the use of the corresponding coefficients $\tilde{a}(z;n)$, $\tilde{b}(z;n)$, $\tilde{\beta}(z;n)$, and $\tilde{\gamma}(z;n)$, associated with the sequence $\{\Psi_n\}_{n\geq 0}$ (when they exist), must lead us to an alternative way to obtain $\mathcal{R}(z;n)$ and $\mathcal{S}(z;n)$ in Theorem 2. This also provides a compact form of the coefficients in the corresponding second-order differential Equation (17). As a consequence, we have the following result.

Proposition 1 (Alternative expression for $\mathcal{R}(z;n)$ and $\mathcal{S}(z;n)$)**.** *Assume that the sequence $\{\Psi_n\}_{n\geq 0}$ satisfies*

$$\begin{aligned} [\Psi_n(z)]' &= \tilde{a}(z;n)\Psi_n(z) + \tilde{b}(z;n)\Psi_{n-1}(z), \\ \Psi_{n+1}(z) &= \tilde{\beta}(z;n)\Psi_n(z) + \tilde{\gamma}(z;n)\Psi_{n-1}(z), \end{aligned}$$

where $\tilde{a}(z;n)$, $\tilde{b}(z;n)$, $\tilde{\beta}(z;n)$, and $\tilde{\gamma}(z;n)$ are rational functions on z. Then, $\{\Psi_n\}_{n\geq 0}$ satisfies the second-order differential equation

$$[\Psi_n(z)]'' + \mathcal{R}(z;n)[\Psi_n(z)]' + \mathcal{S}(z;n)\Psi_n(z) = 0,$$

where $\mathcal{R}(z;n)$ and $\mathcal{S}(z;n)$ have the alternative general expressions

$$\begin{aligned} \mathcal{R}(z;n) &= -\tilde{a}(z;n-1) - \tilde{a}(z;n) + \tilde{b}(z;n-1)\frac{\tilde{\beta}(z;n-1)}{\tilde{\gamma}(z;n-1)} - \frac{\tilde{b}'(z;n)}{\tilde{b}_n(z;n)}, \\ \mathcal{S}(z;n) &= \tilde{a}(z;n)\left[\frac{\tilde{b}'(z;n)}{\tilde{b}(z;n)} + \tilde{a}(z;n-1)\right] \\ &\quad - \frac{\tilde{b}(z;n-1)\,\left[\tilde{a}(z;n)\tilde{\beta}(z;n-1) + \tilde{b}(z;n)\right]}{\tilde{\gamma}(z;n-1)} - \tilde{a}'(z;n). \end{aligned}$$

If $\{\Psi_n\}_{n\geq 0}$ satisfies (4), it is not difficult to obtain the corresponding pairs of coefficients $\tilde{a}(z;n)$ and $\tilde{b}(z;n)$, and $\tilde{\beta}(z;n)$ and $\tilde{\gamma}(z;n)$ in terms of the coefficients associated with $\{\Phi_n\}_{n\geq 0}$, by using the computations in the proof of Theorem 2, as follows.

First, Equation (15) constitutes itself the structure relation type-equation for the sequence $\{\Psi_n\}_{n\geq 0}$, because it relates the expression of the first z-derivative of the polinomial $\Psi_n(z)$ with the two consecutive polynomials $\Psi_n(z)$ and $\Psi_{n-1}(z)$. This leads to $\tilde{a}(z;n) = \Xi(z;n,2)$ and $\tilde{b}(z;n) = -\Xi(z;n,1)$, or equivalently

$$\begin{aligned} \tilde{a}(z;n) &= \frac{1}{\Lambda(z;n)} \begin{vmatrix} C_1(z;n) & A_2(z;n) \\ D_1(z;n) & B_2(z;n) \end{vmatrix}, \\ \tilde{b}(z;n) &= \frac{-1}{\Lambda(z;n)} \begin{vmatrix} C_1(z;n) & A_1(z;n) \\ D_1(z;n) & B_1(z;n) \end{vmatrix}. \end{aligned}$$

Moreover, we can express these quantities in terms of the three pairs of coefficients at the entry of the algorithm. Denoting

$$\mathfrak{D}(z;n) = B_1(z;n-1)\left[A_1(z;n)\beta(z;n-1) + B_1(z;n)\right] - A_1(z;n-1)A_1(z;n)\gamma(z;n-1),$$

we have the explicit formulas

$$\begin{aligned} \tilde{a}(z;n) = {} & \frac{A_1(z;n)\left[B_1(z;n-1)\left(a(z;n)\beta(z;n-1) + b(z;n)\right) - a(z;n)A_1(z;n-1)\gamma(z;n-1)\right]}{\mathfrak{D}(z;n)} \\ & - \frac{A_1(z;n-1)\left[\gamma(z;n-1)A_1'(z;n) + b(z;n-1)B_1(z;n)\right]}{\mathfrak{D}(z;n)} \\ & + \frac{B_1(z;n-1)\left[a(z;n-1)B_1(z;n) + \beta(z;n-1)A_1'(z;n) + B_1'(z;n)\right]}{\mathfrak{D}(z;n)}, \end{aligned}$$

and

$$\begin{aligned} \tilde{b}(z;n) = {} & \frac{b(z;n-1)B_1(z;n)\left[A_1(z;n)\beta(z;n-1) + B_1(z;n)\right]}{\mathfrak{D}(z;n)} \\ & - \frac{\gamma(z;n-1)\left\{A_1(z;n)\left[\left(a(z;n-1) - a(z;n)\right)B_1(z;n) + B_1'(z;n)\right] + A_1^2(z;n)b(z;n) - B_1(z;n)A_1'(z;n)\right\}}{\mathfrak{D}(z;n)}. \end{aligned}$$

On the other hand, to obtain $\tilde{\beta}(z;n)$ and $\tilde{\gamma}(z;n)$, we use the ladder Equations (15) and (16) as well. Shifting $n \to n+1$ in (16), and adding the two equations yields

$$\begin{aligned} \tilde{\beta}(z;n) &= \frac{\Theta(z;n+1,1) + \Xi(z;n,2)}{\Theta(z;n+1,2)}, \\ \tilde{\gamma}(z;n) &= \frac{-\Xi(z;n,1)}{\Theta(z;n+1,2)}, \end{aligned}$$

or, equivalently,

$$\Psi_{n+1}(z) = \frac{\Theta(z;n+1,1) + \Xi(z;n,2)}{\Theta(z;n+1,2)}\Psi_n(z) - \frac{\Xi(z;n,1)}{\Theta(z;n+1,2)}\Psi_{n-1}(z),$$

which constitutes a recurrence formula for $\{\Psi_n(z)\}$. More explicitly, in terms of the quantities at the entry of the algorithm, we also have

$$\begin{aligned} \tilde{\beta}(z;n) = {} & \frac{A_1(z;n+1)B_1(z;n-1)\gamma(z;n)}{\mathfrak{D}(z;n)} \\ & + \frac{\left[A_1(z;n+1)\beta(z;n) + B_1(z;n+1)\right]\left[B_1(z;n-1)\beta(z;n-1) - A_1(z;n-1)\gamma(z;n-1)\right]}{\mathfrak{D}(z;n)} \end{aligned}$$

and

$$\tilde{\gamma}(z;n) = \frac{\gamma(z;n-1)\left\{B_1(z;n)\left[(A_1(z;n+1)\beta(z;n) + B_1(z;n+1))\right] - A_1(z;n)A_1(z;n+1)\gamma(z;n)\right\}}{\mathfrak{D}(z;n)}.$$

3. Holonomic ODE for Spectral Perturbations of OPUC

3.1. OPUC and Canonical Spectral Transformations

Let μ be a nontrivial probability measure with support on the unit circle $\mathbb{T} = \{z : |z| = 1\}$, and its associated inner product

$$\langle p(z), q(z)\rangle_\mu = \int_{\mathbb{T}} p(z)\overline{q(z)}d\mu(z), \quad p,q \in \mathbb{P}, \tag{18}$$

where $\mathbb{P}$ is the set of all complex polynomials. A (unique) sequence of polynomials $\{\phi_n\}_{n\geqslant 0}$ of the form $\phi_n(z) = \kappa_n z^n + \ldots$, $\kappa_n > 0$, such that

$$\langle \phi_n, \phi_m\rangle_\mu = \int_{\mathbb{T}} \phi_n(z)\overline{\phi_m(z)}d\mu(z) = \delta_{m,n}, \tag{19}$$

can be obtained by applying the Gram–Schmidt orthonormalization process to $\{z^n\}_{n\geq 0}$. We say that $\{\phi_n\}_{n\geqslant 0}$ is orthonormal with respect to μ. The corresponding monic sequence will be denoted by $\{\Phi_n\}_{n\geqslant 0}$. The associated reproducing kernel is defined by

$$K_n(z,y) = \sum_{k=0}^{n} \frac{\Phi_k(z)\overline{\Phi_k(y)}}{\|\Phi_k\|^2},$$

and it satisfies

$$\int_{\mathbb{T}} K_n(z,y)\overline{r(z)}d\mu(z) = \overline{r(y)},$$

for any polynomial $r(z)$ such that $\deg r \leq n$. On the other hand, the functions of the second kind associated with μ are defined by

$$q_n(z) = \int_{\mathbb{T}} \frac{\overline{\phi_n(t)}}{z-t}d\mu(t), \quad z \notin \mathbb{T}, \quad n \geq 0.$$

We also denote

$$Q_n(z) = \int_{\mathbb{T}} \frac{\overline{\Phi_n(t)}}{z-t}d\mu(t) = q(z)/\kappa_n, \quad n \geq 0.$$

The study of OPUC initiated with a series of papers by G. Szegő at the beginning of the twentieth century (see [24]). More recently, the monograph [25,26] by B. Simon constitutes the most comprehensive summary of the state of the art on this subject. We summarize some very well known results in the following proposition. They will be useful in the following sections.

Proposition 2. *Let $\{\Phi_n(z)\}_{n\geq 0}$ be a sequence of monic OPUC. Then, the following statements hold.*

1. *Forward and backward recurrence relations. For $n \geq 1$ and $\Phi_0(z) = 1$ we have*

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_{n-1}(z), \tag{20}$$
$$\Phi_n(z) = \left(1 - |\Phi_n(0)|^2\right) z\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_n(z), \tag{21}$$

where

$$\Phi^*_n(z) = z^n\overline{\Phi_n(z^{-1})},$$

is called the reciprocal (reversed) polynomial. Notice that from (21) we have

$$\Phi_n^*(z) = \frac{1}{\Phi_n(0)}\Phi_n(z) - \frac{1-|\Phi_n(0)|^2}{\Phi_n(0)} z\Phi_{n-1}(z), \tag{22}$$

provided $\Phi_n(0) \neq 0$.

2. *Structure relation (see [19] (Th. 8.3.1)). For every* $n \in \mathbb{N}$,

$$[\Phi_n(z)]' = a(z;n)\Phi_n(z) + b(z;n)\Phi_{n-1}(z), \tag{23}$$

where the coefficients $a(z;n)$ *and* $b(z;n)$ *are given in [19] (Th. 8.3.1).*

3. *Three-term recurrence formula (see [25] (1.5.46)), which we will rewrite for convenience as*

$$\Phi_{n+1}(z) = \beta(z;n)\Phi_n(z) + \gamma(z;n)\Phi_{n-1}(z), \tag{24}$$

where

$$\beta(z;n) = z + \frac{\Phi_{n+1}(0)}{\Phi_n(0)}, \tag{25}$$

$$\gamma(z;n) = \frac{-\Phi_{n+1}(0)}{\Phi_n(0)}\left(1-|\Phi_n(0)|^2\right) z, \tag{26}$$

and $\Phi_n(0) \neq 0$. *It is well known that (24) comes after replacing (21) into (20) to eliminate* Φ_{n-1}^*.

4. *Christoffel–Darboux formula*

$$K_{n-1}(z,y) = \frac{\Phi_n^*(z)\overline{\Phi_n^*(y)} - \Phi_n(z)\overline{\Phi_n(y)}}{||\Phi_n||^2(1-\bar{y}z)}, \quad \bar{y}z \neq 1. \tag{27}$$

Notice that from (22) we obtain

$$K_{n-1}(z,y) = W_n^1(z,y)\Phi_n(z) + W_n^2(z,y)\Phi_{n-1}(z), \tag{28}$$

with

$$W_n^1(z,y) = \frac{\overline{\Phi_n^*(y)}}{\|\Phi_n\|^2(1-\bar{y}z)\Phi_n(0)} - \frac{\overline{\Phi_n(y)}}{\|\Phi_n\|^2(1-\bar{y}z)},$$

$$W_n^2(z,y) = \frac{-\overline{\Phi_n^*(y)}(1-|\Phi_n(0)|^2)}{\|\Phi_n\|^2(1-\bar{y}z)\Phi_n(0)} z,$$

for $\Phi_n(0) \neq 0$.

Given a measure μ supported on $\mathbb{T}$, the following perturbations have been studied in the literature.

(i) Christoffel transformation [27]

$$d\mu_C = |z-\alpha|^2 d\mu, \quad \alpha \in \mathbb{C}.$$

(ii) Uvarov transformation with one mass [27]

$$d\mu_U = d\mu + m\delta(z-\alpha), \quad |\alpha| = 1, \quad m \in \mathbb{R}.$$

(iii) Uvarov transformation with two masses [27]

$$d\mu_U = d\mu + m\delta(z-\alpha) + \bar{m}\delta(z-\bar{\alpha}^{-1}), \quad |\alpha| \in \mathbb{R}_+ \setminus \{1,0\}, \quad m \in \mathbb{C}.$$

(iv) Geronimus transformation [28]

$$d\mu_G = \frac{1}{|z-\alpha|^2} d\mu + m\delta(z-\alpha) + \bar{m}\delta(z-\bar{\alpha}^{-1}), \quad |\alpha| > 1, \quad m \in \mathbb{C}.$$

It is easy to see that an application of the Geronimus transformation followed by a Christoffel transformation, yields the original measure. On the other hand, reversing the order, we will obtain the Uvarov transformation. Furthermore, the Carathéodory function associated with μ is defined by

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k,$$

where the constants $c_n = \int_{\mathbb{T}} z^{-n} d\mu(z)$ and $n \in \mathbb{Z}$ are called the *moments* associated with μ. We say that another Carathéodory function $\widetilde{F}$ is a linear spectral transformation of F if

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}, \tag{29}$$

for some polynomials $A(z), B(z)$, and $D(z)$. It turns out (see [29]) that the above perturbations can be expressed in terms of the corresponding Carathéodory functions as in (29), where the explicit polynomials A, B, and D are known for each case. As a consequence, these transformations are called *linear*.

An important aspect of the study of these transformations is the existence of *connection* formulas, i.e., expressions that relate the sequences of orthogonal polynomials associated with the perturbed and original measures. We will restrict our attention to the particular case $m = 0$ for the Geronimus transformation. In this regard, the following connection formulas appear in [27,28].

Proposition 3 ([27,28])**.** *Let μ be a nontrivial probability measure supported on $\mathbb{T}$, and denote its associated MOPS by $\{\Phi_n\}_{n\geqslant 0}$. Denote by $\{C_n\}_{n\geqslant 0}$, $\{U_n\}_{n\geqslant 0}$, $\{V_n\}_{n\geqslant 0}$, and $\{G_n\}_{n\geqslant 0}$ the MOPS associated with the Christoffel, Uvarov (one mass), Uvarov (two masses), and Geronimus (with $m = 0$) transformations of μ, respectively. Then, we have*

$$C_n(z) = \frac{1}{z-\alpha}\left[\Phi_{n+1}(z) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)} K_n(z,\alpha)\right], \quad n \geqslant 1, \tag{30}$$

$$U_n(z) = \Phi_n(z) - \frac{m\Phi_n(\alpha)}{1 + mK_{n-1}(\alpha,\alpha)} K_{n-1}(z,\alpha), \quad n \geqslant 1, \tag{31}$$

$$V_n(z) = \Phi_n(z) - M_n(m,\alpha)K_{n-1}(z,\alpha) - M_n(\bar{m},\bar{\alpha}^{-1})K_{n-1}(z,\bar{\alpha}^{-1}), \quad n \geqslant 1, \tag{32}$$

$$G_n(z) = (z-\alpha)\Phi_{n-1}(z) + \frac{\overline{Q_{n-1}(\alpha)}}{\varepsilon_{n-2}(\alpha)} S_{n-1}(z,\alpha), \quad n \geqslant 2, \tag{33}$$

where

$$M_n(m,\alpha) = \frac{\bar{m}\Phi_n(\bar{\alpha}^{-1})\left[1 + mK_{n-1}(\alpha,\bar{\alpha}^{-1})\right] - m\bar{m}\Phi_n(\alpha)K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})}{\left[1 + mK_{n-1}(\alpha,\bar{\alpha}^{-1})\right]\left[1 + \bar{m}K_{n-1}(\bar{\alpha}^{-1},\alpha)\right] - m\bar{m}K_{n-1}(\alpha,\alpha)K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})},$$

$Q_n(z)$ and $q_n(z)$ are the associated functions of the second kind with respect to μ, $\varepsilon_{n-2} = \|\mu_G\|^2 - \sum_{k=0}^{n-2} |q_k(\alpha)|^2$, and

$$S_{n-1}(z,\alpha) = \int_{\mathbb{T}} \frac{z-t}{\alpha-t} K_{n-2}(z,t)d\mu(t).$$

Notice that expressions (30)–(33) are written here using the monic normalization. Also, (32) is written in a slightly different form. In the following section, we will need some alternative connection formulas as a tool to derive differential operators whose solutions are the perturbed orthogonal polynomials. Such formulas are deduced in the following lemma.

Lemma 4. *Let $\{C_n\}_{n\geqslant 0}$, $\{U_n\}_{n\geqslant 0}$, $\{V_n\}_{n\geqslant 0}$, $\{G_n\}_{n\geqslant 0}$, and $\{\Phi_n\}_{n\geqslant 0}$ be as in the previous proposition. Then, for any n such that $\Phi_n(0) \neq 0$, we have*

(i) Christoffel transformation

$$C_n(z) = A_1(z;n)\Phi_n(z) + B_1(z;n)\Phi_{n-1}(z), \tag{34}$$

where

$$\begin{aligned} A_1(z;n) &= \frac{1}{z-\alpha}\left[\beta(z;n) - \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)}\left(W^1_{n+1}(z,\alpha)\beta(z;n) + W^2_{n+1}(z,\alpha)\right)\right], \\ B_1(z;n) &= \frac{\gamma(z;n)}{z-\alpha}\left[1 + \frac{\Phi_{n+1}(\alpha)}{K_n(\alpha,\alpha)} W^1_{n+1}(z,\alpha)\right]. \end{aligned}$$

(ii) Uvarov transformation with one mass

$$U_n(z) = A_1(z;n)\Phi_n(z) + B_1(z;n)\Phi_{n-1}(z),$$

where

$$\begin{aligned} A_1(z;n) &= 1 - \frac{m\Phi_n(\alpha)W^1_n(z,\alpha)}{1+mK_{n-1}(\alpha,\alpha)}, \\ B_1(z;n) &= -\frac{m\Phi_n(\alpha)W^2_n(z,\alpha)}{1+mK_{n-1}(\alpha,\alpha)}. \end{aligned}$$

(iii) Uvarov transformation with two masses

$$V_n(z) = A_1(z;n)\Phi_n(z) + B_1(z;n)\Phi_{n-1}(z),$$

where

$$\begin{aligned} A_1(z;n) &= 1 - M_n(\alpha)W^1_n(z,\alpha) - M_n(\bar{\alpha}^{-1})W^1_n(z,\bar{\alpha}^{-1}), \\ B_1(z;n) &= -M_n(\alpha)W^2_n(z,\alpha) - M_n(\bar{\alpha}^{-1})W^2_n(z,\bar{\alpha}^{-1}). \end{aligned}$$

(iv) Geronimus transformation ($m = 0$)

$$G_n(z) = A_1(z;n)\Phi_n(z) + B_1(z;n)\Phi_{n-1}(z),$$

where

$$
\begin{aligned}
A_1(z;n) &= \frac{S_{n-1,1}(z,\alpha)}{\Phi_n(0)}, \\
B_1(z;n) &= z-\alpha-\frac{\overline{Q_{n-1}(\alpha)}}{\varepsilon_{n-2}(\alpha)}\left(\frac{S_{n-1,1}(z,\alpha)}{\Phi_n(0)}z+S_{n-1,2}(z,\alpha)\right),
\end{aligned}
$$

with

$$
\begin{aligned}
S_{n-1,1}(z,\alpha) &= \frac{1}{\|\Phi_{n-1}\|^2}\int_{\mathbb{T}}\frac{z-t}{\alpha-t}\frac{\overline{\Phi^*_{n-1}(t)}}{(1-\bar{t}z)}d\mu(t), \\
S_{n-1,2}(z,\alpha) &= \frac{1}{\|\Phi_{n-1}\|^2}\int_{\mathbb{T}}\frac{z-t}{\alpha-t}\frac{\overline{\Phi_{n-1}(t)}}{(1-\bar{t}z)}d\mu(t).
\end{aligned}
$$

Proof. The expressions follow easily by considering the connection formulas in Proposition 3, and Equations (20), (22), (24), and (28). □

Remark 2. *Observe that the condition $\Phi_n(0)\neq 0$ is required for the connection formulas presented on the previous lemma, in order for $\beta(z,n)$ and $\gamma(z,n)$ in (24) to be well defined. Nevertheless, some well known examples of measures in $\mathbb{T}$ do not satisfy that condition for an infinite number of values of n. However, from (20), it is clear that we can take $\beta(z,n)=z$ and $\gamma(z,n)=0$ whenever $\Phi_{n+1}(0)=0$, and therefore it is possible to obtain connection formulas for these cases. We present two examples for illustrative purposes.*

- *For the Lebesgue measure $d\mu=\frac{d\theta}{2\pi}$, we have (see [25]) $\Phi_n(z)=z^n$, $n\geqslant 0$, and therefore $\Phi_n(0)=0$ for every $n\geqslant 1$. This means that we have $\Phi_{n+1}(z)=z\Phi_n(z)$, and thus $\beta(z,n)=z$ and $\gamma(z,n)=0$ for every $n\geqslant 0$. On the other hand, since $\Phi_n^*(z)=1$, $n\geqslant 0$, we have that the connection formula for the Christoffel transformation becomes*

$$
C_n(z) = \frac{1}{z-\alpha}\left[z\Phi_n(z)-\frac{\alpha^{n+1}}{\sum_{k=0}^{n}|\alpha|^{2k}}\left(\frac{1-\bar{\alpha}^{n+1}z\Phi_n(z)}{1-\bar{\alpha}z}\right)\right],
$$

which means that in this particular case, we have

$$
\begin{aligned}
A_1(z,n) &= \frac{z}{z-\alpha}\left(1+\frac{|\alpha|^{2n+2}}{(1-\bar{\alpha}z)\sum_{k=0}^{n}|\alpha|^{2k}}\right), \\
B_1(z,n) &= -\frac{\alpha^{n+1}}{z^{n-1}(z-\alpha)(1-\bar{\alpha}z)\sum_{k=0}^{n}|\alpha|^{2k}}.
\end{aligned}
$$

- *For the normalized Bernstein–Szegő measure with parameter β, with $|\beta|<1$, defined by $d\mu=\frac{1-|\beta|^2}{|1-\bar{\beta}z|^2}\frac{d\theta}{2\pi}$, we have (see [25]) $\Phi_n(z)=z^{n-1}(z-\beta)$ for $n\geqslant 1$, so that $\Phi_n(0)=0$ for every $n\geqslant 2$. Notice that if we choose $\beta=\alpha$, where α is the parameter of the Christoffel transformation, then we have $\Phi_n(\alpha)=0$ for every $n\geqslant 1$ and, as a consequence, we have in the corresponding connection formula*

$$
A_1(z,n)=\frac{z}{z-\alpha},\quad B_1(z,n)=0,\quad n\geqslant 1,
$$

directly from (30).

Notice that, by using the algorithm presented in the preceding section, we can obtain a second-order differential equation satisfied by the orthogonal polynomials $\{\Psi_n\}_{n\geqslant 0}$ associated with canonical transformations of measures on the unit circle, as the corresponding MOPS $\{\Phi_n\}_{n\geqslant 0}$ satisfies a three-term recursion formula and a structure relation, and the previous Lemma provides connection formulas between $\{\Psi_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$. Next, we present two illustrative examples.

3.2. An Uvarov Perturbation of the Circular Jacobi Polynomials

Let $\{\Phi_n\}_{n\geq 0}$ be the monic circular Jacobi polynomials of parameter a, orthogonal with respect to the measure (see [19] (Ex. 8.2.5))

$$d\mu(\theta) = \frac{\Gamma^2(a+1)}{2\pi\Gamma(2a+1)}|1-e^{i\theta}|^{2a}d\theta, \quad a > -1.$$

We will denote by $\{\Psi_n^{\alpha}\}_{n\geq 0}$ the MOPS corresponding to an Uvarov modification of the above measure $\mu(\theta)$ with a unique mass point at $z=\alpha$, with $|\alpha|=1$. For the sake of brevity, throughout this example, we will only consider the particular case $a=1$ and $\alpha=1$. In this situation, the n-th degree monic polynomial is given by (see [19,25])

$$\Phi_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}(k+1)z^k,$$

and its norm is given by $\|\Phi_n(z)\| = \sqrt{\frac{n+2}{2(n+1)}}$. The reversed polynomial is given by

$$\Phi_n^*(z) = \frac{1}{n+1}\sum_{k=0}^{n}(k+1)z^{n-k}.$$

As a consequence, we have $\Phi_n(0) = \frac{1}{n+1}$ and $\Phi_n(1) = \Phi_n^*(1) = \frac{n+2}{2}$. Using the previous expressions we can now compute the coefficients that we need to compute the second-order differential equation. First, the coefficients in (28) are given by $W_n^1(z,1) = \frac{n(n+1)}{1-z}$ and $W_n^2(z,1) = -\frac{n(n+2)}{1-z}z$, and $K_{n-1}(1,1) = \frac{1}{6}n(n+1)(n+2)$. Thus, from (ii) in Lemma 4, we obtain

$$A_1(z,n) = 1 - \frac{3mn(n+1)(n+2)}{[6+mn(n+1)(n+2)](1-z)} \quad \text{and} \quad B_1(z,n) = \frac{3mn(n+2)^2 z}{[6+mn(n+1)(n+2)](1-z)}.$$

On the other hand, for the coefficients in (3), using (25) and (26), we get

$$\beta(z;n) = z + \frac{n+1}{n+2} \quad \text{and} \quad \gamma(z;n) = -\frac{n}{n+1}z,$$

and, finally, the coefficients in the structure relation are (see [19], Ex. 8.3.2)

$$a(z;n) = -\frac{n}{1-z} \quad \text{and} \quad b(z;n) = \frac{n(n+2)}{(n+1)(1-z)}.$$

Using Mathematica® we compute the coefficients $\mathcal{R}(z;n)$ and $\mathcal{S}(z;n)$ in Theorem 2:

$$\mathcal{R}(z;n) = \frac{4}{z-1} - \frac{n}{z} + \frac{r_1(m,n)}{r_2(m,n) - r_1(m,n)z}$$

with $r_1(m,n) = -m^2n^6 - 9m^2n^5 - 31m^2n^4 - 51m^2n^3 - 40m^2n^2 - 12m^2n - 3mn^3 - 27mn^2 - 60mn - 36m + 18$ and $r_2(m,n) = -m^2n^6 - 9m^2n^5 - 31m^2n^4 - 51m^2n^3 - 40m^2n^2 - 12m^2n + 21mn + 18m + 18$, and

$$\begin{aligned}\mathcal{S}(z;n) &= \frac{n\left(2mn^3+3mn^2-5mn-6m-6\right)}{z\left(mn^3+3mn^2+2mn-3\right)} + \frac{-mn^4-6mn^3-11mn^2-6mn-24n}{9(z-1)} \\ &+ \frac{ns_1(m,n)}{9\left(mn^3+3mn^2+2mn-3\right)r_2(m,n)-r_1(m,n)z}.\end{aligned}$$

with $s_1(m,n) = m^4n^{12} + 18m^4n^{11} + 143m^4n^{10} + 660m^4n^9 + 1959m^4n^8 + 3906m^4n^7 + 5297m^4n^6 + 4824m^4n^5 + 2824m^4n^4 + 960m^4n^3 + 144m^4n^2 + 6m^3n^9 + 108m^3n^8 + 792m^3n^7 + 3132m^3n^6 + 7362m^3n^5 + 10584m^3n^4 + 9120m^3n^3 + 4320m^3n^2 + 864m^3n - 27m^2n^6 - 162m^2n^5 - 27m^2n^4 + 1620m^2n^3 + 4104m^2n^2 + 3888m^2n + 1296m^2 - 108mn^3 - 972mn^2 - 2160mn - 1296m + 324$.

It is easy to see that, when we take $m = 0$, we recover the differential equation satisfied by the sequence $\{\Phi_n(z)\}_{n\geq 0}$ studied by Ismail in [19].

3.3. A Christoffel Transformation of the Bernstein–Szegő Polynomials

Following the Remark 2, we consider a Christoffel perturbation with parameter α of the Bernstein–Szegő orthogonal polynomials, i.e., $\Phi_n(z) = z^{n-1}(z-\beta)$, $n \geq 1$ and $\Phi_0(z) = 1$ (see [25]). If we take $\beta = \alpha$, we clearly have $\Psi_n(z) = z^n$, $n \geq 0$, i.e., $\{\Psi_n\}_{n\geq 0}$ is the MOPS associated with the normalized Lebesgue measure. Then, for the connection and recurrence formulas we get

$$A_1(z;n) = \frac{z}{z-\alpha} \quad \text{and} \quad B_1(z;n) = 0,$$

$$\beta(z;n) = z \quad \text{and} \quad \gamma(z;n) = 0,$$

and it is not difficult to show that taking

$$a(z;n) = \frac{1}{z-\alpha} \quad \text{and} \quad b(z;n) = n-1,$$

the structure relation (23) holds. Using Mathematica® we obtain the coefficients

$$\mathcal{R}(z;n) = \frac{1-n}{z} \quad \text{and} \quad \mathcal{S}(z;n) = 0,$$

for the second-order differential equation, which is clearly satisfied by $\{\Psi_n\}_{n\geq 0}$.

4. Coherent Pairs of Measures on $\mathbb{T}$

The notion of coherent pairs of orthogonality measures on the unit circle was introduced (in the more general framework of linear functionals defined in the space of Laurent polynomials) in [30], where the authors considered the Sobolev inner product

$$\langle f,g\rangle_S = \int f(z)\overline{g(z)}d\mu_0 + \lambda \int f'(z)\overline{g'(z)}d\mu_1, \qquad \lambda > 0, \quad f,g \in \mathbb{P}. \tag{35}$$

If the monic sequences $\{\Phi_n(z)\}_{n\geq 0}$ and $\{\Psi_n(z)\}_{n\geq 0}$, orthogonal with respect to μ_0 and μ_1, respectively, satisfy

$$\Psi_n(z) = \frac{\Phi'_{n+1}(z)}{n+1} + c_n\frac{\Phi'_n(z)}{n}, \quad a_n \neq 0, \ n \geq 1, \tag{36}$$

then (μ_0, μ_1) is said to be a (1,0)-coherent pair on the unit circle. It was shown in [30] that (36) constitutes a sufficient condition for

$$\Phi_{n+1}(z) + c_n \frac{n+1}{n} \Phi_n(z) = S_{n+1}(z;\lambda) + \nu_{n,\lambda} S_n(z;\lambda), \quad n \geq 1,$$

where $\{S_n(z;\lambda)\}_{n\geq 0}$ is orthogonal with respect to (35) and the sequence $\{\nu_{n,\lambda}\}_{n\geq 1}$ is given by

$$\nu_{n,\lambda} = \frac{\varrho_{n-1}(\lambda)}{\varrho_n(\lambda)}, \quad n \geq 1,$$

where the polynomials $\{\varrho(\lambda)\}_{n\geq 0}$ are orthogonal with respect to some measure supported in $\mathbb{R}$. The cases when either μ_0 or μ_1 is the Lebesgue measure were analyzed, showing that if (μ_0, μ_1) is an (1,0)-coherent pair, we have

- if μ_0 is the Lebesgue measure, then

$$d\mu_1 = \frac{d\theta}{2\pi |z+\alpha|^2}, \quad |\alpha| < 1, z = e^{i\theta},$$

  i.e., the Bernstein–Szegő measure;
- if μ_1 is the Lebesgue measure, then

$$d\mu_0 = |z-\alpha|^2 \frac{d\theta}{2\pi}, \quad z = e^{i\theta},$$

  i.e., a Christoffel transformation of μ_1, where α is given in terms of the first two moments of μ_0.

Furthermore, they proved that if μ_0 and μ_1 constitute a (1,0)-coherent pair, then the linear functional associated with μ_1 must be a rational perturbation of the linear functional associated with μ_0. This means that there exists a connection formula between the orthogonal sequences associated with μ_0 and μ_1, and therefore we can use the algorithm proposed in Section 2 to compute a second-order differential equation satisfied by $\{\Psi_n(z)\}_{n\geq 0}$. Such a connection formula can be obtained as follows.

From (2), we get

$$\begin{aligned} [\Phi_{n+1}(z)]' &= a(z;n+1)\Phi_{n+1}(z) + b(z;n+1)\Phi_n(z), \\ [\Phi_n(z)]' &= a(z;n)\Phi_n(z) + b(z;n)\Phi_{n-1}(z), \end{aligned}$$

and thus from (36) we have

$$\Psi_n(z) = \frac{a(z;n+1)\Phi_{n+1}(z) + b(z;n+1)\Phi_n(z)}{n+1} + c_n \frac{a(z;n)\Phi_n(z) + b(z;n)\Phi_{n-1}(z)}{n}$$

and by using the recurrence relation we arrive at

$$\begin{aligned} \Psi_n(z) &= \frac{a(z;n+1)\beta(z;n)\Phi_n(z) + a(z;n+1)\gamma(z;n)\Phi_{n-1}(z) + b(z;n+1)\Phi_n(z)}{n+1} \\ &+ c_n \frac{a(z;n)\Phi_n(z) + b(z;n)\Phi_{n-1}(z)}{n}. \end{aligned}$$

As a consequence, we obtain

$$\Psi_n(z) = A_1(z;n)\Phi_n(z) + B_1(z;n)\Phi_{n-1}(z),$$

with

$$A_1(z;n) = \frac{a(z;n+1)\beta(z;n)+b(z;n+1)}{(n+1)} + c_n\frac{a(z;n)}{n},$$
$$B_1(z;n) = \frac{a(z;n+1)\gamma(z;n)}{(n+1)} + c_n\frac{b(z;n)}{n}.$$

As an example, consider the case when μ_0 is the Lebesgue measure, and thus μ_1 is the Bernstein–Szegő measure. In this case, from [19] (Ex. 8.2.5) (with $a = 0$), we have for the recurrence relation

$$\beta(z;n) = \frac{n}{n+1} + z \quad \text{and} \quad \gamma(z;n) = -\frac{nz}{n+1},$$

and for the structure relation we get

$$a(z;n) = \frac{n}{z-1} \quad \text{and} \quad b(z;n) = -\frac{n}{z-1}.$$

Thus, for these families we have

$$A_1(z;n) = \frac{(n+1)\alpha + nz + z - 1}{(n+1)(z-1)} \quad \text{and} \quad B_1(z;n) = -\frac{(n+1)\alpha + nz}{(n+1)(z-1)}.$$

Using Mathematica® we obtain the coefficients

$$\mathcal{R}(z;n) = -\frac{nz - n - 4z + 2}{(z-1)z} + \frac{n(-\alpha) - 2\alpha - n}{n\alpha^2 + \alpha^2 + n\alpha - \alpha + nz\alpha + 2z\alpha + nz},$$

$$\mathcal{S}(z;n) = -\frac{2(n-1)}{(z-1)z} - \frac{2(-\alpha + nz\alpha + 2z\alpha + nz)}{(z-1)z(n\alpha^2 + \alpha^2 + n\alpha - \alpha + nz\alpha + 2z\alpha + nz)},$$

of the second-order differential equation satisfied by the Bernstein-Szegő polynomials.

5. Conclusions and Open Problems

We have generalized a known algorithm to compute second-order differential equations whose solutions are some sequences of polynomials. This method has been used before in the literature, especially in the framework of orthogonality in the real line, and it is based on some algebraic and differential properties connecting two sequences of polynomials. This generalization allows us to deal with more general frameworks of orthogonality, such as the so-called nonstandard orthogonality (the case when the associated multiplication operator is not self-adjoint). As a consequence, we have obtained differential equations whose solutions are orthogonal polynomials associated with spectral transformations of measures supported on the unit circle, as well as orthogonal polynomials associated with coherent pairs of measures on the unit circle. We point out that this approach could also be used to obtain differential equations for para-orthogonal polynomials associated with spectral transformations of measures and develop electrostatic models of their zeros, extending the results in [23]. This problem will be addressed in a future contribution.

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