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On Second Order q-Difference Equations Satisfied by Al-Salam–Carlitz I-Sobolev Type Polynomials of Higher Order

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Abstract: This contribution deals with the sequence $\{\mathbb{U}_n^{(a)}(x;q,j)\}_{n\geq 0}$ of monic polynomials in x, orthogonal with respect to a Sobolev-type inner product related to the Al-Salam–Carlitz I orthogonal polynomials, and involving an arbitrary number j of q-derivatives on the two boundaries of the corresponding orthogonality interval, for some fixed real number $q \in (0,1)$. We provide several versions of the corresponding connection formulas, ladder operators, and several versions of the second order q-difference equations satisfied by polynomials in this sequence. As a novel contribution to the literature, we provide certain three term recurrence formula with rational coefficients satisfied by $\mathbb{U}_n^{(a)}(x;q,j)$, which paves the way to establish an appealing generalization of the so-called J-fractions to the framework of Sobolev-type orthogonality.

Keywords: Al-Salam–Carlitz I polynomials; Al-Salam–Carlitz I-Sobolev type polynomials; second order linear q-difference equations; structure relations; recurrence relations; basic hypergeometric series

MSC: 33D45; 05A30; 39A13

1. Introduction

The Al-Salam–Carlitz I and II orthogonal polynomials of degree n, usually denoted in the literature as $U_n^{(a)}(x;q)$ and $V_n^{(a)}(x;q)$ respectively, are two systems of one parameter q-hypergeometric polynomials introduced in 1965 by W. A. Al-Salam and L. Carlitz, in their seminal work [1]. Here, $q \in \mathbb{R}$ stands for a fixed parameter, being the polynomials written in the variable x. There is a straightforward relationship between $U_n^{(a)}(x;q)$ and $V_n^{(a)}(x;q)$ (see [2] (Chapter VI, §10, pp. 195–198))

$$U_n^{(a)}(x;q^{-1}) = V_n^{(a)}(x;q),$$

and they are known to be positive definite orthogonal polynomial sequences for $a < 0$, and $a > 0$ respectively. Here, and throughout the paper, we assume the parameter q is such that $0 < q < 1$, which implies that these two families belong to the class of orthogonal polynomial solutions of certain second order q-difference equations, known in the literature as the Hahn class (see [3,4]). In fact, as we show later on, they can be explicitly given in terms of basic hypergeometric series. Given the close relation between these two families, and for the sake of clarity, in this paper we will focus on the Al-Salam–Carlitz I orthogonal polynomials $\{U_n^{(a)}(x;q)\}_{n\geq 0}$. The results obtained can be also stated for the Al-Salam–Carlitz II orthogonal polynomials by replacing the parameter q by q^{-1}, so we omit explicit details concerning this second family of orthogonal polynomials.

The Al-Salam–Carlitz I polynomials are orthogonal on the interval $[a,1]$, with a quite simple q-lattice, which makes them suitable for the study to be carried out hereafter, and also they are of interest in their own right. For example, they are known to be proportional to the eigenfunctions of certain quantum mechanical q-harmonic oscillators. In [5], it is clearly shown that many properties of this q-oscillators can be obtained from the properties of the Al-Salam–Carlitz I orthogonal polynomials. They are also known to be birth and death process polynomials ([6] (Section 18.2)), with birth rate aq^n and death rate $1-q^n$, and for $a=0$ they become the well known Rogers-Szegő polynomials, of deep implications in the study of the celebrated Askey–Wilson integral (see, for example [7,8]).

On the other hand, in the last decades, the so called Sobolev orthogonal polynomials have attracted the attention of many researchers. Firstly, this name was given to those families of polynomials orthogonal with respect to inner products involving positive Borel measures supported on infinite subsets of the real line, and also involving regular derivatives. When these derivatives appear only on function evaluations on a finite discrete set, the corresponding families are called Sobolev-type or discrete Sobolev orthogonal polynomial sequences. For a recent and comprehensive survey on the subject, see [9] and the references therein. In the last decade of the past century, H. Bavinck introduced the study of inner products involving differences (instead of regular derivatives) in uniform lattices on the real line (see [10–12], and also [13] for recent results on this topic). By analogy with the continuous case, these are also called Sobolev-type or discrete Sobolev inner products. In contrast, they are defined on uniform lattices. As a generalization of this last matter, here we focus on a particular Sobolev-type inner product defined on a q-lattice, instead of on a uniform lattice. This has already been considered in other works (see, for example in [14] for only one q-derivative). In the present study, we consider an arbitrary number $j\in\mathbb{N}$, $j\geq 1$ of q-derivatives in the discrete part of the inner product. For an interesting related work to this paper, see, for example, the preprint [15], which appeared just a few days ago while we were giving the finishing touches to the present manuscript. There, the authors generalize the action of an arbitrary number of q-derivatives for general orthogonality measures, using the same techniques as for example in [16], and also in the present paper. It is also worth mentioning the nice variation considering special non-uniform lattices (snul), instead of uniform or q-lattices, studied in the recent work [17].

Having said all that, and to the best of our knowledge, an arbitrary number of q-derivatives acting at the same time on the two boundaries of a bounded orthogonality interval, has never been previously considered in the literature, and the present work is intended to be a first step in this direction. This reveals some small differences of the corresponding polynomial sequences, for example, related with the parity of the polynomials, with respect to what happens considering only one mass point (as in [15]), and that we have right now under study. To be more precise, this paper deals with the sequence of monic q-polynomials $\{\mathbb{U}_n^{(a)}(x;q,j)\}_{n\geq 0}$, orthogonal with respect to the Sobolev-type inner product

$$\begin{aligned}\langle f,g\rangle_{\lambda,\mu} \;=\; & \int_a^1 f(x;q)g(x;q)(qx,a^{-1}qx;q)_\infty d_qx \\ & +\lambda(\mathscr{D}_q^j f)(a;q)\,(\mathscr{D}_q^j g)(a;q)+\mu(\mathscr{D}_q^j f)(1;q)\,(\mathscr{D}_q^j g)(1;q)\,,\end{aligned} \tag{1}$$

where $(qx,a^{-1}qx;q)_\infty d_qx$ is the orthogonality measure associated to the Al-Salam–Carlitz I orthogonal polynomials, $a<0$, $\lambda,\mu\in\mathbb{R}_+$ and $(\mathscr{D}_q f)$ denotes the q-derivative operator, as defined below in (2). It is worth noting that the above inner product involves an arbitrary number of q-derivatives on function evaluations on the discrete points $x=a$ and $x=1$, exclusively. We observe such points conform the boundary of the orthogonality interval of the Al-Salam–Carlitz I orthogonal polynomials. Thus, as an extension of the language used in literature, throughout this manuscript we will refer to $\mathbb{U}_n^{(a)}(x;q,j)$ as Al-Salam–Carlitz I-Sobolev type orthogonal polynomials of higher order, and for the sake of brevity, in what follows we just write $\mathbb{U}_n^{(a)}(x;q,j) = \mathbb{U}_n^{(a)}(x;q)$. We provide here two explicit representations for $\mathbb{U}_n^{(a)}(x;q)$, one as a linear combination of two consecutive Al-Salam–Carlitz I orthogonal polynomials $U_n^{(a)}(x;q)$ and $U_{n-1}^{(a)}(x;q)$, and the other one as a ${}_3\phi_2$ q-hypergeometric series, which was unknown so far. This basic hypergeometric character is always ${}_3\phi_2$, with independence of the number j of q-derivatives considered in (1). Next, we obtain two different versions of the structure relation satisfied by the Sobolev-type q-orthogonal polynomials in $\mathbb{U}_n^{(a)}(x;q)$, and next we use them to obtain closed expressions for the corresponding ladder (creation and annihilation) q-difference operators. As an application of these ladder q-difference operators, we obtain a three-term recurrence formula with rational coefficients, which allows us to find every polynomial $\mathbb{U}_{n+1}^{(a)}(x;q)$ of precise degree $n+1$, in terms of the previous two consecutive polynomials of the same sequence $\mathbb{U}_n^{(a)}(x;q)$ and $\mathbb{U}_{n-1}^{(a)}(x;q)$, and up to four different versions of the linear second order q-difference equation satisfied by $\mathbb{U}_n^{(a)}(x;q)$.

In the work, we provide four different versions of the second order q-difference equations satisfied by the family of orthogonal polynomials under consideration. Also, two different representations of such polynomials are determined: one as a linear combination of standard Al-Salam–Carlitz I orthogonal polynomials, and a second one as ${}_3\phi_2$ series. Two versions of structure relations are obtained, in contrast to standard results, giving rise to four second order q-difference equations satisfied by the elements of this family. The previous results, which clarify the enriching structure of such polynomials, are followed by novel results using a non-standard technique to achieve a three-term recurrence formula, and leading to the appearance of the polynomials under consideration as the numerators and denominators of the convergents of certain J-fractions.

The manuscript is organized as follows. In Section 2, we recall some basic definitions and notations of the q-calculus theory, as well as the basic properties of the Al-Salam–Carlitz I polynomials. In Section 3, we obtain some connection formulas and the basic hypergeometric representation of the Al-Salam–Carlitz I-Sobolev type orthogonal polynomials of higher order. Section 4 is focused on two structure relations for the sequence $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$, as well as the two different versions of the aforementioned three term recurrence formula with rational coefficients that $\mathbb{U}_n^{(a)}(x;q)$ satisfies. In Section 5, combining the connection formula for $\mathbb{U}_n^{(a)}(x;q)$, and the structure relations obtained in the preceeding sections, we provide the q-difference ladder operators and four versions of the second linear q-difference equation that the Al-Salam–Carlitz I-Sobolev type polynomials of higher order satisfy. The work ends with two brief sections on further results. The first one describes results relating Al-Salam–Carlitz I-Sobolev type polynomials with Jacobi fractions, and the second illustrates the form of such polynomials together with some important remarks. A final section on conclusions and future research problems is also included.

2. Definitions and Notations

This first part of the section is twofold. A first subsection provides the main tools used in the framework of q-calculus, in order to make our exposition be self-contained. Afterwards, we describe known facts on Al-Salam–Carlitz I polynomials.

2.1. q-Calculus Review

For every $q \neq 0$ and $q \neq 1$, the q-number, q-bracket, or simply the basic number $[n]_q$, is defined by [4,18]

$$[0]_q = 0, \quad [n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad n = 1,2,3,\ldots,$$

which comes from the equality

$$\lim_{q\to 1} \frac{1-q^n}{1-q} = n.$$

In this framework, a q-analogue of the factorial of n is given by

$$[0]_q! = 1, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad n = 1, 2, 3, \ldots,$$

and we can also give a q-analogue of the well known Pochhammer symbol, or shifted factorial (see [4]). For $n = 1, 2, 3, \ldots$, we have

$$(a; q)_0 = 1, \quad (a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{i=1}^{n}(1-aq^{i-1}).$$

Moreover, we use the following notation

$$(a_1, \ldots, a_r; q)_n = \prod_{k=1}^{r}(a_k; q)_n.$$

The following definitions, also in the framework of the q-calculus, can be found in [4]. The basic hypergeometric, or q-hypergeometric series ${}_r\phi_s$, is defined as follows. Let $\{a_i\}_{i=1}^{r}$ and $\{b_i\}_{i=1}^{s}$ be complex numbers such that $b_i \neq q^{-n}$ for $n \in \mathbb{N}$. We write

$${}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^k}{(q; q)_k}.$$

The q-binomial coefficient is given by

$$\begin{bmatrix} k \\ n \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \frac{[n]_q!}{[k]_q![n-k]_q!} = \begin{bmatrix} k \\ n-k \end{bmatrix}_q, \quad k = 0, 1, \ldots, n,$$

where n denotes a nonnegative integer.

Concerning the q-analog of the derivative operator, we have the q-derivative, or the Euler–Jackson q-difference operator

$$(\mathscr{D}_q f)(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & \text{if } z \neq 0,\ q \neq 1, \\ \\ f'(z), & \text{otherwise,} \end{cases} \tag{2}$$

where $\mathscr{D}_q^0 f = f$, $\mathscr{D}_q^n f = \mathscr{D}_q(\mathscr{D}_q^{n-1} f)$, for $n \geq 1$, and

$$\lim_{q\to 1} \mathscr{D}_q f(z) = f'(z).$$

Moreover, one has the following properties

$$\mathscr{D}_q[f(\gamma z)] = \gamma(\mathscr{D}_q f)(\gamma z), \quad \forall\, \gamma \in \mathbb{C}, \tag{3}$$

$$\mathscr{D}_q f(z) = \mathscr{D}_{q^{-1}} f(qz) \Leftrightarrow \mathscr{D}_{q^{-1}} f(z) = \mathscr{D}_q f(q^{-1}z), \tag{4}$$

$$\begin{aligned} \mathscr{D}_q[f(z)g(z)] &= f(qz)\mathscr{D}_q g(z) + g(z)\mathscr{D}_q f(z) \\ &= f(z)\mathscr{D}_q g(z) + g(qz)\mathscr{D}_q f(z), \end{aligned} \tag{5}$$

and the following interesting property which can be found in [19] (p. 104)

$$\mathscr{D}_{q^{-1}}(\mathscr{D}_q f)(z) = q\mathscr{D}_q(\mathscr{D}_{q^{-1}} f)(z), \tag{6}$$

This q-derivative operator leads to define a q-analogue of Leibniz' rule

$$\mathscr{D}_q^n[f(z)g(z)] = \sum_{k=0}^{n} \begin{bmatrix} k \\ n \end{bmatrix}_q (\mathscr{D}_q^k f)(z) \cdot (\mathscr{D}_q^{n-k} g)(q^k z), \quad n = 0,1,2,\ldots. \tag{7}$$

Of special interest is the way in which the integral form of the inner product (1) is defined, corresponding to the so called Jackson q-integral, given by

$$\int_0^z f(x)d_q x = (1-q)z\sum_{k=0}^{\infty} q^k f(q^k z),$$

which in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x.$$

The following definition will also be needed throughout the paper. The Jackson–Hahn–Cigler q-subtraction is given by (see, for example [20] (Definition 6), and the references given there)

$$(x \boxminus_q y)^n = \prod_{j=0}^{n-1}\left(x - yq^j\right) = x^n (y/x;q)_n = \sum_{k=0}^{n} \begin{bmatrix} k \\ n \end{bmatrix}_q q^{\binom{k}{2}} (-y)^k x^{n-k}.$$

Finally, we recall here the q-Taylor formula (see [21] (Theorem 6.3)), with the Cauchy remainder term, which is defined by

$$f(x) = \sum_{k=0}^{n} \frac{(\mathscr{D}_q^k f)(a)}{[k]_q!}(x \boxminus_q a)^k + \frac{1}{[n]_q!}\int_a^x (\mathscr{D}_q^{n+1} f)(t)\cdot(x \boxminus_q qt)^n d_q t.$$

The promising idea of q-derivatives is also potentially applicable in rings, see [22,23].

2.2. Al-Salam–Carlitz I Orthogonal Polynomials

After the above q-calculus introduction, we continue by giving several aspects and properties of the Al-Salam–Carlitz I polynomials $\{U_n^{(a)}(x;q)\}_{n\geq 0}$. All the elements presented in the remaining of this section can be found in [2] (Chapter VI, §10), [6] (Section 18.2), [4] (Section 14.24), and [24], among other references. Such polynomials are orthogonal with respect to the inner product on $\mathbb{P}$, the linear space of polynomials with real coefficients

$$\langle f,g\rangle = \int_a^1 f(x;q)g(x;q)d\alpha^{(a)}, \quad a<0,$$

where $d\alpha^{(a)} = (qx, a^{-1}qx;q)_\infty\, d_q x$, which implies that $\alpha^{(a)}$ is a step function on $[a,1]$ with jumps

$$\frac{q^k}{(aq;q)_\infty (q,q/a;q)_k} \quad \text{at the points } x = q^k, k = 0,1,2,\ldots,$$

and jumps

$$\frac{-aq^k}{(q/a;q)_\infty (q,aq;q)_k} \quad \text{at the points } x = aq^k, k = 0,1,2,\ldots.$$

It can be easily checked that, when $a = -1$ the above inner product becomes the inner product associated to the discrete q-Hermite orthogonal polynomials.

The Al-Salam–Carlitz I polynomials can be explicitly given by

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} ; q, a^{-1}qx \right), \quad a < 0, \tag{8}$$

satisfying the orthogonality relation

$$\int_a^1 U_m^{(a)}(x;q)U_n^{(a)}(x;q)d\alpha^{(a)} = (1-a)(-a)^n(q;q)_n q^{\binom{n}{2}}\delta_{m,n}$$

Proposition 1. *Let $\{U_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I polynomials of degree n. The following statements hold:*

1. *The recurrence relation [4] (Section 14.24)*

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + \beta_n U_n^{(a)}(x;q) + \gamma_n U_{n-1}^{(a)}(x;q), \tag{9}$$

with initial conditions $U_{-1}^{(a)}(x;q) = 0$ and $U_0^{(a)}(x;q) = 1$. Here, $\beta_n = (a+1)q^n$ and $\gamma_n = -aq^{n-1}(1-q^n)$.

2. *Structure relation [19]. For every $n \in \mathbb{N}$,*

$$\sigma(x)\mathscr{D}_{q^{-1}}U_n^{(a)}(x;q) = \overline{\alpha}_n U_{n+1}^{(a)}(x;q) + \overline{\beta}_n U_n^{(a)}(x;q) + \overline{\gamma}_n U_{n-1}^{(a)}(x;q), \tag{10}$$

where $\sigma(x) = (x-1)(x-a)$, $\overline{\alpha}_n = q^{1-n}[n]_q$, $\overline{\beta}_n = (a+1)q[n]_q$ and $\overline{\gamma}_n = aq^n[n]_q$.

3. *Squared norm [4] (Section 14.24). For every $n \in \mathbb{N}$,*

$$||U_n^{(a)}||^2 = (-a)^n(1-q)(q;q)_n\left(q, a, a^{-1}q; q\right)_\infty q^{\binom{n}{2}}.$$

4. *Forward shift operator [4] (Section 14.24)*

$$\mathscr{D}_q^k U_n^{(a)}(x;q) = [n]_q^{(k)} U_{n-k}^{(a)}(x;q), \tag{11}$$

where

$$[n]_q^{(k)} = \frac{(q^{-n};q)_k}{(q-1)^k} q^{kn-\binom{k}{2}},$$

denote the q-falling factorial [25].

5. *Second-order q-difference equation [26]*

$$\sigma(x)\mathscr{D}_q\mathscr{D}_{q^{-1}}U_n^{(a)}(x;q) + \tau(x)\mathscr{D}_q U_n^{(a)}(x;q) + \lambda_{n,q}U_n^{(a)}(x;q) = 0,$$

where $\tau(x) = (x-a-1)/(1-q)$ and $\lambda_{n,q} = [n]_q([1-n]_q\sigma''/2 - \tau')$.

Proposition 2 (Christoffel–Darboux formula)**.** *Let $\{U_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I polynomials. If we denote the n-th reproducing kernel by*

$$K_{n,q}(x,y) = \sum_{k=0}^{n} \frac{U_k^{(a)}(x;q) \cdot U_k^{(a)}(y;q)}{||U_k^{(a)}||^2}.$$

Then, for all $n \in \mathbb{N}$, it holds that

$$K_{n,q}(x,y) = \frac{U_{n+1}^{(a)}(x;q) \cdot U_n^{(a)}(y;q) - U_{n+1}^{(a)}(y;q) \cdot U_n^{(a)}(x;q)}{(x-y)\,||U_n^{(a)}||^2}. \tag{12}$$

Concerning the partial q-derivatives of $K_{n,q}(x,y)$, we use the following notation

$$\begin{aligned} K_{n,q}^{(i,j)}(x,y) &= \mathscr{D}_{q,y}^i(\mathscr{D}_{q,x}^j K_{n,q}(x,y)) \\ &= \sum_{k=0}^{n} \frac{\mathscr{D}_q^i U_k^{(a)}(x;q) \cdot \mathscr{D}_q^j U_k^{(a)}(y;q)}{||U_k^{(a)}||^2}. \end{aligned}$$

Next, we provide a technical result that will be useful later on.

Lemma 1. *Let $\{U_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I polynomials of degree n. Then following statements hold, for all $n \in \mathbb{N}$,*

$$K_{n-1,q}^{(0,j)}(x,y) = \mathcal{A}_n(x,y)U_n^{(a)}(x;q) + \mathcal{B}_n(x,y)U_{n-1}^{(a)}(x;q), \tag{13}$$

where

$$\mathcal{A}_n(x,y) = \frac{[j]_q!}{||U_{n-1}^{(a)}||^2\,(x \boxminus_q y)^{j+1}} \sum_{k=0}^{j} \frac{\mathscr{D}_q^k U_{n-1}^{(a)}(y;q)}{[k]_q!}(x \boxminus_q y)^k,$$

and

$$\mathcal{B}_n(x,y) = -\frac{[j]_q!}{||U_{n-1}^{(a)}||^2\,(x \boxminus_q y)^{j+1}} \sum_{k=0}^{j} \frac{\mathscr{D}_q^k U_n^{(a)}(y;q)}{[k]_q!}(x \boxminus_q y)^k.$$

Proof. Applying the j-th q-derivative to (12) with respect to y yields

$$K_{q,n-1}^{(0,j)}(x,y) = \frac{1}{||U_{n-1}^{(a)}||^2} \times$$

$$\left(U_n^{(a)}(x;q)\mathscr{D}_{q,y}^j\left(\frac{U_{n-1}^{(a)}(y;q)}{x-y}\right) - U_{n-1}^{(a)}(x;q)\mathscr{D}_{q,y}^j\left(\frac{U_n^{(a)}(y;q)}{x-y}\right)\right). \tag{14}$$

Then, using the q-analogue of Leibniz' rule (7) and

$$\mathscr{D}_{q,y}^n\left(\frac{1}{x-y}\right) = \frac{[n]_q!}{(x \boxminus_q y)^{n+1}},$$

we deduce

$$\begin{aligned} \mathscr{D}_{q,y}^j\left(\frac{U_{n-1}^{(a)}(y;q)}{x-y}\right) &= \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q \mathscr{D}_q^k U_{n-1}^{(a)}(y;q) \cdot \mathscr{D}_{q,y}^{j-k}\left(\frac{1}{x-q^k y}\right) \\ &= \sum_{k=0}^{j} \frac{[j]_q!}{[k]_q!} \frac{\mathscr{D}_q^k U_{n-1}^{(a)}(y;q)}{(x \boxminus_q q^k y)^{j-k+1}}. \end{aligned}$$

Next, it is easy to check that

$$(x \boxminus_q q^k y)^{j-k+1} = \frac{(x \boxminus_q y)^{j+1}}{(x \boxminus_q y)^k},$$

and therefore we have

$$\mathscr{D}^{j}_{q,y}\left(\frac{U^{(a)}_{n-1}(y;q)}{x-y}\right)=\frac{[j]_q!}{(x\boxminus_q y)^{j+1}}\sum_{k=0}^{j}\frac{\mathscr{D}^{k}_{q}U^{(a)}_{n-1}(y;q)}{[k]_q!}(x\boxminus_q y)^{k}.$$

Finally, combining all the above with (14) we obtain (13). This completes the proof. □

Remark 1. *We observe that*

$$K^{(0,j)}_{q,n-1}(x,\alpha)=\frac{[j]_q!}{||U^{(a)}_{n-1}||^2\,(x\boxminus_q y)^{j+1}}\left(U^{(a)}_{n}(x;q)\cdot Q_{q,j}(x,\alpha,U^{(a)}_{n-1})-U^{(a)}_{n-1}(x;q)\cdot Q_{q,j}(x,\alpha,U^{(a)}_{n})\right),$$

where $Q_{q,j}(x,\alpha,U^{(a)}_{n-1})$ and $Q_{q,j}(x,\alpha,U^{(a)}_{n})$ denote the q-Taylor polynomials of degree j of the polynomials $U^{(a)}_{n-1}(x;q)$ y $U^{(a)}_{n}(x;q)$ at $x=\alpha$, respectively.

3. Connection Formulas and Hypergeometric Representation

In this section we define the Al-Salam–Carlitz I-Sobolev type polynomials of higher order $\{\mathbb{U}^{(a)}_{n}(x;q)\}_{n\geq 0}$, and describe different relations which relate them to the Al-Salam–Carlitz I polynomials. These links will be useful in the sequel. Al-Salam–Carlitz I-Sobolev type polynomials are defined to be orthogonal with respect to Sobolev-type inner product

$$\begin{aligned}\langle f,g\rangle_{\lambda,\mu} &= \int_a^1 f(x;q)g(x;q)(qx,a^{-1}qx;q)_\infty d_qx \\ &\quad +\lambda(\mathscr{D}^{j}_{q}f)(a;q)\cdot(\mathscr{D}^{j}_{q}g)(a;q)+\mu(\mathscr{D}^{j}_{q}f)(1;q)\cdot(\mathscr{D}^{j}_{q}g)(1;q),\end{aligned} \tag{15}$$

where $a<0$, $\lambda,\mu\in\mathbb{R}_+$, and $j\in\mathbb{N}$, $j\geq 1$.

In a first approach, we express $\{\mathbb{U}^{(a)}_{n}(x;q)\}_{n\geq 0}$ in terms of the Al-Salam–Carlitz I polynomials $\{U^{(a)}_{n}(x;q)\}_{n\geq 0}$, the kernel polynomials and their corresponding derivatives. Moreover, we obtain a representation of the proposed polynomials as hypergeometric functions. Let us depart from the Fourier expansion

$$\mathbb{U}^{(a)}_{n}(x;q)=U^{(a)}_{n}(x;q)+\sum_{k=0}^{n-1}a_{n,k}U^{(a)}_{k}(x;q).$$

In view of (15), and considering the orthogonality properties for $U^{(a)}_{n}(x;q)$, for $0\leq k\leq n-1$, the coefficients in the previous expansion are given by

$$a_{n,k}=-\frac{\lambda\mathscr{D}^{j}_{q}\mathbb{U}^{(a)}_{n}(a;q)\cdot\mathscr{D}^{j}_{q}U^{(a)}_{n}(a;q)+\mu\mathscr{D}^{j}_{q}\mathbb{U}^{(a)}_{n}(1;q)\cdot\mathscr{D}^{j}_{q}U^{(a)}_{n}(1;q)}{||U^{(a)}_{k}||^2}.$$

Thus

$$\mathbb{U}^{(a)}_{n}(x;q)=U^{(a)}_{n}(x;q)-\lambda\mathscr{D}^{j}_{q}\mathbb{U}^{(a)}_{n}(a;q)\cdot K^{(0,j)}_{n-1,q}(x,a)-\mu\mathscr{D}^{j}_{q}\mathbb{U}^{(a)}_{n}(1;q)\cdot K^{(0,j)}_{n-1,q}(x,1).$$

After some manipulations, we obtain a linear system $AX=b$ with two unknowns, namely $\mathscr{D}^{j}_{q}\mathbb{U}^{(a)}_{n}(a;q)$ and $\mathscr{D}^{j}_{q}\mathbb{U}^{(a)}_{n}(1;q)$, where

$$A=\begin{pmatrix}1+\lambda K^{(j,j)}_{n-1,q}(a,a) & \mu K^{(j,j)}_{n-1,q}(a,1)\\ \lambda K^{(j,j)}_{n-1,q}(1,a) & 1+\mu K^{(j,j)}_{n-1,q}(1,1)\end{pmatrix},$$

and

$$X = (\mathscr{D}_q^j \mathbb{U}_n^{(a)}(a;q), \mathscr{D}_q^j \mathbb{U}_n^{(a)}(1;q))^T, \quad b = (\mathscr{D}_q^j U_n^{(a)}(a;q), \mathscr{D}_q^j U_n^{(a)}(1;q))^T.$$

Cramer's rule yields

$$\mathbb{U}_n^{(a)}(x;q) = U_n^{(a)}(x;q) - \lambda K_{n-1,q}^{(0,j)}(x,a) \cdot \Delta_{j,n}^{(1)}(a) - \mu K_{n-1,q}^{(0,j)}(x,1) \cdot \Delta_{j,n}^{(2)}(a), \tag{16}$$

where

$$\Delta_{j,n}^{(i)}(a) = \begin{cases} \det(A)^{-1} \det \begin{pmatrix} \mathscr{D}_q^j U_n^{(a)}(a;q) & \mu K_{n-1,q}^{(j,j)}(a,1) \\ \mathscr{D}_q^j U_n^{(a)}(1;q) & 1 + \mu K_{n-1,q}^{(j,j)}(1,1) \end{pmatrix}, & \text{if } i = 1, \\ \\ \det(A)^{-1} \det \begin{pmatrix} 1 + \lambda K_{n-1,q}^{(j,j)}(a,a) & \mathscr{D}_q^j U_n^{(a)}(a;q) \\ \lambda K_{n-1,q}^{(j,j)}(1,a) & \mathscr{D}_q^j U_n^{(a)}(1;q) \end{pmatrix}, & \text{if } i = 2, \end{cases}$$

Hence, we obtain a first connection formula, namely

$$\mathbb{U}_n^{(a)}(x;q) = \mathcal{C}_{1,n}(x) \cdot U_n^{(a)}(x;q) + \mathcal{D}_{1,n}(x) \cdot U_{n-1}^{(a)}(x;q), \tag{17}$$

where

$$\mathcal{C}_{1,n}(x) = 1 - \lambda \Delta_{j,n}^{(1)}(a) \mathcal{A}_n(x,a) - \mu \Delta_{j,n}^{(2)}(a) \mathcal{A}_n(x,1),$$

and

$$\mathcal{D}_{1,n}(x) = -\lambda \Delta_{j,n}^{(1)}(a) \mathcal{B}_n(x,a) - \mu \Delta_{j,n}^{(2)}(a) \mathcal{B}_n(x,1).$$

The previous connection formula is obtained after the application of Lemma 1, which yields

$$K_{n-1,q}^{(0,j)}(x,a) = \mathcal{A}_n(x,a) \cdot U_n^{(a)}(x;q) + \mathcal{B}_n(x,a) \cdot U_{n-1}^{(a)}(x;q),$$

together with

$$K_{n-1,q}^{(0,j)}(x,1) = \mathcal{A}_n(x,1) \cdot U_n^{(a)}(x;q) + \mathcal{B}_n(x,1) \cdot U_{n-1}^{(a)}(x;q).$$

Therefore, from (16) we get

$$\begin{aligned} \mathbb{U}_n^{(a)}(x;q) &= U_n^{(a)}(x;q) - \lambda[\mathcal{A}_n(x,a) \cdot U_n^{(a)}(x;q) + \mathcal{B}_n(x,a) \cdot U_{n-1}^{(a)}(x;q)]\Delta_{j,n}^{(1)}(a) \\ &\quad -\mu[\mathcal{A}_n(x,a) \cdot U_n^{(a)}(x;q) + \mathcal{B}_n(x,a) \cdot U_{n-1}^{(a)}(x;q)]\Delta_{j,n}^{(2)}(a) \\ &= [1 - \lambda \Delta_{j,n}^{(1)}(a) \mathcal{A}_n(x,a) - \mu \Delta_{j,n}^{(2)}(a) \mathcal{A}_n(x,1)] U_n^{(a)}(x;q) \\ &\quad +[-\lambda \Delta_{j,n}^{(1)}(a) \mathcal{B}_n(x,a) - \mu \Delta_{j,n}^{(2)}(a) \mathcal{B}_n(x,1)] U_{n-1}^{(a)}(x;q), \end{aligned}$$

leading to (17). At this point, we provide another relation between the two families of polynomials, which will be applied in Theorem 1. More precisely, from (17) and the recurrence relation (9) we have that

$$\mathbb{U}_{n-1}^{(a)}(x;q) = \mathcal{C}_{2,n}(x) \cdot U_n^{(a)}(x;q) + \mathcal{D}_{2,n}(x) \cdot U_{n-1}^{(a)}(x;q), \tag{18}$$

where

$$\mathcal{C}_{2,n}(x) = -\frac{\mathcal{D}_{1,n-1}(x)}{\gamma_{n-1}},$$

and

$$\mathcal{D}_{2,n}(x) = \mathcal{C}_{1,n-1}(x) + \mathcal{C}_{2,n}(x)\,(\beta_{n-1} - x)\,.$$

From (17) and (18) we deduce

$$U_n^{(a)}(x;q) = \det(B_n(x))^{-1} \det \begin{pmatrix} \mathbb{U}_n^{(a)}(x;q) & \mathbb{U}_{n-1}^{(a)}(x;q) \\ \mathcal{D}_{1,n}(x) & \mathcal{D}_{2,n}(x) \end{pmatrix} \tag{19}$$

and

$$U_{n-1}^{(a)}(x;q) = -\det(B_n(x))^{-1} \det \begin{pmatrix} \mathbb{U}_n^{(a)}(x;q) & \mathbb{U}_{n-1}^{(a)}(x;q) \\ \mathcal{C}_{1,n}(x) & \mathcal{C}_{2,n}(x) \end{pmatrix} \tag{20}$$

where

$$B_n(x) = \begin{pmatrix} \mathcal{C}_{1,n}(x) & \mathcal{C}_{2,n}(x) \\ \mathcal{D}_{1,n}(x) & \mathcal{D}_{2,n}(x) \end{pmatrix}.$$

Finally, we focus our attention on the representation of $\mathbb{U}_n^{(a)}(x;q)$ as hypergeometric functions. A similar analysis to that carried out in [14] (Theorem 2), yields the following result

Proposition 3 (Hypergeometric character). *For $a<0$, the Al-Salam–Carlitz I-Sobolev type polynomials of higher order $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$, have the following hypergeometric representation:*

$$\mathbb{U}_n^{(a)}(x;q) = (-a)^n \frac{\mathcal{D}_{1,n}(x)(1-\psi_n(x)q^{-1})q^{\binom{n}{2}-n+2}}{a[n]_q\psi_n(x)(1-q)} {}_3\phi_2\left(\begin{matrix} q^{-n}, x^{-1}, \psi_n(x) \\ 0, \psi_n(x)q^{-1} \end{matrix} ; q, a^{-1}qx\right) \tag{21}$$

where $\psi_n(x) = ((1-q)\vartheta_n(x)+1)^{-1}$ and

$$\vartheta_n(x) = \frac{aq^{n-2}[n]_q\mathcal{C}_{1,n}(x)}{\mathcal{D}_{1,n}(x)} - [n-1]_q.$$

Proof. For $n=0$, a trivial verification shows that (21) yields $\mathbb{U}_0^{(a)}(x;q)=1$. For $n\geq 1$, combining (8) with (17) and the relations

$$(q^{1-n};q)_k = -\frac{q}{[n]_q}([k-1]_q - [n-1]_q)(q^{-n};q)_k \tag{22}$$

where, in order to improve the compactness and readability of the expressions involved, we define

$$[-1]_q := \frac{1-q^{-1}}{1-q} = -q^{-1},$$

and

$$(q^{-n};q)_k = 0, \quad n<k, \tag{23}$$

yields

$$\mathbb{U}_n^{(a)}(x;q) = -(-a)^{n-1}q^{\binom{n}{2}-n+2}\frac{\mathcal{D}_{1,n}(x)}{[n]_q}\sum_{k=0}^{n}([k-1]_q+\vartheta_n(x))(q^{-n};q)_k(x^{-1};q)_k\frac{(a^{-1}qx)^k}{(q;q)_k}. \tag{24}$$

More precisely, (8) together with (17) yield

$$\begin{aligned}\mathbb{U}_n^{(a)}(x;q) &= (-a)^n q^{\binom{n}{2}}\mathcal{C}_{1,n}(x)\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k(x^{-1};q)_k}{(q;q)_k}(a^{-1}qx)^k \\ &\quad +(-a)^{n-1}q^{\binom{n}{2}-n+1}\mathcal{D}_{1,n}(x)\sum_{k=0}^{\infty}\frac{(q^{1-n};q)_k(x^{-1};q)_k}{(q;q)_k}(a^{-1}qx)^k.\end{aligned}$$

Taking into account (23), the previous expression turns into

$$\begin{aligned}\mathbb{U}_n^{(a)}(x;q) &= (-a)^n q^{\binom{n}{2}} \mathcal{C}_{1,n}(x) \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} (a^{-1}qx)^k \\ &\quad + (-a)^{n-1} q^{\binom{n}{2}-n+1} \mathcal{D}_{1,n}(x) \sum_{k=0}^{n-1} \frac{(q^{1-n};q)_k (x^{-1};q)_k}{(q;q)_k} (a^{-1}qx)^k,\end{aligned}$$

and (22) now leads to

$$\begin{aligned}\mathbb{U}_n^{(a)}(x;q) &= (-a)^n q^{\binom{n}{2}} \mathcal{C}_{1,n}(x) \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} (a^{-1}qx)^k \\ &\quad - (-a)^{n-1} q^{\binom{n}{2}-n+2} \frac{\mathcal{D}_{1,n}(x)}{[n]_q} \sum_{k=0}^{n} ([k-1]_q - [n-1]_q) \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} (a^{-1}qx)^k.\end{aligned}$$

Finally, a rearrangement of the terms in the sums of the previous expression, leads to (24).

On the other hand, after some straightforward calculations we get

$$[k-1]_q + \vartheta_n(x) = \frac{1-\psi_n(x)q^{-1}}{\psi_n(x)(1-q)} \frac{(\psi_n(x);q)_k}{(\psi_n(x)q^{-1};q)_k}.$$

Therefore

$$\mathbb{U}_n^{(a)}(x;q) = -(-a)^{n-1} \frac{\mathcal{D}_{1,n}(x)(1-\psi_n(x)q^{-1})q^{\binom{n}{2}-n+2}}{[n]_q \psi_n(x)(1-q)} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k (\psi_n(x);q)_k}{(\psi_n(x)q^{-1};q)_k} \frac{(a^{-1}qx)^k}{(q;q)_k}$$

which coincides with (21). This completes the proof. □

Remark 2. *Notice that one recovers (8) from (21) after $\lambda \to 0$ and $\mu \to 0$. One also recovers [14] ((23), p. 13) for $j = 1$, and $\lambda, \mu > 0$ in (21).*

4. Ladder Operators and a Three Term Recurrence Formula

In this section we find several structure relations associated to $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$. It is worth mentioning that such relations can be grouped in two depending on nature of the action of the q-derivative involved in the relation (see Theorem 1). In two of them, such q-derivative is constructed by means of a q-dilation operator ($\ell = -1$) whether in the other two, a q-contraction operator determines the q-derivative ($\ell = 1$). The ladder (creation and annihilation) operators are obtained in Proposition 4, as well as the three-term recurrence relations of Theorem 2, satisfied by $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$. These two results are also stated in terms of the duality provided by the choice of a q-dilating or q-contracting derivation.

The structure relations stated in Theorem 1 lean on the following result.

Lemma 2. *Let $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I-Sobolev type polynomials of degree n. Then, following statements hold, for $\ell = -1, 1$,*

$$\sigma_\ell(x) \mathscr{D}_{q^\ell} \mathbb{U}_n^{(a)}(x;q) = \mathcal{E}_{1+2\delta_{\ell,1},n}(x) \cdot U_n^{(a)}(x;q) + \mathcal{F}_{1+2\delta_{\ell,1},n}(x) \cdot U_{n-1}^{(a)}(x;q), \tag{25}$$

and

$$\sigma_\ell(x) \mathscr{D}_{q^\ell} \mathbb{U}_{n-1}^{(a)}(x;q) = \mathcal{E}_{2+2\delta_{\ell,1},n}(x) \cdot U_n^{(a)}(x;q) + \mathcal{F}_{2+2\delta_{\ell,1},n}(x) \cdot U_{n-1}^{(a)}(x;q), \tag{26}$$

where $\sigma_\ell(x) = \sigma(x)$ for $\ell = -1$ and $\sigma_\ell(x) = 1$ otherwise. Moreover,

$$\mathcal{E}_{1,n}(x) = (\overline{\alpha}_n(x-\beta_n) + \overline{\beta}_n)\mathcal{C}_{1,n}(q^{-1}x) + \sigma(x)\mathscr{D}_{q^{-1}}\mathcal{C}_{1,n}(x) + (\overline{\alpha}_{n-1} - \overline{\gamma}_{n-1}\gamma_{n-1}^{-1})\mathcal{D}_{1,n}(q^{-1}x),$$

$$\mathcal{F}_{1,n}(x) = (\overline{\gamma}_n - \overline{\alpha}_n\gamma_n)\mathcal{C}_{1,n}(q^{-1}x) + \sigma(x)\mathscr{D}_{q^{-1}}\mathscr{D}_{1,n}(x) + (\overline{\beta}_{n-1} + \overline{\gamma}_{n-1}\gamma_{n-1}^{-1}(x-\beta_{n-1}))\mathcal{D}_{1,n}(q^{-1}x),$$

$$\mathcal{E}_{3,n}(x) = \mathscr{D}_q\mathcal{C}_{1,n}(x) - [n-1]_q\gamma_{n-1}^{-1}\mathcal{D}_{1,n}(qx),$$

$$\mathcal{F}_{3,n}(x) = [n]_q\mathcal{C}_{1,n}(qx) + [n-1]_q\gamma_{n-1}^{-1}(x-\beta_{n-1})\mathcal{D}_{1,n}(qx) + \mathscr{D}_q\mathcal{D}_{1,n}(x),$$

$$\mathcal{E}_{2+2\delta_{\ell,1},n}(x) = -\frac{\mathcal{F}_{1+2\delta_{\ell,1},n-1}(x)}{\gamma_{n-1}},$$

and

$$\mathcal{F}_{2+2\delta_{\ell,1},n}(x) = \mathcal{E}_{1+2\delta_{\ell,1},n-1}(x) + \mathcal{E}_{2+2\delta_{\ell,1},n}(x)(\beta_{n-1}-x).$$

Here, $\delta_{m,n}$ denote the Kronecker delta function.

Proof. It is a direct consequence of the connection Formulas (17)–(20), the three-term recurrence relation (9) satisfied by $\{U_n^{(a)}(x;q)\}_{n\geq 0}$, and the structure relation (10). To be more precise, applying the q-derivative operator $\mathscr{D}_{q^\ell}$ to (17) for $\ell = -1, 1$, together with the property (5) yields

$$\begin{aligned} \mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) \;=\; & \mathcal{C}_{1,n}(q^\ell x)\cdot\mathscr{D}_{q^\ell}U_n^{(a)}(x;q) + U_n^{(a)}(x;q)\cdot\mathscr{D}_{q^\ell}\mathcal{C}_{1,n}(x) \\ & +\mathcal{D}_{1,n}(q^\ell x)\cdot\mathscr{D}_{q^\ell}U_{n-1}^{(a)}(x;q) + U_{n-1}^{(a)}(x;q)\cdot\mathscr{D}_{q^\ell}\mathcal{D}_{1,n}(x). \end{aligned}$$

Thus, multiplying the above expression by $\sigma_\ell(x)$ for $\ell = -1, 1$, and next combining (10) with (11) and (9), we deduce (25). Finally, shifting the index in (25) as $n \to n-1$ and using the recurrence relation (9) we get (26). This completes the proof. □

As a direct consequence of the previous result, we next obtain the following structure relations for the Al-Salam–Carlitz I-Sobolev type polynomials of higher order.

Theorem 1. *The Al-Salam–Carlitz I-Sobolev type polynomials of high order $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ satisfy the following structure relations for $\ell = -1, 1$,*

$$\Theta_{\ell,n}(x)\cdot\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) = \Xi_{2,1+2\delta_{\ell,1},n}(x)\cdot\mathbb{U}_n^{(a)}(x;q) + \Xi_{1,1+2\delta_{\ell,1},n}(x)\cdot\mathbb{U}_{n-1}^{(a)}(x;q), \tag{27}$$

and

$$\Theta_{\ell,n}(x)\cdot\mathscr{D}_{q^\ell}\mathbb{U}_{n-1}^{(a)}(x;q) = \Xi_{2,2+2\delta_{\ell,1},n}(x)\cdot\mathbb{U}_n^{(a)}(x;q) + \Xi_{1,2+2\delta_{\ell,1},n}(x)\cdot\mathbb{U}_{n-1}^{(a)}(x;q), \tag{28}$$

where $\Theta_{\ell,n}(x) = \sigma_\ell(x)\det(B_n(x))$ and

$$\Xi_{i,k,n}(x) = (-1)^i \det\begin{pmatrix} \mathcal{E}_{k,n}(x) & \mathcal{C}_{i,n}(x) \\ \mathcal{F}_{k,n}(x) & \mathcal{D}_{i,n}(x) \end{pmatrix}, \quad i = 1,2, \quad k = 1,2,3,4.$$

Proof. The result follows in a straightforward way from (19) and (20), together with the application of the previous Lemma 2. □

In the next result, we provide ladder operators associated to the Al-Salam–Carlitz I-Sobolev type polynomials. Its proof is quite involved, leaning on the use of Theorem 1, and following the same technique as in [16].

Proposition 4. *Let $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I-Sobolev type polynomials defined by (21), and let I be the identity operator. Then, the ladder (destruction and creation) operators $\mathfrak{a}_\ell$ and $\mathfrak{a}_\ell^\dagger$, respectively, are defined by*

$$\mathfrak{a}_\ell = \Theta_{\ell,n}(x)\mathscr{D}_{q^\ell} - \Xi_{2,1+2\delta_{\ell,1},n}(x)I, \tag{29}$$

$$\mathfrak{a}_\ell^\dagger = \Theta_{\ell,n}(x)\mathscr{D}_{q^\ell} - \Xi_{1,2+2\delta_{\ell,1},n}(x)I, \tag{30}$$

which verify

$$\mathfrak{a}_\ell \left(\mathbb{U}_n^{(a)}(x;q)\right) = \Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q), \tag{31}$$

$$\mathfrak{a}_\ell^\dagger \left(\mathbb{U}_{n-1}^{(a)}(x;q)\right) = \Xi_{2,2+2\delta_{\ell,1},n}(x)\mathbb{U}_n^{(a)}(x;q), \tag{32}$$

where $\ell = -1, 1$.

We complete this section with a straightforward application of the above ladder operators $\mathfrak{a}_\ell$ and $\mathfrak{a}_\ell^\dagger$. We use these operators to obtain two versions (one for $\ell = -1$ and other for $\ell = 1$) of certain three term recurrence formula with rational coefficients which provides $\mathbb{U}_{n+1}^{(a)}(x;q)$ in terms of the former two consecutive polynomials $\mathbb{U}_n^{(a)}(x;q)$ and $\mathbb{U}_{n-1}^{(a)}(x;q)$. We recall the importance of such recurrence formula, which allows to give further properties of the family of orthogonal polynomials, as described in classical references such as [2]. The proof of the next result can be followed from the aforementioned technique, which has been recently generalized in [16] (p. 8). However, we have decided to include it below for the sake of completeness.

Theorem 2. *The Al-Salam–Carlitz I-Sobolev type polynomials of high order* $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ *satisfy the following three-term recurrence relations for* $\ell = -1, 1$,

$$\alpha_{\ell,n}(x)\mathbb{U}_{n+1}^{(a)}(x;q) = \beta_{\ell,n}(x)\mathbb{U}_n^{(a)}(x;q) + \gamma_{\ell,n}(x)\mathbb{U}_{n-1}^{(a)}(x;q), \tag{33}$$

where

$$\alpha_{\ell,n}(x) = \Theta_{\ell,n}(x)\Xi_{2,2+2\delta_{\ell,1},n+1}(x),$$

$$\beta_{\ell,n}(x) = \Theta_{\ell,n+1}(x)\Xi_{2,1+2\delta_{\ell,1},n}(x) - \Theta_{\ell,n}(x)\Xi_{1,2+2\delta_{\ell,1},n+1}(x),$$

and

$$\gamma_{\ell,n}(x) = \Theta_{\ell,n+1}(x)\Xi_{1,1+2\delta_{\ell,1},n}(x).$$

Proof. Shifting the index in (28) as $n \to n+1$, yields

$$\Theta_{\ell,n+1}(x)\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) = \Xi_{2,2+2\delta_{\ell,1},n+1}(x)\mathbb{U}_{n+1}^{(a)}(x;q) + \Xi_{1,2+2\delta_{\ell,1},n+1}(x)\mathbb{U}_n^{(a)}(x;q).$$

Next, multiplying the above expression by $-\Theta_{\ell,n}(x)$, and multiplying (27) by $\Theta_{\ell,n+1}(x)$, adding and simplifying the resulting equations, we obtain (33). This completes the proof. □

Remark 3. *Notice that (33) becomes (9) when* $\lambda = \mu = 0$.

5. Holonomic Second Order q-Difference Equations

In this section, we find up to four different q-difference equations of second order that $\mathbb{U}_n^{(a)}(x;q)$ satisfies. It is worth noting that these q-difference equations are no longer classical, in the sense that all their polynomial coefficients depend on n, so we have different coefficients for every different degree n that we consider. When dealing with differential equations, these kind of equations are known in the literature as Holonomic second order differential equations (see, for example [16]), so by natural extension we refer to them as second order linear q-difference equations.

It is also worth remarking the different nature of these four equations. The four of them are linear second order difference equations. However, in two of them (see Proposition 5), a unique difference operator appears. We also point out the appearance of four regular singular points in one of these two equations, more precisely for the choice $\ell = -1$. Such points are $1, a, q$ and aq, which appear to be intimately related to the problem. The two second difference equations (see Proposition 6) an analogous disquisition can be made concerning the singular points, which are now the points $1, a, q^{-1}$ and aq^{-1}.

Moreover, the two last equations involve the two q-difference operators previously mentioned, in contrast to the two first ones.

Proposition 5 (2nd order holonomic equation I)**.** *Let $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I-Sobolev type polynomials defined by (21). Then, the following statement holds, for $\ell = -1, 1$,*

$$\mathcal{R}_{\ell,n}(x)\mathscr{D}_{q^\ell}^2\mathbb{U}_n^{(a)}(x;q) + \mathcal{S}_{\ell,n}(x)\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) + \mathcal{T}_{\ell,n}(x)\mathbb{U}_n^{(a)}(x;q) = 0, \quad n \geq 0, \tag{34}$$

where

$$\mathcal{R}_{\ell,n}(x) = \Theta_{\ell,n}(x)\Theta_{\ell,n}(q^\ell x),$$

$$\begin{aligned}\mathcal{S}_{\ell,n}(x) &= \Theta_{\ell,n}(x)\left[\mathscr{D}_{q^\ell}\Theta_{\ell,n}(x) - \Xi_{2,1+2\delta_{\ell,1},n}(q^\ell x) - \Xi_{1,2+2\delta_{\ell,1},n}(x)\right] \\ &\quad - \frac{\Theta_{\ell,n}(x)\left[\Theta_{\ell,n}(x) + (q^\ell - 1)x\,\Xi_{1,2+2\delta_{\ell,1},n}(x)\right]\mathscr{D}_{q^\ell}\Xi_{1,1+2\delta_{\ell,1},n}(x)}{\Xi_{1,1+2\delta_{\ell,1},n}(x)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_{\ell,n}(x) &= \Xi_{1,2+2\delta_{\ell,1},n}(x)\Xi_{2,1+2\delta_{\ell,1},n}(x) - \Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}\Xi_{2,1+2\delta_{\ell,1},n}(x) \\ &\quad + \frac{\Xi_{2,1+2\delta_{\ell,1},n}(x)\left[\Theta_{\ell,n}(x) + (q^\ell - 1)x\,\Xi_{1,2+2\delta_{\ell,1},n}(x)\right]\mathscr{D}_{q^\ell}\Xi_{1,1+2\delta_{\ell,1},n}(x)}{\Xi_{1,1+2\delta_{\ell,1},n}(x)} \\ &\quad - \Xi_{1,1+2\delta_{\ell,1},n}(q^\ell x)\Xi_{2,2+2\delta_{\ell,1},n}(x).\end{aligned}$$

Proof. In fact, from (31) we have

$$\mathfrak{a}_\ell^\dagger\left[\mathfrak{a}_\ell\left(\mathbb{U}_n^{(a)}(x;q)\right)\right] = \mathfrak{a}_\ell^\dagger\left[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\right]. \tag{35}$$

Then, applying (29) and (30) to left hand member of (35) we have

$$\begin{aligned}\mathfrak{a}_\ell^\dagger\left[\mathfrak{a}_\ell\left(\mathbb{U}_n^{(a)}(x;q)\right)\right] &= \mathfrak{a}_\ell^\dagger\left[\Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) - \Xi_{2,1+2\delta_{\ell,1},n}(x)\mathbb{U}_n^{(a)}(x;q)\right] \\ &= \Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}\left[\Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) - \Xi_{2,1+2\delta_{\ell,1},n}(x)\mathbb{U}_n^{(a)}(x;q)\right] \\ &\quad - \Xi_{1,2+2\delta_{\ell,1},n}(x)\Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) + \Xi_{1,2+2\delta_{\ell,1},n}(x)\Xi_{2,1+2\delta_{\ell,1},n}(x)\mathbb{U}_n^{(a)}(x;q).\end{aligned}$$

Next, using (5) we arrive

$$\begin{aligned}\mathfrak{a}_\ell^\dagger\left[\mathfrak{a}_\ell\left(\mathbb{U}_n^{(a)}(x;q)\right)\right] &= \Theta_{\ell,n}(x)\Theta_{\ell,n}(q^\ell x)\mathscr{D}_{q^\ell}^2\mathbb{U}_n^{(a)}(x;q) \\ &\quad + \Theta_{\ell,n}(x)\left[\mathscr{D}_{q^\ell}\Theta_{\ell,n}(x) - \Xi_{2,1+2\delta_{\ell,1},n}(q^\ell x) - \Xi_{1,2+2\delta_{\ell,1},n}(x)\right]\mathscr{D}_{q^\ell}\mathbb{U}_n^{(a)}(x;q) \\ &\quad + \left[\Xi_{1,2+2\delta_{\ell,1},n}(x)\Xi_{2,1+2\delta_{\ell,1},n}(x) - \Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}\Xi_{2,1+2\delta_{\ell,1},n}(x)\right]\mathbb{U}_n^{(a)}(x;q).\end{aligned} \tag{36}$$

Applying (30) to the right hand side of (35) we have

$$\begin{aligned}\mathfrak{a}_\ell^\dagger\left[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\right] &= \Theta_{\ell,n}(x)\mathscr{D}_{q^\ell}[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)] \\ &\quad - \Xi_{1,2+2\delta_{\ell,1},n}(x)\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q).\end{aligned}$$

From property (5) we deduce

$$\begin{aligned}\mathfrak{a}_{\ell}^{\dagger}\left[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\right] &= \Xi_{1,1+2\delta_{\ell,1},n}(q^{\ell}x)\Theta_{\ell,n}(x)\mathscr{D}_{q^{\ell}}\mathbb{U}_{n-1}^{(a)}(x;q)\\ &+[\Theta_{\ell,n}(x)\mathscr{D}_{q^{\ell}}\Xi_{1,1+2\delta_{\ell,1},n}(x)-\Xi_{1,2+2\delta_{\ell,1},n}(x)\Xi_{1,1+2\delta_{\ell,1},n}(x)]\mathbb{U}_{n-1}^{(a)}(x;q).\end{aligned}$$

Therefore, from (28) we get

$$\begin{aligned}\mathfrak{a}_{\ell}^{\dagger}\left[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\right] &= \Xi_{1,1+2\delta_{\ell,1},n}(q^{\ell}x)\Xi_{2,2+2\delta_{\ell,1},n}(x)\mathbb{U}_{n}^{(a)}(x;q)\\ &+\Xi_{1,1+2\delta_{\ell,1},n}(q^{\ell}x)\Xi_{1,2+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\\ &+[\Theta_{\ell,n}(x)\mathscr{D}_{q^{\ell}}\Xi_{1,1+2\delta_{\ell,1},n}(x)-\Xi_{1,2+2\delta_{\ell,1},n}(x)\Xi_{1,1+2\delta_{\ell,1},n}(x)]\mathbb{U}_{n-1}^{(a)}(x;q).\end{aligned}$$

Property (2) leads to

$$\begin{aligned}\mathfrak{a}_{\ell}^{\dagger}\left[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\right] &= \Xi_{1,1+2\delta_{\ell,1},n}(q^{\ell}x)\Xi_{2,2+2\delta_{\ell,1},n}(x)\mathbb{U}_{n}^{(a)}(x;q)\\ &+[\Theta_{\ell,n}(x)+(q-1)x\Xi_{1,2+2\delta_{\ell,1},n}(x)]\mathscr{D}_{q^{\ell}}\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q).\end{aligned}$$

Next, we apply (27) to arrive at

$$\mathbb{U}_{n-1}^{(a)}(x;q)=\frac{\Theta_{\ell,n}(x)}{\Xi_{1,1+2\delta_{\ell,1},n}(x)}\mathscr{D}_{q^{\ell}}\mathbb{U}_{n}^{(a)}(x;q)-\frac{\Xi_{2,1+2\delta_{\ell,1},n}(x)}{\Xi_{1,1+2\delta_{\ell,1},n}(x)}\mathbb{U}_{n}^{(a)}(x;q),$$

and therefore

$$\begin{aligned}&\mathfrak{a}_{\ell}^{\dagger}\left[\Xi_{1,1+2\delta_{\ell,1},n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)\right]\\ &=\frac{\Theta_{\ell,n}(x)\left[\Theta_{\ell,n}(x)+(q-1)x\,\Xi_{1,2+2\delta_{\ell,1},n}(x)\right]\mathscr{D}_{q^{\ell}}\Xi_{1,1+2\delta_{\ell,1},n}(x)}{\Xi_{1,1+2\delta_{\ell,1},n}(x)}\mathscr{D}_{q^{\ell}}\mathbb{U}_{n}^{(a)}(x;q)\\ &+\Big[\Xi_{1,1+2\delta_{\ell,1},n}(q^{\ell}x)\Xi_{2,2+2\delta_{\ell,1},n}(x)\\ &-\frac{\Xi_{2,1+2\delta_{\ell,1},n}(x)\left[\Theta_{\ell,n}(x)+(q-1)x\,\Xi_{1,2+2\delta_{\ell,1},n}(x)\right]\mathscr{D}_{q^{\ell}}\Xi_{1,1+2\delta_{\ell,1},n}(x)}{\Xi_{1,1+2\delta_{\ell,1},n}(x)}\Big]\mathbb{U}_{n}^{(a)}(x;q).\end{aligned}\tag{37}$$

Finally, equaling (36) and (37) we arrived to the desired result. □

Proposition 6 (2nd order holonomic equation II). *Let $\{\mathbb{U}_{n}^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I-Sobolev type polynomials defined by (21). Then, the following statement holds, for $\ell=-1,1$,*

$$\overline{\mathcal{R}}_{\ell,n}(x)\mathbb{D}_{q^{\ell}}^{2}\mathbb{U}_{n}^{(a)}(x;q)+\overline{\mathcal{S}}_{\ell,n}(x)\mathscr{D}_{q^{-\ell}}\mathbb{U}_{n}^{(a)}(x;q)+\overline{\mathcal{T}}_{\ell,n}(x)\mathbb{U}_{n}^{(a)}(x;q)=0,\quad n\geq 0,\tag{38}$$

where

$$\mathbb{D}_{q^{\ell}}^{2}=\begin{cases}\mathscr{D}_{q}\mathscr{D}_{q^{-1}}, & \ell=-1,\\[2ex] \mathscr{D}_{q^{-1}}\mathscr{D}_{q}, & \ell=1,\end{cases}$$

and

$$\overline{\mathcal{R}}_{\ell,n}(x)=\mathcal{R}_{\ell,n}(q^{-\ell}x),\quad \overline{\mathcal{S}}_{\ell,n}(x)=\mathcal{S}_{\ell,n}(q^{-\ell}x)+(q^{-\ell}-1)x\mathcal{T}_{\ell,n}(q^{-\ell}x),\quad \overline{\mathcal{T}}_{\ell,n}(x)=\mathcal{T}_{\ell,n}(q^{-\ell}x).$$

Proof. Combining (4) with (34), and then using (3) we get

$$q^{\ell}\mathcal{R}_{\ell,n}(x)\mathbb{D}^2_{q^{-\ell}}(\mathbb{U}_n^{(a)})(q^{\ell}x;q)+\mathcal{S}_{\ell,n}(x)\mathscr{D}_{q^{-\ell}}\mathbb{U}_n^{(a)}(q^{\ell}x;q)+\mathcal{T}_{\ell,n}(x)\mathbb{U}_n^{(a)}(x;q)=0.$$

Next, replacing x by $q^{-\ell}x$ and using (6), yields

$$\mathcal{R}_{\ell,n}(q^{-\ell}x)\mathbb{D}^2_{q^{\ell}}\mathbb{U}_n^{(a)}(x;q)+\mathcal{S}_{\ell,n}(q^{-\ell}x)\mathscr{D}_{q^{-\ell}}\mathbb{U}_n^{(a)}(x;q)+\mathcal{T}_{\ell,n}(q^{-\ell}x)\mathbb{U}_n^{(a)}(q^{-\ell}x;q)=0,$$

which is (38). This completes the proof. □

6. Jacobi Fractions and Al-Salam–Carlitz I-Sobolev Type Polynomials

In this section, we state some fresh results relating the elements in the family of orthogonal polynomials $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ and Jacobi fractions. Basic concepts and the main properties related to continued fractions, in particular with Jacobi fractions, and orthogonal polynomials can be found in [2] (Chapter 3), for instance. In this section, we use our original result (30) to consider J-fractions in a more general sense than in the previous text.

Given two sequences of polynomials with complex coefficients $\{a_n(x)\}_{n\geq 1}$ and $\{b_n(x)\}_{n\geq 0}$, the J-fraction associated to the previous sequences is the formal expression

$$b_0(x)+\cfrac{a_1(x)}{b_1(x)+\cfrac{a_2(x)}{b_2(x)+\cfrac{a_3(x)}{b_3(x)+\cdots}}}. \tag{39}$$

For all $n\geq 0$, the n-th convergent associated to the previous J-fraction is given by

$$b_0(x)+\frac{a_1(x)\mid}{\mid b_1(x)}+\cdots+\frac{a_n(x)\mid}{\mid b_n(x)}:=b_0(x)+\cfrac{a_1(x)}{b_1(x)+\cfrac{a_2(x)}{b_2(x)+\cdots+\cfrac{a_n(x)}{b_n(x)}}}.$$

The formal continued fraction (39) is usually denoted by

$$b_0(x)+\frac{a_1(x)\mid}{\mid b_1(x)}+\cdots+\frac{a_n(x)\mid}{\mid b_n(x)}+\cdots$$

Proposition 7. *Let $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I-Sobolev type polynomials defined by (21). Then, $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ is the sequence of denominators of the sequence of n-th convergents of the J-fraction*

$$\hat{\beta}_{\ell,0}(x)+\frac{\hat{\gamma}_{\ell,1}(x)\mid}{\mid\hat{\beta}_{\ell,1}(x)}+\frac{\hat{\gamma}_{\ell,2}(x)\mid}{\mid\hat{\beta}_{\ell,2}(x)}+\cdots+\frac{\hat{\gamma}_{\ell,n}(x)\mid}{\mid\hat{\beta}_{\ell,n}(x)}+\cdots, \tag{40}$$

where $\hat{\beta}_{\ell,n}(x)=\beta_{\ell,n-1}/\alpha_{\ell,n-1}$ and $\hat{\gamma}_{\ell,n}(x)=\gamma_{\ell,n-1}/\alpha_{\ell,n-1}$ for all $n\geq 1$, and any fixed $\hat{\beta}_{0,n}(x)\in\mathbb{C}[x]$.

Proof. We recall from (33) that the sequence of Al-Salam–Carlitz I-Sobolev type polynomials defined by (21) satisfy the following recurrence formula:

$$\mathbb{U}_n^{(a)}(x;q)=\hat{\beta}_{\ell,n}(x)\mathbb{U}_{n-1}^{(a)}(x;q)+\hat{\gamma}_{\ell,n}(x)\mathbb{U}_{n-2}^{(a)}(x;q),\quad \mathbb{U}_{-1}^{(a)}(x;q)=0,\qquad \mathbb{U}_0^{(a)}(x;q)=1,\quad n\geq 1,$$

where $\hat{\beta}_{\ell,n}(x)$ and $\hat{\gamma}_{\ell,n}(x)$ are defined as in the statements of the result. We define the sequence of polynomials $\{\mathcal{N}_n^{(a)}(x)\}_{n\geq -2}$ by the shifted recursion

$$\mathcal{N}_n^{(a)}(x;q)=\hat{\beta}_{\ell,n}(x)\mathcal{N}_{n-1}^{(a)}(x;q)+\hat{\gamma}_{\ell,n}(x)\mathcal{N}_{n-2}^{(a)}(x;q),\quad \mathcal{N}_{-2}^{(a)}(x;q)=0,\quad \mathcal{N}_{-1}^{(a)}(x;q)=1,\quad n\geq 0.$$

Consequently, one has that
$$\mathcal{N}_0^{(a)}(x;q) = \hat{\beta}_{\ell,0}(x), \quad \mathbb{U}_0^{(a)}(x;q) = 1,$$
Also, for $n = 1$ we have
$$\mathcal{N}_1^{(a)}(x;q) = \hat{\beta}_{\ell,0}(x)\hat{\beta}_{\ell,1}(x) + \hat{\gamma}_{\ell,1}(x), \quad \mathbb{U}_1^{(a)}(x;q) = \hat{\beta}_{\ell,1}(x),$$
which entail that
$$\frac{\mathcal{N}_1^{(a)}(x;q)}{\mathbb{U}_1^{(a)}(x;q)} = \hat{\beta}_{\ell,0}(x) + \frac{\hat{\gamma}_{\ell,1}(x)}{\hat{\beta}_{\ell,1}(x)}.$$
Let $n = 2$. Then, it holds that
$$\mathcal{N}_2^{(a)}(x;q) = \hat{\beta}_{\ell,2}(x)\mathcal{N}_1^{(a)}(x;q) + \hat{\gamma}_{\ell,2}(x)\hat{\beta}_{\ell,2}(x)\mathcal{N}_0^{(a)}(x;q) = \hat{\beta}_{\ell,0}(x)[\hat{\beta}_{\ell,1}(x)\hat{\beta}_{\ell,2}(x) + \hat{\gamma}_{\ell,2}(x)] + \hat{\beta}_{\ell,2}(x)\hat{\gamma}_{\ell,1}(x),$$
and
$$\mathbb{U}_2^{(a)}(x;q) = \hat{\beta}_{\ell,2}(x)\mathbb{U}_1^{(a)}(x;q) + \hat{\gamma}_{\ell,2}(x)\mathbb{U}_0^{(a)}(x;q) = \hat{\beta}_{\ell,1}(x)\hat{\beta}_{\ell,2}(x) + \hat{\gamma}_{\ell,2}(x).$$
Therefore, we derive
$$\frac{\mathcal{N}_2^{(a)}(x;q)}{\mathbb{U}_2^{(a)}(x;q)} = \hat{\beta}_{\ell,0}(x) + \cfrac{\hat{\gamma}_{\ell,1}(x)}{\hat{\beta}_{\ell,1}(x) + \cfrac{\hat{\gamma}_{\ell,2}(x)}{\hat{\beta}_{\ell,2}(x)}}.$$
A recursion argument determines the n-th convergent of the J-fraction in (40), which coincides with $\frac{\mathcal{N}_n^{(a)}(x;q)}{\mathbb{U}_n^{(a)}(x;q)}$, and the proof can conclude. □

An analogous result can be stated relating the denominators of the convergents of a J-fraction and the sequence of Al-Salam–Carlitz I-Sobolev type polynomials.

Proposition 8. *Let $\{\mathbb{U}_n^{(a)}(x;q)\}_{n\geq 0}$ be the sequence of Al-Salam–Carlitz I-Sobolev type polynomials defined by (21). Then, the polynomial $\mathbb{U}_{n+1}^{(a)}(x;q)$ is the numerator of the n-th convergent of the J-fraction*
$$\tilde{\beta}_{\ell,0}(x) + \frac{\tilde{\gamma}_{\ell,1}(x) \mid}{\mid \tilde{\beta}_{\ell,1}(x)} + \frac{\tilde{\gamma}_{\ell,2}(x) \mid}{\mid \tilde{\beta}_{\ell,2}(x)} + \cdots + \frac{\tilde{\gamma}_{\ell,n}(x) \mid}{\mid \tilde{\beta}_{\ell,n}(x)} + \cdots,$$
for all $n \geq 0$. Here, $\tilde{\beta}_{\ell,n}(x) = \hat{\beta}_{\ell,n+1}(x) = \frac{\beta_{\ell,n}(x)}{\alpha_{\ell,n}(x)}$ and $\tilde{\gamma}_{\ell,n}(x) = \hat{\gamma}_{\ell,n+1}(x) = \frac{\gamma_{\ell,n}(x)}{\alpha_{\ell,n}(x)}$, for all $n \geq 0$.

Proof. The recursion (33) can be written in the form
$$\mathbb{U}_{n+1}^{(a)}(x;q) = \tilde{\beta}_{\ell,n}(x)\mathbb{U}_n^{(a)}(x;q) + \tilde{\gamma}_{\ell,n}(x)\mathbb{U}_{n-1}^{(a)}(x;q), \quad \mathbb{U}_{-1}^{(a)}(x;q) = 0, \qquad \mathbb{U}_0^{(a)}(x;q) = 1, \quad n \geq 0.$$
We define the sequence $\{\mathcal{M}_n^{(a)}\}_{n\geq -1}$ by
$$\mathcal{M}_n^{(a)}(x;q) = \tilde{\beta}_{\ell,n}(x)\mathcal{M}_{n-1}^{(a)}(x;q) + \tilde{\gamma}_{\ell,n}(x)\mathcal{M}_{n-2}^{(a)}(x;q), \quad \mathcal{M}_{-1}^{(a)}(x;q) = 0, \quad \mathcal{M}_0^{(a)}(x;q) = 1, \quad n \geq 1.$$
Consequently, for $n = 0$ we have
$$\mathbb{U}_1^{(a)}(x;q) = \tilde{\beta}_{\ell,0}(x) \quad \mathcal{M}_0^{(a)}(x;q) = 1,$$
whereas for $n = 1$ we have
$$\mathbb{U}_2^{(a)}(x;q) = \tilde{\beta}_{\ell,0}(x)\tilde{\beta}_{\ell,1}(x) + \tilde{\gamma}_{\ell,1}(x), \quad \mathcal{M}_1^{(a)}(x;q) = \tilde{\beta}_{\ell,1}(x).$$

Therefore, one can write the quotient

$$\frac{\mathbb{U}_2^{(a)}(x;q)}{\mathcal{M}_1^{(a)}(x;q)} = \tilde{\beta}_{\ell,0}(x) + \frac{\tilde{\gamma}_{\ell,1}(x)}{\tilde{\beta}_{\ell,1}(x)}.$$

For $n = 2$ we have

$$\mathbb{U}_3^{(a)}(x;q) = \tilde{\beta}_{\ell,0}(x)[\tilde{\beta}_{\ell,1}(x)\tilde{\beta}_{\ell,2}(x) + \tilde{\gamma}_{\ell,2}(x)] + \tilde{\beta}_{\ell,2}(x)\tilde{\gamma}_{\ell,1}(x), \quad \mathcal{M}_2^{(a)}(x;q) = \tilde{\beta}_{\ell,1}(x)\tilde{\beta}_{\ell,2}(x) + \tilde{\gamma}_{\ell,2}(x).$$

Thus,

$$\frac{\mathbb{U}_3^{(a)}(x;q)}{\mathcal{M}_2^{(a)}(x;q)} = \tilde{\beta}_{\ell,0}(x) + \frac{\tilde{\gamma}_{\ell,1}(x)}{\tilde{\beta}_{\ell,1}(x) + \frac{\tilde{\gamma}_{\ell,2}(x)}{\tilde{\beta}_{\ell,2}(x)}}.$$

A recursion argument allows to conclude the result. □

Following a similar argument, the next result holds.

Corollary 1. *Let $n \geq 1$. Then, for $\ell = -1, 1$ one has*

$$\frac{\mathbb{U}_{n+1}^{(a)}(x;q)}{\mathbb{U}_n^{(a)}(x;q)} = \sum_{i=1}^{n} \tilde{\gamma}_{\ell,i}(x) \prod_{h=i+1}^{n} \tilde{\beta}_{\ell,h}(x),$$

with the last term in the previous sum being reduced to $\tilde{\gamma}_{\ell,n}$.

Proof. It can be derived from the fact that $\omega_n = \frac{\mathbb{U}_{n+1}^{(a)}(x;q)}{\mathbb{U}_n^{(a)}(x;q)}$ satisfies that $\omega_{n+1} = \tilde{\beta}_{\ell,n+1}(x) + \tilde{\gamma}_{\ell,n+1}(x)/\omega_n$, for all $n \geq 0$, and a recursion argument. □

7. Examples and Further Comments

In this section, we illustrate the theory with some explicit first elements in $\{\mathbb{U}_n^{(a)}(x;q,j)\}_{n\geq 0}$. Let $j = 2$. We have

$$\mathbb{U}_0^{(a)}(x;q,2) = 1, \quad \mathbb{U}_1^{(a)}(x;q,2) = x - a - 1,$$

$$\mathbb{U}_2^{(a)}(x;q,2) = x^2 + (-aq - a - q - 1)x + a^2q + aq + a + q,$$

$$\mathbb{U}_3^{(a)}(x;q,2) = x^3 + a_2x^2 + a_1x + a_0,$$

where

$$a_2 = -\frac{Z_qa^3q^4\,(q;q)_2 + Z_qa^2q^4\,(q;q)_2 - Z_qa^3q\,(q;q)_2 - a\lambda\,q^4}{Z_qa^2q^2\,(q;q)_2 - Z_qa^2q\,(q;q)_2 - \lambda\,q^2 - \mu\,q^2 - 2\,\lambda\,q - 2\,\mu\,q - \lambda - \mu}$$
$$- \frac{-Z_qa^2q\,(q;q)_2 - 3\,a\lambda\,q^3 - \mu\,q^4 - 4\,a\lambda\,q^2 - 3\,\mu\,q^3 - 3\,a\lambda\,q - 4\,\mu\,q^2 - a\lambda - 3\,\mu\,q - \mu}{Z_qa^2q^2\,(q;q)_2 - Z_qa^2q\,(q;q)_2 - \lambda\,q^2 - \mu\,q^2 - 2\,\lambda\,q - 2\,\mu\,q - \lambda - \mu},$$

$$a_1 = \frac{Z_q a^4 q^5 (q;q)_2 + Z_q a^3 q^5 (q;q)_2 + Z_q a^3 q^4 (q;q)_2 + Z_q a^2 q^5 (q;q)_2 - Z_q a^4 q^2 (q;q)_2 - a^2 \lambda q^5}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu}$$
$$\frac{-Z_q a^3 q^2 (q;q)_2 - 3 a^2 \lambda q^4 + a^2 \mu q^4 - Z_q a^3 q (q;q)_2 - Z_q a^2 q^2 (q;q)_2 - 4 a^2 \lambda q^3 + 3 a^2 \mu q^3 - \mu q^5}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu}$$
$$\frac{-3 a^2 \lambda q^2 + 4 a^2 \mu q^2 + \lambda q^4 - 3\mu q^4 - a^2 \lambda q + 3 a^2 \mu q + 3\lambda q^3 - 4\mu q^3 + a^2 \mu + 4\lambda q^2}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu}$$
$$\frac{-3\mu q^2 + 3\lambda q - \mu q + \lambda}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu},$$

and

$$a_0 = -\frac{Z_q a^5 q^5 (q;q)_2 - Z_q a^5 q^4 (q;q)_2 + Z_q a^4 q^5 (q;q)_2 + Z_q a^3 q^5 (q;q)_2 + Z_q a^2 q^5 (q;q)_2 - a^3 \lambda q^5}{Z a^2 q^2 (q;q)_2 - Z_q a^2 q (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu}$$
$$- \frac{-Z_q a^4 q^2 (q;q)_2 - Z_q a^2 q^4 (q;q)_2 - 2 a^3 \lambda q^4 + a^3 \mu q^4 - Z_q a^3 q^2 (q;q)_2 - a^3 \lambda q^3 + 3 a^3 \mu q^3}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 q (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu}$$
$$- \frac{+a^2 \mu q^4 + 3 a^3 \mu q^2 + 3 a^2 \mu q^3 + a\lambda q^4 - \mu q^5 + a^3 \mu q + 4 a^2 \mu q^2 + 3 a\lambda q^3 + \lambda q^4}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 q (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu}$$
$$- \frac{-2\mu q^4 + 3 a^2 \mu q + 4 a\lambda q^2 + 3\lambda q^3 - \mu q^3 + a^2 \mu + 3 a\lambda q + 3\lambda q^2 + a\lambda + \lambda q}{Z_q a^2 q^2 (q;q)_2 - Z_q a^2 q (q;q)_2 - \lambda q^2 - \mu q^2 - 2\lambda q - 2\mu q - \lambda - \mu},$$

with $Z_q = \left(q, a, a^{-1}q; q\right)_\infty$.

Let $j = 3$. We have

$$\mathbb{U}_0^{(a)}(x;q,3) = 1, \quad \mathbb{U}_1^{(a)}(x;q,2) = x - a - 1,$$
$$\mathbb{U}_2^{(a)}(x;q,3) = x^2 + (-aq - a - q - 1)x + a^2 q + aq + a + q,$$

$$\mathbb{U}_3^{(a)}(x;q,3) = x^3 + (-aq^2 - aq - q^2 - a - q - 1)x^2 + (a^2 q^3 + a^2 q^2 + aq^3 + a^2 q + 2aq^2$$
$$q^3 + 2aq + q^2 + a + q)x - a^3 q^3 - a^2 q^3 - a^2 q^2 - aq^3 - a^2 q - aq^2 - q^3 - aq,$$

We observe that in the case of $j = 2, 3$, it holds that the nature of the polynomials is much simpler for $j > n$. This is due to the definition of the polynomials. Indeed, observe that

$$\mathcal{D}_q^j x^n(x;q) \equiv 0, \qquad \text{for } j > n.$$

This can be proved directly from the definition of the differential operator $\mathcal{D}_q$ applied on any monomial. Therefore, given $j \geq 1$, one has that $\mathbb{U}_n^{(a)}(x;q,j)$ coincides with $U_n^{(a)}(x;q)$ for all $0 \leq n < j$. The values of λ, μ are irrelevant for these first polynomials in the sequence. This phenomenon can also be observed in different points through the work. More precisely, the representation (16) of $\mathbb{U}_n^{(a)}(x;q)$ in terms of $U_n^{(a)}(x;q)$ is made in terms of the quantities $\Delta_{j,n}^{(i)}(a)$, for $i = 1, 2$. The definition of these two elements is made in terms of the determinant of a matrix with a null column for $j > n$. Therefore, Formula (16) states the coincidence of Al-Salam–Carlitz I polynomials and the Sobolev type polynomials. This property is also directly observed at the Fourier coefficients $a_{n,k}$.

We also remark that the choice of $\lambda = \mu = 0$ provides $\mathbb{U}_n^{(a)}(x;q,j) = U_n^{(a)}(x;q)$ for every n. We observe this is the case for the polynomial $\mathbb{U}_3^{(a)}(x;q,2)$, which is given by

$$\mathbb{U}_3^{(a)}(x;q,2) = x^3 + (-aq^2 - aq - q^2 - a - q - 1)x^2 + (a^2 q^3 + a^2 q^2 + aq^3$$
$$+ a^2 q + 2\,aq^2 + q^3 + 2\,aq + q^2 + a + q)x - a^3 q^3 - a^2 q^3 - a^2 q^2 - aq^3 - a^2 q - aq^2 - q^3 - aq,$$

after evaluation at $\lambda = \mu = 0$. Figure 1 illustrates some of these polynomials.

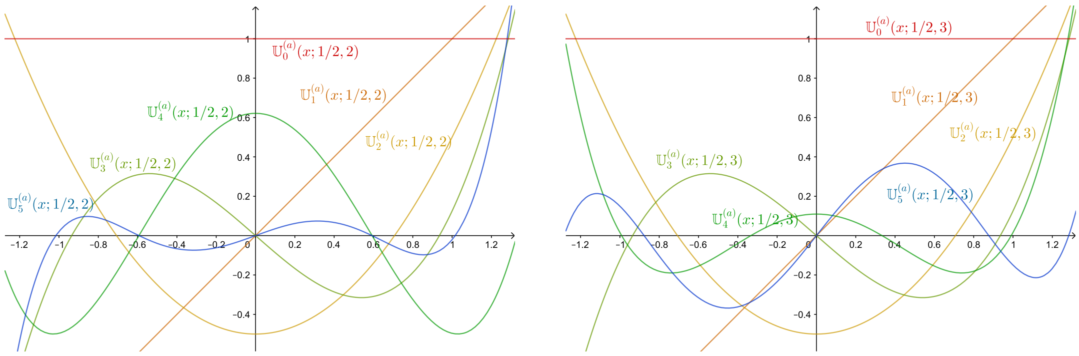

Figure 1. $\mathbb{U}_n^{(-1)}(x;\frac{1}{2},j)$ for $j=2$ (**left**) and $j=3$ (**right**) for $n=0,1,\ldots,5$.

8. Conclusions and Open Problems

Using well known techniques, we provide up to four different versions of the second order q-difference equations satisfied by the monic polynomials $\{\mathbb{U}_n^{(a)}(x;q,j)\}_{n\geq 0}$, orthogonal with respect to a Sobolev-type inner product associated to the Al-Salam–Carlitz I orthogonal polynomials. The studied inner product involves an arbitrary number of q-derivatives, evaluated on the two boundaries of the orthogonality interval of the Al-Salam–Carlitz I orthogonal polynomials. We gave two representations for $\mathbb{U}_n^{(a)}(x;q,j)$, one as a linear combination of two consecutive Al-Salam–Carlitz I orthogonal polynomials, and other as a ${}_3\phi_2$ series. We state not only as usual, but two different versions of structure relations, which lead to the corresponding ladder operators, which help us to find up to four different versions of the second order linear q-difference equation satisfied by $\mathbb{U}_n^{(a)}(x;q,j)$. Finally, as a truly original contribution to the literature, we obtained a three term recurrence formula with rational coefficients satisfied by $\mathbb{U}_n^{(a)}(x;q,j)$, which is the key point to establish an appealing generalization of the so-called J-fractions to the framework of Sobolev-type orthogonality. As problems to be addressed in a future contribution, we consider to analyze the effect of having two mass points, each one on a different side of the bounded orthogonality interval, in the parity of the corresponding Sobolev-type orthogonal sequence. We also wish to carry out an in-depth analysis on the zero behavior of these polynomials, as well as to study several of their asymptotic properties.

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